

Asymptotic Theory of Turbulent Bluff-Body Separation: A Novel Shear Layer Scaling Deduced from an Investigation of the Unsteady Motion

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Abstract. A rational treatment of time-mean separation of a nominally steady turbulent boundary layer from a smooth surface in the limit $Re \rightarrow \infty$, where Re denotes the globally defined Reynolds number, is presented. As a starting point, it is outlined why the ‘classical’ concept of a small streamwise velocity deficit in the main portion of the oncoming boundary layer does not provide an appropriate basis for constructing an asymptotic theory of separation. Amongst others, the suggestion that the separation points on a two-dimensional blunt body is shifted to the rear stagnation point of the impressed potential bulk flow as $Re \rightarrow \infty$ – expressed in a previous related study – is found to be incompatible with a self-consistent flow description. In order to achieve such a description, a novel scaling of the flow is introduced, which satisfies the necessary requirements for formulating a self-consistent theory of the separation process that distinctly contrasts former investigations of this problem. As a rather fundamental finding, it is demonstrated how the underlying asymptotic splitting of the time-mean flow can be traced back to a minimum of physical assumptions and, to a remarkably large extent, be derived rigorously from the unsteady equations of motion. Furthermore, first analytical and numerical results displaying some essential properties of the local rotational/irrotational interaction process of the separating shear layer with the external inviscid bulk flow are presented.

Key words: coherent motion, gross separation, perturbation methods, turbulent boundary layers.

1. Introduction

The rational description of break-away separation of a statistically steady and two-dimensional incompressible turbulent boundary layer flow past an impermeable rigid and smooth surface in the high-Reynolds-number limit represents a long-standing unsolved hydrodynamical problem. Needless to say that an accurate prediction of the position of separation, in combination with the local behaviour of the skin friction, has great relevance for many engineering applications, where e.g. internal flows, like those through diffuser ducts, or flows past airfoils play a crucial role.

1.1. PROBLEM FORMULATION AND GOVERNING EQUATIONS

The picture of such flows near separation is sketched in Figure 1. As a basic assumption, the suitably formed global Reynolds number Re is taken to be asymptotically large,

$$Re = \tilde{U}\tilde{L}/\tilde{\nu} \rightarrow \infty, \quad \nu = Re^{-1} \rightarrow 0. \quad (1)$$

Herein $\tilde{\nu}$, \tilde{L} , and \tilde{U} denote, respectively, the (constant) kinematic viscosity of the fluid, a reference length, typical for the geometry of the portion of the surface under consideration, and a characteristic value of the surface slip velocity impressed by the limiting inviscid stationary and two-dimensional irrotational bulk flow, hereafter formally indicated by $\nu = 0$. All flow quantities are suitably non-dimensionalised with \tilde{L} , \tilde{U} , and the (uniform) fluid density. Let t , p , $\mathbf{x} = (s, n, z)$, and $\mathbf{u} = (u, v, w)$ be the time, the fluid pressure, and the position and the velocity vector. Here u , v , and w are the components of \mathbf{u} in directions of the natural coordinates s , n , and z , respectively, along, normal to, and projected onto the separating streamline \mathcal{S} , given by $n = 0$, of the flow in the limit $\nu = 0$. Furthermore, $u_e(s)$ denotes the surface slip velocity in that limit. The origin $s = n = 0$ is chosen as the location S where \mathcal{S} departs from the surface. Thus, \mathcal{S} coincides with the surface contour for $s \leq 0$. Also, note that \mathcal{S} has, in general, a curvature of $O(1)$ for $|s| = O(1)$.

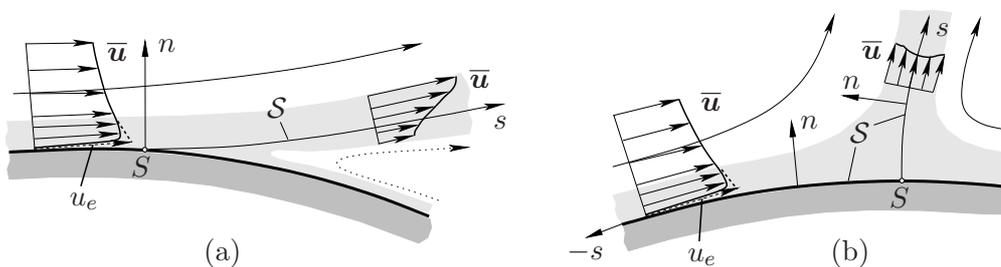


Figure 1. Time-mean flow near (a) smooth separation (the dotted streamline indicates possible backflow), (b) separation due to stagnation of the bulk flow, cf. [5]. The inviscid limit of $\bar{\mathbf{u}}$ is shown dashed, and the turbulent shear flow is indicated by a shading.

In coordinate-free form the Navier–Stokes equations then are written as

$$\nabla \cdot \mathbf{u} = 0, \quad (2a)$$

$$D_t \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u}, \quad D_t = \partial_t + \mathbf{u} \cdot \nabla, \quad \Delta = \nabla \cdot \nabla, \quad (2b)$$

where ∇ is the gradient with respect to \mathbf{x} . They are subject to the common no-slip condition $\mathbf{u} = \mathbf{0}$ holding at the surface. As a well-known characteristic, the stationary Reynolds-averaged turbulent flow can be expressed in terms of the time-averaged motion. In the following we employ the conventional Reynolds decomposition of any (in general, tensorial) flow quantity q into its time-mean component \bar{q} , here regarded as independent of z , and the (in time and space) stochastically fluctuating contribution q' ,

$$q(\mathbf{x}, t, \dots) = \bar{q}(\mathbf{x}, y, \dots) + q'(\mathbf{x}, t, \dots), \quad \bar{q} = \lim_{\Theta \rightarrow \infty} \frac{1}{\Theta} \int_{-\Theta/2}^{\Theta/2} q(\mathbf{x}, t + \theta, \dots) d\theta. \quad (3)$$

Herein the dots indicate any further dependences of q apart from \mathbf{x} and t . Reynolds-averaging of (2) then yields the well-established Reynolds equations (in the case $\partial_z \equiv 0$ of planar time-mean flow),

$$\nabla \cdot \bar{\mathbf{u}} = 0, \quad (4a)$$

$$\bar{D}_t \bar{\mathbf{u}} = -\nabla \bar{p} - \nabla \cdot \overline{\mathbf{u}'\mathbf{u}'} + \nu \Delta \bar{\mathbf{u}}, \quad \bar{D}_t = \bar{\mathbf{u}} \cdot \nabla. \quad (4b)$$

It is further presumed in the subsequent analysis, that all components of the Reynolds stress tensor $-\overline{\mathbf{u}'\mathbf{u}'}$ are, in general, of asymptotically comparable magnitude (assumption of locally isotropic turbulence). Most important, we disregard any effects due to free-stream turbulence. That is, the turbulent motion originates from the relatively thin fully turbulent boundary layer adjacent to the surface, which near S passes into an accordingly slender separated free shear layer along \mathcal{S} for $s > 0$.

1.2. MOTIVATION

From an asymptotic point of view, three outstanding contributions to the solution of the problem under consideration have to be mentioned.

Sychev [11, 12] was the first who elucidated the question of the asymptotic structure of the oncoming boundary layer by proposing a three-layer splitting of the latter, sufficiently far ahead of S . This scaling, however, is at variance with the classical finding of a two-tiered boundary layer that is found to hold for firmly attached flow only (see, for instance, the pioneering work by Mellor [3]). We start the outline of both formulations by noting that each of them adopts the familiar description of the viscous wall layer close to the surface; the same holds for the flow descriptions discussed subsequently. On top of that region the Reynolds shear stress $-\overline{u'v'}$ asymptotically equals the (local) wall shear stress, given by the square of the skin friction velocity u_* , and the streamwise velocity component \bar{u} satisfies the celebrated logarithmic law of the wall. By using the conventional notation, it reads

$$\bar{u}/u_* \sim \kappa^{-1} \ln n^+ + C^+, \quad n^+ = n u_* Re \rightarrow \infty, \quad (5)$$

where the well-known constants κ and C^+ are quantities of $O(1)$. The match of the wall region with the adjacent layer then shows that the expansion

$$[\bar{u}, -\overline{u'v'}/u_*^2] \sim [\bar{u}_0, T_0](s, \eta) - \gamma[U_1, T_1](s, \eta) + O(\gamma^2), \quad \eta = n/\delta, \quad (6)$$

holds in the latter. Here, δ is a measure for the thickness of that layer, and, by introducing the so-called slip velocity u_s , the gauge function γ is seen to satisfy the skin-friction law

$$\gamma = u_*/u_s \sim \kappa/\ln Re, \quad d\gamma/ds = O(\gamma^2), \quad u_s(s) = \bar{u}_0(s, 0). \quad (7)$$

In the classical two-tiered description of the boundary layer, cf. [3], it is assumed that in the fully turbulent main region the (positive) streamwise velocity ‘defect’ with respect to the external potential flow, $u_e - u$, is asymptotically small. In turn, $\bar{u}_0(s, \eta) \equiv u_s(s) \equiv u_e(s)$, and in the boundary layer limit the momentum balance (4b) reduces to a balance between the linearised convective terms and $\partial_n(-\overline{u'v'})$ in leading order, showing that the boundary layer thickness δ is of $O(\gamma)$. In contrast, according to [11, 12] the expansion (6) holds in the additionally introduced middle layer which meets the requirement that the velocity defect $u_e - u$ and, consequently, $u_e - u_s$ are quantities of $O(1)$. Thus, in the boundary layer approximation to (4b) the convective terms balance both $\partial_n(-\overline{u'v'})$ and the imposed (adverse) pressure gradient $-u_e du_e/ds$, such that the thickness δ of the middle layer is of $O(\gamma^2)$. This wake-type flow structure then allows for a significant decrease in the wall shear stress according to (7) when u_s tends to zero as $s \rightarrow 0_-$ and, moreover, for the occurrence of flow reversal further downstream by adopting a local turbulent/irrotational interaction strategy (without the need of a specific turbulence closure).

One readily finds that the gradients $\partial_n \bar{u}$ in the viscous wall layer and the adjacent layer, described by (6), match on the basis of (5) provided that $\partial_\eta \bar{u}_0 \equiv 0$. Unfortunately, this again gives $\bar{u}_0(s, \eta) \equiv u_s(s) \equiv u_e(s)$ and thus contradicts the original assumption of a large velocity defect in the middle layer. That inherent mismatch of the wall layer and the wake region was first noted by Melnik [4], who used mixing length arguments, in the second work to be highlighted. Therefore, Sychev's approach can hardly be accepted as a self-consistent theory. However, Melnik also proposed a non-classical initially three-tiered boundary layer where the outermost part plays the role of the aforementioned middle layer. But most important, and in striking difference to any previous asymptotic treatment of turbulent shear flows, in [4] the slenderness of the latter is measured by some small non-dimensional parameter, denoted by α , which is regarded to be essentially independent of Re . Melnik's motivation for the resultant two-parameter matched asymptotic expansions of the flow quantities merely relies upon the observation that any commonly employed shear stress closure includes a small number (a most familiar example is the so-called Clauser 'constant' $\alpha \approx 0.0168$ in the algebraic Cebeci–Smith model) which is seen to measure the boundary layer thickness if the velocity defect in the fully turbulent flow regime is taken to be of $O(1)$. This idea has been followed up and substantiated by order-of-magnitude reasoning by Scheichl and Kluwick [7, 6], where it is shown to provide a sound basis for developing a self-consistent theory of turbulent marginal separation. On the other hand, it is found that Melnik's theory cannot be extended in order to describe the global separation process due to two serious shortcomings: (i) the proposed flow structure is strongly associated with the adopted coupling $\alpha^{1/2} \ln Re = O(1)$, which is apparently inconsistent with the original assumption on α and, hence, does not allow for a correct formulation of the gradual transition from attachment to separation of the flow inside the wall layer; (ii) the impressed potential flow does not exhibit a free streamline departing smoothly from the surface, in order to avoid a Goldstein-type singularity encountered by the boundary layer solution that is evidently unsurmountable by assuming a firmly attached external bulk flow, cf. [7].

A different viewpoint was taken up in the third contribution to be noticed, by Neish and Smith [5]. They considered the streamwise development of a classical small-defect boundary layer where the irrotational external flow is indeed presumed to be strictly attached; that is, it exhibits a rear stagnation point, see Figure 1 (b). Interestingly, this concept is fully consistent with the following important finding elucidated in the subsequent analysis: in the case of smooth inviscid flow detachment, as depicted in Figure 1 (a), the associated singular behaviour of the surface pressure immediately upstream of the (a priori unknown) position of S does not trigger a significant change in the order of magnitude of the (initially small) velocity defect, which would be necessary to render smooth boundary layer separation possible. Consequently, within the framework of classical turbulent boundary layer theory separation is suggested to occur asymptotically close to the rear stagnation point as $Re \rightarrow \infty$.

Unfortunately, however, it is not addressed satisfactorily in [5] whether and how the small velocity defect may rather abruptly become of $O(1)$ due to the retardation of the potential flow as the stagnation point S is approached in order to ensure an uniformly valid flow description. As pointed out in the first part of the present study, the inviscid vortex flow induced in the immediate vicinity of the stagnation point

S indeed appears to hamper severely the construction of a self-consistent asymptotic theory. This finding represents the starting point for the subsequent analysis, where it is shown how the closure-independent asymptotic formulation of a turbulent boundary layer having a finite thickness of $O(\alpha)$, $\alpha \ll 1$, as $Re \rightarrow \infty$ and which may undergo marginal separation, see [7], can be adapted to that of massive separation. Unlike the theories presented in [4, 5], here the formal limit $\alpha = Re^{-1} = 0$ corresponds to the required class of inviscid flows with free streamlines. Furthermore, we demonstrate how the asymptotic scaling of the (oncoming) flow, which in [7] was based on rather heuristic arguments from a time-averaged point of view, can be deduced by means of a multiple-scales analysis of the equations of motion (2).

We commence the investigation by considering the evolution of the boundary layer immediately upstream of the surface position S , indicating inviscid separation.

2. Limitations of the Small-Defect Approach

The case where the streamwise velocity defect in the fully turbulent main region of the boundary layer is small, say, $u_e - u = O(\epsilon)$, $\epsilon \ll 1$, is considered first. To be more precise, we assume that $\epsilon = \gamma$, according to (6) and (7) (although the more general assumption $\gamma/\epsilon = O(1)$, including $o(1)$, would not alter the following analysis substantially). Therefore, the boundary layer thickness δ is of $O(\gamma)$ and expanded as

$$\delta/\gamma = \Delta_0(x) + \gamma\Delta_1(x) + \dots. \quad (8)$$

By setting $U_1/u_e = F'_0(s, \eta)$, $\eta = O(1)$, the leading-order streamwise momentum equation, supplemented with appropriate boundary and matching conditions, then reads

$$u_e[d(u_e\Delta_0)/ds]\eta F''_0 - \Delta_0\partial_s(u_e^2 F'_0) = u_e^2 T'_0, \quad (9a)$$

$$F_0(s, 0) = T_0(s, 0) - 1 = 0, \quad F'_0 \sim -\kappa^{-1} \ln \eta + O(1), \quad \eta \rightarrow 0, \quad (9b)$$

$$F'_0(s, 1) = F''_0(s, 1) = T_0(s, 1) = 0. \quad (9c)$$

We note, that in this connection primes denote derivatives with respect to η . Also, it will prove convenient to integrate (9a) with respect to η by using (9b), which gives

$$u_e^2[d(u_e\Delta_0)/ds]\eta F'_0 - \partial_s(u_e^3\Delta_0 F_0) = u_e^3(T_0 - 1). \quad (10)$$

Finally, evaluation of (10) at the boundary layer edge and subsequent integration from some value $s_0 < 0$ to $s < 0$ yields

$$d[u_e^3\Delta_0 F'_0(s, 1)]/ds = u_e^3, \quad u_e^3(\sigma)\Delta_0(\sigma)F'_0(\sigma, 1) \Big|_{\sigma=s_0}^{\sigma=s} = \int_{s_0}^s u_e^3(\sigma) d\sigma. \quad (11)$$

In order to assess the assumption of a small velocity defect holding in the oncoming flow with respect separation, we analyse (9) in the limit $s \rightarrow 0_-$ for the two different cases indicated by Figure 1 (a) and (b), respectively. Without adopting a specific turbulence closure, we begin the analysis by considering the first case.

2.1. FLOW SLIGHTLY UPSTREAM OF SMOOTH SEPARATION

It is well known that, under rather general conditions concerning the flow in the stagnant or backflow region where $s > 0$ and $n < 0$,

$$u_e(s)/u_e(0) \sim 1 + 2k(-s)^{1/2} + (10k^2/3)(-s) + O[(-s)^{3/2}], \quad s \rightarrow 0_-, \quad (12)$$

in the inviscid limit $\nu = 0$, cf. [1], for instance. Here the non-negative parameter k parametrises the class of smoothly separating flows as it depends on the position of S on the body contour. It gives rise to a locally adverse and unbounded pressure gradient $-u_e du_e/ds \sim k(-s)^{-1/2}$. Therefore, the question arises if the latter provokes a significant increase of the velocity defect in the oncoming boundary layer, which is required for a correct description of flow reversal further downstream.

In order to keep the analysis as general as possible, we only assume that

$$u_e(s)/u_e(0) \sim 1 + \chi(s) + \dots, \quad |d\chi/ds| \rightarrow \infty, \quad s \rightarrow 0_-. \quad (13)$$

This singular behaviour is expected to provoke a considerable growth of the turbulent velocity scale u_* (and, in turn, of the fluctuations), expressed through a gauge function $\varphi(s)$,

$$F_0 \sim \varphi(s)G(\eta) + \dots, \quad T_0 \sim \varphi^2(s)R(\eta) + \dots, \quad \varphi \rightarrow \infty, \quad s \rightarrow 0_-. \quad (14)$$

From (11), (14), and the fact that u_e in (13) admits a finite limit, there follows a (intuitively rather unexpected) decrease of the boundary layer thickness of the form $\Delta_0 \sim D/\varphi$, where D is a (positive) constant. Also note, that the term $\partial_s(u_e^3 \Delta_0 F_0)$ in (10) is bounded for $s \rightarrow 0_-$. Since u_e is bounded too, the first term in (10) asymptotically equals $-Du_e^3(0)\eta G'(\eta) d(\ln \varphi)/ds$. As the velocity defect and, in turn, G' are non-negative, that expression tends to $-\infty$ for $s \rightarrow 0_-$. Then φ is seen to be proportional to $(-s)^{-1/2}$, as (10) reduces to a balance between that negative term and $u_e^3 \varphi^2(s)R(\eta)$. The latter term, however, is non-negative, as is the Reynolds stress T_0 in the oncoming flow. From this contradiction one then infers that F_0 , T_0 , and Δ_0 are finite for $s \rightarrow 0_-$. Consequently, inspection of (10) and (13), subject to condition (9c), shows that (14) is to be replaced by a sub-expansion of expansion (6),

$$[F_0, T_0, \Delta_0] \sim [F_{00}(\eta), T_{00}(\eta), \Delta_{00}] + \chi(s)[F_{01}(\eta), T_{01}(\eta), \Delta_{01}] + \dots. \quad (15)$$

Therefore, the velocity defect does not change its order of magnitude. One then concludes that, by specifying $\chi(s)$ in (13) in accordance with (12), the small-defect formulation represents an inadequate description of a turbulent boundary layer approaching smooth separation. Note, that the same conclusion can be drawn for turbulent separation at a trailing edge under angle of attack, where the external velocity admits a square-root behaviour akin to (12). More generally spoken, (15) holds if u_e admits a finite limit according to (13). We add that it has been demonstrated numerically in [8] that even in case of a rather sharp step-like decrease of $u_e(s)$ the velocity defect characterised by F_0 , T_0 , and Δ_0 , remains bounded.

Summarising, it is possible to give a rather comprehensive answer to an interesting question raised by Degani [2], namely, how the small-defect structure responds to different limiting forms of $u_e(s)$ as $s \rightarrow 0_-$: apparently, the only scenario that is compatible with a change of magnitude of the velocity defect, as it is required for an asymptotic description of separation, is that of a boundary layer approaching a stagnation point of the (otherwise attached) flow in the inviscid limit $\nu = 0$. This is exactly the picture of separation originally proposed by Neish and Smith [5].

2.2. FLOW IN THE VICINITY OF A REAR STAGNATION POINT

Close to a rear stagnation point, see Figure 1 (b), the potential flow is linearly retarded as $u \sim -cs$, $v \sim cn$, where $s, n \rightarrow 0$ and c is a positive constant. Then $u_e \sim -cs$, in contrast to (13). Substitution of this relationship into (10) and (11) then predicts a growth of both the boundary layer thickness and the velocity defect, as expressed by (14). Specifically,

$$\Delta_0 \sim D[-\ln(-s)]^{1/2}/(-s), \quad \varphi = D/\{2[-\ln(-s)]^{1/2}(-s)^2\}, \quad s \rightarrow 0_-, \quad (16)$$

where D again is a positive constant, cf. [5, 2]. It then follows from (16) that (10) reduces to the equation $\eta G'(\eta) = R(\eta)$ for $\eta = O(1)$. Since (16) is incompatible with the inhomogeneous boundary conditions (9b) required by the match with the viscous wall layer, on top of the latter a sublayer where $\eta = O(\varphi^{-2})$ has to be introduced. However, as that flow region appears to behave passively with respect to the further analysis, it is disregarded here.

As a consequence of the growth of Δ , see (16), the boundary layer approximation ceases to be valid close to the stagnation point S when the distance $-s$ and δ are of comparable magnitude. From (8) it then follows that this region is characterised by suitably rescaled coordinates $(X, Y) = (s, n)/\tau$, where $\tau = (D\gamma)^{1/2}[-(\ln \gamma)/2]^{1/4}$. The resulting asymptotic splitting of the flow is depicted in Figure 2 (a). In the new ‘square’ domain II of extent τ the flow quantities are expanded in the form

$$\left[\frac{\bar{u}}{c\tau}, \frac{\bar{v}}{c\tau}, \frac{\bar{p} - \bar{p}_S}{(c\tau)^2} \right] \sim \left[-X, Y, \frac{X^2 + Y^2}{2} \right] + \frac{1}{\ln \gamma} [\partial_Y \Psi, -\partial_X \Psi, P] + O\left[\frac{1}{\ln^2 \gamma} \right], \quad (17)$$

where \bar{p}_S is the (time-mean) pressure in S . Here the magnitude of the velocity defect is still asymptotically small and varies only logarithmically with γ . As an important implication, the presence of the logarithmic terms in (16) is seen to prevent the Reynolds stresses, which are of $O(\tau^2/\ln^2 \gamma)$, to affect even the perturbed flow in leading order. Indeed, substitution of (17) into the momentum equation (4b) shows that the perturbation stream function $\Psi(X, Y)$ and the pressure disturbance $P(X, Y)$ satisfy the Euler equations, linearised about the stagnant potential flow,

$$\partial_X(X \partial_Y \Psi) - Y \partial_{YY} \Psi = -\partial_X P, \quad -X \partial_{XX} \Psi + \partial_Y(Y \partial_X \Psi) = -\partial_Y P. \quad (18)$$

By introducing the vorticity $\Omega = (\partial_{XX} + \partial_{YY})\Psi$, elimination of P in (18) yields the vorticity transport equation, $(X\partial_X - Y\partial_Y)\Omega = 0$. Finally, integration gives

$$(\partial_{XX} + \partial_{YY})\Psi = \Omega(-XY), \quad (19a)$$

expressing the well-known property of two-dimensional steady inviscid flows that the vorticity is constant along a streamline. The match with the oncoming boundary layer flow according to (14), (16), and (17) then fixes both the vorticity Ω and the boundary conditions supplementing (19a),

$$\Omega = G''(\eta), \quad \eta = -XY, \quad (19b)$$

$$\Psi(X, 0) = 0, \quad (19c)$$

$$\Psi \sim G(\eta)/X^2, \quad \eta = O(1), \quad X \rightarrow -\infty. \quad (19d)$$

Also, the reuse of the boundary layer coordinate η introduced before in (19b) shows that the edge $n = \delta$ of the turbulent flow region II here is given by $\delta \sim \tau/(-X)$, see

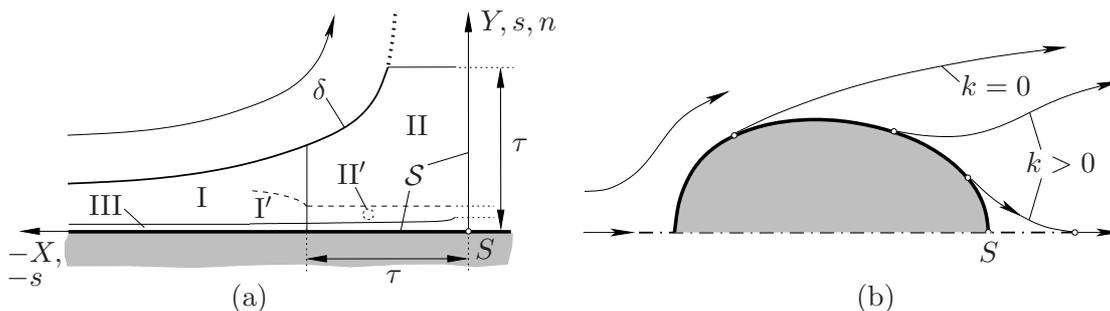


Figure 2. (a) Asymptotic flow splitting near rear stagnation point S : oncoming boundary layer I with emerging sublayer I', resulting 'square' region II with sublayer II' (not considered in text), viscous wall layer III (increase of thickness proportional to $1/s$, $s \rightarrow 0$, not discussed here), separating streamline S of the stagnant potential flow; the dotted lines indicate the connection to the regions not considered in the analysis. (b) Smooth inviscid separation from a (here symmetrical) cylindrical body for different values of k in (12); note the flow showing a cusp-shaped closed cavity which neighbours the attached flow characterised by a rear stagnation point S .

Figure 2 (a). Stated equivalently, the curve $-XY = \eta \sim 1$ disjoins the turbulent from the (approximately) irrotational external region as $\Omega = 0$ for $\eta \geq 1$.

We seek a solution of the Poisson problem (19) for Ψ for $X < 0$, $Y \geq 0$. That is, in the present investigation we do not take into account the 'collision' of the oncoming flow with that approaching S for $s \rightarrow 0_+$, cf. Figures 1 (b) and 2 (a). We set $\Psi = \Psi_p + \Psi_h$, where $\Psi_p(X, Y)$ is a particular solution of (19a)–(19c) and the homogeneous contribution $\Psi_h(X, Y)$ satisfies Laplace's equation, $(\partial_{XX} + \partial_{YY})\Psi_h = 0$, subject to (19c). By defining $G(-\eta) := -G(\eta)$, $\eta \geq 0$, and using standard methods, one obtains after integration by parts and some manipulations

$$\Psi_p = \frac{1}{2\pi} \int_{-1}^1 G'(\eta) \oint_{-\infty}^{\infty} \frac{|\sigma|Y - \eta}{\sigma[\sigma^2(X - \sigma)^2 + (|\sigma|Y - \eta)^2]} d\sigma d\eta \quad (|X| \geq 0). \quad (20)$$

The function $\Psi_p(X, Y)$ is found to vary with R^{-2} for $R^2 = X^2 + Y^2 \rightarrow \infty$ and fixed values of $\vartheta := \arctan(Y/X)$. On the other hand, $\Psi_p \sim H(\eta)/X^2$ for $X \rightarrow -\infty$ where η and, in turn, the function $H(\eta)$ (which is not stated explicitly here) are kept fixed. Since $H \not\equiv G$, however, $\Psi_h(X, Y)$ must behave as

$$\Psi_h \sim [G(\eta) - H(\eta)]/X^2, \quad X \rightarrow -\infty, \quad \eta = O(1), \quad (21)$$

such that Ψ satisfies (19d). An asymptotic investigation of Laplace's equation then shows that $\Psi_h \sim R^{-2}[A \cos(2\vartheta) + B \sin(2\vartheta)]$, where A and B are constants, is the only possible behaviour for $R \rightarrow \infty$. Unfortunately, this relationship does not meet the required match with (21) as $\vartheta \rightarrow \pi_-$, $X \rightarrow -\infty$. Thus, the problem (19) has no solution. Therefore, the asymptotic picture of separation taking place close to a rear stagnation point, as proposed in [5], must be regarded as at least questionable.

The formal inconsistency outlined before has not been addressed by Neish and Smith [5]. Apparently, this is due to the neglect of the logarithmic terms in (16) in their discussion of the match with the 'square' region II. In turn, they propose a vortex flow there which exhibits a velocity defect relative to the stagnating external flow of $O(1)$, in striking contrast to the expansion (17). Consequently, in [5, 2] both the magnitude of the velocities and the extent of the emerging region II are of $\gamma^{1/2}$.

Thus, the flow there is governed by the full Reynolds equations (1), rather than their linearised form (18). It is that fully nonlinear stage which prompted the authors of [5, 2] to conclude that separation would occur a distance of $O(\gamma^{1/2})$ upstream of S . Also, it is not explained in these papers how the flow region II is transformed into a turbulent shear layer along the separated streamline \mathcal{S} , which then coincides with the Y -axis, see Figure 2 (a).

A further uncertainty is raised by another issue put forward in [5]: it is argued that the position of smooth flow detachment approaches the rear stagnation point if one considers the limit $k \rightarrow \infty$ in (12). The flow situation for different values of k is sketched in Figure 2 (b), cf. [1]: from a topological point of view, the only candidate for a flow exhibiting free streamlines around a cylindrical body that neighbours the completely attached potential flow with a rear stagnation point S is the one which embeds a vanishingly small interior (cusp-shaped) cavity/eddy in the vicinity of S . However, it has not been demonstrated convincingly so far that such a solution is associated with correspondingly large values of k . We note that the class of inviscid flows having free streamlines is currently under investigation.

3. The Large-Defect Boundary Layer and Smooth Separation

The picture of separation considered in [5, 2] is apparently not in accordance with experimental findings. In fact, separation from a cylindrical body takes place a relatively short distance downstream of the location of its maximum cross-section, even for very large values of Re . This finding, together with the serious difficulties discussed in the previous section, then strongly suggests to abandon the assumption of a small-defect boundary layer in favour of a flow description where a streamwise velocity deficit of $O(1)$ is stipulated. As outlined in the introduction, such an asymptotic concept that (i) surmounts the difficulties in the matching procedure due to the logarithmic velocity distribution (5) encountered in Sychev's theory [11, 12], and (ii) is corroborated by any commonly used turbulence closure, has already been proven successful in the description of turbulent marginal separation, see [7, 6].

In this novel flow description the boundary layer thickness δ is measured by a small parameter α which is independent of Re as $Re \rightarrow \infty$. This most remarkable characteristic anticipates the existence of a turbulent shear layer of finite width with a wake-type flow, even in the formal limit $\alpha \rightarrow 0$, $\nu = 0$, included in (1). In that limit the unsteady flow in the wake region is presumably not affected significantly by the periodically occurring well-known wall layer bursts. As will turn out, this characteristic allows for an investigation of some properties associated with the unsteady motion on the basis of (2).

3.1. THE SLENDER-WAKE LIMIT

In the wake region the Reynolds stresses are quantities of $O(\alpha)$. Then the nonlinearities in the momentum equation (2b) suggest the expansions

$$[u, v, w, p] \sim [\bar{u}_0, 0, 0, \bar{p}_0](s, N) + \alpha^{1/2} [u'_1, v'_1, w'_1, p'_1](t, s, N, \dots) \\ + \alpha \{ [\bar{u}_2, \bar{v}_2, 0, \bar{p}_2](s, N) + [u'_2, v'_2, w'_2, p'_2](t, s, N, \dots) \} + O(\alpha^{3/2}), \quad (22a)$$

$$\delta/\alpha = \delta_0(s) + O(\alpha^2). \quad (22b)$$

In (22a) a suitable shear layer coordinate $N = n/\alpha$ is introduced, and the dots indicate dependences on inner spatial and time scales, which are specified later. Inserting (22) into (4) then gives rise to the shear layer approximation

$$\bar{p}_0(s, N) = p_0(s), \quad -dp_0/ds = u_e du_e/ds, \quad (23a)$$

$$\partial_s \bar{u}_0 + \partial_N \bar{v}_2 = 0, \quad \bar{u}_0 \partial_s \bar{u}_0 + \bar{v}_2 \partial_N \bar{u}_0 = -dp_0/ds - \partial_N(\overline{u'_1 v'_1}). \quad (23b)$$

The equations (23a) and (23b) govern the turbulent motion along the separating streamline \mathcal{S} to leading order sufficiently far from S , i.e. for $|s| = O(1)$, see Figure 1 (a). They are subject to the wake-type boundary conditions

$$\bar{v}_2(s, 0) = \overline{u'_1 v'_1}(s, 0) = 0, \quad \bar{u}_0(s, \delta_0(s)) - u_e(s) = \overline{u'_1 v'_1}(s, \delta_0(s)) = 0. \quad (23c)$$

By excluding the apparent trivial solution $\bar{u}_0 \equiv u_e(s)$, $\bar{v}_2 \equiv \overline{u'_1 v'_1} \equiv 0$, which implies a velocity defect of $o(1)$, we seek non-trivial solutions $\bar{u}_0, \bar{v}_2, \delta_0$ of (23) with respect to separation. To this end, it is useful to consider (23b) and (23a) evaluated for $N = 0$,

$$d(u_s^2 - u_e^2)/ds = -2[\partial_N(\overline{u'_1 v'_1})](s, 0). \quad (24)$$

Herein $u_s(s) = \bar{u}_0(s, 0)$ again denotes the slip velocity. Note, that separation is associated with flow reversal further downstream, which, in turn, requires $u_s(0) = 0$. To gain first insight how the boundary layer behaves as $s \rightarrow 0_-$, the problem (23) has been solved numerically, by adopting the same algebraic shear stress closure that was employed successfully for the boundary layer calculations in [7].

We again discard the possibility that the impressed potential flow exhibits a rear stagnation point S , since inspection of (24), confirmed by the numerical study, shows that then u_s not necessarily approaches zero in the vicinity of S . Therefore, the picture of a ‘collision’ of two boundary layers is apparently not appropriate for describing turbulent separation. Consequently, separation is seen to be associated with a smoothly separating inviscid flow, according to the situation sketched in Figure 2 (b). As outlined in [1], only flows having $k \geq 0$ are topologically possible. A suitable model for the surface velocity $u_e(s)$ that exhibits the then required local behaviour (12) is given by $u_e(s) = (3/2 + s)^m [1 + k(-2s)^{1/2}]/(1 + k)$, $-1/2 \leq s < 0$, such that $u_e(-1/2) = 1$. Here the exponent m represents an eigenvalue of the self-preserving solution for a given value of $u_s(-1/2)$, which serves as the initial condition for the downstream integration of (23), cf. [7]. Specifically, the value $u_s(0) = 0.95$ has been imposed, yielding $m \doteq -0.3292$. The distributions for the impressed adverse difference pressure $p_0(s) - p_0(0)$ and the resulting slip velocity $u_s(s)$ are plotted in Figure 3 for different values of the control parameter k . It is found that for sufficiently small values of k the integration terminates in a singular manner at $s = 0$ where u_s assumes a finite limit, i.e. $u_s(0) > 0$. For increasing values of k this threshold decreases, such that it finally vanishes for a critical value of k , say, $k = k_c$. We note that near $k = k_c$ the numerical calculations are very sensitive to slight variations in the value of k ; for the specific choice of $u_e(s)$ adopted here one finds that $k_c \doteq 2.7$. For $k > k_c$, however, the solution admits a Goldstein-type singularity at a position upstream of $s = 0$ which is discussed in more detail in [7]. Here we add, that a thorough analytical study of the numerically observed singular behaviour of the boundary layer solutions, also expressed through (24), is a task of the current research.

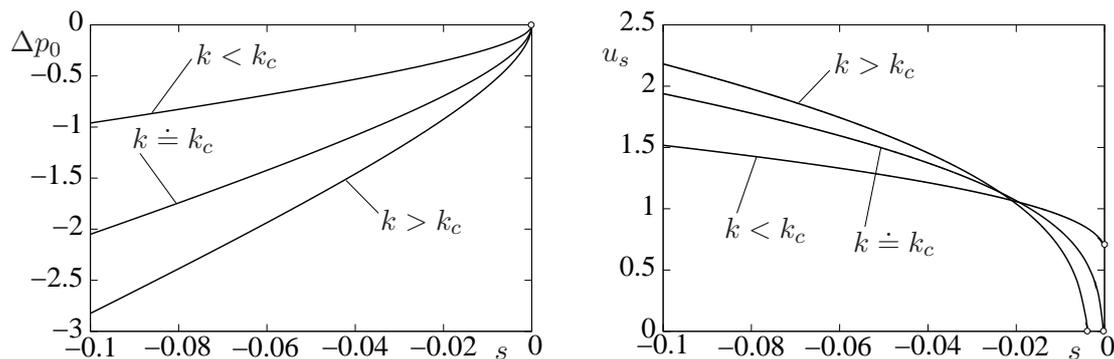


Figure 3. Local distributions of $\Delta p_0 = p_0(s) - p_0(0)$ and $u_s(s)$ for $k = 1.5$, $k = 2.7 \doteq k_c$, and $k = 3.4$; the circles indicate the occurrence of singular points.

As a first, rather remarkable, result, the location of turbulent break-away separation in the double limit $\alpha = \nu = 0$ is seen to be associated with a positive, presumably single-valued, value k_c of k , which has to be found by means of iterative boundary layer calculations. This strikingly contrasts its laminar counterpart, where the so-called Brillouin-Villat condition fixes the position of inviscid flow detachment by the requirement $k = 0$, see [10, 9]. Furthermore, the downstream shift of that point for increasing values of k , sketched in Figure 2 (b), explains why, in general, turbulent separation from a cylindrical body takes place further downstream as it is the case when the flow is still laminar. Moreover, first investigations indicate that in the turbulent case the more precise determination of the location of separation for small but finite values of both α and ν is determined by a locally strong rotational/irrotational interaction mechanism, analogous to that proposed in [11, 12].

3.2. INTERNAL STRUCTURE ‘DERIVED’ FROM FIRST PRINCIPLES

As the starting point, we consider the well-known transport equation for the time-mean value of the specific turbulent kinetic energy $\varkappa = \mathbf{u}' \cdot \mathbf{u}' / 2 \equiv (u'^2 + v'^2 + w'^2) / 2$, which results from Reynolds-averaging the inner product of \mathbf{u}' with (2b) by substituting (2a),

$$\overline{\Delta_t \bar{\varkappa}} + \nabla \cdot (\overline{\varkappa + p'} \mathbf{u}') - \nu \Delta \bar{\varkappa} + \varepsilon_p = -\overline{\mathbf{u}' \mathbf{u}'} : \nabla \bar{\mathbf{u}}, \quad \varepsilon_p := \nu \overline{\nabla \mathbf{u}' : \nabla \mathbf{u}'}. \quad (25)$$

Herein ε_p is commonly referred to as turbulent (pseudo-)dissipation. By taking into account (22), the least-degenerate shear layer approximation of (25) in the double limit $\alpha \rightarrow 0$, $\nu \rightarrow 0$ is found to be

$$\partial_N (\overline{p'_1 v'_1}) + \varepsilon_p \sim -\overline{u'_1 v'_1} \partial_N \bar{u}_0. \quad (26)$$

We integrate (26) across the shear layer thickness, i.e. from $N = 0$ to $N = \delta_0$. Then the net contribution of the diffusive term on the left-hand side of (26) is seen to vanish, whereas the resulting net turbulent ‘production’ on the right-hand side is positive and of $O(1)$ since both the Reynolds shear stress $-\overline{u'_1 v'_1}$ and the shear rate $\bar{u}_{0,N}$ are apparently non-negative. Remarkably, then ε_p is a quantity of $O(1)$ in the formal limit $\nu = 0$.

The quantity ε_p is obtained by Reynolds-, or equivalently, time-averaging according to (3), the stochastically varying quadratic form $\nabla \mathbf{u}' : \nabla \mathbf{u}'$. By adopting

the very weak assumption that the averaging process leaves its order of magnitude unchanged, we find, with some reservation, that

$$\nabla \mathbf{u}' = O(\nu^{-1/2}) \quad (27)$$

holds for the predominant fraction of intervals of the time t . As the most simple description of the fluctuating motion, we next assume that the turbulent fluctuations are governed by a single spatial ‘micro scale’, denoted by λ (together with a correspondingly small time scale) apart from the ‘macro scales’, represented by a streamwise length of $O(1)$ and the shear layer thickness of $O(\alpha)$. It then follows from the estimate (27) in combination with (22) that appropriate ‘micro variables’ are given by $(t', \mathbf{x}') = (t, \mathbf{x})/\lambda$ where $\mathbf{x}' = (s', n', z')$ and $\lambda = (\nu\alpha)^{1/2}$. That is, the smallest scales are measured by λ . Interestingly, they are asymptotically larger than the (non-dimensional) celebrated Kolmogorov length scale, which is commonly associated with the dissipative small-scale structure of turbulence and given by $(\nu^3/\varepsilon_p)^{1/4}$. Hence, the equations of motion (2) are expanded in the sequence of ‘inviscid’ linear equations

$$\nabla' \cdot \mathbf{u}'_i = 0, \quad (28a)$$

$$D'_t \mathbf{u}'_i + \mathbf{N}'_{i-1} = -\nabla' p'_i, \quad \mathbf{N}'_0 = \mathbf{0}, \quad D'_t = \partial_{t'} + \bar{u}_0(s, N) \partial_{s'}. \quad (28b)$$

Here and in the following $i = 1, 2, \dots$, $\mathbf{u}'_i = (u'_i, v'_i, w'_i)$, and ∇' denotes the gradient with respect to \mathbf{x}' . The inhomogeneous terms \mathbf{N}'_i in (28b) are defined by expanding the nonlinear convective operator in (2b) according to (22),

$$(\mathbf{u} \cdot \nabla' - \bar{u}_0 \partial_{s'}) \mathbf{u}' \sim \alpha^{1/2} \mathbf{N}'_1 + \alpha \mathbf{N}'_2 + \dots \quad (28c)$$

Then the vector \mathbf{N}'_i depends on the velocity fluctuations \mathbf{u}'_j where $j = 1, 2, \dots, i$. By eliminating the pressure fluctuations p'_i in (28b), the vorticity fluctuations $\boldsymbol{\omega}'_i$ are seen to satisfy the equations

$$D'_t \boldsymbol{\omega}'_i = -\nabla' \times \mathbf{N}'_{i-1}, \quad \boldsymbol{\omega}'_i = \nabla' \times \mathbf{u}'_i. \quad (28d)$$

Thus, $D'_t \boldsymbol{\omega}'_1 = \mathbf{0}$, so that $\boldsymbol{\omega}'_1$ depends on the ‘micro variables’ $\xi' = s' - \bar{u}_0 t'$, n' , and z' , but not explicitly on t' . In principle, \mathbf{u}'_1 then can be calculated from its Helmholtz decomposition, given by the distribution of $\boldsymbol{\omega}'_1$ together with the vanishing divergence as expressed by (28a). Therefore, \mathbf{u}'_1 and, in turn, \mathbf{N}'_1 also show no explicit dependence on t' , giving $\boldsymbol{\omega}'_2 = \mathbf{C}' - (\nabla' \times \mathbf{N}'_1)t'$, where \mathbf{C}' is a ‘constant’ of integration. The requirement that the expansion (22a) must be uniformly valid with respect to the ‘micro time’ t' then gives rise to the solvability condition $\nabla' \times \mathbf{N}'_1 = \mathbf{0}$. As a result, one recursively finds that $D'_t \boldsymbol{\omega}'_i = \mathbf{0}$ in general, such that the velocity and pressure fluctuations \mathbf{u}'_i and, p'_i , respectively, depend on ξ' , s' , n' , and z' , but, most important, not explicitly on t' , and are determined by

$$\nabla' \times \mathbf{N}'_i = \mathbf{0}, \quad \nabla' p'_i = -\mathbf{N}'_{i-1}. \quad (29)$$

Equations (29) describe a stationary motion with respect to ξ' , i.e. in a frame of reference which moves with the time-mean streamwise velocity $\bar{u}_0(s, N)$ along the separating streamline \mathcal{S} of the flow in the formal limit $\nu = 0$, see Figure 1 (a). Note, that they comprise the full nonlinear steady Euler equations, satisfied by \mathbf{u}'_1 and p'_2 .

This transport of the stochastic fluctuations along with the time-averaged flow found from the ‘micro-scales’ analysis is commonly termed as ‘coherent motion’. As a

further consequence of these considerations, the process of time-averaging according to (3) is seen to provide a filtering of the fluctuating motion with respect to ξ' and, in turn, rather not only with respect to the ‘micro time’ t' but also to the streamwise ‘micro variable’ s' . The view that the statistically stationary turbulent flow depends on the spatial ‘macro variables’ s and N only is, therefore, supported by an asymptotic investigation of the Navier–Stokes equations (2) and time-averaging.

The relationships (28) are valid for $i < I$ where the index I signifies contributions to (22) of $O(\nu^{1/2})$. For $i = I$ it follows from (2) that the dynamics of these contributions are affected by the viscous term on the right-hand side of (2b). Also, the normal gradient $\partial_N \bar{u}_0$ then enters the momentum balance as a consequence of the ‘macro scale’ α which describes the time-mean shear layer approximation. This in turn suggests the introduction of a further set $(t^\alpha, \mathbf{x}^\alpha) = (t, \mathbf{x})/\alpha$ of ‘micro variables’. Let ∇^α denote the gradient with respect to \mathbf{x}^α and $\mathbf{e}_s, \mathbf{e}_n$, and \mathbf{e}_z the unit vectors in the respective directions indicated by the subscripts. We then find

$$\nabla' \cdot \mathbf{u}'_I = -\nabla^\alpha \cdot \mathbf{u}'_1, \quad (30a)$$

$$\begin{aligned} D'_t \mathbf{u}'_I + \mathbf{N}'_{I-1} + D_t^\alpha \mathbf{u}'_1 + \mathbf{e}_s v'_1 \partial_N \bar{u}_0 &= -\nabla' p'_I - \nabla^\alpha p'_1 + \Delta' \mathbf{u}'_1, \\ D_t^\alpha &= \partial_{t^\alpha} + \bar{u}_0(s, N) \partial_{s^\alpha}, \quad \Delta' = \nabla' \cdot \nabla'. \end{aligned} \quad (30b)$$

From (29) it follows that p'_I is independent of \mathbf{x}' since \mathbf{N}'_0 vanishes according to (28b). By taking the curl with respect to \mathbf{x}' one then obtains from (30b)

$$D'_t \boldsymbol{\omega}'_I = -\nabla' \times \mathbf{N}'_{I-1} - D_t^\alpha \boldsymbol{\omega}'_1 - (\partial_N \bar{u}_0)(\mathbf{e}_n \partial_{z'} - \mathbf{e}_z \partial_{n'}) v'_1 + \Delta' \boldsymbol{\omega}'_1. \quad (30c)$$

The right-hand sides of both (30c) and (30a) do not explicitly depend on t' . With the same arguments leading to (29), then the Helmholtz decomposition of $\boldsymbol{\omega}'_I$ suggests that \mathbf{u}'_I and, as a consequence of (30b), p'_I exhibit no explicit t' -dependence too. In turn, the right-hand side of (30c) must vanish. Therefore, (30c) not only determines the quantity \mathbf{u}'_{I-1} , but can also be interpreted as a linear transport equation for the leading-order contribution $\boldsymbol{\omega}'_1$ to the vorticity with respect to the newly introduced time t^α and \mathbf{x}' . However, the motion which is affected by the viscous term in (2b) is presumably also governed by convective terms which are nonlinear in the leading-order contribution \mathbf{u}'_1 to the velocity fluctuations. But, in view of (30b), this is only possible by introducing a set of ‘intermediate micro variables’ $(\hat{t}, \hat{\mathbf{x}}) = (t, \mathbf{x})/\alpha^{3/2}$. Thus, the associated new length scale of $O(\alpha^{3/2})$ is much larger than the viscosity-affected one, λ , but still smaller than the shear layer thickness of $O(\alpha)$. We close the analysis by noting that this new length scale serves as a measure for the size of the large eddies in the wake region, and, in turn, of the mixing length. This fully agrees with the scaling of the latter found from the time-mean analysis, cf. [7].

4. Conclusions and Further Outlook

We have demonstrated that turbulent bluff-body separation requires a streamwise velocity defect of $O(1)$ as $Re \rightarrow \infty$ in the fully turbulent main region of the oncoming boundary layer, as the classical assumption of a small velocity deficit is intrinsically tied to the idea of a firmly attached external potential flow, and, in turn, leads to a serious inconsistency in the asymptotic hierarchy of the flow, which originates from an asymptotically small vicinity of the rear stagnation point. On the other hand, for

the large-defect boundary layer the limiting inviscid solution must be sought in the class of flows exhibiting a free streamline which departs smoothly from the surface. As one remarkable result strikingly contrasting a well-known finding in the theory of laminar separation, here the Brillouin-Villat condition is not met at the separation point. The formulation of the locally strong rotational/irrotational interaction of the separating flow with the external bulk flow is a topic of the current research. Future research activities include, amongst others, the asymptotic investigation of the unsteady motion. Most important, as a first step in this direction, it has been shown here how the underlying boundary layer concept is strongly supported by such an analysis. As one physically appealing result, an inner length of $O(\alpha^{3/2})$ reflecting the size of the large-scale eddies in the wake flow regime is found, which interestingly equals that of the mixing length given in [7].

Acknowledgements

This research was granted by the Austrian Science Fund (project no. P16555-N12), which is gratefully acknowledged.

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