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A comparison of approximate response functions in structural reliability analysis

Christian Bucher[a,*,1], Thomas Most[b]

[a] Center of Mechanics and Structural Dynamics, Vienna University of Technology, Austria
[b] Institute of Structural Mechanics, Bauhaus-University Weimar, Germany

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Abstract

In order to reduce computational costs in structural reliability analysis, it has been suggested to utilize approximate response functions for reliability assessment. One well-established class of methods to deal with suitable approximations is the *Response Surface Method*. The basic idea in utilizing the response surface method is to replace the true limit state function by an approximation, the so-called response surface, whose function values can be computed more easily. The functions are typically chosen to be first- or second-order polynomials. Higher-order polynomials on the one hand tend to show severe oscillations, and on the other hand they require too many support points. This may be overcome by applying smoothing techniques such as the moving least-squares method. An alternative approach is given by *Artificial Neural Networks*. In this approach, the input and output parameters are related by means of relatively simple yet flexible functions, such as linear, step, or sigmoid functions which are combined by adjustable weights. The main feature of this approach lies in the possibility of adapting the input–output relations very efficiently. A further possibility lies in the utilization of *radial basis functions*. This method also allows for a flexible adjustment of the interpolation scheme. In all approaches as presented it is essential to achieve high quality of approximation primarily in the region of the random variable space which contributes most significantly to the probability of failure. The paper presents an overview of these approximation methods and demonstrates their potential by application to several examples of nonlinear structural analysis.

Keywords: Response surface method; Structural reliability; Artificial neural network; Moving least squares; Radial basis functions

1. Introduction

Structural reliability assessment requires the estimation of probabilities of failure, which, in general, are of rather small magnitude. Moreover, structural failure is most typically assessed by means of nonlinear, possibly time-variant analyses of complex structural models. In such cases the computational cost incurred for one single analysis – to decide whether or not a structure is safe – may become quite demanding. Consequently, the application of direct (or even advanced) Monte Carlo simulation – being the most versatile solution technique available – is quite often not feasible. It has been suggested to utilize the response surface method for structural reliability assessment [1–3] in order to reduce computational costs.

Let us assume that the reliability assessment problem under consideration is governed by a vector $\mathbf{X}$ of n basic random variables X_i ($i = 1, 2, \ldots, n$), i.e.,

$$\mathbf{X} = (X_1, X_2, \ldots, X_n)' \tag{1}$$

where $(\cdot)'$ is transpose. Assuming furthermore that the random variables $\mathbf{X}$ have a joint probability density function $f(\mathbf{x})$, the probability of failure $P(\mathcal{F})$ – i.e., the probability that a limit state will be reached – is defined by

$$P(\mathcal{F}) = \int \cdots \int_{g(\mathbf{x})\le 0} f(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{2}$$

whereby $g(\mathbf{x})$ is the limit state function which divides the n-dimensional probability space into a failure domain $\mathcal{F} = \{\mathbf{x} : g(\mathbf{x}) \le 0\}$ and a safe domain $\mathcal{S} = \{\mathbf{x} : g(\mathbf{x}) > 0\}$.

* Corresponding author.
E-mail address: christian.bucher@tuwien.ac.at (C. Bucher).
[1] Formerly: Institute of Structural Mechanics, Bauhaus-University Weimar, Germany.

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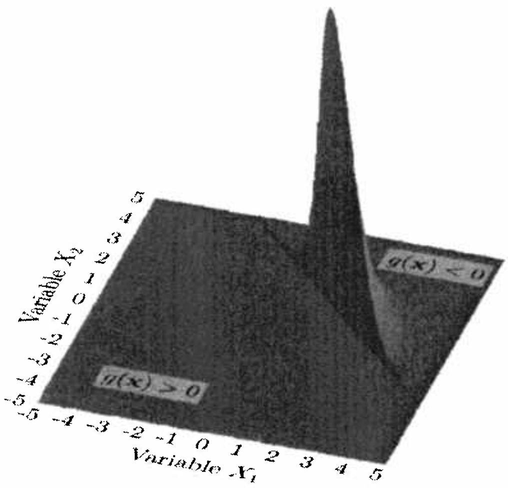

Fig. 1. Integrand for calculating the probability of failure for $g(x_1, x_2) = 3 - x_1 - x_2$.

As already mentioned above, the computational challenge in determining the integral of Eq. (2) lies in evaluating the limit state function $g(\mathbf{x})$, which for nonlinear systems usually requires an incremental/iterative numerical approach. The basic idea in utilizing the response surface method is to replace the true limit state function $g(\mathbf{x})$ by an approximation $\eta(\mathbf{x})$, the so-called response surface, whose function values can be computed more easily.

In this context, it is essential to realize that the limit state function $g(\mathbf{x})$ serves the sole purpose of defining the bounds of integration in Eq. (2). As such, it is quite important that the function $\eta(\mathbf{x})$ approximates this boundary sufficiently well, in particular in the region which contributes most to the failure probability $P(\mathcal{F})$. As an example, consider a two-dimensional problem with standard normal random variables X_1 and X_2, and a limit state function $g(x_1, x_2) = 3 - x_1 + x_2$. In Fig. 1 the integrand of Eq. (2) in the failure domain is displayed. It is clearly visible, that only a very narrow region around the so-called design point $\mathbf{x}^*$ really contributes to the value of the integral, i.e., the probability of failure $P(\mathcal{F})$. Even a relatively small deviation of the response surface $\eta(\mathbf{x})$ from the true limit state function $g(\mathbf{x})$ in this region may therefore lead to significantly erroneous estimates of the probability of failure. In order to avoid this type of error, it must be ensured that the important region is sufficiently well covered when designing the response surface.

The response surface method has been a topic of extensive research in many different application areas since the influential paper by Box and Wilson in 1951 [4]. Whereas in the formative years the general interest was on experimental designs for polynomial models (see, e.g., [4,5]), in the following years nonlinear models, optimal design plans, robust designs and multi-response experiments – to name just a few – came into focus. A fairly complete review on the existing techniques and the research directions of the response surface methodology can be found in [6–9]. However, traditionally the application area of the response surface method is not structural engineering, but, e.g., chemical or industrial engineering. Consequently, the above-mentioned special requirement for structural reliability analysis – i.e., the high degree of accuracy required in a very narrow region – is usually not reflected upon in the standard literature on the response surface method [10–12].

One of the earliest suggestions to utilize the response surface method for structural reliability assessment was made in [1]. Therein, Lagrangian interpolation surfaces and second-order polynomials are rated as useful response surfaces. Moreover, the importance of reducing the number of basic variables and error checking is emphasized. Support points for estimating the parameters of the response surface are determined by spherical design. In [13] first-order polynomials with interaction terms are utilized as response surfaces to analyse the reliability of soil slopes. The design plan for the support points is saturated—either by full or by fractional factorial design. Another analysis with a saturated design scheme is given in [2], where quadratic polynomials without interaction terms are utilized to solve problems from structural engineering. Polynomials of different order in combination with regression analysis are proposed in [14], whereby fractional factorial designs are utilized to obtain a sufficient number of support points. The validation of the chosen response surface model is done by means of analysis of variance.

In [15] it has been pointed out that for reliability analysis it is most important to obtain support points for the response surface very close to or exactly at the limit state $g(\mathbf{x}) = 0$. This finding has been further extended in [16,17]. In [18] the response surface concept has been applied to problems involving random fields and nonlinear structural dynamics. Besides polynomials of different order, piecewise continuous functions such as hyperplanes or simplices can also be utilized as response surface models. For the class of reliability problems defined by a convex safe domain, secantial hyperplane approximations such as presented by [19,20] yield conservative estimates for the probability of failure. Several numerical studies indicate, however, that in these cases the interpolation method converges slowly from above to the exact result with increasing number of support points. The effort required for this approach is thereby comparable to Monte Carlo simulation based on directional sampling [21].

It should be emphasized that in the following the discussion will be focused on the mathematical representation of the limit state functions. An alternative way is the choice of approximate and typically simple *physical* models representing the behavior of complex structures near failure. Such a method is the so-called Model Correction Factor Method as developed by Ditlevsen and his co-workers [22,23]. This method is particularly attractive if the collapse mechanisms of a complex structure can be well approximated by simple mechanical models (e.g. plastic hinge theory).

2. Response surface models

2.1. Basic formulation

Response surface models as discussed here are more or less simple mathematical models which are designed to describe the possible experimental outcome (e.g., the structural response in terms of displacements, stresses, etc.) of a more or less complex structural system as a function of quantitatively variable factors (e.g., loads or system conditions), which can be controlled by an experimenter. Obviously, the chosen response surface model should give the best possible fit to any collected data. In general, we can distinguish two different types of response surface models: regression models (e.g., polynomials of varying degree or nonlinear functions such as exponentials) and interpolation models (e.g., polyhedra).

Let us denote the response of any structural system to a vector $\mathbf{x}$ of n experimental factors or input variables x_i $(i = 1, 2, \ldots, n)$, i.e., $\mathbf{x} = (x_1, x_2, \ldots, x_n)'$, by $z(\mathbf{x})$. In most applications it is quite likely that the *exact* response function will not be known. Therefore, it has to be replaced by a flexible function $q(\cdot)$ which will express satisfactorily the relation between the response z and the input variables $\mathbf{x}$. Taking into account a (random) error term ε, then the response can be written over the region of experimentation as

$$z = q(\theta_1, \theta_2, \ldots, \theta_p; x_1, x_2, \ldots, x_n) + \varepsilon \tag{3}$$

whereby θ_j $(j = 1, 2, \ldots, p)$ are the parameters of the approximating function $q(\cdot)$. Taking now expectations, i.e.,

$$\eta = \mathrm{E}[z] \tag{4}$$

then the surface represented by

$$\begin{aligned} \eta &= q(\theta_1, \theta_2, \ldots, \theta_p; x_1, x_2, \ldots, x_n) \\ &= q(\boldsymbol{\theta}; \mathbf{x}) \end{aligned} \tag{5}$$

is called a *response surface*. The vector of parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)'$ has to be estimated from the experimental data in such a way that Eq. (4) is fulfilled. In the following we will investigate the most common response surface models and the methods to estimate their respective parameters.

2.2. Local interpolation models

Besides the above-mentioned regression models, there are also interpolation models available for describing response surfaces. A special type of such models are polyhedra, i.e., an assemblage of continuous functions. The given rationale for utilizing such models is that they are more flexible to local approximations and should, therefore, indeed converge in the long run to the exact limit state function [19,20]. This flexibility is an important feature in reliability assessment, where the estimated reliability measure depends quite considerably on a sufficiently accurate representation of the true limit state function in the vicinity of the design point. Therefore, these models utilize only points on the limit state surface $g(\mathbf{x}) = 0$, which separates the failure domain from the safe domain.

A useful Shepard-type [24] interpolation strategy for reliability problems with convex limit state functions has been described by [25]. This involves an approximation of the distance r from the mean value to the limit state surface $g(\mathbf{x}) = 0$ in terms of the distances R_i $(i = 1 \ldots m)$ of given support points $\mathbf{x}_i$ on the surface. It appears to be helpful to carry out the interpolation in non-dimensional space (e.g. standard Gaussian space $\mathbf{u}$) (cf. Fig. 2).

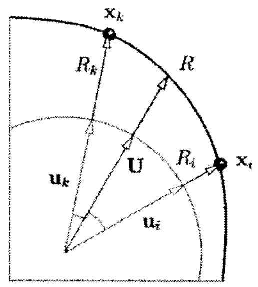

Fig. 2. Limit state surface.

All Shepard-type models allow, by including additional points on the limit state surface, an adaptive refinement of the approximating response surface and, consequently, an improvement of the estimator of the failure probability. However, this refinement is often seriously restricted by an excessive need for additional points on the limit state surface.

2.3. Moving least-squares regression

Based on the Shepard scheme the Moving Least-Squares (MLS) approach was introduced by [26]. In the MLS method the constant basis, which is used in the Shepard scheme, can be replaced by any polynomial or other basis function. By doing so this method obtains the best local fit of the basis function for the actual interpolation point. Furthermore the MLS method can represent the basis function exactly.

If we do polynomial interpolation an arbitrary function u is interpolated at a point $\mathbf{x}$ as

$$\begin{aligned} u^h(\mathbf{x}) &= \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & \cdots \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \\ &= \mathbf{p}'(\mathbf{x})\mathbf{a} \end{aligned} \tag{6}$$

where $\mathbf{p(x)}$ is the base vector and $\mathbf{a}$ contains the coefficients of the polynomial. These coefficients are constant in the interpolation domain and can be determined directly if the number of supporting points m used for the interpolation is equivalent to the number of coefficients n. Within the Moving Least-Squares method the number of supporting points m exceeds the number of coefficients n, which leads to an overdetermined system of equations. This kind of optimization problem can be solved by using a least-squares approach

$$\mathbf{P}\tilde{\mathbf{u}} = \mathbf{P}\mathbf{P}'\mathbf{a}(\mathbf{x}) \tag{7}$$

with changing ("moving") coefficients $\mathbf{a}(\mathbf{x})$. In order to obtain a local regression model in the MLS method distance depending weighting functions $w = w(s)$ have been introduced, where s is the standardized distance between the interpolation point and the considered supporting point

$$s_i = \frac{\|\mathbf{x} - \mathbf{x}_i\|}{D} \tag{8}$$

and D is the influence radius, which is defined as a numerical parameter. All types of functions can be used as weighting function $w(s)$ which have their maximum in $s = 0$ and vanish outside the influence domain specified by $s = 1$.

The final approximation scheme reads

$$u^h(\mathbf{x}) = \mathbf{\Phi}^{MLS}(\mathbf{x})\tilde{\mathbf{u}}; \quad \mathbf{\Phi}^{MLS}(\mathbf{x}) = \mathbf{p}'(\mathbf{x})\mathbf{A}(\mathbf{x})^{-1}\mathbf{B}(\mathbf{x}). \tag{9}$$

with

$$\mathbf{A}(\mathbf{x}) = \mathbf{PW}(\mathbf{x})\mathbf{P}', \quad \mathbf{B}(\mathbf{x}) = \mathbf{PW}(\mathbf{x}). \tag{10}$$

Due to the approximative character of the classical MLS method the support point values cannot be represented exactly, which is a positive aspect only if we have noisy input data. But for the application in the framework of a reliability analysis this property may lead to significant errors in the prediction of the failure probability. Therefore in [27] a new weighting function was presented which enables the fulfillment of the MLS interpolation condition with very high accuracy

$$\Phi_i^{MLS}(\mathbf{x}_j) \approx \delta_{ij}. \tag{11}$$

The weighting function value of a node i at an interpolation point $\mathbf{x}$ is introduced by the following regularized formulation

$$w_R(s_i) = \frac{\tilde{w}_R(s_i)}{\sum_{j=1}^{m} \tilde{w}_R(s_j)} \tag{12}$$

with

$$\tilde{w}_R(s) = (s^2 + \epsilon)^{-2}; \quad \epsilon \ll 1. \tag{13}$$

The regularization parameter ϵ has to be chosen small enough to fulfill Eq. (11) with high accuracy, but large enough to obtain a regular, differentiable function at $s = 0$ within machine precision. This approach allows a very good representation of given support points which is crucial for the accuracy of reliability estimates.

2.4. Artificial neural networks

Recently, approximations based on the concept of artificial neural networks are being introduced into reliability analysis (see, e.g., [28]). The basic network includes nodes and connections which link the nodes. Each link is associated with a weight property which is the principal mechanism how a network stores information. Before a neural network can approximate complex coherences it has to be trained for the specific problem by adjusting these weights. The most widely used network type for approximation problems is the multi-layer perceptron which is also called feed-forward back-propagation network. This network type is used in this study and is shown in Fig. 3.

The network consists of an input layer, several hidden layers and an output layer and all nodes which are called neurons of one layer are connected with each neuron of the previous layer.

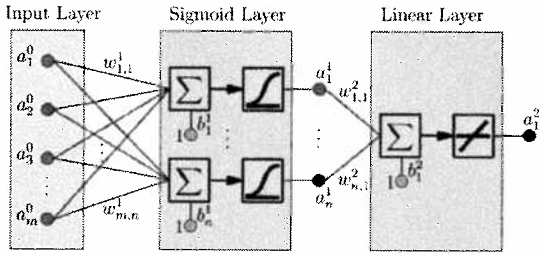

Fig. 3. Neural network with feed-forward architecture and one hidden layer.

This connection is feed-forward and no backward connection is allowed as in recurrent networks. At the neuron level a bias is added to the weighted sum of the inputs and the neuron transfer function is applied, which can be of linear or nonlinear type. The output of a single neuron reads

$$a_i^j = f_i^j(x) = f\left(\sum_{k=1}^{m} w_{k,i}^j a_k^{j-1} + b_i^j\right) \tag{14}$$

where m is the number of input impulses, i is the number of the current neuron in the layer j and $w_{k,i}^j$ is the synaptic weight factor for the connection of the neuron i, j with the neuron $k, j-1$. For the approximation of functions with minor discontinuities generally a combination of layers with sigmoid transfer functions $f(x) = 1/(1 + e^{-x})$ and a linear output layer with $f(x) = x$ are used. Other transfer types are for example hard limit and sign function, which can represent strong discontinuities.

A very important point for a sufficient network approximation is the design of the network architecture. Depending on the number of available training samples the number of neurons in the hidden layers has to be chosen in that way, that the so-called over-fitting is avoided. This phenomenon occurs, if the number of hidden nodes is too large for the number of training samples. Then the network can converge easier and fits well for the training data but it cannot generalize well for other data. In [29] it is mentioned that the number of training samples m should be larger than the number of adjustable parameters

$$(n+2)M + 1 < m \tag{15}$$

where n is the number of input values and M is the number of hidden neurons for a network with single hidden layer.

2.5. Radial basis functions

Radial basis function interpolations [30,31] construct interpolating functions of the form

$$u^h(\mathbf{x}) = \sum_{j=1}^{N} a_j \varphi(\|\mathbf{x} - \mathbf{x}_j\|) \tag{16}$$

in which $\|.\|$ denotes the Euclidian norm of the vector argument and $\varphi(.)$ are usually taken to be simple functions of their arguments, e.g. linear: $\varphi(r) = cr$, "thin plate spline": $\varphi(r) = cr^2 \log r$ or Gaussian: $\varphi(r) = \exp(-cr^2)$ [30]. In the following applications, the Gaussian radial basis function is used. The vectors $\mathbf{x}_j$ denote the support points and the coefficients a_j have to be adjusted such as to fulfill the interpolation condition.

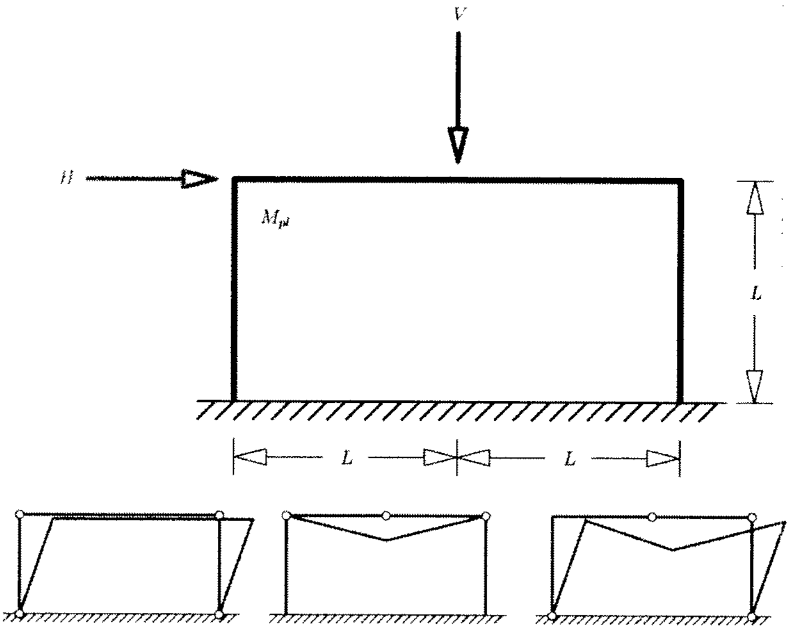

Fig. 4. Simple frame with collapse mechanisms.

Table 1
Calculated mean and maximum relative errors in failure probability from 50 simulation runs

	Number of sampling points				
	10	20	50	100	200
Directional sampling					
Mean error $\bar{\epsilon}$	0.5578	0.4332	0.2457	0.1935	0.1318
Max. error ϵ_{max}	1.6984	1.3174	0.6856	0.9506	0.5614
Polynomial regression					
Mean error $\bar{\epsilon}$	0.3438	0.1982	0.1461	0.1458	0.1480
Max. error ϵ_{max}	0.9556	0.6267	0.3368	0.3222	0.2833
MLS interpolation					
Mean error $\bar{\epsilon}$	0.3117	0.1008	0.0300	0.0100	0.0027
Max. error ϵ_{max}	0.8348	0.7185	0.1080	0.0467	0.0106
RBF interpolation					
Mean error $\bar{\epsilon}$	29.641	0.2383	0.0350	0.0126	0.0071
Max. error ϵ_{max}	1254.8	2.3927	0.1877	0.0640	0.0252
ANN approximation					
Number of neurons	2	2	5	5	10
Mean error $\bar{\epsilon}$	164.77	0.2558	0.0636	0.0125	0.0028
Max. error ϵ_{max}	8191.9	1.3033	0.8205	0.0405	0.0134

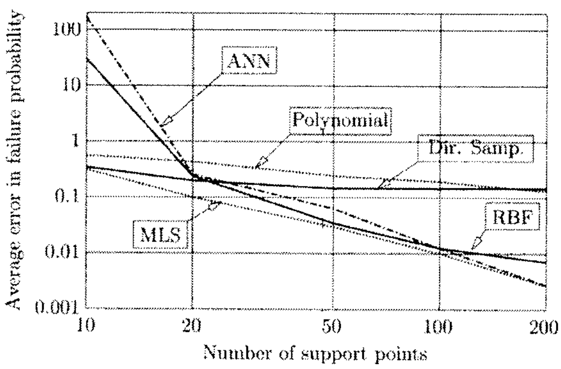

Fig. 5. Calculated mean relative errors in failure probability from 50 simulation runs.

3. Application examples

3.1. One-bay one-storey frame

This is a simple analytical example representing the failure of a one-bay one-storey frame. The failure is assumed to be described by first-order rigid-plastic hinge theory. The plastic moment of the cross-sections is assumed to be equal to M_{pl}. Due to the presence of horizontal and vertical loads H and V (cf. Fig. 4) there are three relevant collapse mechanisms. The loads are assumed to be normally distributed with mean values $\bar{H} = \bar{V} = 1.7\frac{M_{pl}}{L}$. Their standard deviations are $\sigma_H = \sigma_V = 0.5\frac{M_{pl}}{L}$. Using directional sampling with 1000 samples, the failure probability is determined as $P(\mathcal{F}) = 1.208 \times 10^{-4}$ corresponding to a value of the safety index $\beta = 3.67$. This value is used as a reference for comparison to the results based on various response surface approaches. These results are given in Table 1, whereby as an error measure the relative error in the estimated failure probability $\epsilon = |\tilde{P}(\mathcal{F}) - P(\mathcal{F})|/P(\mathcal{F})$ is used. The application of approximations to the limit state function using different approaches is compared in terms of choice of method and number of support points as shown in Fig. 5. Comparing the results for $P(\mathcal{F})$ it is seen that although there are errors of quite different magnitude, the accuracy as typically desired in reliability analysis is quite comparable for a given number of support points larger than 50, i.e. for a specific moderate level of numerical effort. Although the geometrical representation of the limit state function differs somewhat (cf. Fig. 6), the computed safety index is not severely affected by this difference.

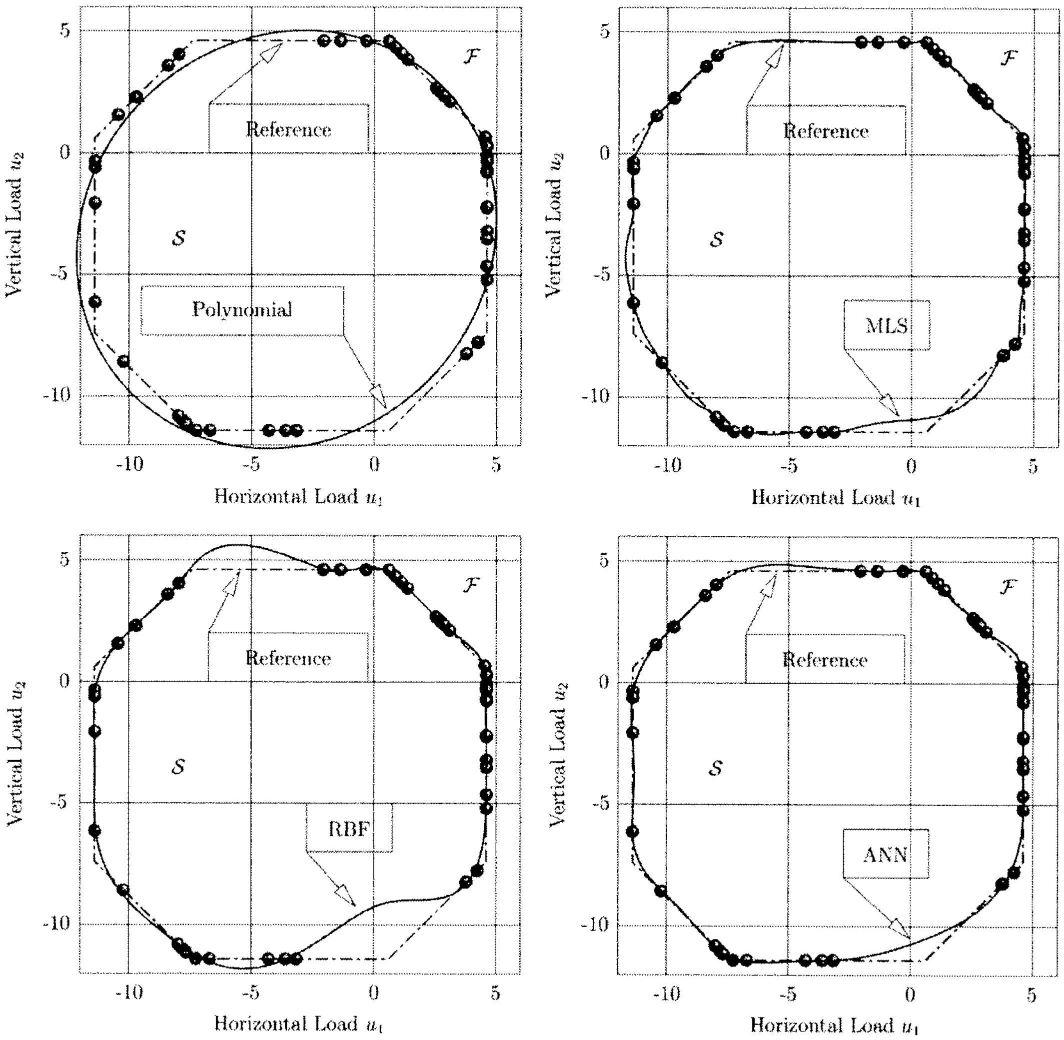

Fig. 6. Approximated limit state of the simple frame structure for the investigated approximation schemes using 50 support points ($\mathcal{F}$ denotes failure domain, $\mathcal{S}$ is safe domain).

3.2. Application to nonlinear finite element structure

A simple three-dimensional steel frame subjected to three random loadings is considered as shown in Fig. 7.

The three-dimensional frame is modeled by 24 physically nonlinear beam elements (linear elastic-ideally plastic material law, elasticity modulus $E = 2.1 \times 10^{11}$ N/m^2, yield stress $\sigma_Y = 2.4 \times 10^8$ N/m^2). For numerical stabilization, a post-yielding stiffness with a magnitude equal to 10^{-5}-times the initial value is chosen. The cross-section for the girder is a box (width 0.2 m, height 0.15 m, wall thickness 0.005 m) and the columns are I-sections (flange width 0.2 m, web height 0.2 m, thickness 0.005 m). The columns are fully clamped at the supports. The static loads acting on the system p_z, F_x, and F_y are assumed to be Gaussian random variables (Mean values: 12.0 kN/m, 30.0 kN, 40.0 kN and standard deviations: 1.2 kN/m, 2.4 kN, 4.8 kN).

The failure condition is given by total or partial collapse of the structure. Numerically, this is checked either by tracking the smallest eigenvalue of the global tangent stiffness matrix (it becomes zero at the collapse load) or by failure in the global Newton iteration indicating loss of equilibrium. Since this type of collapse analysis is typically based on a discontinuous function (convergence vs. non-convergence), it is imperative that the support points for the response surface be located exactly at the limit state. A bisection procedure is utilized to determine collapse loads with high precision (to the accuracy of 1% of the respective standard deviation). It should be mentioned that this type of problem can be conveniently dealt with using the *support vector machine* concept. e.g. [32].

The geometry of the limit state separating the safe from the failure domain is shown in Fig. 8. The limit points were obtained from directional sampling using 20 000 samples. The color of the dots indicates the distance from the mean values expressed in multiples of the standard deviation. It can be easily seen that there is considerable interaction between the random variables F_y and p_z at certain levels. Near the region of most importance for the probability of failure (this is where most of the points from the directional sampling are located) is essentially flat, and mainly governed by the value of F_y.

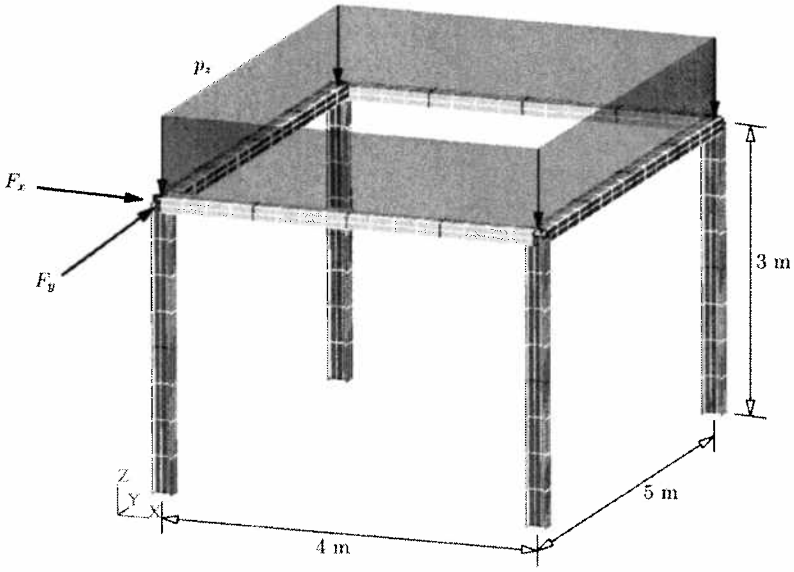

Fig. 7. Three-dimensional steel frame structure.

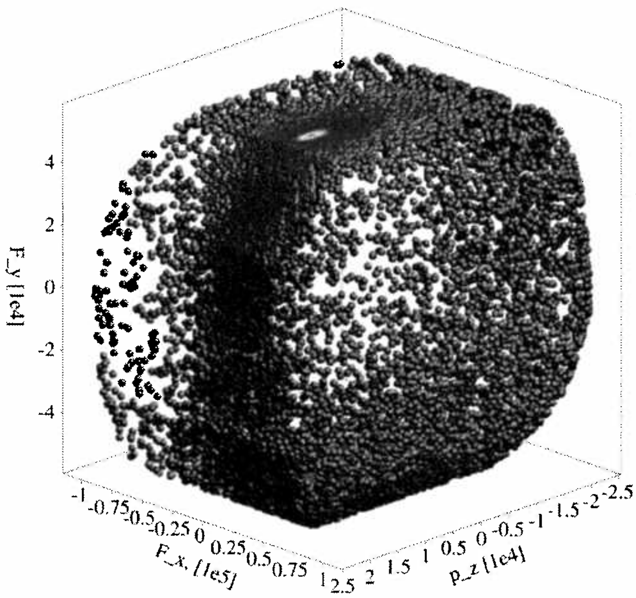

Fig. 8. Limit state function from 20 000 samples.

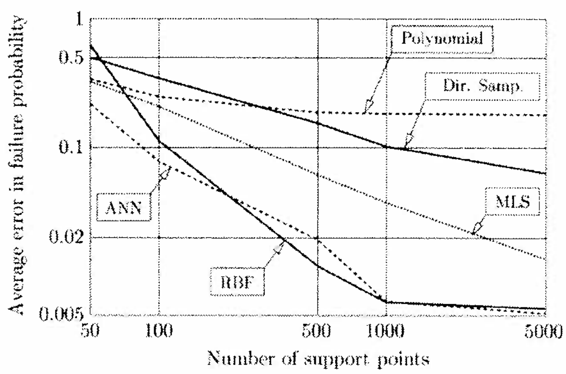

Fig. 9. Calculated mean relative errors in failure probability for the three-dimensional steel frame with three random loads.

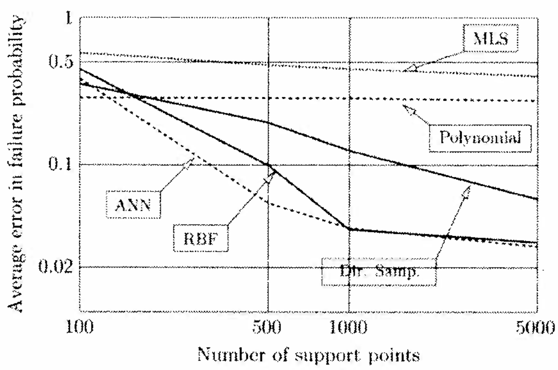

Fig. 10. Calculated mean relative errors in failure probability for the three-dimensional steel frame with five random loads and random yield stress.

The probability of failure obtained from directional sampling (100 000 samples) is $P(\mathcal{F}) = 3.445 \times 10^{-5}$. This corresponds to a safety index of $\beta = 3.98$. Note that this reference value in itself still contains some statistical error. However, with increasing number of samples the estimation error becomes insignificantly small compared to the systematic errors inherent in the response surface models.

The results from the presented response surface approaches are compared in Table 2. The table clearly indicates that the values of the MLS, RBF, and ANN approach converge much faster close to the exact value than these obtained with directional sampling (an average error of 0.1 is obtained with MLS at 300 samples, with RBF at slightly more than 100 samples and with ANN at only 80 samples compared to the required 1000 directional samples). The polynomial regression (due to its very nature) cannot converge to the exact value but can give an acceptably accurate estimate with a small number of samples. For a comparison cf. Fig. 9. In Fig. 11 and Fig. 12 two approximated limit state surfaces are shown which are obtained with the ANN approach with 100 and 1000 support points, respectively.

Table 2
Calculated mean and maximum relative errors in failure probability for the three-dimensional steel frame with three random loads

	Number of support points				
	50	100	500	1000	5000
Number of simulations	2000	1000	200	100	20
Directional sampling					
Mean error $\bar{\epsilon}$	0.4986	0.3442	0.1546	0.1020	0.0636
Max. error ϵ_{max}	2.5333	2.0022	0.7412	0.3804	0.1557
Polynomial regression					
Mean error $\bar{\epsilon}$	0.3400	0.2470	0.1873	0.1834	0.1809
Max. error ϵ_{max}	4.1928	1.4799	0.6992	0.4461	0.2360
MLS interpolation					
Mean error $\bar{\epsilon}$	0.3250	0.2075	0.0615	0.0374	0.0137
Max. error ϵ_{max}	0.9762	0.7613	0.1177	0.0492	0.0155
RBF interpolation					
Mean error $\bar{\epsilon}$	0.6229	0.1123	0.0121	0.0063	0.0057
Max. error ϵ_{max}	53.527	1.9106	0.0753	0.0188	0.0101
ANN approximation					
Number of neurons	9	10	20	50	50
Mean error $\bar{\epsilon}$	0.2173	0.0777	0.0193	0.0064	0.0052
Max. error ϵ_{max}	8.7957	1.5094	0.0641	0.0223	0.0114

Table 3
Calculated mean and maximum relative errors in failure probability for the three-dimensional steel frame with five random loads and random yield stress

	Number of support points			
	100	500	1000	5000
Directional sampling				
Mean error $\bar{\epsilon}$	0.3551	0.1939	0.1251	0.0587
Max. error ϵ_{max}	1.1610	0.4684	0.4138	0.1564
Polynomial regression				
Mean error $\bar{\epsilon}$	0.2851	0.2846	0.2840	0.2742
Max. error ϵ_{max}	0.7674	0.5333	0.4542	0.3425
MLS interpolation				
Mean error $\bar{\epsilon}$	0.5775	0.4773	0.4504	0.4028
Max. error ϵ_{max}	0.8266	0.5860	0.5196	0.4235
RBF interpolation				
Mean error $\bar{\epsilon}$	0.4474	0.0993	0.0364	0.0298
Max. error ϵ_{max}	1.9935	0.3480	0.1252	.
ANN approximation				
Number of neurons	10	20	50	50
Mean error $\bar{\epsilon}$	0.3848	0.0549	0.0371	0.0298
Max. error ϵ_{max}	1.8150	0.2247	0.0770	0.0517

The same three-dimensional steel frame is now assumed to be subjected to five random loadings p_z, F_{x1}, F_{x2}, F_{y1}, and F_{y2} as shown in Fig. 13 (Mean values: 12.0 kN/m, 15.0 kN, 15.0 kN, 20 kN, 20 kN and standard deviations: 1.2 kN/m, 2.0 kN, 2.0 kN, 4.0 kN, 4.0 kN). Additionally, the yield stress of the structural steel is assumed to be a random variable (mean value: 250 N/mm^2, standard deviation: 20 N/mm^2). The reference value of the failure probability is obtained from 250 000 directional samples as $P(F) = 2.1521 \times 10^{-3}$.

The different response surface approaches are compared with respect to the predicted failure probability in Table 3. Again the ANN method enables an efficient prediction compared to directional sampling. Also, the results from the RFB approach are very good. The MLS estimation shows a surprisingly slow converge behavior which is worse than directional sampling and even than polynomial regression. A graphical representation is shown in Fig. 10.

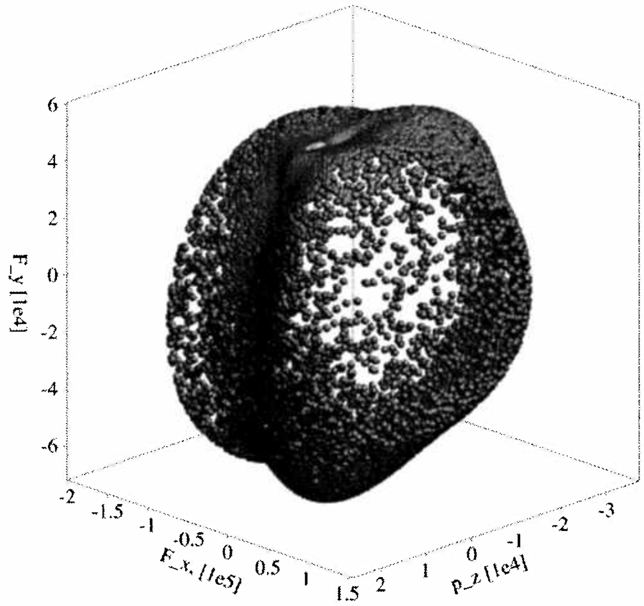

Fig. 11. Approximated limit state function using ANN with 100 directional samples as support points.

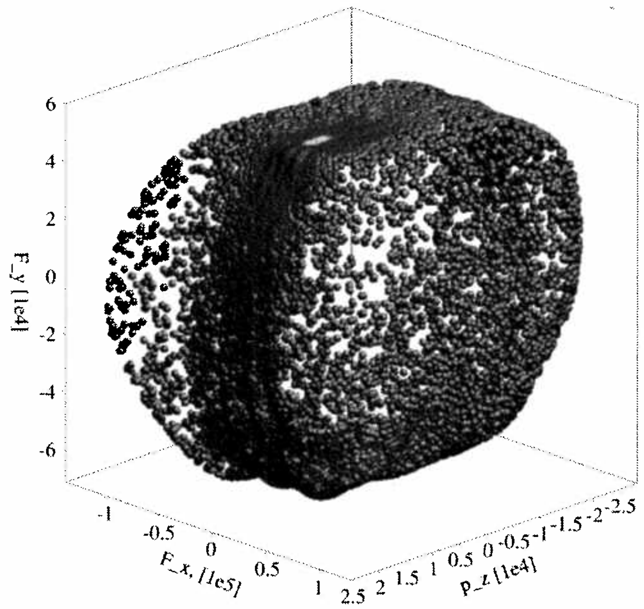

Fig. 12. Approximated limit state function using ANN with 1000 directional samples as support points.

4. Concluding remarks

The relative accuracy of the various approaches as presented depends on the specific problem under consideration. Therefore it appears to be difficult to provide a general recommendation on methods to be used in applications. Considering the examples as shown, response surface approaches based on

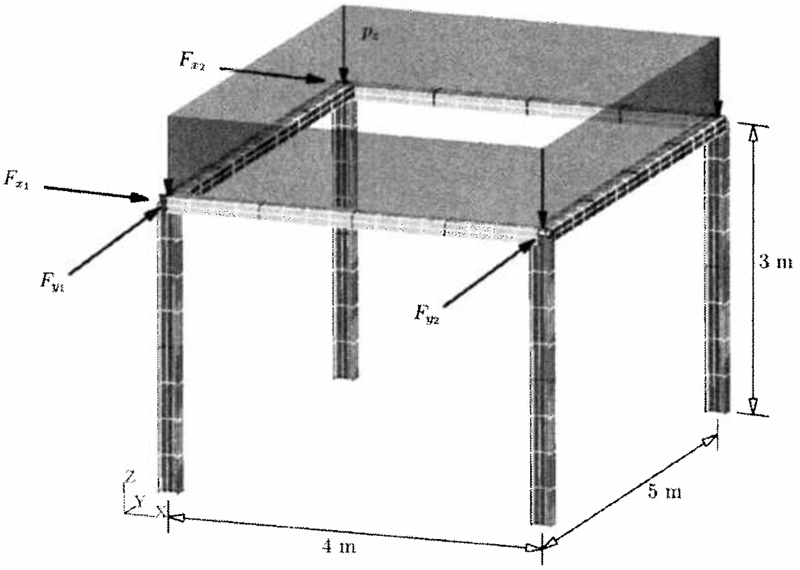

Fig. 13. Three-dimensional steel frame structure with 5 random loads.

polynomial functions, radial basis functions or artificial neural networks appear to be quite capable of representing failure conditions of structural systems in such a way that reliability analysis can be performed with acceptable accuracy. Due to the availability of interpolating functions which can be incrementally augmented by providing additional support points it is possible to construct approximation sequences of approximations whose reliability estimates converge fast to the true value. Based on this property, adaptive strategies can be developed which allow for a cost-effective improvement of the response surface models driven by accuracy requirements for reliability analysis. First ideas of such adaptive approaches have been presented e.g. in [33,34].

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