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Singular solutions of the turbulent boundary layer equations in the case of marginal separation as $Re \to \infty$

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We consider a nominally steady and two-dimensional turbulent boundary layer (BL) of uniform density along a flat surface under the action of an adverse pressure gradient. Classical analysis of the flow in the limit of large Reynolds number, $Re \to \infty$, (a survey is found e.g. in [2]) predicts the well-known asymptotically small velocity defect holding within most of a strictly attached BL. However, as demonstrated in [1], a more general asymptotic approach accounting for a large velocity deficit and, in turn, the possibility of separation apparently requires the existence of a second perturbation parameter, denoted by α. The latter is (i) essentially independent of Re, (ii) serves as a measure for the BL slenderness and is (iii) in fact provided by the empirical constants entering any commonly employed Reynolds shear stress closure.

Including $\alpha \ll 1$ in the theoretical considerations as first put forward by Melnik (1989), referred to in [1], has the important consequence that the BL thickness remains finite and of $O(\alpha)$ in the limit $Re^{-1} = 0$, which will be considered here. To this end, let x, y, ψ, δ and ℓ denote Cartesian coordinates parallel and normal to the wall, the stream function, the BL thickness and the mixing length, non-dimensional with global reference quantities, respectively. As shown in [1], appropriately scaled variables in the outer part of the BL are $Y = y/\alpha$, $\Psi = \psi/\alpha$, $L = \ell/\alpha^{3/2}$, $\Delta = \delta/\alpha$ which upon substitution into the set of Reynolds-averaged Navier–Stokes equations yield the leading-order problem

$$\left.\begin{array}{r} \Psi_Y\Psi_{Yx} - \Psi_x\Psi_{YY} = -p_x + T_Y\,,\ \ p_x = -U_eU_{ex}\,,\ \ T = L^2\,\Psi_{YY}|\Psi_{YY}|\,; \\ Y=0:\ \Psi = T = 0\,,\quad Y = \Delta(x):\ \Psi_Y = U_e(x,\beta),\, T = 0\,, \end{array}\right\} \quad (1)$$

where we require that $L \to L_0(x) = O(1)$ for $Y \to 0$, implying $\Psi_Y = U_s + O\left(Y^{3/2}\right)$. The resulting wall slip $U_s(x,\beta)$ reflects the absence of viscous forces and is assumed to depend on the controlling parameter β used to characterize the potential flow velocity $U_e(x,\beta)$ imposed at the BL edge. Additional sublayers allowing among others to satisfy the no-slip conditions at the wall emerge if the expansions are carried on to higher orders in α and Re^{-1}. Note, however, that the solution in the outer region which comprises most of the BL is completely determined by (1). Here we are interested primarily in the case that U_s vanishes locally, indicating the onset of separation.

In this connection numerical solutions of (1) have been obtained for retarded flows specified by

$$U_e(x,\beta) = (x+1)^m \left[\beta\left(\exp(-5x^2)-1\right)+1\right] \; ; \quad L = I\left(Y/\Delta(x)\right)^{1/2}\Delta(x) \, . \quad (2)$$

Klebanoff's intermittency factor $I(Y/\Delta)$ was implemented to improve the prediction of the flow near the BL edge. It is expected, however, that other choices of $U_e(x,\beta)$ will not affect the flow behaviour near $U_s = 0$ significantly. Also note that problem (1) admits (in addition to the trivial result $\Psi_Y \equiv U_e$) self-similar solutions $\Psi = \Delta U_e F(Y/\Delta)$, $\Delta \propto x$, for external flows of the form $U_e \propto x^m$, where the exponent m is a function on $F'(0)$ and $-1/3 < m < 0$, leading to a wall slip $U_s \propto x^m F'(0)$, $F'(0) < 1$; c.f. [1]. These solutions were used to provide initial conditions at $x = 0$ for the numerical calculations with $U_e(x,\beta)$ given by (2) which were carried out for a range of values of β; see Figure 1: If β is sufficiently small the distribution of U_s is smooth, and $U_s > 0$ throughout. However, if the parameter β exceeds a critical value β_{crit}, $U_s(x,\beta_{crit})$ is found to vanish at a single location $x = x_{crit}$ but is positive elsewhere. A further increase of β will cause a breakdown of the calculations accompanied by the occurrence of a weak singularity slightly upstream at $x = x^*$.

A qualitatively similar behaviour of the wall shear in laminar BLs was observed originally by Ruban (1981), see e.g. [2], and is now commonly referred to as marginal separation. This notion is used also here although the mechanism leading to separation is vastly different from the laminar counterpart.

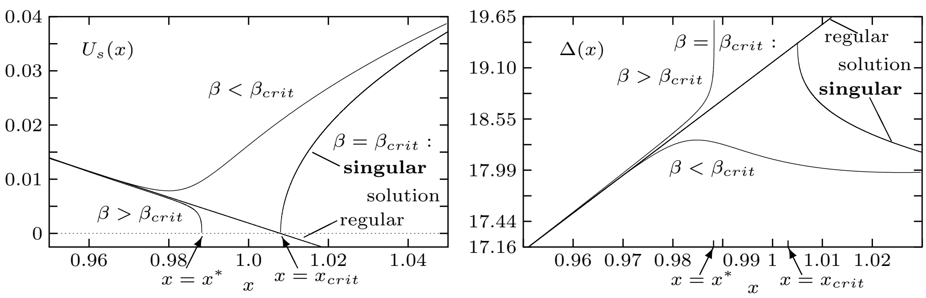

Figure 1: Distributions of U_s and Δ near $x = x_{crit}$.

To study the local flow behaviour near $x = x_{crit}$ the pressure gradient is expanded as $p_x = P_0 + \epsilon + P_1 s + \ldots$ for $s \to 0_{\mp}$ where $s = x - x_{crit}$, $\epsilon = \beta - \beta_{crit} \ll 1$. We first focus on the case $\epsilon = 0$. Since Δ assumes a finite value Δ_0 at $x = x_{crit}$ one infers that $L_0 \to L_{00}$ as $s \to 0$. The balance (1) is retained in flow regimes II$\mp$, see Figure 2. There the appropriately rescaled local quantities $\eta = y/\left({L_{00}}^{2/3}(\mp s)^{1/3}\right)$, $f = \Psi/\left({L_{00}}^{2/3}{P_0}^{1/2}(\mp s)^{5/6}\right)$ suggest the expansions

$$f = f_0(\eta) + (\mp s)^{\lambda} \ln^{\nu}(\mp s)\, f_{1\mp}(\eta) + \ldots \; ; \quad \mp 1/2\, {f_0'}^2 \pm 5/6\, f_0 f_0'' = -1 + \left({f_0''}^2\right)' . \quad (3)$$

The solutions of the resulting differential equations for f_0, f_{1-}, f_{1+}, …, supplemented with boundary conditions following from (1) are sought subject to the requirement of sub-exponential growth for $\eta \to \infty$, in order to provide a match to the solution in the regimes I$\mp$. In case of the upper sign, i.e. $x < x_{crit}$, a numerical treatment indicates that the only acceptable solution of the nonlinear leading-order equation for f_0 is given by $f_0 = 4/15\,\eta^{5/2}$. It expresses the balance between pressure and Reynolds shear stress gradient at the surface for regularly vanishing U_s as $s \to 0_-$. In contrast, numerical calculations for the lower sign, i.e. $x > x_{crit}$, indicate the existence of a further solution having $f_0'(0) \doteq 1.1835$ and $f_0 = 4/15\eta^{5/2} + O\big(\eta^{3/2}\big)$ as $\eta \to \infty$. As a result, the flow exhibits a non-zero wall slip $U_s \sim f_0'(0)(P_0 s)^{1/2}$, which is singular at $s = 0_-$. Hence, the convective term in (1) evaluated at $Y = 0$ jumps from 0 to $U_s U_{sx} \sim P_0 f_0'(0)^2/2$ at $x = x_{crit}$. However, for $s < 0$, the behaviour of U_s is fixed by the homogeneous solution f_{1-} of a linear problem implying the well-known Kummer's equation, together with the inhomogeneous problem arising due to terms of $O\big(\nu(-s)^{\lambda} \ln^{\nu-1}(-s)\big)$; c.f. (3). Non-exponential growth as $\eta \to \infty$ is provided if $\lambda = 1/2, 3/2, \ldots$ and $\nu = 0$. In turn, the expected linear behaviour $U_s \sim -c\,s$, with $c > 0$, is revealed.

Finally, in regimes I$\mp$ where $Y = O(1)$ expansions $\Psi = \Psi_0(Y) + (\mp s)^{r_\mp}\Psi_1(Y) + \ldots$, $r_- = 1$, $r_+ = 1/3$, hold. In turn, $\Delta \sim \Delta_0 + O\big((-s)^{r_\mp}\big)$; c.f. Figures 1, 2.

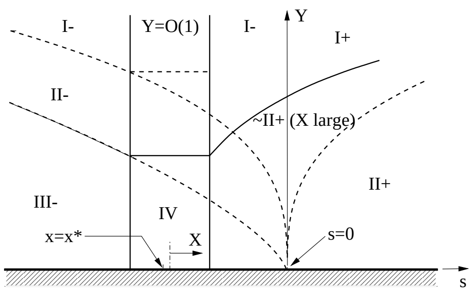

Figure 2: Asymptotic splitting near $x = x_{crit}$ as $\epsilon \to 0$; broken lines: $\epsilon = 0$.

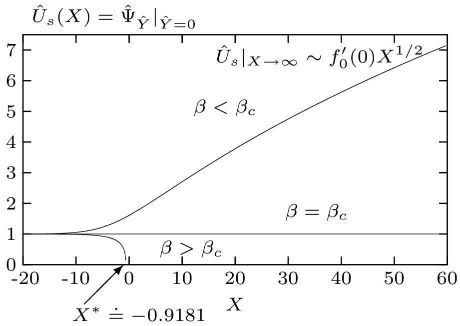

Figure 3: Bifurcating distributions of $\hat{U}_s$, determined by solving (6).

Note that regular terms reflecting the smooth behaviour of p_x have been disregarded since Ψ is near $x = x_{crit}$ mainly governed by local eigensolutions as discussed so far. Therefore, in the case $\epsilon \neq 0$, the above structure for $s < 0$ is perturbed solely due to regular terms, associated with $\lambda = -1/2$ in (3), which cannot explain the excitation of the singular downstream solution as the regular one is obviously suppressed, see Figure 1. However, inspection of the results for $x < x_{crit}$ indicate that the contributions resulting from f_0 and f_1 become of the same size if $\eta \sim (-c^2 s)^{1/3}$, c.f. (3), forcing the formation of a new sublayer III$-$ (see Figure 2), where singular eigensolutions may arise. Introducing the scaled quantities $\hat{\eta} = \eta/(-c^2 s)^{1/3}$, $\hat{f} = f/(-c^2 s)^{5/6}$, it follows that

$$\hat{f} = \hat{f}_0(\hat{\eta}) + O(-s) + \epsilon(-s)^{\mu} d \exp[-\chi/(c^2 s)] g(\hat{\eta}) + \ldots, \quad \hat{f}_0 = 4/15\,\hat{\eta}^{5/2} + \hat{\eta}\,; \quad (4)$$

$$\chi\big[(1+2/3\,\hat\eta^{3/2})g' - \hat\eta^{1/2}g''\big] = 2\big(\hat\eta^{1/2}g\big)', \quad g'(0)=1,\ g(0)=g''(0)=0\,. \quad (5)$$

Thus, the unknown constant $d>0$ measures the perturbation of U_s. A numerical investigation shows that problem (5) allows for an eigensolution g exhibiting non-exponential growth as $\hat\eta\to\infty$ solely if $\chi=1/3$. Only in that case the solution is found analytically. It reads $g = 2/3\,\hat\eta\exp(-z)+(2/9)^{1/3}\hat f_0'(\hat\eta)\int_0^z t^{-1/3}\exp(-t)\,\mathrm{d}t$ where $z = 2/9\,\hat\eta^{3/2}$. An analogue study of the inhomogeneous higher-order problem for the contribution of $O\big((-s)^{\mu-1}\exp(-\chi/s)\big)$ to the expansion (4) gives the single value $\mu=-9/5$. In turn, the perturbation g provokes exponentially small disturbances $\propto\epsilon f_0'$ and $\propto\epsilon\Psi_0'$ in the flow regimes II$-$ and I$-$, respectively.

Expansion (4) ceases to be valid within region IV, Figure 2. Here we define appropriate variables X, $\hat Y = Y/(L_{00}\,\hat\epsilon)^{2/3}$ and $\hat\Psi = \Psi/({P_0}^{1/2}{L_{00}}^{2/3}\hat\epsilon^{5/3})$ where

$$-c\,s = \hat\epsilon - \hat\epsilon^2 c\,X\,, \quad c\,\hat\epsilon = -\chi\big(1-\mu\ln(-c^2\ln|d\epsilon|/\chi)/\ln|d\epsilon|\big)/\ln|d\epsilon|\,.$$

Substitution into (1) yields to leading order the reduced problem

$$\left.\begin{aligned} \hat\Psi_{\hat Y}\hat\Psi_{\hat Y X} - \hat\Psi_X\hat\Psi_{\hat Y\hat Y} = -1+\hat T_{\hat Y}\,,\ \hat T = (\hat\Psi_{\hat Y\hat Y})^2;\ \ \hat\Psi|_{\hat Y=0} = \hat\Psi_{\hat Y\hat Y} = 0\,, \\ \hat T|_{\hat Y\to\infty}\sim\hat Y;\ \ \hat\Psi|_{X\to-\infty}\sim\hat f_0(\hat Y) + j\,g(\hat Y)\exp(\chi X)\,,\ j=\pm1,0\,, \end{aligned}\right\} \quad (6)$$

which has to be solved numerically. The distributions of the rescaled wall slip $\hat U_s = \hat\Psi_{\hat Y}|_{\hat Y=0}$ are depicted in Figure 3. In the subcritical case $\epsilon<0$, i.e. $j=+1$, the solution of (6) asymptotes to the non-trivial downstream solution $\Psi\sim {L_{00}}^{2/3}{P_0}^{1/2}f_0(\eta)$, holding in region II+, for $X\to\infty$, c.f. Figures 2, 3. Likewise, analysis of the flow regime I+ reveals that $\Delta-\Delta_0 = O(X^{1/3})$ as $X\to\infty$ there. For $j=0$ the solution of (6) is $\hat\Psi = \hat f_0(\hat Y)$, which corresponds to the critical case $\epsilon=0$. In the supercritical case $\epsilon>0$, that is $j=-1$, the solution breaks down at a distinct location $X=X^*$, i.e. $x^*<x_{crit}$ in the original scaling.

Again, this behaviour is examined by means of a local analysis: Introducing appropriate local variables $S = X-X^*\to 0_-$, $\zeta = \hat Y/(-S)^{1/3}$, $\hat F = \hat\Psi/(-S)^{5/6}$:

$$\hat F = \hat F_0(\zeta) + (-S)^\sigma\hat F_1(\zeta)+\dots\,;\quad -1/2\,\hat F_0'^2 + 5/6\,\hat F_0\hat F_0'' = -1 + \big(\hat F_0''^2\big)'\,.$$

Here the leading-order term $\hat F_0 = 2^{1/2}\zeta$ gives rise to a GOLDSTEIN-type singularity, i.e. $U_s\sim(-2P_{00}\,s)^{1/2}$ (c.f. [2]). As the existing limiting profile, $\hat\Psi(X^*,\hat Y)$, of the solution in regime IV cannot be matched to the solution in the region $\zeta=O(1)$, a transition layer is introduced where $\hat Y = O\big((-S)^{1/6}\big)$ and inertia terms balance the imposed pressure gradient to leading order. The matching procedure, which concludes the present analysis, then shows that $\sigma=1/4$, $\hat F_1\propto\zeta^{5/2}$, and $\Delta = \Delta_0 + O\big(\gamma,(-S)^{1/6}\big)$ (see the singular branch in Figure 1).

The effect of small finite values of the slenderness parameter α on that singular behaviour is the topic of the current research.

References

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