# The Spherically Symmetric Standard Model with Gravity 

H. Balasin ${ }^{*}$, C. G. Böhmer ${ }^{\dagger}$ and D. Grumiller ${ }^{\ddagger}$<br>${ }^{* \dagger} \ddagger$ Institut für Theoretische Physik<br>Technische Universität Wien<br>Wiedner Hauptstr. 8-10, A-1040 Wien<br>Austria<br>${ }^{\ddagger}$ Institut für Theoretische Physik<br>Universität Leipzig<br>Augustusplatz 10-11, D-04103 Leipzig<br>Germany


#### Abstract

Spherical reduction of generic four-dimensional theories is revisited. Three different notions of "spherical symmetry" are defined. The following sectors are investigated: Einstein-Cartan theory, spinors, (non-)abelian gauge fields and scalar fields. In each sector a different formalism seems to be most convenient: the Cartan formulation of gravity works best in the purely gravitational sector, the Einstein formulation is convenient for the Yang-Mills sector and for reducing scalar fields, and the Newman-Penrose formalism seems to be the most transparent one in the fermionic sector. Combining them the spherically reduced Standard Model of particle physics together with the usually omitted gravity part can be presented as a two-dimensional (dilaton gravity) theory.


Key words: spherical symmetry, Standard Model, $2 d$ theories, dilaton gravity

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## 1 Introduction

Spherical symmetry plays a pivotal role in theoretical physics because the system simplifies such that an exact solution is often possible; this in turn allows for an understanding of some basic principles of the underlying dynamical system and thus can be of considerable pedagogical value. ${ }^{1}$

As compared to other frequently used scenarios, like the ultra-relativistic limit where the rest mass is much smaller than the kinetic energy or the static limit where the rest mass is much larger than the kinetic energy, spherical symmetry has the advantage that it allows for dynamics such as scattering of s-waves as opposed to the static case and for bound states as opposed to the ultra-relativistic limit. Moreover, many physical systems of relevance exhibit at least approximate spherical symmetry - to name a few: the $l=0$ sector of the Hydrogen atom, non-rotating isolated stars, the universe on large scales (actually isotropic with respect to any point), etc. Also semi-classically spherically symmetric modes are often the dominant ones - e.g. ca. $90 \%$ of the Hawking flux of an evaporating black hole is due to this sector (cf. e.g. [1]), the Balmer series stems from it (disregarding the finestructure), etc. From a technical point of view the success of spherical symmetry is related to the fact that systems in two dimensions $d=2$ have many favourable properties (cf. e.g. [2]).

However, the advantages of simplifications due to spherical symmetry become most apparent in the context of (quantum) gravity. As an illustration four selected examples are presented: Krasnov and Solodukhin discussed recently the wave equation on spherically symmetric black hole (BH) backgrounds [3]. They found an intriguing interpretation in terms of Conformal Field Theory, at least in certain limits (near horizon, near singularity and high damping), thus realizing 'tHooft's suggestion [4] of an analogy between strings and BHs. In the framework of canonical quantum gravity recently the concepts of quantum horizons [5] and quantum black holes [6] have been introduced for spherically symmetric systems. While the former work is inspired by the concept of isolated horizons [7], the latter invokes trapped surfaces and thus may be applied to dynamical horizons. Both confirm the heuristic picture that at the quantum level horizons fluctuate. One of the present authors together with Fischer, Kummer and Vassilevich considered scattering of s -waves on their own gravitational self-energy by means of two-dimensional methods, obtaining a simple but nontrivial S-matrix with virtual BHs as intermediate states [8] (for a review cf. [9]), in accordance with 'tHooft's idea that BHs have to be considered in

[^1]the S-matrix together with elementary matter fields [10]. Finally, the seminal numerical work by Choptuik [11] on critical collapse was based upon a study of the spherically symmetric Einstein-massless-Klein-Gordon model. Although similar features were found later in many other systems (for a review cf. [12]) we believe it is no coincidence that the crucial discovery was made first in the simpler spherically symmetric case.

In the first, third and fourth example the coupling to matter degrees of freedom was essential. It is therefore of some interest to study the most general coupling to matter consistent with observation, in particular the Standard Model of particle physics [13] or a recent improvement thereof [14].

The purpose of this work is 1 . to clarify what is meant by spherical symmetry; three different notions will be presented, 2. to review the basic formalisms that are useful in the context of spheric reduction (Cartan-, Geroch-Held-Penrose (GHP) and metric-formalism), 3. to apply them to obtain the spherically symmetric Standard Model plus gravity (SSSMG) in a comprehensive manner, 4 . to present an effective theory in $d=2$ which then in principle can be quantised. As byproducts several sectors will be discussed in technical detail. Necessarily, large part of this work have the character of a review. Nonetheless, several new results are contained in it: It is shown that static perfect fluids can be regarded as generalised dilaton gravity models. In the reduction of the Einstein-Yang-Mills-Dirac system we find an additional contribution that might have been overlooked in previous considerations. We discuss the symmetry restoration of spontaneous symmetry breaking by giving an interpretation to the effective Higgs potential. The Yukawa interaction is spherically reduced without fixing the isospin direction. We spherically reduce the torsion induced four fermion interaction term present in Einstein-Cartan theory. Finally we comment on the quantisation of the SSSMG.

We would also like to point out that one of the main aims of our work is to provide a link between concepts from particle physics (like the matter content inspired by the standard model) and input from general relativity like the spin-coefficient formalism that is particularly adequate for the reduction of spinors in a spherically symmetric context.

This paper is organised as follows: in Section 2 three different notions of spherical symmetry are discussed. Section 3 fixes the notation and introduces the three formalisms (Cartan, GHP, metric) by means of relevant and rather explicit examples. A brief recapitulation of dilaton gravity with matter by means of a discussion of static perfect fluid solutions is given. Collecting them the spherically symmetric Standard Model plus gravity is constructed in Section (4) It is presented as an effective theory in two dimensions (Section [5). The final Section 6 contains some concluding remarks. The appendices provide supplementary material mostly related to
the GHP formalism.

## 2 Three ways of spherical symmetry/reduction

In the following we will define different notions of spherical symmetry.
0 ) In order to be able to talk about spherical symmetry one needs an action of the rotation group $S O(3)$ on the spacetime manifold $\mathcal{M}$ under consideration. For the geometry we require that the vector-fields of the action $\xi$ leave the metric $g_{a b}$ unchanged, i.e., ${ }^{2}$

$$
\begin{equation*}
\mathcal{L}_{\xi} g_{a b}=\nabla_{(a} \xi_{b)}=0, \quad \xi \in S O(3) \tag{2.1}
\end{equation*}
$$

or equivalently that they are Killing. (Let us remark that the action is given by space-like vector-fields). This property will be assumed subsequently. It entails the form of the symmetry generators, a basis of $S O(3)$, and the metric

$$
\begin{array}{r}
\xi_{\phi}=i \partial_{\phi}, \quad \xi_{ \pm}=\frac{1}{\sqrt{2}} e^{ \pm i \phi}\left(\partial_{\theta} \pm i \cot \theta \partial_{\phi}\right) \\
\mathrm{d} s^{2}=g_{\alpha \beta}\left(x^{\alpha}\right) \mathrm{d} x^{\alpha} \otimes \mathrm{d} x^{\beta}-X\left(x^{\alpha}\right) \mathrm{d} \Omega^{2}, \quad \alpha, \beta=0,1 \tag{2.3}
\end{array}
$$

in adapted coordinates, where $\phi, \theta$ are the standard coordinates of (the round) $S^{2}, \mathrm{~d} \Omega^{2}=\mathrm{d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \phi^{2}$. The Killing vectors (2.2) obey the angular momentum algebra $\left[\xi_{ \pm}, \xi_{\phi}\right]= \pm \xi_{ \pm},\left[\xi_{+}, \xi_{-}\right]=\xi_{\phi}$.

The state of the physical systems under consideration will in addition to the metric also contain various matter fields which we denote by $\phi_{\alpha}$ and which are taken to be sections of various (vector-)bundles over spacetime. As long as those bundles are naturally tied to $\mathcal{M}$, i.e., are tensor products of the tangent- and cotangent bundles $T \mathcal{M}$ and $T^{*} \mathcal{M}$, the action of $\xi$ on their sections is well-defined.

1) For these matter fields strict spherical symmetry is defined by $\mathcal{L}_{\xi} \phi_{\alpha}=0$.

Example 1.1 (Reduction of scalar matter I)
The action for scalar matter in $d=4$ reads

$$
\begin{equation*}
L^{(4)}=\int\left(G^{\mu \nu} \nabla_{\mu} \phi \nabla_{\nu} \phi\right) \omega_{G} \tag{2.4}
\end{equation*}
$$

where $\phi$ is the scalar field and $\omega_{G}$ is the $4 d$ volume form. $\mathcal{L}_{\xi} \phi=0$ implies that in adapted coordinates $\phi=\phi(t, r)$ and hence the action (2.4) after

[^2]integrating out the angular part simply leads to the reduced action with $2 d$ volume form $\omega_{g}$
\[

$$
\begin{equation*}
L^{(2)}=4 \pi \int\left(g^{\alpha \beta} \nabla_{\alpha} \phi \nabla_{\beta} \phi\right) X \omega_{g} \tag{2.5}
\end{equation*}
$$

\]

However, in general we are also interested in bundles more loosely tied to the spacetime manifold (like Spin-bundles and principal $S U(N)$-bundles). For their sections the $S O(3)$ action on $\mathcal{M}$ does not automatically extend. However, in the above mentioned cases there exist certain "natural" notions, which allow an action to be defined on the sections of these bundles. In general strict invariance will not be possible (or too restrictive).

Example 1.2 (No strictly spherically symmetric spinors)
Let $k^{A}=k^{0} o^{A}+k^{1} \iota^{A}$ be an arbitrary spinor. This spinor would be called strictly spherically symmetric if one could solve $\mathcal{L}_{\xi} k^{A}=0$ for non-trivial $k^{0}$ and $k^{1}$. Direct calculation easily shows that this is impossible.
2) For these fields only "covariant" transformation behaviour is possible. They will be called spherically symmetric if

$$
\begin{equation*}
\mathcal{L}_{\xi} \phi_{\alpha}=D(\xi) \phi_{\alpha} \tag{2.6}
\end{equation*}
$$

where $D$ refers to a typically linear transformation, e.g. a derivative operator.

Example 2.1 (Gauge fields)
A gauge field obeying $\mathcal{L}_{\xi_{i}} A=D W_{i}$, where $D$ is the gauge covariant derivative; thus, the field $A$ itself need not be strictly spherically symmetric, only up to gauge transformations.

Finally an even less stringent form, which we call weak spherical symmetry, may be defined by expanding the fields with respect to a complete set of eigenfunctions of the spherical Laplacian.
3) For these fields of $\operatorname{spin} s$, we decompose

$$
\begin{equation*}
\Delta_{S^{2} s} Y_{j, m}=-(j-s)(j+s+1)_{s} Y_{j, m}, \quad \phi_{\alpha}=\sum_{j m} \phi_{\alpha, j m s} Y_{j, m} \tag{2.7}
\end{equation*}
$$

where ${ }_{s} Y_{j, m}$ are the spin-weighted spherical harmonics (for $s=0$ they coincide with the standard spherical harmonics while for higher spin we refer the reader to subsection 3.3). On the dynamical level, i.e., upon insertion into the action and integration of the angular part, this yields a spherically reduced (two-dimensional) system.

Example 3.1 (Reduction of scalar matter II)
The action for scalar matter reads

$$
\begin{equation*}
L=\int\left(G^{\mu \nu} \nabla_{\mu} \phi \nabla_{\nu} \phi\right) \omega_{G} \tag{2.8}
\end{equation*}
$$

where $\phi$ is the scalar field. Expanding $\phi$ in terms of spherical harmonics $\phi=\sum_{l m} \phi_{l m} Y_{l m}$, the scalar action (2.8) upon integration of the angular part leads to

$$
\begin{equation*}
L=\sum_{l m} \int\left(g^{\alpha \beta} \nabla_{\alpha} \phi_{l m} \nabla_{\beta} \phi_{l m}+\frac{l(l+1)}{X} \phi_{l m} \phi_{l m}\right) X \omega_{g} \tag{2.9}
\end{equation*}
$$

where the s-wave sector $l=0$ corresponds to (2.5).
Example 3.2 (Spherically reduced gravity)
Spherically reduced gravity (SRG) emerges from averaging over the angular part,

$$
<R_{\mu \nu}>-1 / 2<g_{\mu \nu} R>=\kappa<T_{\mu \nu}>
$$

where $R_{\mu \nu}$ is the Ricci tensor, $T_{\mu \nu}$ is the energy-momentum tensor, $\kappa$ is the gravitational coupling and the bracket denotes integration over the angular part - this system of averaged equations of motion can be deduced from an action in $d=2$, the geometric part of which is just the spherically reduced Einstein-Hilbert action. For the Einstein-massless-Klein-Gordon model the matter part (2.9) contains an infinite tower of scalar fields with dilaton dependent (and l-dependent) mass.

Each of these notions is weaker than its predecessor:
strict sph. sym. $\geq$ spherical symmetry $\geq$ weak sph. sym.
For the rest of this paper we assume spherical symmetry according to the second notion, unless stated otherwise.

Having defined spherical symmetry we would like to focus on spherical reduction. By this procedure we mean the derivation of a reduced action in $d=2$ the equations of motion of which are equivalent (in a well-defined way) to the equations of motion of the original theory if the latter are restricted to spherical symmetry. The fact that such a procedure works is not trivial in general (i.e., if the isometry group is different from $S O(3)$ ). Due to the compactness of $S O(3)$, however, one can immediately apply Theorem 5.17 (or proposition 5.11) of [15] and employ the "principle of symmetric criticality" [16] which guarantees the (classical) equivalence of the reduced theory to the original one (cf. also [17]). The main advantage of spherical reduction is the possibility to exploit the simplicity of two dimensional field theories.

## 3 Three formalisms

The purpose of this section is threefold: the three relevant formalisms are reviewed together with their respective advantages, relevant examples are considered and en passant our notation is fixed in detail.

### 3.1 Cartan's form calculus and Gravity

In the Cartan formalism one works in an anholonomic frame and uses the vielbein 1-form and connection 1-form as independent variables. With these variables one can use the advantages of the form calculus, where diffeomorphism invariance is implied automatically, see e.g. [18].

### 3.1.1 The 2-2 split

In the Cartan formalism the line element can be written as

$$
\begin{equation*}
\mathrm{d} s^{2}=g_{\mu \nu} \mathrm{d} x^{\mu} \otimes \mathrm{d} x^{\nu}=e_{\mu}^{m} e_{\nu}^{n} \eta_{m n} \mathrm{~d} x^{\mu} \otimes \mathrm{d} x^{\nu}=\eta_{m n} e^{m} \otimes e^{n} \tag{3.1}
\end{equation*}
$$

Greek letters are used for holonomic indices and Latin letters for anholonomic ones. $\eta_{m n}$ is the flat (Minkowski) metric with signature $(+,-,-,-)$. The vielbein is denoted by $E_{m}^{\mu}$,

$$
\begin{equation*}
\left.E_{m}^{\mu} e_{\mu}^{n}=\delta_{m}^{n}, \quad E_{m}\right\lrcorner e^{n}=\delta_{m}^{n} \tag{3.2}
\end{equation*}
$$

where $\lrcorner$ means contraction. One similarly writes the vector field $E_{m}=$ $E_{m}^{\mu} \partial_{\mu}$. The covariant derivative is written as

$$
\begin{equation*}
\tilde{D}^{m}{ }_{n}=\delta_{n}^{m} \mathrm{~d}+\tilde{\omega}^{m}{ }_{n} \tag{3.3}
\end{equation*}
$$

with the skew-symmetric connection 1-from $\tilde{\omega}_{m n}=-\tilde{\omega}_{n m}$, because of metricity. The connection 1-form may be split accordingly

$$
\begin{equation*}
\tilde{\omega}^{m}{ }_{n}=\omega^{m}{ }_{n}+K^{m}{ }_{n}, \tag{3.4}
\end{equation*}
$$

where $\omega^{m}{ }_{n}$ is the torsion free part and $K^{m}{ }_{n}$ is the contortion.
Acting with (3.3) on the vielbein $e^{m}$ and on the connection 1-form $\tilde{\omega}^{m}{ }_{n}$ defines the torsion 2 -form and the curvature 2-form, respectively

$$
\begin{align*}
T^{m} & =(\tilde{D} e)^{m}=\mathrm{d} e^{m}+\tilde{\omega}^{m}{ }_{n} e^{n}=K^{m}{ }_{n} e^{n}=\frac{1}{2} T^{m}{ }_{\mu \nu} \mathrm{d} x^{\mu} \mathrm{d} x^{\nu}  \tag{3.5}\\
R^{m}{ }_{n} & =\left(\tilde{D}^{2}\right)^{m}{ }_{n}=\mathrm{d} \tilde{\omega}^{m}{ }_{n}+\tilde{\omega}^{m}{ }_{l} \tilde{\omega}^{l}{ }_{n}=\frac{1}{2} R^{m}{ }_{n \mu \nu} \mathrm{~d} x^{\mu} \mathrm{d} x^{\nu} \tag{3.6}
\end{align*}
$$

Note that we avoid writing out the wedge product explicitly.

In case of spherical symmetry one can separate the metric (3.1)

$$
\begin{equation*}
\mathrm{d} s^{2}=\eta_{m n} e^{m} \otimes e^{n}=\eta_{a b} e^{a} \otimes e^{b}-\delta_{r s} e^{s} \otimes e^{r} \tag{3.7}
\end{equation*}
$$

where the indices $(\alpha, \beta, \ldots ; a, b, \ldots)$ denote quantities of the two-dimensional manifold $L$ and letters $(\rho, \sigma, \ldots ; r, s, \ldots)$ quantities connected with the sphere $S^{2}$. Moreover the dilaton $X$ from (2.3) has been redefined as $X=\Phi^{2}$ in order to avoid square-roots in subsequent formulas. Barred ("intrinsic") and unbarred quantities are related by

$$
\begin{align*}
e^{a} & =\bar{e}^{a}, & & e^{r} \\
E_{a} & =\bar{E}_{a}, & & E_{r} \tag{3.8}
\end{align*}=\frac{1}{\Phi} \bar{E}_{r} .
$$

The torsion free connection 1-form $\omega^{m}{ }_{n}$ is given by

$$
\begin{align*}
\omega^{a}{ }_{b}=\bar{\omega}^{a}{ }_{b}, & \omega^{a}{ }_{r}=\left(\bar{E}^{a} \Phi\right) \bar{e}_{r}, \\
\omega^{r}{ }_{s}=\bar{\omega}^{r}{ }_{s}, & \omega^{r}{ }_{a}=\left(\bar{E}_{a} \Phi\right) \bar{e}^{r} \tag{3.9}
\end{align*}
$$

### 3.1.2 Reduction of torsion and curvature

Gauge covariant transformation behaviour under spherical symmetry restricts the possible contributions of torsion [19] according to

$$
\begin{equation*}
\mathcal{L}_{\xi} T^{a}=0, \quad \mathcal{L}_{\xi} T^{r}=\Omega^{r}{ }_{s} T^{s}, \tag{3.10}
\end{equation*}
$$

where the skew-symmetric matrix $\Omega$ is defined by $\mathcal{L}_{\xi} e^{r}=\Omega^{r}{ }_{s} e^{s}$. This together with the non-existence of a rotationally invariant vector field on $S^{2}$ - entails the decomposition

$$
\begin{equation*}
T^{a}=\bar{T}^{a}+\frac{1}{2} T_{r s}^{a} e^{r} e^{s}, \quad T^{r}=T_{a s}^{r} e^{a} e^{s} \tag{3.11}
\end{equation*}
$$

Consequently, the sphere $S^{2}$ has to be torsion free intrinsically, i.e., $\left.\bar{E}_{t}\right\lrcorner K^{r}{ }_{s}=$ 0.

The connection need not be strictly spherically symmetric but only symmetric up to gauge transformations (much like in the Yang-Mills case below). Expanding (3.10 according to our 2-2 split the conditions read

$$
\begin{align*}
& \mathrm{d} \mathcal{L}_{\xi} e^{a}+\mathcal{L}_{\xi}\left(\tilde{\omega}^{a}{ }_{b} e^{b}\right)+\mathcal{L}_{\xi}\left(\tilde{\omega}^{a}{ }_{r} e^{r}\right)=0  \tag{3.12}\\
& \mathrm{~d} \mathcal{L}_{\xi} e^{r}+\mathcal{L}_{\xi}\left(\tilde{\omega}^{r}{ }_{a} e^{a}\right)+\mathcal{L}_{\xi}\left(\tilde{\omega}^{r}{ }_{s} e^{s}\right)=\Omega^{r}{ }_{s} T^{s} . \tag{3.13}
\end{align*}
$$

Because of $\mathcal{L}_{\xi} e^{a}=0=\mathcal{L}_{\xi} \tilde{\omega}^{a}{ }_{b}$ equation (3.12) establishes

$$
\begin{equation*}
\mathcal{L}_{\xi} K^{a}{ }_{r}=\tilde{k}^{a} e_{r}+\tilde{h}^{a} \varepsilon_{r s} e^{s}, \quad \tilde{k}^{a}=\tilde{k}^{a}\left(x^{\alpha}\right), \tilde{h}^{a}=\tilde{h}^{a}\left(x^{\alpha}\right) . \tag{3.14}
\end{equation*}
$$

Plugging (3.4) and (3.9) into (3.13) yields

$$
\begin{equation*}
\mathcal{L}_{\xi} K^{r}{ }_{a}=\Omega^{r}{ }_{s} K^{s}{ }_{a}, \tag{3.15}
\end{equation*}
$$

This means that the right hand side of (3.14) is not only valid for $\mathcal{L}_{\xi_{i}} K^{r}{ }_{a}$ but also for $K^{a}{ }_{r}$, which suggests the useful definitions of vector valued scalars

$$
\begin{equation*}
\left.\left.k_{a}:=\bar{E}_{r}\right\lrcorner K_{a}^{r}, \quad h_{a}:=\bar{E}^{s}\right\lrcorner K_{a}^{r} \varepsilon_{r s} . \tag{3.16}
\end{equation*}
$$

The following observation is helpful: The intrinsic torsion $\bar{T}^{a}$ of the reduced $2 d$ theory is irrelevant as there is no way to couple sources to it (because the $2 d$ connection is not present in the action of $2 d$ fermions). Thus, it can be demanded always $\left.\bar{E}_{c}\right\lrcorner K^{a}{ }_{b}=0$. Obviously, due to spherical symmetry also $\left.\bar{E}_{r}\right\lrcorner K^{a}{ }_{b}=0$. Thus, $K^{a}{ }_{b}$ can be set to zero.

The curvature 2 -form (3.6) can be calculated using (3.4), (3.9) and taking into account the previous considerations on torsion. The $2-2$ split for curvature and torsion yields the result

$$
\begin{array}{rlrl}
R_{b}^{a} & =\bar{R}^{a}{ }_{b}+\mathcal{R}^{a}{ }_{b}, & R^{r}{ }_{s} & =\bar{R}_{s}^{r}\left(1+\left(\bar{E}_{a} \Phi\right)\left(\bar{E}^{a} \Phi\right)\right)+\mathcal{R}^{r}{ }_{s}, \\
R_{r}^{a} & =-\eta^{a b} R_{r b}, & R_{a}^{r} & =\left(\bar{E}_{b} \bar{E}_{a} \Phi\right) \bar{e}^{b} \bar{e}^{r}-\left(\bar{E}_{b} \Phi\right) \omega^{b}{ }_{a} \bar{e}^{r}+\mathcal{R}^{r}{ }_{a}, \\
T^{a} & =K^{a}{ }_{r} e^{r}, & T^{r} & =K^{r}{ }_{a} e^{a}+K^{r}{ }_{s} e^{s}, \\
K^{a}{ }_{b} & =0, & K^{r}{ }_{s} & =\varepsilon^{r}{ }_{s} s_{a} e^{a}, \\
K_{r}^{a} & =-\eta^{a b} K_{r b}, & & K^{r}{ }_{a}  \tag{3.21}\\
=-\frac{1}{2}\left(k_{a} \bar{e}^{r}+h_{a} \varepsilon^{r s} \bar{e}_{s}\right),
\end{array}
$$

with $\bar{R}^{r}{ }_{s}=\bar{e}^{r} \bar{e}_{s}$ being the curvature 2-form of $S^{2}$ and $\bar{R}^{a}{ }_{b}$ being the intrinsic curvature in $2 d$. The contortion contributions to curvature read

$$
\begin{align*}
\mathcal{R}^{r}{ }_{s}= & \omega^{r}{ }_{a} K^{a}{ }_{s}+K^{r}{ }_{a} \omega^{a}{ }_{s}+K^{r}{ }_{a} K^{a}{ }_{s} \\
& +\mathrm{d} K^{r}{ }_{s}+\omega^{r}{ }_{t} K^{t}{ }_{s}+K^{r}{ }_{t} \omega^{t}{ }_{s}  \tag{3.22}\\
\mathcal{R}^{r}{ }_{a}= & \mathrm{d} K^{r}{ }_{a}+K^{r}{ }_{b} \omega^{b}{ }_{a}+K^{r}{ }_{s} K^{s}{ }_{a} \\
& +\varepsilon^{r}{ }_{s} s_{b} \bar{e}^{b}\left(E_{a} \Phi\right) \bar{e}^{s}-\frac{1}{2} \omega^{r}{ }_{s}\left(k_{a} \bar{e}^{s}+h_{a} \varepsilon^{s r} \bar{e}_{r}\right),  \tag{3.23}\\
\mathcal{R}^{a}{ }_{b}= & 0 \\
& -\frac{1}{4} \eta^{a c}\left(2\left(\bar{E}_{[c} \Phi\right)+k_{[c}\right) h_{b]} \bar{e}_{r} \varepsilon^{r s} \bar{e}_{s}, \tag{3.24}
\end{align*}
$$

where we use $T_{[\mu \nu]}:=T_{\mu \nu}-T_{\nu \mu}$. Note that in each equation the second line does not contribute to the curvature scalar because the corresponding contractions vanish. Thus, for instance, the contortion contribution $\mathcal{R}^{a}{ }_{b}$ does not produce any terms in the Einstein-Hilbert action.

As compared to the torsionless case additional effective fields are obtained: three vector valued scalars (depending on $x^{\alpha}$ ), $k_{a}, h_{a}$ and $s_{a}$. Depending on the original action in $d=4$ some of these fields might drop.

### 3.1.3 Reduction of the Einstein-Hilbert action

Double contraction,

$$
\begin{equation*}
\left.\left.\left.\left.\tilde{R}=R+2 E^{a}\right\lrcorner E_{r}\right\lrcorner \mathcal{R}^{r}{ }_{a}+E^{s}\right\lrcorner E_{r}\right\lrcorner \mathcal{R}^{r}{ }_{s}, \tag{3.25}
\end{equation*}
$$

yields the torsion free curvature scalar $R$ in terms of the two-dimensional one $R^{L}$, terms coming from intrinsic and extrinsic curvature of $S^{2}$ and torsion terms

$$
\begin{align*}
\tilde{R}=R^{L}-\frac{2}{\Phi^{2}}(1 & \left.+\left(\nabla_{a} \Phi\right)\left(\nabla^{a} \Phi\right)\right)-\frac{4}{\Phi}\left(\nabla_{a} \nabla^{a} \Phi\right) \\
& +\frac{1}{\Phi^{2}} \frac{1}{2}\left(h_{a} h^{a}-k_{a} k^{a}\right)+\frac{2}{\Phi} s_{a} h^{a}+\frac{2}{\Phi^{2}} \nabla_{a}\left(k^{a} \Phi\right) \tag{3.26}
\end{align*}
$$

The first line coincides with the torsion-free result (e.g. equation (A.8) of reference [20]). Note that in (3.26) $\nabla_{a}$ is the covariant derivative operator with respect to $L$. The last term together with the volume form produces just a surface term. Thus, as can be expected on general grounds [21] torsion is not propagating in the Einstein-Hilbert action.

For easier comparability with later results the anholonomic components of the contortion 1-form $K^{m}{ }_{n}=K_{l}{ }^{m}{ }_{n} e^{l}$ are decomposed into the contortion vector $k_{a}$ like in (3.16),

$$
\begin{equation*}
k_{a}=\Phi K_{r a}^{r}, \quad A^{a}=\frac{1}{3!} \varepsilon^{a l m n} K_{l m n} \tag{3.27}
\end{equation*}
$$

and the axial contortion vector $A^{a}$. The remaining components of the contortion tensor are denoted by $U_{l m n}$. Then the curvature scalar (3.26) can alternatively be written

$$
\begin{align*}
\tilde{R}=R^{L}-\frac{2}{\Phi^{2}}\left(1+\left(\nabla_{a} \Phi\right)\right. & \left.\left(\nabla^{a} \Phi\right)\right)-\frac{4}{\Phi}\left(\nabla_{a} \nabla^{a} \Phi\right) \\
& -\frac{1}{\Phi^{2}} \frac{1}{2} k^{2}-U^{2}-6 A^{2}+\frac{2}{\Phi^{2}} \nabla_{a}\left(k^{a} \Phi\right) \tag{3.28}
\end{align*}
$$

where

$$
\begin{align*}
6 A^{2} & =-\frac{2}{3}\left(s_{a} s^{a}+\frac{1}{\Phi^{2}} h_{a} h^{a}+\frac{2}{\Phi} s_{a} h^{a}\right)  \tag{3.29}\\
U^{2} & =\frac{2}{3}\left(s_{a} s^{a}+\frac{1}{4 \Phi^{2}} h_{a} h^{a}-\frac{1}{\Phi} s_{a} h^{a}\right) \tag{3.30}
\end{align*}
$$

This second form of the curvature scalar is used in section 4.5. Moreover the separation of the contortion tensor into its irreducible parts is often found in literature [22].

In the absence of torsion spheric reduction [23] of the Einstein-Hilbert action $L_{\mathrm{EH}}=\int_{M} R \omega_{G}$ yields the dilaton gravity action

$$
\begin{equation*}
L_{\mathrm{dil}}\left[g_{\alpha \beta}, X\right]=4 \pi \int_{L}\left(X R^{L}+(\nabla X)^{2} /(2 X)-2\right) \omega_{g} \tag{3.31}
\end{equation*}
$$

where $\omega_{G}=\Phi^{2} \omega_{g} \mathrm{~d}^{2} \Omega$. $M$ denotes the four-dimensional manifold and $L$ its two-dimensional Lorentzian part.

It is convenient to reformulate this second order action ${ }^{3}$ as a first order one [26] and to rescale the dilaton as $X \rightarrow \lambda^{2} X$ in order to make it dimensionless,

$$
\begin{equation*}
L_{\mathrm{FOG}}\left[e^{a}, \omega, X, X^{a}\right]=\frac{2 \pi}{\lambda^{2}} \int_{L}\left[X_{a}(D \wedge e)^{a}+X \mathrm{~d} \omega+\mathcal{V}\left(X, X^{a} X_{a}\right) \epsilon\right] \tag{3.32}
\end{equation*}
$$

with $\mathcal{V}=-X_{a} X^{a} /(4 X)-\lambda^{2}$. Whenever a first order action in $d=2$ is presented for sake of compatibility with [20] the following notation is used: in accordance with above $e^{a}$ is the zweibein 1-form, $\epsilon=e^{+} \wedge e^{-}$is the volume 2-form. The 1-form $\omega$ represents the spin-connection ${ }^{4} \omega^{a}{ }_{b}=\varepsilon^{a}{ }_{b} \omega$ with the totally antisymmetric Levi-Civitá symbol $\varepsilon_{a b}\left(\varepsilon_{01}=+1\right)$. With the flat metric $\eta_{a b}$ in light-cone coordinates ( $\eta_{+-}=1=\eta_{-+}, \eta_{++}=$ $0=\eta_{--}$) the first ("torsion") term of (3.32) is given by $X_{a}(D \wedge e)^{a}=$ $\eta_{a b} X^{b}(D \wedge e)^{a}=X^{+}(\mathrm{d}-\omega) \wedge e^{-}+X^{-}(\mathrm{d}+\omega) \wedge e^{+}$. Signs and factors of the Hodge- $*$ operation are defined by $* \epsilon=1$. The auxiliary fields $X, X^{a}$ can be interpreted as Lagrange multipliers for geometric curvature and torsion, respectively. $X^{ \pm}$correspond to the expansion spin coefficients $\rho, \rho^{\prime}$ (both are real in case of spherical symmetry, see below).

All classical solutions can be obtained with particular ease from (3.32) not only locally, but globally [27].

Even in the presence of torsion the reduced equations of motion enforce vanishing torsion unless matter couplings to torsion exist. Such a discussion will be postponed because fermion fields - which are the topic of the next section - will be needed as sources. It will turn out that the field $k_{a}$, the contortion vector, decouples from the theory even in the presence of fermions.

### 3.2 Dilaton gravity with matter

In this subsection dilaton gravity with matter is discussed. Although we specialise to spherically reduced gravity the following is still valid for generic

[^3]dilaton gravity theories, which means for generic functions $U(X)$ and $V(X)$, combined in the potential $\mathcal{V}=X^{+} X^{-} U+V$.

Spherical reduction produces (3.32) with $\mathcal{V}=-X^{+} X^{-} /(2 X)-\lambda^{2}$, where $\lambda$ is a physical parameter which can be scaled to 1 by redefining the units. By a conformal transformation $e^{a} \rightarrow \tilde{e}^{a}=e^{a} \Omega$ with conformal factor $\Omega=X^{1 / 4}$ the transformed dilaton potential $\tilde{V}=-2 \lambda^{2} \sqrt{X}$ becomes independent of $X^{ \pm}$. Choosing such a conformal frame is often helpful, however we will not specify the conformal frame for the time being.

It will be assumed that $X^{+} \neq 0$ in a given patch. If $X^{+}=0$ and $X^{-} \neq 0$ everything can be repeated with $+\leftrightarrow-$. If both $X^{+}=0=X^{-}$ in an open region a constant dilaton vacuum is encountered, which will not be discussed here (but they are rather trivial anyhow). If $X^{+}=0=X^{-}$ at an isolated point typically this corresponds to a bifurcation 2-sphere. This slight complication will be neglected here as it is not essential for the present discussion. ${ }^{5} X^{+} X^{-}=0$ corresponds to an apparent horizon, which in the static case is a Killing horizon.

The generic Ansatz for the energy-momentum 1-form is

$$
\begin{equation*}
W^{ \pm}=W_{X}^{ \pm} \mathrm{d} X+W_{Z}^{ \pm} Z \tag{3.33}
\end{equation*}
$$

where the 1 -form $Z$ is defined by

$$
\begin{equation*}
Z:=\frac{e^{+}}{X^{+}} \tag{3.34}
\end{equation*}
$$

For the following it will make sense to further specify (3.33):

$$
\begin{array}{ll}
W_{X}^{+}=X^{+} T_{1}, & W_{Z}^{+}=X^{+} T_{2} \\
W_{X}^{-}=X^{-} T_{3}, & W_{Z}^{-}=X^{-} T_{4} \tag{3.35}
\end{array}
$$

which is only allowed in the absence of horizons. The EOM

$$
\begin{align*}
& \mathrm{d} X+X^{-} e^{+}-X^{+} e^{-}=0  \tag{3.36}\\
& (\mathrm{~d} \pm \omega) X^{ \pm} \mp \mathcal{V} e^{ \pm}+W^{ \pm}=0  \tag{3.37}\\
& \mathrm{~d} \omega+\frac{\partial \mathcal{V}}{\partial X} \epsilon+W \epsilon=0  \tag{3.38}\\
& (\mathrm{~d} \pm \omega) \wedge e^{ \pm}+\frac{\partial \mathcal{V}}{\partial X^{\mp}} \epsilon=0 \tag{3.39}
\end{align*}
$$

Let us emphasise that $\omega$ is the Levi-Civitá connection only in a conformal frame with $U=0$, i.e., $\mathcal{V}=V(X)$. Together with (3.33) and (3.35)

[^4]immediately imply the following relations:
\[

$$
\begin{align*}
& e^{-}=\frac{\mathrm{d} X}{X^{+}}+X^{-} Z  \tag{3.40}\\
& \epsilon=e^{+} \wedge e^{-}=Z \wedge \mathrm{~d} X  \tag{3.41}\\
& \omega=-\frac{\mathrm{d} X^{+}}{X^{+}}+\mathcal{V} Z-\frac{W^{+}}{X^{+}}  \tag{3.42}\\
& \mathrm{d} Z=\left(T_{1}+U(X)\right) \mathrm{d} X \wedge Z \tag{3.43}
\end{align*}
$$
\]

where in addition

$$
\begin{align*}
\mathrm{d}\left(X^{+} X^{-}\right)+V & (X) \mathrm{d} X+X^{+} X^{-} U(X) \mathrm{d} X \\
& +X^{+} X^{-}\left(T_{1}+T_{3}\right) \mathrm{d} X+X^{+} X^{-}\left(T_{2}+T_{4}\right) Z=0 \tag{3.44}
\end{align*}
$$

indicates the existence of a conserved quantity. The line element can easily be computed to be

$$
\begin{equation*}
\mathrm{d} s^{2}=2 e^{+} \otimes e^{-}=2 X^{+} X^{-} Z \otimes Z+2 Z \otimes \mathrm{~d} X \tag{3.45}
\end{equation*}
$$

which follows from (3.34) and (3.40) and takes the usual Eddington-Finkelstein gauge,

$$
g_{\alpha \beta}=\left(\begin{array}{cc}
2 X^{+} X^{-} & 1  \tag{3.46}\\
1 & 0
\end{array}\right), \quad g^{\alpha \beta}=\left(\begin{array}{cc}
0 & 1 \\
1 & -2 X^{+} X^{-}
\end{array}\right)
$$

By virtue of the previous relations one obtains the minus part of (3.39)

$$
\begin{align*}
\mathrm{d}\left(X^{+} X^{-}\right) \wedge Z+V(X) \mathrm{d} X+X^{+} & X^{-} U(X) \mathrm{d} X \\
& +\left(2 X^{+} X^{-} T_{1}-T_{2}\right) \mathrm{d} X \wedge Z=0 \tag{3.47}
\end{align*}
$$

which together with (3.44) implies

$$
\begin{equation*}
T_{2}=X^{+} X^{-}\left(T_{1}-T_{3}\right) \tag{3.48}
\end{equation*}
$$

Therefore we are left with three independent functions. This is of course expected since any symmetric two-dimensional energy-momentum tensor has only three independent components.

For later use it is important to relate the generic energy-momentum 1-form (3.33) with the energy-momentum tensor $T^{\alpha \beta}$ obtained by varying the matter Lagrangian with respect to the metric. It would be tempting to vary the matter Lagrangian with respect to the 1 -form $e^{a}$ and relate this object directly with the energy-momentum tensor. However, in the four dimensional case variation of the matter Lagrangian with respect to
the 1 -form $e^{m}$ gives a 3 -form and its dual defines the energy-momentum tensor. Therefore one finds

$$
\begin{align*}
\delta L^{m}=\int \delta e^{a} \wedge W_{a}=\frac{1}{2} \int \delta_{\alpha \beta}^{\gamma \delta} \delta e_{\gamma}^{a} W_{a \delta} \mathrm{~d} x^{\alpha} \wedge \mathrm{d} x^{\beta} & \\
& =\int W_{a \alpha} \varepsilon^{\alpha \beta} \delta e_{\beta}^{a} \epsilon=\int T_{a}^{\alpha} \delta e_{\alpha}^{a} \epsilon \tag{3.49}
\end{align*}
$$

where $\delta_{\alpha \beta}^{\gamma \delta}$ is the permutation symbol, and which together with (3.48) implies the following form the energy-momentum tensor $T^{\alpha \beta}$ are related by

$$
\begin{equation*}
T^{\alpha \beta}=\varepsilon^{\gamma \alpha} W_{\gamma}^{a} E_{a}^{\beta} \tag{3.50}
\end{equation*}
$$

Hence we find the following energy-momentum tensor

$$
T^{\alpha \beta}=\left(\begin{array}{cc}
T_{1} & -T_{2}  \tag{3.51}\\
-T_{2} & X^{+} X^{-}\left(T_{2}-T_{4}\right)
\end{array}\right) .
$$

Note again, that we are working in the Eddington-Finkelstein gauge.
So far all EOMs have been exploited except for two; one of them yields the local Lorentz angle (i.e., it determines the ratio of $X^{-} / X^{+}$), which is not of interest here, while the other one yields the dilaton current $W$. In addition to the equations of motion one has one more equation, namely the covariant conservation of the energy-momentum tensor. In the non-static case we cannot do much more but if in addition staticity is assumed, we can solve the equations of motion.

### 3.2.1 Static and spherically symmetric matter

In the following staticity is assumed. Then the equations of motion simplify considerably and the conservations equation (3.44) can be integrated. For static solutions of generic dilaton gravity models cf. e.g. [29-31]. Staticity implies that $X^{+} X^{-}=X^{+} X^{-}(X)$ and $T_{i}=T_{i}(X)$. Putting this into (3.44) immediately leads to

$$
\begin{equation*}
T_{2}+T_{4}=0 . \tag{3.52}
\end{equation*}
$$

Equation (3.38), which yields the dilaton current, simplifies to

$$
\begin{equation*}
T_{2}^{\prime}+T_{1}\left(T_{2}-V+X^{+} X^{-} U\right)+W=0, \tag{3.53}
\end{equation*}
$$

where the prime means differentiation with respect to the dilaton. Furthermore the covariant conservation of the energy-momentum 1-form takes the following form

$$
\begin{equation*}
\left.E_{a}\right\lrcorner\left(\mathrm{d} W^{a}+\varepsilon^{a}{ }_{b} \omega \wedge W^{b}\right)=\left(W+X^{+} X^{-} U\left(T_{1}+T_{3}\right)\right) \mathrm{d} X, \tag{3.54}
\end{equation*}
$$

where the above relation (3.48) implied the vanishing of the $Z$ direction and (3.52), (3.53) were used for simplifications. It should be noted that the $4 d$ energy-momentum conservation equation is given by (3.54). Thus, one concludes that the non-conservation of the $2 d$ energy-momentum tensor is essentially given by the dilaton current $W$.

The conservation equation (3.44) reads

$$
\begin{equation*}
\mathrm{d}\left(X^{+} X^{-}\right)+V(X) \mathrm{d} X+X^{+} X^{-}\left(U(X)+T_{1}+T_{3}\right) \mathrm{d} X=0 \tag{3.55}
\end{equation*}
$$

which suggests the definitions

$$
\begin{align*}
I(X) & :=\exp \int^{X}\left(U(y)+T_{1}(y)+T_{3}(y)\right) \mathrm{d} y \\
w(X) & :=\int^{X} I(y) V(y) \mathrm{d} y \tag{3.56}
\end{align*}
$$

and the total conserved quantity can be integrated to

$$
\begin{equation*}
C=X^{+} X^{-} I(X)+w(X)=\text { const. } \tag{3.57}
\end{equation*}
$$

which is precisely the form of ordinary dilaton gravity. The difference is, of course, that $I$ and $w$ depend on functions present in the energy-momentum tensor. Note: If the term $T_{1}+T_{3}$ scales with $1 / X^{+} X^{-}$one should redefine the potential $V \mapsto V+T_{1}+T_{3}$ and leave $U$ unchanged. In the absence of horizons $X^{+} X^{-} \neq 0$ this redefinition is well defined. This point does not change the integrability feature.

From (3.43) one finds the $Z$ can be written as

$$
\begin{equation*}
Z=e^{Q} \mathrm{~d} u, \quad Q=\int^{X}\left(T_{1}(y)+U(y)\right) \mathrm{d} y \tag{3.58}
\end{equation*}
$$

which in turn gives

$$
\begin{equation*}
\mathrm{d} X=e^{-Q} \mathrm{~d} r \tag{3.59}
\end{equation*}
$$

Therefore the line element simplifies to

$$
\begin{equation*}
\mathrm{d} s^{2}=2 \mathrm{~d} u \mathrm{~d} r+K(X) \mathrm{d} u^{2} \tag{3.60}
\end{equation*}
$$

where the Killing norm is given by

$$
\begin{align*}
& K(X)=2 e^{2 Q} X^{+} X^{-} \\
& \quad=2 \exp \left(\int^{X}\left(U(y)+T_{1}(y)-T_{3}(y)\right) \mathrm{d} y\right)(C-w(X)) \tag{3.61}
\end{align*}
$$

This is nothing but the most general solution of dilaton gravity (cf. eq. (3.26) of [20]). The aspect that static matter solutions can be mapped on ordinary solutions of dilaton gravity is discussed in subsection 3.2.3 where matter is assumed to be a static and spherically symmetric perfect fluid.

### 3.2.2 Spherically symmetric perfect fluids

A perfect fluid is characterised by

$$
\begin{equation*}
T^{\mu \nu}=(\rho+P) u^{\mu} u^{\nu}-P g^{\mu \nu} \tag{3.62}
\end{equation*}
$$

where $\rho$ and $P$ denote the energy density and pressure respectively with respect to the equal time frame (momentaneous) defined by $u^{\mu}$, the fluid's four-velocity. The gravitational field equations imply the vanishing of the covariant derivative of the energy-momentum tensor

$$
\begin{equation*}
\nabla_{\nu} T^{\mu \nu}=0 \tag{3.63}
\end{equation*}
$$

Its form is best known in spherically symmetric four-dimensional gravity in diagonal gauge $\mathrm{d} s^{2}=e^{\nu} \mathrm{d} t^{2}-e^{a} \mathrm{~d} r^{2}-r^{2} \mathrm{~d} \Omega^{2}$. By suppressing the spherical components it reads

$$
T^{\mu \nu}=\left(\begin{array}{cc}
\rho e^{-\nu} & 0  \tag{3.64}\\
0 & P e^{-a}
\end{array}\right),
$$

which in the Eddington-Finkelstein gauge becomes

$$
T^{\mu \nu}=\left(\begin{array}{cc}
\frac{\rho+P}{K} & -P  \tag{3.65}\\
-P & P K
\end{array}\right) .
$$

Comparing the above energy-momentum tensor with 3.51) leads to the following identifications

$$
\begin{array}{ll}
T_{1}=\frac{\rho+P}{2 X^{+} X^{-}}, & T_{2}=P \\
T_{3}=\frac{\rho-P}{2 X^{+} X^{-}}, & T_{4}=-P \tag{3.66}
\end{array}
$$

Therefore one immediately finds that $T_{1}+T_{3}$ scales with $1 / X^{+} X^{-}$which henceforth must be taken into account if the conservation equation (3.44) is considered, see the discussion above.

Lastly we denote the explicit form of the energy-momentum 1-form (3.35) for a perfect fluid

$$
\begin{equation*}
W^{ \pm}=\mp \frac{\rho-P}{2} e^{ \pm} \pm \frac{X^{ \pm}}{X^{\mp}} \frac{\rho+P}{2} e^{\mp} \tag{3.67}
\end{equation*}
$$

which explicitly depends on $X^{ \pm}$. It is not surprising that we cannot recover a perfect fluid action from 3.67). However, with prescribed equation of state the action is given by the pressure and one can well define what is meant by an action principle [32].

### 3.2.3 Static and spherically symmetric perfect fluids

Spherically symmetric static perfect fluid solutions have been studied in several publications, cf. [33] and references therein. As may be expected, the discussion becomes particularly easy within the reduced theory. The fact that a perfect fluid couples minimally to the dilaton also in the reduced theory is a crucial technical ingredient. Assuming staticity the EOMs are solved. Integrability of this system can be deduced from a general discussion [34], but it will be made explicit below.

By virtue of the identification 3.56 one obtains

$$
\begin{equation*}
X^{+} X^{-}\left(T_{1}+T_{3}\right)=\rho \tag{3.68}
\end{equation*}
$$

Assume $\rho>0$, then the latter equation (3.68) implies the absence of Killing horizons, $X^{+} X^{-} \neq 0$, if $\left|T_{1}+T_{3}\right|<\infty$ holds. However at the boundary of a perfect fluid sphere the energy density may vanish $\rho=0$. Since the exterior spacetime is matterless, i.e., $T_{1}+T_{3}=0$, there is no horizon located at the boundary. The condition $X^{+} X^{-} \neq 0$ is weaker than the Buchdahl inequality [35] but suffices to show the non-existence of horizons.

The static conservation equation (3.55) gives

$$
\begin{equation*}
\mathrm{d}\left(X^{+} X^{-}\right)+V(X) \mathrm{d} X+X^{+} X^{-} U(X) \mathrm{d} X+\rho(X) \mathrm{d} X=0 \tag{3.69}
\end{equation*}
$$

where we now see that rather than (3.56) one must choose

$$
\begin{align*}
& I(X):=\exp \int^{X} U(y) \mathrm{d} y \\
& w(X):=\int^{X} I(y)(V(y)+\rho(y)) \mathrm{d} y \tag{3.70}
\end{align*}
$$

which yields the total conserved quantity (3.57 to be

$$
\begin{equation*}
C=X^{+} X^{-} I(X)+w(X) \tag{3.71}
\end{equation*}
$$

The usual energy-momentum conservation (3.63) is encoded in equation (3.53) with vanishing dilaton current, $W=0$. Hence we conclude that a static perfect fluid couples minimally to dilaton gravity. The conservation equation (3.63) reads

$$
\begin{align*}
P^{\prime}+T_{1}\left(P-V+X^{+} X^{-} U\right) & =0 \\
T_{1}\left(P-V+X^{+} X^{-} U\right) & =\frac{K^{\prime}}{2 K}(\rho+P) \tag{3.72}
\end{align*}
$$

where the second relation of (3.72) can be obtained by differentiating (3.61) and using the identification (3.66). The Killing norm for static perfect fluids
becomes

$$
\begin{equation*}
K(X)=2 X^{+} X^{-} \exp \left(2 \int^{X} T_{1}(y)+U(y) \mathrm{d} y\right) \tag{3.73}
\end{equation*}
$$

where we explicitly see that $X^{+} X^{-}=0$ corresponds to a Killing horizon. The identification of $T_{1}$ can be re-expressed with (3.71) and yields

$$
\begin{equation*}
T_{1}=\frac{\rho+P}{2 X^{+} X^{-}}=\frac{1}{2} \frac{I(X)(\rho+P)}{C-\int^{X} I(y)(V(y)+\rho(y)) \mathrm{d} y} \tag{3.74}
\end{equation*}
$$

It should be noted that the energy-momentum conservation equation of an anisotropic perfect fluid reads

$$
\begin{equation*}
P^{\prime}+\frac{K^{\prime}}{2 K}(\rho+P)=\frac{X^{\prime}}{X}\left(P_{\perp}-P\right), \tag{3.75}
\end{equation*}
$$

where $P_{\perp}$ is the orthogonal pressure component of the anisotropic perfect fluid $T_{\nu}^{\mu}=\operatorname{diag}\left(\rho,-P,-P_{\perp},-P_{\perp}\right)$. If $P_{\perp}=P$ then the conservation equations decouples from the dilaton and one is back at the isotropic case. Therefore one concludes that an anisotropic fluid can only be described with non-minimal coupling to the dilaton.

Therefore every static, spherically symmetric, minimally coupled ( $W=$ 0 ) matter solution can be mapped onto solutions of a dilaton gravity model, see the discussion that follows. This in particular includes the discussed perfect fluid case. For a colliding null dust this statement can already be found in [36], in our framework this corresponds to prescribing the pressure to vanish.

Starting from (3.60) and (3.61) with the redefinition

$$
\begin{equation*}
\mathrm{d} r=\exp \left(\int^{\tilde{X}}\left(U(y)+T_{1}(y)-T_{3}(y)\right) \mathrm{d} y\right) \mathrm{d} \tilde{X}=\tilde{I}(\tilde{X}) \mathrm{d} \tilde{X}, \tag{3.76}
\end{equation*}
$$

yields the line element in the following form

$$
\begin{equation*}
\mathrm{d} s^{2}=\tilde{I}(\tilde{X})\left(2 \mathrm{~d} u \mathrm{~d} \tilde{X}+(C-\tilde{w}(\tilde{X})) \mathrm{d} u^{2}\right) \tag{3.77}
\end{equation*}
$$

where we furthermore redefined (3.56) to be

$$
\begin{align*}
& \tilde{w}(\tilde{X})=\int^{\tilde{X}} \tilde{I}(y) \tilde{V}(y) \mathrm{d} y \\
& \tilde{V}(\tilde{X})=V(\tilde{X}) \exp \left(2 \int^{\tilde{X}} T_{3}(y) \mathrm{d} y\right) \tag{3.78}
\end{align*}
$$

Let us now, in contrast to the perfect fluid case, assume that the function $T_{1}(X)$ is given, which corresponds to the introduction of some generating
function [33]. Note that for given $T_{1}(X)$ equation (3.53) yields $T_{2}(X)$ and therefore $T_{4}(X)$ by (3.52) and finally $T_{3}(X)$ is obtained from (3.48).

Hence for each choice of $T_{1}$ in the dilaton gravity sector there is exactly one $\tilde{w}$ in the matter or perfect fluid sector. However, not every $\tilde{w}$ permits a regular representation as a perfect fluid! Only if one allows for singular configurations all ${ }^{6} 2 d$ dilaton gravity theories can be mapped onto a static spherically symmetric perfect fluid model coupled to Einstein gravity in $d=4$. This can be seen most easily be checking that for regular $T_{1}$ the relation between $X$ and $\tilde{X}$ is invertible. The same holds for $r$ and $X$. Thus, these three coordinates can be expressed as monotonous functions with respect to each other (e.g. $X(\tilde{X})$ ). Because the original $V(X)$ is also monotonous, this means that also $\tilde{w}$ is monotonous. Moreover, the function $\tilde{I}$ cannot be zero. Therefore, there can be at most one (nonextremal) Killing horizon, depending on the sign of $C$ and eventual lower or upper bounds of $\tilde{w}$. Thus, the only possibility to express generic $2 d$ dilaton gravity as a perfect fluid model is to allow for singular energy distributions. However, at a singular point of $T_{1}$ all previous coordinate redefinitions are not valid anymore. Only if one simultaneously performs a conformal transformation with compensating singularities-thus changing the causal structure in an essential way - finally all dilaton gravity models can be reproduced.

In this sense, generic $2 d$ dilaton gravity corresponds to a not necessarily regular) perfect fluid solution in a certain (not necessarily regular) conformal frame. However, regardless of this minor interpretational issue the particular ease of this formalism should be emphasised and compared with the usually more involved calculations in $d=4$.

## Further remarks and comments

In order to complete the perfect fluid discussion some remarks are necessary. Firstly one should have in mind that the Einstein field equations for a static and spherically symmetric perfect fluid reduce to a system of two first order differential equations for a given equation of state. Existence and uniqueness of the solution of this system was proved in [38] for an already wide class of equations of state. Many assumptions on the equation of state could later be weakened in [39] and [40].

The power of dilaton gravity is to get equations of motion of the first order, so it seems that in the perfect fluid case only little can be won, namely the total conserved quantity $C$ in (3.71). The disadvantage on the other hand is the more complicated structure of the differential equations if the density, the pressure or an equation of state is specified. Already in

[^5]the constant density case equations get more involved than with the usual approach through the Tolman-Oppenheimer-Volkoff [41, 42] equation. As expected, the three equations (3.71)-(3.72) contain four unknown functions, namely $\rho, P, K$ and $X^{+} X^{-}$, therefore one of these functions can be chosen freely.

### 3.3 Spinor formalism and reduction of fermions

Since spherical symmetry provides a foliation of spacetime by spacelike two-surfaces (round two-spheres) it is natural to adapt the Clifford algebra to this foliation. In particular the Geroch-Held-Penrose (GHP) spincoefficient formalism $[43,44]$ is particularly well suited for this situation. It uses a double-null tetrad $\left(l^{a}, n^{a}, m^{a}, \bar{m}^{a}\right)$ satisfying ${ }^{7}$

$$
\begin{align*}
& l \cdot n=1, m \cdot \bar{m}=-1, \quad l^{2}=0, n^{2}=0, m^{2}=0, \bar{m}^{2}=0  \tag{3.79}\\
& m_{a} \mathrm{~d} x^{a}=-\frac{\Phi}{\sqrt{2}}(\mathrm{~d} \theta-i \sin \theta \mathrm{~d} \phi) \tag{3.80}
\end{align*}
$$

adapted to such a foliation by noticing that the orthogonal complement of the tangent space of the two-surfaces is uniquely spanned by two nullnormals $l^{a}, n^{a}$.


Figure 1: Foliation of spacetime by two-spheres

[^6]In the GHP formalism the null tetrad gives uniquely rise to a spinor basis (dyad) via the identification

$$
\begin{align*}
& l^{a}=o^{A} o^{A^{\prime}}, n^{a}=\iota^{A} \iota^{A^{\prime}}, m^{a}=o^{A} \iota^{A^{\prime}}, \bar{m}^{a}=\iota^{A} o^{A^{\prime}}  \tag{3.81}\\
& g_{a b}=\varepsilon_{A B} \varepsilon_{A^{\prime} B^{\prime}}=l_{a} n_{b}+n_{a} l_{b}-m_{a} \bar{m}_{b}-\bar{m}_{a} m_{b}  \tag{3.82}\\
& \varepsilon^{A B} o_{A} \iota_{B}=o_{A} \iota^{A}=1 \tag{3.83}
\end{align*}
$$

This identification allows us to consider tensor fields as a special case of spinor-fields, by identifying a tensor index $a$ with a pair of primed and unprimed spinor indices $A A^{\prime}$. Note that in the previous section Latin letters were used for anholonomic indices, in this section they are used as abstract indices.

The covariant derivatives along the null directions of the tetrad define the 12 complex spin coefficients (taking into account priming and complex conjugation)

$$
\begin{align*}
D o^{A} & =-\gamma^{\prime} o^{A}-\kappa \iota^{A}, & D \iota^{A} & =\gamma^{\prime} \iota^{A}-\tau^{\prime} o^{A}  \tag{3.84}\\
\delta o^{A} & =\beta o^{A}-\sigma \iota^{A}, & \delta \iota^{A} & =-\beta \iota^{A}-\rho^{\prime} o^{A} \tag{3.85}
\end{align*}
$$

where $D=l^{a} \nabla_{a}$ and $\delta=m^{a} \nabla_{a}$. The GHP formalism and Cartan's form calculus can be linked by noting appendix B and especially equations B.2 and B.3). They can be used to read of the spin coefficients for a given null tetrad.

In a spherically symmetric spacetime 6 of the 12 spin coefficients vanish, see [45] for the static and spherically symmetric case. The vanishing coefficients are $\kappa=\sigma=\tau=0$, together with their primed counterparts. Furthermore $\gamma$ and $\gamma^{\prime}$ are real quantities describing the $2 d$ spacetime only. $\rho$ and $\rho^{\prime}$, which are also real, describe the expansion of the sphere. As already said in the end of subsection 3.1.3 they correspond to $X^{ \pm}$respectively. The remaining two spin coefficients are not independent and are explicitly given by $\beta=\bar{\beta}^{\prime}=(\cot \theta) /(2 \sqrt{2} \Phi)$.

The two-spinor equivalent Dirac action functional [46] can be written

$$
\begin{equation*}
L=\int(i \overline{\mathbf{\Psi}} \nabla \Psi-m \overline{\mathbf{\Psi}} \Psi) \omega_{G} \tag{3.86}
\end{equation*}
$$

where $\overline{\mathbf{\Psi}}=\boldsymbol{\Psi}^{\dagger} \gamma^{0}$ is the Dirac conjugate and $\nabla=\gamma^{a} \nabla_{a}$. We take the space of Dirac 4 -spinors $\boldsymbol{\Psi}$ to be of the form $\boldsymbol{\Psi}=\left(\psi_{A}, \chi^{A^{\prime}}\right)$, i.e., the direct sum of the dual with the complex conjugate 2-spinor space. Note that $\psi_{A}$ and $\chi^{A^{\prime}}$ are left- and right-handed fermions respectively. In this combination the Dirac conjugate is simply given by $\overline{\mathbf{\Psi}}=\left(\bar{\chi}^{A}, \bar{\psi}_{A^{\prime}}\right)$. The Clifford algebra associated with $g_{a b}$ is follows from the identification

$$
\nLeftarrow=\sqrt{2}\left(\begin{array}{cc}
0 & v_{A B^{\prime}}  \tag{3.87}\\
v^{A^{\prime} B} & 0
\end{array}\right), \quad\langle\psi \psi+\psi \psi=2(u \cdot v) \mathbb{1}
$$

Identifying the vectors of the null tetrad (3.81) leads to

$$
\begin{equation*}
l n+n l=2 \mathbb{1}, \quad m n \hbar+m m=-2 \mathbb{1} \tag{3.88}
\end{equation*}
$$

Thus the four-dimensional Clifford algebra is generated by two two-dimensional Clifford algebras. The first is generated by the orthonormal basis $l^{a}$ and $n^{a}$, the second by the basis vectors of the two-sphere $S^{2}, m^{a}$ and $\bar{m}^{a}$.

Furthermore we find

$$
\begin{equation*}
S_{I}=\left\{\binom{o_{A}}{0},\binom{0}{\iota^{A^{\prime}}}\right\}, \quad S_{I I}=\left\{\binom{0}{o^{A^{\prime}}},\binom{\iota_{A}}{0}\right\}, \tag{3.89}
\end{equation*}
$$

as invariant subspaces of the two-dimensional Clifford algebra generated by $l^{a}$ and $n^{a}$, hence one can write $S=S_{I} \oplus S_{I I}$. With respect to this basis the two-dimensional Clifford algebra is represented by

$$
l \rightarrow \gamma_{I, I I}^{-}= \pm \sqrt{2}\left(\begin{array}{ll}
0 & 1  \tag{3.90}\\
0 & 0
\end{array}\right), \quad h \rightarrow \gamma_{I, I I}^{+}= \pm \sqrt{2}\left(\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right)
$$

from which the $\gamma^{a}$-matrices in a local Lorentz frame are given by

$$
\begin{array}{r}
\gamma_{I, I I}^{0}= \pm\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right), \quad \gamma_{I, I I}^{1}=\mp\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right) \\
\gamma^{\star}=\gamma^{0} \gamma^{1}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) \tag{3.92}
\end{array}
$$

where we used $\sqrt{2} \gamma^{0}=\gamma^{+}+\gamma^{-}$. The upper and lower signs refer to the invariant subspaces $S_{I}$ and $S_{I I}$ respectively.

From the above the Dirac action functional (3.86) can be written in two-spinor form

$$
\begin{equation*}
L=\int\left(i \sqrt{2}\left(\bar{\psi}_{A^{\prime}} \nabla^{A A^{\prime}} \psi_{A}+\bar{\chi}^{A} \nabla_{A A^{\prime}} \chi^{A^{\prime}}\right)-m \bar{\psi}_{A^{\prime}} \chi^{A^{\prime}}-m \bar{\chi}^{A} \psi_{A}\right) \omega_{G} \tag{3.93}
\end{equation*}
$$

which by variation with respect to the spinors $\bar{\psi}_{A^{\prime}}$ and $\bar{\chi}^{A}$ leads to the Dirac equation [47] in two-spinor form

$$
\begin{equation*}
i \sqrt{2} \nabla^{A A^{\prime}} \psi_{A}-m \chi^{A^{\prime}}=0, \quad i \sqrt{2} \nabla_{A A^{\prime}} \chi^{A^{\prime}}-m \psi_{A}=0 \tag{3.94}
\end{equation*}
$$

The Dirac two-spinors are expanded in terms of the basis spinors $\psi^{A}=$ $A o^{A}+P \iota^{A}, \chi^{A^{\prime}}=B \iota^{A^{\prime}}+Q o^{A^{\prime}}$, where $A, B, P$ and $Q$ are functions of all four spacetime coordinates. The functions $A, B, P$ and $Q$ have spin weights $-1 / 2,-1 / 2,1 / 2$ and $1 / 2$ respectively. Therefore one can rewrite
the first term of the Dirac action (3.93) in terms of weighted derivative operators [44]

$$
\begin{equation*}
\bar{\psi}_{A^{\prime}} \nabla^{A A^{\prime}} \psi_{A}=\bar{A}(\mathrm{p}-\rho) A+\bar{A} \check{ð}^{\prime} P+\bar{P}\left(\mathrm{p}^{\prime}-\rho^{\prime}\right) P+\bar{P} \check{\partial} A \tag{3.95}
\end{equation*}
$$

where the weighted operators are given by

$$
\begin{align*}
& \mathrm{p} \eta=D \eta+\frac{w}{2}\left(\gamma^{\prime}+\bar{\gamma}^{\prime}\right) \eta+\frac{s}{2}\left(\gamma^{\prime}-\bar{\gamma}^{\prime}\right) \eta  \tag{3.96}\\
& \mathrm{\partial} \eta=\delta \eta-\frac{w}{2}\left(\beta-\bar{\beta}^{\prime}\right) \eta-\frac{s}{2}\left(\beta+\bar{\beta}^{\prime}\right) \eta \tag{3.97}
\end{align*}
$$

when acting on a weighed quantity $\eta$ with spin weight $s$ and boost weight $w$. It was taken into account that the spin coefficients $\kappa, \sigma$ and $\tau$ together with their primed counterparts vanish in case of spherical symmetry.

Since the action (3.93) must be a real functional the real spin coefficients $\rho$ and $\rho^{\prime}$ drop out because of the factor $i$ in front. This yields

$$
\begin{align*}
& L=\int\left(i \sqrt{2}\left(\bar{A} \mathrm{p} A+\bar{A} ð^{\prime} P+\bar{B} \mathrm{~b}^{\prime} B+\bar{B} ð^{\prime} Q\right)\right. \\
& \qquad \begin{array}{l}
i \sqrt{2}\left(\bar{Q} \mathrm{p} Q+\bar{Q} \precsim B+\bar{P} \mathrm{~b}^{\prime} P+\bar{P} \precsim A\right) \\
\\
\quad-m(\bar{A} B+\bar{B} A)+m(\bar{Q} P+\bar{P} Q)) \omega_{g} \Phi^{2} \mathrm{~d}^{2} \Omega
\end{array}
\end{align*}
$$

Next the weighed functions $A, B, P$ and $Q$ are expanded in terms of spin weighted spherical harmonics with the appropriate spin weights, $A=$ $\sum_{j m} A_{j m-\frac{1}{2}} Y_{\frac{1}{2} \frac{1}{2}}$, etc. These are the eigenfunctions of the operator $\partial^{\prime} \varnothing$ for each spin weight $s$. They are defined by $[44,48]$

$$
\begin{align*}
& ð^{\prime} ð_{s} Y_{j, m}=-\frac{(j+s+1)(j-s)}{2 \Phi^{2}}{ }_{s} Y_{j, m}  \tag{3.99}\\
& ð_{s} Y_{j, m}=-\frac{\sqrt{(j+s+1)(j-s)}}{\sqrt{2} \Phi}_{s+1} Y_{j, m}  \tag{3.100}\\
& ð_{s}^{\prime} Y_{j, m}=\frac{\sqrt{(j-s+1)(j+s)}}{s-1}{ }^{2} \Phi  \tag{3.101}\\
& Y_{j, m}
\end{align*}
$$

and, for each spin weight $s$, enjoy the orthogonality condition

$$
\begin{equation*}
\left\langle{ }_{s} Y_{j, m},{ }_{s} Y_{j^{\prime}, m^{\prime}}\right\rangle=\frac{1}{4 \pi} \delta_{j j^{\prime}} \delta_{m m^{\prime}}, \quad\langle f, g\rangle=\frac{1}{4 \pi} \int \bar{f} g \mathrm{~d}^{2} \Omega \tag{3.102}
\end{equation*}
$$

Hence the spherical dependence of (3.98) can be integrated out. In particular we obtain

$$
\begin{align*}
\int \bar{A} \mathrm{p} A \mathrm{~d}^{2} \Omega & =\sum_{j m} \bar{A}_{j m} \mathrm{p} A_{j m}  \tag{3.103}\\
\int \bar{A} \check{ð}^{\prime} P \mathrm{~d}^{2} \Omega & =\frac{j+\frac{1}{2}}{\sqrt{2} \Phi} \sum_{j m} \bar{A}_{j m} P_{j m} \tag{3.104}
\end{align*}
$$

Then the spherically reduced fermion action reads

$$
\begin{align*}
& L_{\mathrm{D}}=\sqrt{2} \sum_{j m} \int\left(i\left(\bar{A}_{j m} \mathrm{p} A_{j m}+\frac{j+\frac{1}{2}}{\sqrt{2} \Phi} \bar{A}_{j m} P_{j m}\right)+i\left(\bar{B}_{j m} \mathrm{p}^{\prime} B_{j m}+\frac{j+\frac{1}{2}}{\sqrt{2} \Phi} \bar{B}_{j m} Q_{j m}\right)\right. \\
& \quad+i\left(\bar{Q}_{j m} \mathrm{p} Q_{j m}-\frac{j+\frac{1}{2}}{\sqrt{2} \Phi} \bar{Q}_{j m} B_{j m}\right)+i\left(\bar{P}_{j m} \mathrm{p}^{\prime} P_{j m}-\frac{j+\frac{1}{2}}{\sqrt{2} \Phi} \bar{P}_{j m} A_{j m}\right) \\
& \left.-\frac{m}{\sqrt{2}}\left(\bar{A}_{j m} B_{j m}+\bar{B}_{j m} A_{j m}\right)+\frac{m}{\sqrt{2}}\left(\bar{P}_{j m} Q_{j m}+\bar{Q}_{j m} P_{j m}\right)\right) \Phi^{2} \omega_{g} \tag{3.105}
\end{align*}
$$

## Two-spinor representation

Let the two-spinors with respect to their invariant subspace $S_{I, I I}$ respectively be ${ }^{8}$

$$
\begin{align*}
& \Psi_{j m}^{I}=\binom{A_{j m}}{B_{j m}}, \quad \bar{\Psi}_{j m}^{I}=\Psi_{j m}^{I \dagger} \gamma_{I}^{0}=\left(\bar{B}_{j m}, \quad \bar{A}_{j m}\right),  \tag{3.106}\\
& \Psi_{j m}^{I I}=\binom{Q_{j m}}{P_{j m}}, \quad \bar{\Psi}_{j m}^{I I}=\Psi_{j m}^{I \dagger} \gamma_{I I}^{0}=-\binom{\bar{P}_{j m},}{\bar{Q}_{j m}} . \tag{3.107}
\end{align*}
$$

The weighted derivative operators p and $\mathrm{p}^{\prime}$ are simply given by $\mathrm{b}=l^{a} \nabla_{a}$ and $\mathrm{p}^{\prime}=n^{a} \nabla_{a}$. Therefore the dyads $E_{+}^{a}=l^{a}$ and $E_{-}^{a}=n^{a}$ in light cone form are introduced.

Thus the reduced action 3.105 written in an intrinsically $2 d$ form becomes

$$
\begin{align*}
& L_{\mathrm{D}}=\sum_{j m} \int\left(\bar{\Psi}_{j m}^{I}\left(i E_{+}^{a} \nabla_{a} \gamma_{I}^{+}+i E_{-}^{a} \nabla_{a} \gamma_{I}^{-}-m \mathbb{1}\right) \Psi_{j m}^{I}+\right. \\
& \bar{\Psi}_{j m}^{I I}\left(i E_{+}^{a} \nabla_{a} \gamma_{I I}^{+}+i E_{-}^{a} \nabla_{a} \gamma_{I I}^{-}-m \mathbb{1}\right) \Psi_{j m}^{I I}+ \\
&  \tag{3.108}\\
& \left.\frac{j+\frac{1}{2}}{\Phi}\left(\bar{\Psi}_{j m}^{I I} \gamma^{\star} I \Psi_{j m}^{I}+\bar{\Psi}_{j m}^{I} \gamma^{\star} I^{-1} \Psi_{j m}^{I I}\right)\right) \Phi^{2} \omega_{g}
\end{align*}
$$

where $I$ is the intertwiner between the representations of the two-dimensional Clifford algebra in $S_{I}$ and $S_{I I}$ respectively. The unity matrices in the first and second line should also carry an index with respect to their spinorspace, which was avoided for clarity. With respect to the bases chosen in $S_{I}$ and $S_{I I}$ we have

$$
I: S_{I} \rightarrow S_{I I}, \quad I=\left(\begin{array}{cc}
i & 0  \tag{3.109}\\
0 & -i
\end{array}\right)=i \gamma^{\star}
$$

[^7]This allows us to identify $S_{I}$ with the two-dimensional (irreducible) representation space $S$ of the (two-dimensional) Clifford algebra, whereas the representation in $S_{I I}$ is equivalent under the action of the intertwiner $I$. We rewrite the reduced action by denoting $\Psi_{j m}^{I}=\eta$ (this means identifying $S$ with $S_{I}$ ) and $\Psi_{j m}^{I I}=I \lambda$ where $\lambda \in S$. Then the above expression 3.108 turns into

$$
\begin{equation*}
L=\int\left(\bar{\eta}(i \nabla-m \mathbb{1}) \eta+\bar{\lambda}(i \nabla-m \mathbb{1}) \lambda+\frac{j+\frac{1}{2}}{\Phi}\left(\bar{\lambda} \gamma^{\star} \eta-\bar{\eta} \gamma^{\star} \lambda\right)\right) \Phi^{2} \omega_{g} \tag{3.110}
\end{equation*}
$$

where the summation over the modes is understood and henceforth the kinetic term is abbreviated using $\nabla=E_{+}^{a} \nabla_{a} \gamma_{I, I I}^{+}+E_{-}^{a} \nabla_{a} \gamma_{I, I I}^{-}$when acting on $\eta$ or $\lambda$, respectively. In this formulation both spinors $\eta$ and $\lambda$ belong to the same spinor space. Finally we introduce an internal index $u$ and write $\psi_{u}=(\eta, \lambda)$

$$
\begin{equation*}
L=\int\left(\delta^{u v} \bar{\psi}_{u}(i \nabla-m \mathbb{1}) \psi_{v}-\frac{j+\frac{1}{2}}{\Phi} \varepsilon^{u v}\left(\bar{\psi}_{u} \gamma^{\star} \psi_{v}\right)\right) \Phi^{2} \omega_{g} \tag{3.111}
\end{equation*}
$$

which display that the action is in a $S O(2) \simeq U(1)$ covariant form.
This remaining freedom is the two parameter subgroup of the Lorentz group at each point and can be understood from the following. One can rescale the basis spinors by a complex scalar field $o^{A} \mapsto \Lambda o^{A}$ and $\iota^{A} \mapsto(1 / \Lambda) \iota^{A}$, which leaves the null directions invariant. By writing $\Lambda^{2}=R \exp (i \phi)$ one finds that the null directions $l^{a}$ and $n^{a}$ are boosted, whereas $m^{a}$ and $\bar{m}^{a}$ are rotated by an angle $\pm \phi$, respectively.

Graphically the above spaces and their embeddings are summarised in figure 2

### 3.4 Reduction of Yang-Mills fields

In this subsection we use the metric formalism to spherically reduce YangMills fields, where Latin letters correspond to abstract indices. This formalism uses a generic metric and does not specify the signature, in contrast to the GHP formalism, in which the signature is fixed.

Before starting with the nonabelian case it is worthwhile to consider $U(1)$. In standard notation [44] the skew-symmetric field tensor

$$
\begin{equation*}
F_{a b}=\phi_{A B^{\prime}} \varepsilon_{A^{\prime} B^{\prime}}+\varepsilon_{A B} \bar{\phi}_{A^{\prime} B^{\prime}} \tag{3.112}
\end{equation*}
$$

can be decomposed in terms of a complex, symmetric bispinor $\phi_{A B}$. If a potential exists the relation $\phi_{A B}=\nabla_{A^{\prime}(A} A_{B)}{ }^{A^{\prime}}$ implies $F_{a b}=2 \nabla_{[a} A_{b]}$.

According to our notion, spherical symmetry means that the Lie-derivative taken into the Killing-directions acting on the (in the present case abelian)


Figure 2: Embedding diagram for the spin-spaces of $\mathcal{M}=M, L . T \mathcal{M}$ denotes the tangent bundle and $C l\left(\mathcal{M}, g_{\mathcal{M}}\right)$ the corresponding Clifford bundle with Clifford-map $\gamma_{\mathcal{M}} . S_{\mathcal{M}}$ refers to the respective spin-bundles, i.e., representation spaces of $C l\left(\mathcal{M}, g_{\mathcal{M}}\right)$.

Yang-Mills action yields only surface terms. A sufficient, but by no means necessary condition is strict spherical symmetry, $\mathcal{L}_{\xi} F_{a b} \stackrel{!}{=} 0$, implying $\mathcal{L}_{\xi} \phi_{A B}=0$ (since $\mathcal{L}_{\xi} \varepsilon_{A B}=0$ by construction). More explicitly this condition reads

$$
\begin{equation*}
(\xi \cdot \nabla) \phi_{A B}+\Phi_{A}^{C} \phi_{C B}+\Phi_{B}^{C} \phi_{A C}=0 \tag{3.113}
\end{equation*}
$$

Applying the decomposition with respect to the basis, $\phi_{A B}=\phi^{00} o_{A} O_{B}+$ $\phi^{01} o_{(A} \iota_{B)}+\phi^{11} \iota_{A} \iota_{B}$, yields three conditions for the three coefficients $\phi^{i j}$. In the same way in which the nonexistence of strictly spherically symmetric spinors can be proved the relation $\phi^{i i}=0$ can be shown. However, as opposed to spinors this does not imply a trivial field configuration. Indeed, the equation

$$
\begin{equation*}
(\xi \cdot \nabla)\left(\phi^{01} o_{\left(A \iota_{B}\right)}\right)=0 \tag{3.114}
\end{equation*}
$$

after contraction with $o^{B} \iota^{A}$ reduces to

$$
\begin{equation*}
(\xi \cdot \nabla) \phi^{01}=0 \tag{3.115}
\end{equation*}
$$

allowing for nontrivial field configurations (namely electric monopoles and their dual). This is of course not unexpected, since the Coulomb-solution
is well-known for exhibiting spherical symmetry. This result can be generalised to the Yang-Mills case. However, the condition of strictness can be relaxed.

Since $s u(2)$ is the building block of all other Lie-Algebras (and for sake of simplicity) we will restrict ourselves to the gauge group $S U(2)$. It can be expected that spherical reduction yields a non-trivial result due to the fact that the $S U(2) \equiv S^{3}$ allows for a Hopf-fibration $U(1)$ over $S^{2}$ and because the isometry group of the metric by construction contains $S O(3)$ as subgroup with $S^{2}$-orbits. Spherical reduction of $S U(2)$-Yang-Mills theory has been performed by several authors during the 1970's - the most prominent is probably reference [49]. We will follow in our description closely the approach of Forgács and Manton [50]. The condition

$$
\begin{equation*}
\mathcal{L}_{\xi_{m}} A_{a} \stackrel{!}{=} D_{a} W_{m}, \tag{3.116}
\end{equation*}
$$

provides spherical symmetry up to gauge transformations. For each Killing vector $\xi_{m}$ a Lie-algebra valued scalar field $W_{m}=W_{m}^{i} T^{i}$ is introduced, with $T^{i}$ being the generators of $S U(2)$. Note that $m$ is not a usual abstract index, but just a label for the $m^{\text {th }}$ Killing vector. $D_{a}$ is the gauge-covariant derivative, i.e., $D_{a} W=\nabla_{a} W-i g\left[A_{a}, W\right]$. Equation. (3.116) is equivalent to gauge-covariant transformation behaviour of the nonabelian field tensor $\mathcal{L}_{\xi} F_{a b}=i g\left[F_{a b}, W\right]$.

Applying the commutator of two Lie-derivatives establishes the (WessZumino) consistency condition

$$
\begin{equation*}
2 \mathcal{L}_{\xi[m} W_{n]}=\left[W_{m}, W_{n}\right]+f_{m n l} W_{l}, \tag{3.117}
\end{equation*}
$$

with the structure constants $\left[\xi_{m}, \xi_{n}\right]=i f_{m n l} \xi_{l}$
The general idea to solve (3.116) as advocated in [50] is as follows: instead of solving this equation on the coset-space $S^{2}=S O(3) / S O(2)$, it is solved using the whole symmetry group $S O(3)$, thus introducing an additional dimension. By a gauge transformation one can simplify (3.116) such that the right hand side vanishes. Equations of this type can be solved easily (they must be fulfilled separately for each generator). Afterwards a gauge transformation is used to effectively project the gauge-field to the coset space. Then one proceeds to find the most general solution to the consistency conditions.

What has been explained in words will now be presented briefly in formulas. The abstract index-set $\{\tilde{a}, i, w\}$ is split into the isometry part $\tilde{a}$ (" $r-t$-part"), the coset part $i$ (" $\theta-\phi$ "-part) and the phase part $w$ (the third Euler angle $\chi=x^{w}$ ). A generic index of the subset $\{i, w\}$ is denoted with $\hat{a}$. A generic index of the subset $\{\tilde{a}, i\}$ is denoted by $a$. The solution of

$$
\begin{equation*}
\mathcal{L}_{\xi_{m}} A_{\hat{a}}^{i}=0, \quad \forall m, i, \tag{3.118}
\end{equation*}
$$

reads

$$
\begin{equation*}
A_{\hat{a}}^{i}=\Phi_{m}^{i} \tilde{\xi}_{m \hat{a}} \tag{3.119}
\end{equation*}
$$

where $\tilde{\xi}$ denotes left-translations (as opposed to the usual right-translations $\xi)$. The scalars $\phi_{m}^{i}$ are independent of the phase $x^{w}$. The $\tilde{a}$-component is trivially given by

$$
\begin{equation*}
A_{\tilde{a}}^{i}=A_{\tilde{a}}^{i}\left(x^{\tilde{a}}\right) \tag{3.120}
\end{equation*}
$$

Afterwards a gauge is chosen such that $A_{w}=0$ and the other components be independent of $x^{w}$ (so in fact the third Euler angle decouples and one can restrict to the coset space).

For explicit calculations one assumes that the generator of the $S O(2)$ subgroup (defining the coset space $S^{2}$ ) is $T^{3}$. There are three possible solutions of the consistency equations

$$
\begin{align*}
\nabla_{a} \Phi_{3}^{i}-g \varepsilon^{i \beta \gamma} A_{a}^{\beta} \Phi_{3}^{\gamma} & =0  \tag{3.121}\\
\varepsilon_{m 3 l} \Phi_{l}^{i}+g \varepsilon^{i \beta \gamma} \Phi_{m}^{\beta} \Phi_{3}^{\gamma} & =0 \tag{3.122}
\end{align*}
$$

It is convenient to fix a gauge where $\Phi_{3}^{1}=\Phi_{3}^{2}=0$.
There exists a degenerate solution when all $\Phi_{m}^{i}$ vanish, which produces simply pure $S U(2)$ gauge theory after reduction (this is the analogue of the $U(1)$-example discussed before). Another solution is obtained for vanishing $\Phi_{1}=\Phi_{2}$ and constant $\Phi_{3}^{3}$, corresponding to an abelian monopole of arbitrary charge.

The solution of the constraint equations in the general case is given by

$$
\begin{align*}
A_{a}^{i} & =\frac{1}{g}\left(0,0, a_{a}\right),  \tag{3.123}\\
\Phi_{1}^{i} & =\frac{1}{g}\left(\Phi_{1}, \Phi_{2}, 0\right)  \tag{3.124}\\
\Phi_{2}^{i} & =\frac{1}{g}\left(\Phi_{2},-\Phi_{1}, 0\right),  \tag{3.125}\\
\Phi_{3}^{i} & =\frac{1}{g}(0,0,1) \tag{3.126}
\end{align*}
$$

A gauge rotation which makes $A_{w}^{i}$ equal to zero for all $i$ establishes an
equivalent form, the so-called Witten ansatz:

$$
\begin{align*}
A_{t}^{i} & =\frac{1}{g}\left(0,0, a_{0}\right)  \tag{3.127}\\
A_{r}^{i} & =\frac{1}{g}\left(0,0, a_{1}\right)  \tag{3.128}\\
A_{\theta}^{i} & =\frac{1}{g}\left(-\Phi_{1},-\Phi_{2}, 0\right)  \tag{3.129}\\
A_{\phi}^{i} & =\frac{1}{g}\left(\Phi_{2} \sin \theta,-\Phi_{1} \sin \theta, \cos \theta\right) \tag{3.130}
\end{align*}
$$

It leads after reduction to an abelian gauge theory, supplemented by a complex scalar field. The corresponding scalar fields $W_{m}^{i}$ read (cf. equation (3.116))

$$
\begin{equation*}
W_{1}^{3}=\frac{\sin \phi}{\sin \theta}, \quad W_{2}^{3}=\frac{\cos \phi}{\sin \theta}, \quad W_{m}^{i}=0 \text { otherwise } \tag{3.131}
\end{equation*}
$$

The real scalars $\Phi_{1}$ and $\Phi_{2}$ are combined to one complex quantity $w$. Most conveniently [51], the spherically symmetric Lie algebra valued 1form can be written as follows: Let us denote $a=a_{0} \mathrm{~d} t+a_{1} \mathrm{~d} r$ and for $S U(2)$ we have $\mathbf{T}_{i}=\sigma_{i} / 2$, where $\sigma_{i}$ are the Pauli matrices. For $S U(3)$ the above Witten ansatz has to be supplemented by just one additional term $(b / 2 g) \lambda_{8}$, where $b=b_{0} \mathrm{~d} t+b_{1} \mathrm{~d} r, \mathbf{T}_{i}=\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right) / 2$ and $\lambda_{i}$ are the standard Gell-Mann matrices. Therefore

$$
\begin{align*}
A=\mathbf{T}_{i} A_{\mu}^{i} \mathrm{~d} x^{\mu}= & \frac{a}{g} \mathbf{T}_{3}+\frac{1}{g}\left(\operatorname{Im} w \mathbf{T}_{1}+\operatorname{Re} w \mathbf{T}_{2}\right) \mathrm{d} \theta+ \\
& \frac{1}{g}\left(\operatorname{Im} w \mathbf{T}_{2}-\operatorname{Re} w \mathbf{T}_{1}+\cot \theta \mathbf{T}_{3}\right) \sin \theta \mathrm{d} \phi+\frac{b}{2 g} \lambda_{8} \tag{3.132}
\end{align*}
$$

We already introduced an additional contribution in the simple $S U(3)$ case, where the $S U(2)$ generators are the trivially embedded into the $S U(3)$ group, called isospin- $1 / 2$ embedding. There is a second, more involved, isospin-1 embedding of the $S U(2)$ group into the $S U(3)$ group which will not be used here [52,53].

Equation (3.132) is invariant under $\mathrm{U}(1)$ gauge transformations generated by $\mathrm{U}=\exp \left(i \Omega(t, r) \mathbf{T}_{3}\right)$, under which

$$
\begin{equation*}
a_{a} \mapsto a_{a}+\partial_{a} \Omega, \quad w \mapsto e^{i \Omega} w, \quad b \mapsto b \tag{3.133}
\end{equation*}
$$

The four dimensional Yang-Mills action reads

$$
\begin{equation*}
L=-\int \frac{1}{2} \operatorname{Tr}(F * F) \tag{3.134}
\end{equation*}
$$

where the gauge field strength is given by $F=\mathrm{d} A-i g[A, A]$ and $g$ is the gauge coupling constant. This action is invariant under gauge transformations of $A$, under which $A \mapsto U A U^{-1}+(1 / g) U \mathrm{~d} U^{-1}$ and $F \mapsto U F U^{-1}$.

Then the spherically reduced Yang-Mills action of the gauge field ansatz (3.132) reads [51]

$$
\begin{equation*}
L_{\mathrm{YM}}=4 \pi \int\left(-\frac{1}{4 g^{2}} f^{2}-\frac{1}{4 g^{2}} \mathfrak{f}^{2}+\frac{1}{g^{2}} \frac{|D w|^{2}}{\Phi^{2}}-\frac{1}{g^{2}} \frac{\left(|w|^{2}-1\right)^{2}}{2 \Phi^{4}} \omega_{g}\right) \Phi^{2} \tag{3.135}
\end{equation*}
$$

where $f^{2}=f * f,|D w|^{2}=|D w * D w|$ and the $2 d$ abelian field strengths are given by

$$
\begin{align*}
f & =\mathrm{d} a  \tag{3.136}\\
f & =\mathrm{d} b \tag{3.137}
\end{align*}
$$

The gauge covariant derivative is $D=\mathrm{d}-i a$, when acting on scalars. In case of arbitrary magnetic and electric charge the Yang-Mills equations imply $w=0$. On the other hand, $w \neq 0$ only allows a magnetic monopole with unit charge but still an arbitrary electric one [51].

The field strength (3.137) does not emerge from the commutator of the gauge covariant derivative $D=\mathrm{d}-i a$. Thus there is no coupling between $w$ and $b$. This can be understood from the fact that $\lambda_{8}$ in (3.132) commutes with the generators $\lambda_{1}, \ldots, \lambda_{3}$ of the $S U(2)$ Lie sub-algebra.

Nonetheless the gauge covariant derivative defined below, equation (4.22), when acting on spinors, has an contribution due to $b$.

## 4 The spherically symmetric Standard Model

The aim of the last section was to review the three formalisms needed for spherical reduction. This section uses these to spherically reduce the remaining parts of the SM of particle physics. Furthermore, with all the machinery already at hand, we spherically reduce torsion generated by fermions, which yields a four fermion interaction term.

Our procedure of spherical reduction can easily be extended to also include the new terms of the recently proposed New Minimal Standard Model [14]. For example, dark matter necessitates the introduction of a new real scalar field, which was already spherically reduced in example 3.1 equation (2.9).

### 4.1 Reduction of the $S U(2)$ Yang-Mills Dirac system

The spherical reduction of the interaction term was often performed by an ansatz for the Dirac spinors $[54,55]$. With our methods we show that
an additional term appears that may have been overlooked in previous calculations.

The interaction term is described by

$$
\begin{equation*}
L=\int\left(\bar{\Psi}_{\alpha}^{\Lambda}\left(\gamma^{\mu}\right)^{\alpha}{ }_{\beta} e_{\mu}^{a}\left(g A_{a}\right)_{\Lambda \Delta} \Psi^{\Delta \beta}\right) \omega_{g} \tag{4.1}
\end{equation*}
$$

where $\Lambda$ and $\Delta$ denote the group indices of the Yang-Mills field $A_{a}$, whereas the indices $\alpha$ and $\beta$ describe the group indices of the Dirac four-spinors $\Psi$.

Writing action (4.1) in the spinor formalism of section 3.3 yields

$$
\begin{equation*}
L=\int\left(\sqrt{2} \bar{\psi}^{\Lambda A^{\prime}} g A_{A A^{\prime} \Lambda \Delta} \psi^{\Delta A}\right) \omega_{G} \tag{4.2}
\end{equation*}
$$

since only left-handed fermions are interacting in the SM and where the minimal substitution

$$
\begin{equation*}
\nabla_{A A^{\prime}} \mapsto \nabla_{A A^{\prime}}-i g A_{A A^{\prime} \Lambda \Delta} \tag{4.3}
\end{equation*}
$$

was used. The Yang-Mills field in terms of spinor components become

$$
\begin{align*}
\iota^{A} \iota^{A^{\prime}} A_{A A^{\prime}} & =\frac{1}{g}\left(n^{a} a_{a}\right) \mathbf{T}_{3}  \tag{4.4}\\
o^{A} o^{A^{\prime}} A_{A A^{\prime}} & =\frac{1}{g}\left(l^{a} a_{a}\right) \mathbf{T}_{3}  \tag{4.5}\\
o^{A} \iota^{A^{\prime}} A_{A A^{\prime}} & =\frac{-i}{\sqrt{2} g \Phi}\left(\bar{w} \mathbf{T}_{-}-\cot \theta \mathbf{T}_{3}\right)  \tag{4.6}\\
\iota^{A} o^{A^{\prime}} A_{A A^{\prime}} & =\frac{i}{\sqrt{2} g \Phi}\left(w \mathbf{T}_{+}-\cot \theta \mathbf{T}_{3}\right) \tag{4.7}
\end{align*}
$$

where

$$
\begin{equation*}
\mathbf{T}_{ \pm}=\mathbf{T}_{1} \pm i \mathbf{T}_{2} \tag{4.8}
\end{equation*}
$$

In analogy to subsection 3.3 the left-handed spinor $\psi^{\Delta A}$ is written as

$$
\begin{equation*}
\psi^{\Delta A}=A^{\Delta} o^{A}+P_{\iota}^{\Delta} \iota^{A} \tag{4.9}
\end{equation*}
$$

where the two-component objects $P^{\Delta}$ and $A^{\Delta}$ are written as

$$
\begin{equation*}
P^{\Delta}=\binom{P^{1}}{P^{2}}, \quad A^{\Delta}=\binom{A^{1}}{A^{2}} \tag{4.10}
\end{equation*}
$$

Putting in the components of the Yang-Mills fields in (4.2) leads to

$$
\begin{array}{r}
L=\sqrt{2} \int\left(\bar{A} l^{a} a_{a} \mathbf{T}_{3} A+\bar{P} n^{a} a_{a} \mathbf{T}_{3} P+\bar{P} \frac{-i}{\sqrt{2} \Phi}\left(\bar{w} \mathbf{T}_{-}-\cot \theta \mathbf{T}_{3}\right) A\right. \\
\left.+\bar{A} \frac{i}{\sqrt{2} \Phi}\left(w \mathbf{T}_{+}-\cot \theta \mathbf{T}_{3}\right) P\right) \omega_{G} \tag{4.11}
\end{array}
$$

Next we expand the functions $A^{\Delta}$ and $P^{\Delta}$ in terms of spin weighted spherical harmonics and integrate out the sphere. To simplify the following calculations the spin weighted spherical harmonics are restricted to the cases $j=1 / 2$ and $m= \pm 1 / 2$

$$
\begin{equation*}
A^{\Delta}=\sum_{m= \pm \frac{1}{2}} A_{\frac{1}{2} m-\frac{1}{2}}^{\Delta} Y_{\frac{1}{2} m}, \quad P^{\Delta}=\sum_{m= \pm \frac{1}{2}} P_{\frac{1}{2} m \frac{1}{2}}^{\Delta} Y_{\frac{1}{2} m} \tag{4.12}
\end{equation*}
$$

Spherical reduction of the first two terms yields

$$
\begin{align*}
& L_{1}=\frac{\sqrt{2}}{2} \sum_{m= \pm \frac{1}{2}} \int\left(\bar{A}_{\frac{1}{2} m}^{1} l^{a} a_{a} A_{\frac{1}{2} m}^{1}-\bar{A}_{\frac{1}{2} m}^{2} l^{a} a_{a} A_{\frac{1}{2} m}^{2}\right. \\
&\left.+\bar{P}_{\frac{1}{2} m}^{1} n^{a} a_{a} P_{\frac{1}{2} m}^{1}-\bar{P}_{\frac{1}{2} m}^{2} n^{a} a_{a} P_{\frac{1}{2} m}^{2}\right) \Phi^{2} \omega_{g} \tag{4.13}
\end{align*}
$$

In the remaining two terms of 4.11) the $\cot \theta$ terms vanish if the integration over the sphere is performed. $\mathbf{T}_{+}$and $\mathbf{T}_{-}$project out one component of $A^{\Delta}$ and $P^{\Delta}$, respectively. Integrating out the sphere in the remaining terms gives $\pm \pi / 4$. Thus one finds

$$
\begin{align*}
L_{2}=i \frac{\pi}{4} \int\left(\bar{P}_{\frac{1}{2} \frac{1}{2}}^{2} \frac{\bar{w}}{\Phi} A_{\frac{1}{2} \frac{1}{2}}^{1}\right. & -\bar{P}_{\frac{1}{2}-\frac{1}{2}}^{2} \frac{\bar{w}}{\Phi} A_{\frac{1}{2}-\frac{1}{2}}^{1} \\
& \left.-\bar{A}_{\frac{1}{2} \frac{1}{2}}^{1} \frac{w}{\Phi} P_{\frac{1}{2} \frac{1}{2}}^{2}+\bar{A}_{\frac{1}{2}-\frac{1}{2}}^{1} \frac{w}{\Phi} P_{\frac{1}{2}-\frac{1}{2}}^{2}\right) \Phi^{2} \omega_{g} \tag{4.14}
\end{align*}
$$

where we needed the explicit form of the harmonics (c.f. appendix A), to evaluate the integral over the sphere. $L_{1}+L_{2}$ represent the spherically reduced $S U(2)$ Yang-Mills-Dirac interaction term.

## Two-spinor representation of the $S U(2)$ interaction term

Following the notation of the former sections the two spinors are defined by

$$
\begin{equation*}
\Psi_{j m}^{I}=\binom{A_{j m}}{0}, \quad \Psi_{j m}^{I I}=\binom{0}{P_{j m}} \tag{4.15}
\end{equation*}
$$

where $B=Q=0$ was taken because only left-handed fermions couple to $S U(2)$-Yang-Mills fields in the SM. Rewriting (4.13) in terms of these two-spinors leads to

$$
\begin{equation*}
L_{1}=\sum_{m= \pm \frac{1}{2}} \int\left(\bar{\Psi}_{\frac{1}{2} m}^{I} l^{a} a_{a} \mathbf{T}_{3} \gamma_{I}^{-} \Psi_{\frac{1}{2} m}^{I}+\bar{\Psi}_{\frac{1}{2} m}^{I I} n^{a} a_{a} \mathbf{T}_{3} \gamma_{I I}^{+} \Psi_{\frac{1}{2} m}^{I I}\right) \Phi^{2} \omega_{g} \tag{4.16}
\end{equation*}
$$

whereas for (4.14) we find

$$
\begin{equation*}
L_{2}=\frac{\pi}{4} \sum_{m= \pm \frac{1}{2}}(-)^{\frac{1}{2}+m} \int\left(\bar{\Psi}_{\frac{1}{2} m}^{I} \frac{w}{\Phi} \mathbf{T}_{+} I^{-1} \bar{\Psi}_{\frac{1}{2} m}^{I I}+\bar{\Psi}_{\frac{1}{2} m}^{I I} \frac{\bar{w}}{\Phi} \mathbf{T}_{-} I \bar{\Psi}_{\frac{1}{2} m}^{I}\right) \Phi^{2} \omega_{g} \tag{4.17}
\end{equation*}
$$

Before fully writing out the reduced interaction term, we study the $S U(3)$ case.

### 4.2 Reduction of the $S U(3)$ Yang-Mills Dirac system

The spherical reduction of the interaction of fermions and $S U(3)$ Yang-Mills fields is very similar to the $S U(2)$ case. The additional terms in (3.132) are just

$$
\begin{equation*}
\iota^{A} \iota^{A^{\prime}} A_{A A^{\prime}}=\frac{1}{2 g} n^{a} b_{a} \lambda_{8}, \quad o^{A} o^{A^{\prime}} A_{A A^{\prime}}=\frac{1}{2 g} l^{a} b_{a} \lambda_{8} \tag{4.18}
\end{equation*}
$$

and all equations of the former subsections hold if $\mathbf{T}_{i}$ denote the first three $S U(3)$ generators. This depends on the fact that we only consider the simple isospin- $1 / 2$ embedding of the $S U(2)$ group into $S U(3)$. In the isospin-1 case [53] things change considerably, since spacetime and group indices mix and hence the spherical reduction is much more involved. However, the presented procedure can be applied straightforwardly.
$P^{\Lambda}$ is now a three-component object and the only additional term in the action reads

$$
\begin{equation*}
L=\frac{\sqrt{2}}{2} \int\left(\bar{A} l^{a} b_{a} \lambda_{8} A+\bar{P} n^{a} b_{a} \lambda_{8} P\right) \omega_{G} \tag{4.19}
\end{equation*}
$$

where the sphere can be integrated out easily to give

$$
\begin{equation*}
L=\frac{\sqrt{2}}{2} \sum_{j m} \int\left(\bar{A}_{j m} l^{a} b_{a} \lambda_{8} A_{j m}+\bar{P}_{j m} n^{a} b_{a} \lambda_{8} P_{j m}\right) \Phi^{2} \omega_{g} \tag{4.20}
\end{equation*}
$$

Combining the left-handed part of (3.108 with the reduced terms finally leads to the reduced Dirac-Yang-Mills action

$$
\begin{align*}
L_{\mathrm{DYM}}=\sum_{m= \pm \frac{1}{2}} & \int\left(\bar{\Psi}_{\frac{1}{2} m}^{I} i E_{+}^{a} \gamma_{I}^{+} D_{a} \Psi_{\frac{1}{2} m}^{I}+\bar{\Psi}_{\frac{1}{2} m}^{I I} i E_{-}^{a} \gamma_{I I}^{-} D_{a} \Psi_{\frac{1}{2} m}^{I I}\right. \\
& +\frac{1}{\Phi} \bar{\Psi}_{\frac{1}{2} m}^{I I}\left(\gamma^{\star}+(-)^{\frac{1}{2}+m} \frac{\pi}{4} \bar{w} \mathbf{T}_{-}\right) I \Psi_{\frac{1}{2} m}^{I} \\
& \left.+\frac{1}{\Phi} \bar{\Psi}_{\frac{1}{2} m}^{I}\left(\gamma^{\star}+(-)^{\frac{1}{2}+m} \frac{\pi}{4} w \mathbf{T}_{+}\right) I^{-1} \Psi_{\frac{1}{2} m}^{I I}\right) \Phi^{2} \omega_{g} \tag{4.21}
\end{align*}
$$

with the gauge covariant derivative

$$
\begin{equation*}
D_{a}=\nabla_{a}-i a_{a} \mathbf{T}_{3}-i \frac{b_{a}}{2} \lambda_{8} \tag{4.22}
\end{equation*}
$$

when acting on fermions. In case of $S U(2)$ Yang-Mills theory, i.e., $b=$ $0, \mathbf{T}_{i}=\sigma_{i} / 2$, the above action (4.21) is in agreement with references [54,55] if only one value of the 'magnetic' quantum number $m$ is considered.

Exact solutions of the $S U(2)$ and $S U(4)$ Einstein-Yang-Mills-Dirac systems by reduction methods were found in [56]. In addition to spherical symmetry these authors also assumed homogeneity, hence considered cosmological solutions.

### 4.3 The Higgs model

In the action of the Higgs model one considers a complex scalar field with mass and self-interaction term

$$
\begin{equation*}
L=\int\left(G^{\mu \nu}\left(D_{\mu} H\right)^{\dagger} D_{\nu} H-\frac{\lambda}{4}\left(H^{\dagger} H-v^{2}\right)^{2}\right) \omega_{G} \tag{4.23}
\end{equation*}
$$

where the gauge covariant derivative reads $D_{\mu} H=\nabla_{\mu} H-i g A_{\mu} H$. In spherical symmetry $[51,55,57,58]$ the Higgs field is given by ${ }^{9}$

$$
\begin{equation*}
H=\frac{v}{g} \varphi \exp \left(i \xi \mathbf{T}_{r}\right)|a\rangle \tag{4.24}
\end{equation*}
$$

where $\varphi=\varphi\left(x^{\alpha}\right)$ and $\xi=\xi\left(x^{\alpha}\right)$ are real functions and $|a\rangle$ is a constant unit spinor, $\langle a \mid a\rangle=1$. The radial Pauli matrix $\mathbf{T}_{r}$ is defined by

$$
\begin{equation*}
\mathbf{T}_{r}=\sin \theta \cos \phi \mathbf{T}_{1}+\sin \theta \sin \phi \mathbf{T}_{2}+\cos \theta \mathbf{T}_{3} \tag{4.25}
\end{equation*}
$$

Note that the Higgs field ansatz (4.24) differs from the standard parametrisation in particle physics. There the Higgs field is usually parametrised by its shift around the vacuum expectation value $H \mapsto H_{0}+H^{\prime}$, where $H_{0}$ denotes the vacuum expectation value and $H^{\prime}$ is the shifted field. The classical potential in (4.23) vanishes for $H_{0}^{\dagger} H_{0}=v^{2}$. Therefore one sees that the function $\varphi$ represents the deviation around the minimum of the potential, but in contrast to the above, by multiplication rather than addition.

The exponential $\exp \left(i \xi \mathbf{T}_{r}\right)$ written explicitly yields

$$
\exp \left(i \xi \mathbf{T}_{r}\right)=\cos \frac{\xi}{2} \mathbb{1}+i \sin \frac{\xi}{2}\left(\begin{array}{cc}
\cos \theta & e^{-i \phi} \sin \theta  \tag{4.26}\\
e^{i \phi} \sin \theta & -\cos \theta
\end{array}\right)
$$

The second term contains the spherical harmonics with $s=0$ and $l=1$ (see appendix A). Before performing the spherical reduction it should be noted

[^8]that one could set $\xi=0$ and fix the isospin direction. The Higgs field (4.24) is still spherically symmetric is some sense but not spherically symmetric up to gauge transformations, so not according to our second notion. When we analyse the effective theory in two dimensions in section ${ }^{5}$ we will choose the gauge $\xi=0$ to simplify the further calculation.

Spherical reduction of the Higgs action (4.23) using the ansatz (4.24) leads to

$$
\begin{equation*}
L_{\mathrm{H}}=\int\left(\frac{v^{2}}{g^{2}}|D h|^{2}-\frac{v^{2}}{2 g^{2} \Phi^{2}} \varphi^{2}\left|w-e^{i \xi}\right|^{2}-\frac{\lambda v^{4}}{4 g^{4}}\left(\varphi^{2}-g^{2}\right)^{2}\right) \Phi^{2} \omega_{g} \tag{4.27}
\end{equation*}
$$

where $h=\varphi \exp (i \xi / 2)$ and the gauge covariant derivative reads $D_{\alpha}=$ $\nabla_{\alpha}-i a_{\alpha} / 2$. (For $w$ recall the remark after equation (3.131). The Higgs mass and the vector boson mass are given by $M_{H}^{2}=2 \lambda v^{2}$ and $M_{W}^{2}=g^{2} v^{2}$, respectively.

The spherical reduction procedure yields an additional dilaton dependent term in the Higgs potential. Hence the effective potential in 4.27) reads

$$
\begin{equation*}
V=\frac{v^{2}}{g^{2}}\left(\frac{\varphi^{2}}{2 \Phi^{2}}\left|w-e^{i \xi}\right|^{2}+\frac{\lambda v^{2}}{4 g^{2}}\left(\varphi^{2}-g^{2}\right)^{2}\right) \tag{4.28}
\end{equation*}
$$

which has a global minimum if

$$
\begin{equation*}
\Phi \leq \sqrt{2} \frac{\left|w-e^{i \xi}\right|}{M_{H}} \tag{4.29}
\end{equation*}
$$

and its usual symmetry breaking form otherwise. Following e.g. [57] finite energy solutions require $|w|^{2}<1$ and hence $\left|w-e^{i \xi}\right|<2$. Therefore from (4.29) we conclude

$$
\begin{equation*}
\Phi \leq \frac{2 \sqrt{2}}{M_{H}} \tag{4.30}
\end{equation*}
$$

For $M_{H} \approx 100 \mathrm{GeV}-1 \mathrm{TeV}$ this yields a radius of ca. $10^{16}-10^{17}$ Planck lengths.

Restoration of symmetry at some small radius, e.g. near a black hole horizon [59,60], can be understood from the following consideration. An observer would not see the Higgs field in some vacuum state but rather in a thermal bath of Hawking quanta close to a black hole. Hence, if that temperature is high enough, the potential smears out.

### 4.4 Yukawa couplings

In the SM of particle physics fermion masses are introduced by Yukawa couplings and the Higgs mechanism. Therefore the explicit mass terms in
the fermion sector, section 3.3 can be ignored henceforth. The Yukawa interaction term reads

$$
\begin{equation*}
L=\int \chi_{A^{\prime}} \bar{\psi}^{A^{\prime} \Delta} H_{\Delta} \omega_{G} \tag{4.31}
\end{equation*}
$$

where the internal group index in the Higgs field indicates the presence of the unit spinor $|a\rangle$ in (4.24). Since the action (4.31) is not hermitian one must add the hermitian conjugate, which we do at the end.

Following the procedure of the previous sections we first write the spinors $\chi_{A^{\prime}}$ and $\bar{\psi}^{A^{\prime} \Delta}$ in terms of basis spinors and get

$$
\begin{equation*}
L=\int\left(\frac{v}{g} \varphi\left(Q \bar{P}^{\Delta}-B \bar{A}^{\Delta}\right) \exp \left(i \xi \mathbf{T}_{r}\right)|a\rangle\right) \Phi^{2} \omega_{g} \mathrm{~d}^{2} \Omega \tag{4.32}
\end{equation*}
$$

Next we expand the coefficients $A, B, P, Q$ in terms of spin weighted spherical harmonics and use their explicit form 4.26. Furthermore, to simplify the following, we choose the unit vector to be $\langle a|=(0,1)$. The first term of (4.26) is easily reduced because one can use the orthogonality condition (3.102) since $A, B$ and $P, Q$ have the same spin weights respectively. This yields

$$
\begin{equation*}
L_{1}=\sum_{m= \pm \frac{1}{2}} \int\left(\frac{v}{g} \varphi \cos \frac{\xi}{2}\left(Q_{\frac{1}{2} m} \bar{P}_{\frac{1}{2} m}^{2}-B_{\frac{1}{2} m} \bar{A}_{\frac{1}{2} m}^{2}\right)\right) \Phi^{2} \omega_{g} \tag{4.33}
\end{equation*}
$$

The second and more involved term after spherical reduction reads

$$
\begin{align*}
L_{2} & =\int \frac{v}{g} \varphi i \sin \frac{\xi}{2}\left(\frac{2}{3}\left(\bar{A}_{\frac{1}{2}-\frac{1}{2}}^{1} B_{\frac{1}{2} \frac{1}{2}}-\bar{P}_{\frac{1}{2}-\frac{1}{2}}^{1} Q_{\frac{1}{2} \frac{1}{2}}\right)\right. \\
& \left.+\frac{1}{3}\left(\bar{A}_{\frac{1}{2} \frac{1}{2}}^{2} B_{\frac{1}{2} \frac{1}{2}}-\bar{A}_{\frac{1}{2}-\frac{1}{2}}^{2} B_{\frac{1}{2}-\frac{1}{2}}-\bar{P}_{\frac{1}{2} \frac{1}{2}}^{2} Q_{\frac{1}{2} \frac{1}{2}}+\bar{P}_{\frac{1}{2}-\frac{1}{2}}^{2} Q_{\frac{1}{2}-\frac{1}{2}}\right)\right) \Phi^{2} \omega_{g} \tag{4.34}
\end{align*}
$$

The last two terms plus their hermitian conjugates will be written in terms of two-spinors. The latter are defined by (3.106) and (3.107), moreover (4.10) is taken into account. Since we wish to write the effective $2 d$ action without the unit spinor $|a\rangle$, the expansion coefficients $B$ and $Q$ are embedded in the complex two-spinor space by

$$
\begin{equation*}
B_{j m}=\binom{0}{B_{j m}}, \quad Q_{j m}=\binom{0}{Q_{j m}} \tag{4.35}
\end{equation*}
$$

Then the spherically reduced Yukawa term becomes

$$
\begin{align*}
L_{\mathrm{Y}}=\sum_{m= \pm \frac{1}{2}} & \int \frac{v}{g} \varphi\left(-\cos \frac{\xi}{2}\left(\bar{\Psi}_{\frac{1}{2} m}^{I} \mathbb{1} \Psi_{\frac{1}{2} m}^{I}+\bar{\Psi}_{\frac{1}{2} m}^{I I} \mathbb{1} \Psi_{\frac{1}{2} m}^{I I}\right)\right. \\
& +i \frac{1}{3} \sin \frac{\xi}{2}(-)^{\frac{1}{2}+m}\left(\bar{\Psi}_{\frac{1}{2} m}^{I} \gamma^{\star} \Psi_{\frac{1}{2} m}^{I}-\bar{\Psi}_{\frac{1}{2} m}^{I I} \gamma^{\star} \Psi_{\frac{1}{2} m}^{I I}\right) \\
& +i \frac{2}{3} \sin \frac{\xi}{2}\left(\bar{\Psi}_{\frac{1}{2}-\frac{1}{2}}^{I} \mathbf{T}_{+} P_{-} \Psi_{\frac{1}{2} \frac{1}{2}}^{I}-\bar{\Psi}_{\frac{1}{2} \frac{1}{2}}^{I} \mathbf{T}_{-} P_{+} \Psi_{\frac{1}{2}-\frac{1}{2}}^{I}\right. \\
& \left.\left.\quad+\bar{\Psi}_{\frac{1}{2}-\frac{1}{2}}^{I I} \mathbf{T}_{+} P_{+} \Psi_{\frac{1}{2} \frac{1}{2}}^{I I}-\bar{\Psi}_{\frac{1}{2} \frac{1}{2}}^{I I} \mathbf{T}_{-} P_{-} \Psi_{\frac{1}{2}-\frac{1}{2}}^{I I}\right)\right) \Phi^{2} \omega_{g} \tag{4.36}
\end{align*}
$$

where we added the hermitian conjugate. $P_{ \pm}$are the usual chiral projection operators defined by

$$
\begin{equation*}
P_{ \pm}=\frac{1}{2}\left(\mathbb{1} \pm \gamma^{\star}\right) \tag{4.37}
\end{equation*}
$$

which are needed since left- and right-handed fermions are coupled together.

For later use we set $\xi=0$ in the spherically reduced Yukawa action (4.36) which fixes the isospin direction. Then the last three lines vanish and one is left with the simple term

$$
\begin{equation*}
L_{\mathrm{Y}}=\sum_{m= \pm \frac{1}{2}} \int\left(-\frac{v}{g} \varphi\right)\left(\bar{\Psi}_{\frac{1}{2} m}^{I} \mathbb{1} \Psi_{\frac{1}{2} m}^{I}+\bar{\Psi}_{\frac{1}{2} m}^{I I} \mathbb{1} \Psi_{\frac{1}{2} m}^{I I}\right) \Phi^{2} \omega_{g} \tag{4.38}
\end{equation*}
$$

Therefore the induced mass of the Yukawa coupling reads

$$
\begin{equation*}
m_{\mathrm{Y}}=\frac{v}{g} \varphi \tag{4.39}
\end{equation*}
$$

by comparison with the spherically reduced Dirac action (3.108). A small consistency check is to note that the negative sign of 4.38 is consistent with (3.108).

### 4.5 Einstein-Cartan theory

As in section 3.1 torsion is most naturally included by assuming the existence of a derivative operator $\tilde{\nabla}_{a}$ that is not torsion-free. That derivative operator can be split into a torsion-free part $\nabla_{a}$ and a torsion dependent part by

$$
\begin{equation*}
\tilde{\nabla}_{a} U^{c}=\nabla_{a} U^{c}+K_{a b}^{c} U^{b} \tag{4.40}
\end{equation*}
$$

where $K_{a b}{ }^{c}$ is called the contortion tensor and where we follow the notation of Penrose [44]. $K_{a b}{ }^{c}$ is the holonomic version of (3.4) with an additional
negative sign because of the different index positions used in the different formalisms.

Metricity of both covariant derivative operators immediately implies antisymmetry of $K_{a b}{ }^{c}$ in the last index pair. Contortion and torsion are related by

$$
\begin{equation*}
\tilde{T}_{a b}^{c}=K_{b a}^{c}-K_{a b}{ }^{c} . \tag{4.41}
\end{equation*}
$$

When $\tilde{\nabla}_{a}$ is acting on spinors we write

$$
\begin{align*}
& \tilde{\nabla}_{A A^{\prime}} \psi^{C}=\nabla_{A A^{\prime}} \psi^{C}+\Theta_{A A^{\prime} B}{ }^{C} \psi^{B}  \tag{4.42}\\
& \tilde{\nabla}_{A A^{\prime}} \chi^{C^{\prime}}=\nabla_{A A^{\prime}} \chi^{C^{\prime}}+\bar{\Theta}_{A A^{\prime} B^{\prime}} C^{\prime}  \tag{4.43}\\
& \chi^{B^{\prime}}
\end{align*}
$$

from which the contortion tensor can be reconstructed when the action on a vector $U^{c}=U^{C C^{\prime}}$

$$
\begin{equation*}
K_{a b}^{c}=\Theta_{A A^{\prime} B}^{C} \varepsilon_{B^{\prime}} C^{C^{\prime}}+\bar{\Theta}_{A A^{\prime} B^{\prime}}^{C^{\prime}} \varepsilon_{B}^{C} \tag{4.44}
\end{equation*}
$$

is considered. Torsion can now be incorporated in the former equations by replacing $\nabla_{a}$ by $\tilde{\nabla}_{a}$. Then one uses (4.40) and (4.42), (4.43) and finds additional contributions containing the contortion spinor. The latter can be decomposed further into irreducible parts by

$$
\begin{equation*}
\Theta_{A^{\prime} A B C}=\Theta_{A^{\prime}(A B C)}+\frac{1}{3} \varepsilon_{A B} \Theta_{A^{\prime} C}+\frac{1}{3} \varepsilon_{A C} \Theta_{A^{\prime} B} \tag{4.45}
\end{equation*}
$$

where the trace terms are

$$
\begin{equation*}
\Theta_{A^{\prime} B}=\Theta_{D A^{\prime} B}^{D}, \quad \bar{\Theta}_{A B^{\prime}}=\bar{\Theta}_{D^{\prime} A B^{\prime}}^{D^{\prime}} \tag{4.46}
\end{equation*}
$$

Using the above, a third way of writing the Einstein-Hilbert-Cartan action is

$$
\begin{align*}
L_{\mathrm{EHC}}= & \int \tilde{R} \omega_{G}=\int\left(R+K_{a e}^{b} K_{b}^{a e}-K_{b e}{ }^{b} K_{a}^{a e}\right) \omega_{G} \\
= & \int\left(R+\frac{4}{3} \Theta_{A^{\prime} B} \Theta^{A^{\prime} B}+\frac{4}{3} \bar{\Theta}_{A B^{\prime}} \Theta^{\bar{A}^{\prime} B}\right. \\
& \left.-\Theta_{A^{\prime}(A B C)} \Theta^{A^{\prime}(C A B)}-\bar{\Theta}_{A\left(A^{\prime} B^{\prime} C^{\prime}\right)} \bar{\Theta}^{A\left(C^{\prime} A^{\prime} B^{\prime}\right)}\right) \omega_{G} \tag{4.47}
\end{align*}
$$

where the surface term is omitted, see e.g. [61]. The introduction of $\tilde{\nabla}_{a}$ in the Dirac action functional (3.93) leads to an additional term

$$
\begin{equation*}
L_{\mathrm{DT}}=\frac{i}{\sqrt{2}} \int\left(\Theta_{A^{\prime} B}\left(\bar{\psi}^{A^{\prime}} \psi^{B}-\chi^{A^{\prime}} \bar{\chi}^{B}\right)-\bar{\Theta}_{A B^{\prime}}\left(\psi^{A} \bar{\psi}^{B^{\prime}}-\bar{\chi}^{A} \chi^{B^{\prime}}\right)\right) \omega_{G} \tag{4.48}
\end{equation*}
$$

From (4.47) and (4.48) one can derive the equations of motion for the contortion contribution. Variation with respect to $\delta \Theta_{A^{\prime}(A B C)}$ yields the trivial equation of motion $\Theta_{A^{\prime}(A B C)}=0$. Variation with respect to $\delta \Theta_{A^{\prime} B}$ yields

$$
\begin{equation*}
\frac{\delta L_{E H T}}{\delta \Theta_{A^{\prime} B}}=\frac{8}{3} \Theta^{A^{\prime} B}, \quad \frac{\delta L_{D T}}{\delta \Theta_{A^{\prime} B}}=\frac{i}{\sqrt{2}}\left(\bar{\psi}^{A^{\prime}} \psi^{B}-\chi^{A^{\prime}} \bar{\chi}^{B}\right) \tag{4.49}
\end{equation*}
$$

which implies an algebraic equation of motion

$$
\begin{equation*}
\Theta^{A^{\prime} B}=i \frac{3}{8 \sqrt{2}}\left(\bar{\psi}^{A^{\prime}} \psi^{B}-\chi^{A^{\prime}} \bar{\chi}^{B}\right) \tag{4.50}
\end{equation*}
$$

for the trace of the contortion spinor. We already argued in subsection 3.1.3 that this is expected on general grounds. The contortion spinor is given by the fermion current and is purely imaginary.

In the literature $[21,22,62]$ the statement is often found that Dirac fermions only couple to the axial torsion vector or that the contortion tensor is totally skew-symmetric. This can easily be understood in the spinor formalism since one easily checks that

$$
\begin{equation*}
A^{a}=\frac{2}{3} \operatorname{Im} \Theta^{A A^{\prime}}, \quad k^{a}=-\Phi 2 \operatorname{Re} \Theta^{A A^{\prime}} \tag{4.51}
\end{equation*}
$$

where $A^{a}$ and $k^{a}$ are the holonomic, not yet spherically reduced versions of (3.27). The vector $A^{a}$ given by the equation of motion for contortion clearly has components along the $m^{a}, \bar{m}^{a}$ directions. The minus sign in $k^{a}$ is due to the different conventions, already mentioned in the beginning of this subsection. Since fermions only couple to the axial contortion vector variation with respect to $k^{a}$ and $U_{l m n}$ must vanish. The vanishing of $U_{l m n}$ implies that

$$
\begin{equation*}
s_{a}=\frac{1}{2 \Phi} h_{a} \tag{4.52}
\end{equation*}
$$

as can be seen from (3.30). Since the equation of motion (4.50) is purely algebraic one can eliminate the contortion terms from $L_{\mathrm{EHT}}$ and $L_{\mathrm{DT}}$. Since

$$
\begin{align*}
\frac{4}{3} \Theta^{A^{\prime} B} \Theta_{A^{\prime} B} & =\frac{3}{16}\left(\bar{\psi}^{A^{\prime}} \psi^{B} \chi_{A^{\prime}} \bar{\chi}_{B}\right)  \tag{4.53}\\
\frac{i}{\sqrt{2}} \Theta_{A^{\prime} B}\left(\bar{\psi}^{A^{\prime}} \psi^{B}-\chi^{A^{\prime}} \bar{\chi}^{B}\right) & =\frac{3}{8}\left(\bar{\psi}^{A^{\prime}} \psi^{B} \chi_{A^{\prime}} \bar{\chi}_{B}\right) \tag{4.54}
\end{align*}
$$

one finds that the elimination yields $\tau=2(3 / 16+3 / 8)=9 / 8$

$$
\begin{equation*}
L_{\mathrm{T}}=L_{\mathrm{EHC}}+L_{\mathrm{DT}}=\tau \int\left(\bar{\psi}^{A^{\prime}} \chi_{A^{\prime}} \psi^{B} \bar{\chi}_{B}\right) \omega_{G} \tag{4.55}
\end{equation*}
$$

an effectively four-fermion interaction term. It has the structure of a dilaton deformed Thirring model [63]. If the fermion action only consists of chiral fermions, then either $\psi^{A}$ of $\chi^{A^{\prime}}$ is zero, hence the action 4.55 would vanish. Therefore torsion generated by fermions is nontrivial if and only if both four dimensional chiralities are present. However, only one of the invariant two-spinors (3.106) or (3.107) is needed to generate torsion.

If the Dirac two-spinors are expanded in terms of basis spinors the action 4.55 becomes

$$
\begin{equation*}
L_{\mathrm{T}}=\tau \int(\bar{P} Q-\bar{A} P)(P \bar{Q}-A \bar{B}) \omega_{G} \tag{4.56}
\end{equation*}
$$

As already pointed out the standard model with torsion is characterised by one additional term only, namely the four-fermion interaction term 4.55). This action can be spherically reduced by the above methods. We expand the functions $A, B, P, Q$ in terms of spin weighted spherical harmonics ${ }_{s} Y_{j m}$ with the additional restriction $j=1 / 2$ and $m= \pm 1 / 2$.

Putting the expansion in the action (4.56) gives $2^{6}=64$ terms. Next the spherical dependence can be integrated out and one has to evaluate inner products of four spin weighted spherical harmonics. 40 of these inner product vanish and one is left with 24 non-vanishing terms, which equals the number of independent components of the torsion or contortion tensor. Note that these 24 non-vanishing terms are not independent since for fermions the contortion tensor has only four independent components. The inner products are given in appendix C As before these terms can be written in terms of two spinors (3.106), (3.107), which are also given in appendix C

For the moment we put $\Psi_{j m}^{I I}=0$ and moreover assume that only one 'magnetic' quantum number $m$ is present, say $m=1 / 2$, in C.12 and C.13). Then the simplest non-trivial torsion term becomes

$$
\begin{equation*}
L_{\mathrm{T}}=\frac{\tau}{3 \pi} \int\left(\bar{\Psi}_{\frac{1}{2} \frac{1}{2}}^{I} P_{+} \Psi_{\frac{1}{2} \frac{1}{2}}^{I}\right)\left(\bar{\Psi}_{\frac{1}{2} \frac{1}{2}}^{I} P_{-} \Psi_{\frac{1}{2} \frac{1}{2}}^{I}\right) \Phi^{2} \omega_{g} \tag{4.57}
\end{equation*}
$$

which we simply state to show the general structure of those terms.
Similar to the Yukawa coupling, the projectors 4.37) are needed since left- and right-handed fermions couple together. The factor $1 / 3 \pi$ enters because of the integration of four spin-weighted spherical harmonics.

For sake of completeness we mention some additional aspects of EinsteinCartan theory. The GHP spin-coefficient formalism can be extended to include torsion [64]. The idea is based on equation (4.40), one splits every spin-coefficient into two parts, $\rho^{0}$ which is torsion free and $\rho_{1}$ that depends on the contortion, where we adopted the notation of [64]. Therefore the complete spin-coefficient is just the sum of those two parts. The extended
formalism was then used in [62] to analyse neutrino fields in EinsteinCartan theory. There the torsion spin-coefficients are

$$
\begin{equation*}
\rho_{1}=i k \bar{\phi} \phi, \quad \gamma_{1}=\frac{i k}{2} \bar{\phi} \phi, \tag{4.58}
\end{equation*}
$$

where $\phi^{A}=\phi o^{A}$ is the neutrino field and $k$ is the coupling constant. Furthermore [62] contains some interesting theorems, that we can use directly. One of them states (see $\S(7)$ of [62]) that ghost neutrinos, which have vanishing canonical energy-momentum tensor, cannot be constructed in spherically symmetric spacetimes.

The perfect fluid considered previously in subsection 3.2 .2 can be generalised by a Weyssenhoff fluid [65] which permits a non-vanishing spin density. It is characterised by a classical description of spin, where the source term of torsion is written $s^{\kappa}{ }_{\mu \nu}=u^{\kappa} S_{\mu \nu}$, with $u^{\kappa}$ the fluid's four velocity and $S_{\mu \nu}$ the intrinsic angular momentum tensor. In [66] a Weyssenhoff fluid determining torsion by one function $S$ was considered within the framework of the extended spin-coefficient formalism by the same authors. It was found that the torsion spin-coefficient are

$$
\begin{equation*}
\rho_{1}=-\rho_{1}^{\prime}=2 \gamma_{1}=-2 \gamma_{1}^{\prime}=i S \tag{4.59}
\end{equation*}
$$

By considering a static and spherically symmetric Weyssenhoff fluid in a cosmological context, one of the present authors could suggest a mechanism to solve the sign problem of the cosmological constant [67]. Note that in equation (4.58) the torsion coefficient $\rho_{1}$ appears, although the torsion free part $\rho_{0}$ drops out of the Dirac action (3.98).

Einstein-Cartan theory is derived from the usual Einstein-Hilbert action without restricting to torsionless spacetimes. This action is linear in curvature. However, the term $\varepsilon^{k l m n} \tilde{R}_{k l m n}$ is also linear in curvature and was considered in [68]. In case of vanishing torsion this term identically vanishes which was already mentioned in [69]. If torsion is present then the term gives additional contributions to the field equations. In [68] it was used to analyse parity violating contributions in the action. Fortunately it turns out that this term also vanishes identically if fermions are the source of torsion as in the SM under consideration. The first non-trivial contributions enters the field equations if massive spin one particles are allowed to generate torsion, which is beyond the scope of the present work.

## 5 Effective theory in d=2

The advantage of an effective theory in lower dimensions is twofold: at the level of equations of motion the theory is equivalent to the higherdimensional one, but the classical analysis is much simpler. Thus, exact solutions
can be constructed with particular ease. However, there is more to the lowerdimensional theory than just a convenient scheme for reproduction: it can be treated as a model on its own and semi-classical and quantum aspects can be studied in detail. This can provide valuable insight into the quantum regime of the higherdimensional theory, although one has to be careful with interpreting the results because spherical reduction and renormalisation need not to commute [70].

We now present the SSSMG as an effective $2 d$ theory and address its quantisation.

### 5.1 The SSSMG as a $2 d$ model

We now combine the spherically reduced actions of the former sections into an effectively $2 d$ action which represents the SSSMG in first order form

$$
\begin{equation*}
L=L_{\mathrm{FOG}}+L_{\mathrm{YM}}^{U(1)}+L_{\mathrm{YM}}^{S U(2)}+L_{\mathrm{YM}}^{S U(3)}+L_{\mathrm{DYM}}+L_{\mathrm{H}}+L_{\mathrm{Y}}+L_{\mathrm{T}} \tag{5.1}
\end{equation*}
$$

where the different parts of the action are given by $\left(\epsilon:=e^{+} \wedge e^{-}\right)$

$$
\begin{align*}
& L_{\mathrm{FOG}}= \frac{2 \pi}{\lambda^{2}} \int_{L}\left(X_{a}(D \wedge e)^{a}+X \mathrm{~d} \omega+\epsilon \mathcal{V}\left(X, X^{a} X_{a}\right)\right)  \tag{5.2}\\
& L_{\mathrm{YM}}^{U(1)}= \frac{4 \pi}{g_{1}^{2}} \int_{L}\left(z_{1} \mathrm{~d} a_{1}+\epsilon \frac{z_{1}^{2}}{X}\right)  \tag{5.3}\\
& L_{\mathrm{YM}}^{S U(2)}= \frac{4 \pi}{g_{2}^{2}} \int_{L}\left(z_{2} \mathrm{~d} a_{2}+\epsilon \frac{z_{2}^{2}}{X}+\left|D w_{2} \wedge *\left(D w_{2}\right)\right|-\frac{\left(\left|w_{2}\right|^{2}-1\right)^{2}}{2 X} \epsilon\right),  \tag{5.4}\\
& L_{\mathrm{YM}}^{S U(3)}= \frac{4 \pi}{g_{3}^{2}} \int_{L}\left(z_{3} \mathrm{~d} a_{3}+y \mathrm{~d} b+\epsilon \frac{z_{3}^{2}+y^{2}}{X}\right. \\
&\left.\quad\left|D w_{3} \wedge *\left(D w_{3}\right)\right|-\frac{\left(\left|w_{3}\right|^{2}-1\right)^{2}}{2 X} \epsilon\right),  \tag{5.5}\\
& L_{\mathrm{DYM}}=\sum_{m= \pm \frac{1}{2}} \int_{L}\left(X \sum_{N=I, I}\left(i \bar{\Psi}_{\frac{1}{2} m}^{N} \gamma_{N}^{a} e_{a} \wedge * D \Psi_{\frac{1}{2} m}^{N}\right)\right. \\
& \quad+\left(j+\frac{1}{2}\right) \sqrt{X} \epsilon\left(\bar{\Psi}_{\frac{1}{2} m}^{I}\left(\gamma^{\star}+(-)^{\frac{1}{2}+m} \frac{\pi}{4} \bar{w} \mathbf{T}_{-}\right) I \Psi_{\frac{1}{2} m}^{I}\right. \\
&\left.\left.\quad+\bar{\Psi}_{\frac{1}{2} m}^{I}\left(\gamma^{\star}+(-)^{\frac{1}{2}+m} \frac{\pi}{4} w \mathbf{T}_{+}\right) I^{-1} \Psi_{\frac{1}{2} m}^{I I}\right)\right),  \tag{5.6}\\
& L_{\mathrm{H}=}=\frac{v^{2}}{g_{2}^{2}} \int_{L}\left(X|D h \wedge * D h|-\frac{1}{2} \epsilon \varphi^{2}\left|w_{2}-e^{i \xi}\right|^{2}-X \epsilon \frac{\lambda v^{2}}{4 g_{2}^{2}}\left(\varphi^{2}-g^{2}\right)^{2}\right) . \tag{5.7}
\end{align*}
$$

New auxiliary fields $y, z_{i}$ have been introduced in order to bring the YangMills part (with gauge field 1-forms $b, a_{i}$ ) into first order form. For convenience of the reader we recall the field content: $\left(X, X^{a}, z_{1}, z_{2}, z_{3}, y\right)$ are scalar fields which in the absence of matter can be interpreted as target space coordinated of a Poisson manifold [26]; $\left(\omega, e^{a}, a_{1}, a_{2}, a_{3}, b\right)$ are connection, zweibein, $U(1)$ connection, $S U(2)$ connection, $S U(3)$ connection, respectively. $w_{2}$ and $w_{3}$ are complex scalar fields coming from the reduction of the $S U(2)$ and $S U(3)$ connection, respectively. The complex scalar $h=\varphi \exp (i \xi / 2)$ is the Higgs field. $\Psi$ represents all SM fermions.

The gauge covariant derivatives read

$$
\begin{align*}
(D \wedge e)^{ \pm} & =\mathrm{d} e^{ \pm} \pm \omega \wedge e^{ \pm}  \tag{5.8}\\
D w_{2} & =\mathrm{d} w_{2}-i a_{2} w_{2}  \tag{5.9}\\
D w_{3} & =\mathrm{d} w_{3}-i a_{3} w_{3}  \tag{5.10}\\
\text { neutrinos: } D \Psi & =\mathrm{d} \Psi-i \frac{a_{2}}{2} \sigma_{3} \Psi  \tag{5.11}\\
\text { charged leptons: } D \Psi & =\mathrm{d} \Psi-i a_{1} \Psi-i \frac{a_{2}}{2} \sigma_{3} \Psi  \tag{5.12}\\
\text { quarks: } D \Psi & =\mathrm{d} \Psi-i a_{1} \Psi-i \frac{a_{2}}{2} \sigma_{3} \Psi-i \frac{a_{3}}{2} \lambda_{3} \Psi-i \frac{b}{2} \lambda_{8} \Psi  \tag{5.13}\\
D h & =\mathrm{d} h-i \frac{a_{2}}{2} h \tag{5.14}
\end{align*}
$$

(5.11) - (5.13) cover all covariant derivatives with and without $S U(2)$ couplings.

Different parts of the above actions and permutations thereof were a rich source of analytical and numerical investigations during the last 15 years. The likely starting point was [71]. In recent years the inclusion of the cosmological constant with its non-flat asymptotic structure motivated further studies of the above system, going back probably to [72].

The remaining terms of the standard model are

$$
\begin{align*}
L_{\mathrm{Y}}= & \sum_{m= \pm \frac{1}{2}} \int_{L} \frac{v}{g_{2}} \varphi\left(-\cos \frac{\xi}{2}\left(\bar{\Psi}_{\frac{1}{2} m}^{I} \mathbb{1} \Psi_{\frac{1}{2} m}^{I}+\bar{\Psi}_{\frac{1}{2} m}^{I I} \mathbb{1} \Psi_{\frac{1}{2} m}^{I I}\right)\right. \\
& +i \frac{1}{3} \sin \frac{\xi}{2}(-)^{\frac{1}{2}+m}\left(\bar{\Psi}_{\frac{1}{2} m}^{I} \gamma^{\star} \Psi_{\frac{1}{2} m}^{I}-\bar{\Psi}_{\frac{1}{2} m}^{I I} \gamma^{\star} \Psi_{\frac{1}{2} m}^{I I}\right) \\
& +i \frac{2}{3} \sin \frac{\xi}{2}\left(\bar{\Psi}_{\frac{1}{2}-\frac{1}{2}}^{I} \mathbf{T}_{+} P_{-} \Psi_{\frac{1}{2} \frac{1}{2}}^{I}-\bar{\Psi}_{\frac{1}{2} \frac{1}{2}}^{I} \mathbf{T}_{-} P_{+} \Psi_{\frac{1}{2}-\frac{1}{2}}^{I}\right. \\
& \left.\left.\quad+\bar{\Psi}_{\frac{1}{2}-\frac{1}{2}}^{I I} \mathbf{T}_{+} P_{+} \Psi_{\frac{1}{2} \frac{1}{2}}^{I I}-\bar{\Psi}_{\frac{1}{2} \frac{1}{2}}^{I} \mathbf{T}_{-} P_{-} \Psi_{\frac{1}{2}-\frac{1}{2}}^{I I}\right)\right) X \epsilon \tag{5.15}
\end{align*}
$$

Finally the torsion terms are

$$
\begin{align*}
& L_{\mathrm{T}}=\frac{\tau}{3 \pi} \sum_{m= \pm \frac{1}{2}} \int_{L}\left(\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I, I I} P_{+} \Psi_{\frac{1}{2} \pm m}^{I, I I}\right)\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I, I} P_{-} \Psi_{\frac{1}{2} \pm m}^{I I I}\right)\right. \\
& -\left(\bar{\Psi}_{\frac{1}{2} \mp m}^{I} P_{ \pm} \Psi_{\frac{1}{2} \mp m}^{I I}\right)\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I I} P_{ \pm} \Psi_{\frac{1}{2} \pm m}^{I I}\right)+\frac{1}{2}\left(\bar{\Psi}_{\frac{1}{2} \mp m}^{I, I I} P_{+} \Psi_{\frac{1}{2} \mp m}^{I, I I}\right)\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I, I I} P_{-} \Psi_{\frac{1}{2} \pm m}^{I, I}\right) \\
& +\frac{1}{2}\left(\bar{\Psi}_{\frac{1}{2} \mp m}^{I, I I} P_{+} \Psi_{\frac{1}{2} \pm m}^{I, I I}\right)\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I I I} P_{-} \Psi_{\frac{1}{2} \mp m}^{I, I}\right)-\frac{1}{2}\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I} P_{ \pm} \Psi_{\frac{1}{2} \pm m}^{I}\right)\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I I} P_{ \pm} \Psi_{\frac{1}{2} \pm m}^{I I}\right) \\
&  \tag{5.16}\\
& \left.\quad+\frac{1}{2}\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I} P_{ \pm} \Psi_{\frac{1}{2} \mp m}^{I}\right)\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I I} P_{ \pm} \Psi_{\frac{1}{2} \mp m}^{I I}\right)\right) X \epsilon .
\end{align*}
$$

In case of Riemannian manifolds rather that Lorentzian ones, the action of the torsion free standard model with gravity was formally expressed in terms of Dirac-Yukawa operators in [73].

As an illustration why the reformulation as a 2D model is useful we consider now its classical solutions. Surprisingly, up to the very last step the construction of geometry works exactly as for the matterless case and the relevant details have been spelled out in section 3.2. So assuming that either $X^{+}$or $X^{-}$are non-vanishing in an open region the line element reads

$$
\begin{equation*}
\mathrm{d} s^{2}=2 \mathrm{~d} u \mathrm{~d} r+2 X^{+} X^{-} \mathrm{d} u^{2}-r^{2} \mathrm{~d} \Omega^{2}, \tag{5.17}
\end{equation*}
$$

where $r \propto \sqrt{X}$ and the product $X^{+} X^{-}$fulfils the conservation equation (3.44). Of course, in general it is quite hopeless to integrate that equation which contains the information of all matter contributions described above; nevertheless, the simplicity of (5.17) allows for some general statements, independent from material details: Apparent horizons are encountered for $X^{+} X^{-}=0$. If both $X^{+}=X^{-}=0$ at an isolated point the region around that point behaves like the region around the bifurcation 2-sphere of the Schwarzschild BH; in that case instead of (5.17) one should use a different gauge, e.g. Kruskal gauge or Israel gauge [28]. The construction of an atlas by means of "large" Eddington-Finkelstein patches and "small" Kruskal patches has been introduced by Walker [74]. By tuning the matter contributions in a special way it may be possible to achieve $X^{+}=0=X^{-}$in an open region which implies that also the dilaton field $X$ (and thus the surface area) has to be constant in that region. For minimal coupling to the dilaton it follows from (3.38) that such a region has constant curvature and thus may be only Minkowski, Rindler or (A)dS. Note that in the absence of matter and cosmological constant it is not possible to achieve such a constant dilaton region for finite $X$. Thus, the appearance of such regions is a non-trivial consequence of the presence of matter.

### 5.2 Quantisation of SSSMG

There are two basic strategies: either to quantise first and to impose symmetries later or the other way round. The first one appears to be preferable conceptually, but it is more difficult to implement. Since one of the points of imposing spherical symmetry is to simplify the quantisation procedure itself it is also tempting to take the second route. At least some of the basic conceptual problems arise even in this simplified framework and provided they can be solved one can learn something for the full theory without introducing unnecessary technical difficulties. Actually, the preference for either of the two strategies depends on which kind of question one would like to ask; it is not just a matter of taste. We would like to be more concrete on this: First of all, one should recall that in both cases there are no propagating physical modes in the gravity sector. Thus, if one is interested e.g. in scattering problems where virtual BH s may arise as intermediate states one has to add matter degrees of freedom if one would like to keep spherical symmetry, as there are no spherically symmetric gravitons. Adding matter is much simpler following the second route. On the other hand, if one is interested in questions that may be addressed without matter the first route seems to be the better one as it allows for slightly more structure in the geometrical sector than the more restrictive first one. As we are interested in interactions with SM fields and ensuing questions of information loss, scattering problems, virtual BH production etc. we impose spherical symmetry first and quantise later. Thus, we take (5.1) as our starting point and try to quantise this effective action in two dimensions.

There are still two alternatives: either one fixes a geometric background before quantisation and applies methods from quantum field theory on a curved background (cf. e.g. [75]), thus encountering the phenomenon of Hawking radiation, or one quantises geometry exactly first and applies perturbative methods in the matter sector afterwards (cf. e.g. sect. 7 of [20]). As there exists an extensive amount of literature devoted to the first route (even in the more general case when spherical symmetry is absent) the focus will be on the second path. Along these lines the simple case with a single scalar field in the matter sector has been studied extensively $[76,77]$, for reviews cf. $[9,20]$. The extension to the SSSMG is straightforward, albeit somewhat lengthy. Thus, we will merely present the algorithm and note especially where differences to the previous cases arise, rather than presenting all calculations in detail.

The first step, a Hamiltonian analysis including a discussion of constraints, their algebra and the construction of the BRST charge and the ghost sector fortunately essentially remains the same. The algorithm works as follows: as starting point declare the zero-component as "time" and introduce canonical coordinates $q=\left(\omega_{1}, e_{1}^{-}, e_{1}^{+}, a_{1}, b_{1}\right.$, matter $)$ with associated
momenta $p=\left(X, X^{+}, X^{-}, y, z\right.$, matter $)$ and $\bar{q}=\left(\omega_{0}, e_{0}^{-}, e_{0}^{+}, a_{0}, b_{0}\right)$ producing primary first class constraints $\bar{p} \approx 0$. In addition, there will be the usual second class constraints from the fermions, which may be dealt with in the standard way, i.e., by introducing Dirac brackets. Then, take the Dirac bracket of the first class constraints with the Hamiltonian to calculate the secondary constraints denoted by $G$ (which are also first class). It is then noticed that the Hamiltonian is a sum over constraints, $H=\Sigma \bar{q} G$, as expected for a reparameterisation invariant theory [78]. No ternary constraints arise. Next, one should consider the structure functions arising in the Dirac algebra of first class constraints. They will enter the BRST charge, which may be constructed straightforwardly and does not receive any higher order ghost contributions, i.e., no quartic ghost terms arise. At least in the geometric sector no ordering ambiguities arise, as discussed in appendix B. 2 of [77]. A convenient gauge-fixing fermion is one that leads to "temporal" gauge

$$
\begin{equation*}
\omega_{0}=0, \quad e_{0}^{-}=1, \quad e_{0}^{+}=0, \quad a_{0}=0, \quad b_{0}=0 \tag{5.18}
\end{equation*}
$$

In the geometric sector this amounts to Eddington-Finkelstein gauge. Note that it is not possible to set all zero components to zero because this would amount to a singular metric. The choice 5.18 exploits the maximum amount of simplification and consequently the gauge fixed Hamiltonian simplifies drastically as most of the terms drop out. The most convenient order of path integrations seems to be the following one: all non-geometric gauge fields, their related auxiliary fields and the corresponding ghost sectors are integrated out exactly. Then, the remaining ghost sector is eliminated yielding some (contribution to the Faddeev-Popov-)determinant in the measure. As a next step eventual matter momenta are integrated out, if this can be performed exactly by linear or Gaussian path integration. The ensuing action will be linear in the remaining zweibein and connection components. Thus, path integration over geometry can be performed yielding functional $\delta$-functions. They can be used to perform the integration over the auxiliary fields $X, X^{ \pm}$, cancelling exactly the first contribution to the Faddeev-Popov-determinant mentioned above. Because the functional $\delta$-functions contain first derivatives acting on the auxiliary fields at this point homogeneous solutions arise which have to be fixed conveniently. In ordinary QFT often "natural boundary conditions" are invoked, but clearly they cannot be implemented for all fields as the metric must not vanish asymptotically. Instead, a very natural and simple condition is asymptotic flatness, which indeed fixes the relevant homogeneous contributions. Irrelevant contributions may be absorbed by fixing the scalingand shift-ambiguity of the dilaton field. The path integral measure for the final matter integrations can be adjusted in accordance with [79]. The ensuing effective action will be nonlocal and non-polynomial in the matter
fields. So at the end of this algorithm one has an effective theory depending solely on the propagating physical modes (scalar fields and fermions), but, to emphasise this again as it is very important, this theory is nonlocal and non-polynomial. Physically, non-locality comes from integrating out exactly the gravitational self-energy of the fields.

Perturbation theory may be imposed upon this effective theory and Feynman rules may be derived. Since the theory is non-polynomial tree vertices with an arbitrary number of external legs will emerge. Moreover, these vertices will be non-local, in general. For the special case of the spherically reduced Einstein-massless-Klein-Gordon model explicit results may be found in ref. [8]. These Feynman rules are then the basis of any phenomenological study. It is not very difficult to obtain them, but somewhat tedious.

## 6 Concluding remarks

The three formalisms discussed (Cartan, GHP, metric) were used to spherically reduce the Standard model of particle physics with gravity in a comprehensive manner yielding the SSSMG. One of the main aims of the present work was to link knowledge from particle physics on the one and gravity on the other hand.

We are convinced that the SSSMG as a two-dimensional model as presented in section 5 will be of use for further research. These results can for example be used to study several classical aspects like the existence of solitonic solutions, semi-classical aspects like the Hawking radiation and the no-hair theorem and quantum aspects like the role of the virtual black holes in scattering.

Moreover, we hope that our investigations lay the foundation for the discussion of extensions of the spherically reduced Standard Model, like (local) supersymmetry, non-metricity and the reduction of higher dimensional models.

## Acknowledgement

This work has been supported by projects P-14650-TPH and J-2330-N08 of the Austrian Science Foundation (FWF) and by projects 622-1/2002, 798-1/2003 and 349-1/2004 of the Austrian Exchange Service (ÖAD) and is part of the research project Nr. 01/04 Quantum Gravity, Cosmology and Categorification of the Austrian Academy of Sciences (ÖAW) and the National Academy of Sciences of Ukraine (NASU). Part of this work has been performed in the hospitable atmosphere of the International Erwin Schrödinger Institute.

We deeply thank W. Kummer and D. Vassilevich for useful discussions.

## A Spherical harmonics

Spherical harmonics for $s=0$ and $j=1$ are

$$
\begin{equation*}
{ }_{0} Y_{10}=\frac{1}{2} \sqrt{\frac{3}{\pi}} \cos \theta, \quad{ }_{0} Y_{1 \pm 1}=\mp \frac{1}{2} \sqrt{\frac{3}{2 \pi}} \sin \theta e^{ \pm i \phi} \tag{A.1}
\end{equation*}
$$

The spin weighted spherical harmonics are given for spin weight $s= \pm \frac{1}{2}$ and $j=\frac{1}{2}$. Since $-j \leq m \leq j$ one only has $m= \pm \frac{1}{2}$. With

$$
\begin{equation*}
\overline{{ }_{s} Y_{j, m}}=(-1)^{m+s}{ }_{-s} Y_{j,-m}, \tag{A.2}
\end{equation*}
$$

one finds that there are only two independent spin weighted spherical harmonics if $s= \pm \frac{1}{2}, j=\frac{1}{2}$ and $m= \pm \frac{1}{2}$. These are

$$
\begin{align*}
\frac{1}{2} Y_{\frac{1}{2} \frac{1}{2}} & =\frac{1}{\sqrt{2 \pi}} \cos \frac{\theta}{2} e^{i \frac{\phi}{2}}  \tag{A.3}\\
-\frac{1}{2} Y_{\frac{1}{2} \frac{1}{2}} & =-\frac{1}{\sqrt{2 \pi}} \sin \frac{\theta}{2} e^{i \frac{\phi}{2}} \tag{A.4}
\end{align*}
$$

The other two spin weighted spherical harmonics

$$
\begin{align*}
-\frac{1}{2} Y_{\frac{1}{2}-\frac{1}{2}} & =-\frac{1}{\sqrt{2 \pi}} \cos \frac{\theta}{2} e^{-i \frac{\phi}{2}}  \tag{A.5}\\
\frac{1}{2} Y_{\frac{1}{2}-\frac{1}{2}} & =-\frac{1}{\sqrt{2 \pi}} \sin \frac{\theta}{2} e^{-i \frac{\phi}{2}} \tag{A.6}
\end{align*}
$$

are obtained by using A.2). One can easily check that these functions obey the orthogonality condition (3.102).

## B Linking GHP and Cartan formalism

Using basis 1-forms $\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \overline{\boldsymbol{m}}$ the metric can be rewritten to

$$
\begin{equation*}
\mathrm{d} s^{2}=2 \boldsymbol{l} \otimes \boldsymbol{n}-2 \boldsymbol{m} \otimes \overline{\boldsymbol{m}} \tag{B.1}
\end{equation*}
$$

where the notation is the same as for the null tetrad (3.81). Note that the choice of the basis 1-forms is not unique and can be changed in many more
or less practical ways. The spin coefficients can be read of from

$$
\begin{align*}
\mathrm{d} \boldsymbol{l} & =\boldsymbol{m} \wedge \boldsymbol{l}\left(\beta^{\prime}-\bar{\beta}+\bar{\tau}\right)+\overline{\boldsymbol{m}} \wedge \boldsymbol{l}\left(\bar{\beta}^{\prime}-\beta+\tau\right)+\boldsymbol{l} \wedge \boldsymbol{n}\left(\gamma^{\prime}+\bar{\gamma}^{\prime}\right) \\
& +\boldsymbol{m} \wedge \overline{\boldsymbol{m}}(\rho-\bar{\rho})+\boldsymbol{m} \wedge \boldsymbol{n} \bar{\kappa}+\overline{\boldsymbol{m}} \wedge \boldsymbol{n} \kappa,  \tag{B.2}\\
\mathrm{d} \boldsymbol{m} & =\boldsymbol{m} \wedge \boldsymbol{l}\left(\bar{\gamma}-\gamma+\bar{\rho}^{\prime}\right)+\overline{\boldsymbol{m}} \wedge \boldsymbol{l} \bar{\sigma}^{\prime} \\
& +\boldsymbol{m} \wedge \boldsymbol{n}\left(\gamma^{\prime}-\bar{\gamma}^{\prime}+\rho\right)+\overline{\boldsymbol{m}} \wedge \boldsymbol{n} \sigma \\
& +\boldsymbol{l} \wedge \boldsymbol{n}\left(\bar{\tau}^{\prime}-\tau\right)+\boldsymbol{m} \wedge \overline{\boldsymbol{m}}\left(\beta+\bar{\beta}^{\prime}\right) . \tag{B.3}
\end{align*}
$$

## C Inner products

The notation is shortened by writing $\pm$ for $\pm 1 / 2$. Furthermore the expansion coefficients $A, B, P, Q$ are left out in the integrands because one can read them of by looking at the spin weighted spherical harmonics.

Terms of type $(\bar{P} Q)(P \bar{Q})$ and $(\bar{A} B)(A \bar{B})$ for $s= \pm 1 / 2$ respectively read

$$
\begin{align*}
& \int\left( \pm \bar{Y}_{+ \pm \pm} Y_{+ \pm}\right)\left(_{ \pm} Y_{+ \pm \pm} \bar{Y}_{+ \pm}\right) \mathrm{d} \Omega^{2}=\frac{1}{3 \pi}  \tag{C.1}\\
& \int\left({ }_{ \pm} \bar{Y}_{+ \pm \pm} Y_{+ \pm}\right)\left({ }_{ \pm} Y_{+\mp \pm} \bar{Y}_{+\mp}\right) \mathrm{d} \Omega^{2}=\frac{1}{6 \pi}  \tag{C.2}\\
& \int\left({ }_{ \pm} \bar{Y}_{+ \pm \pm} Y_{+\mp}\right)\left(_{ \pm} Y_{+ \pm \pm} \bar{Y}_{+\mp}\right) \mathrm{d} \Omega^{2}=\frac{1}{6 \pi} \tag{C.3}
\end{align*}
$$

Terms of type $(\bar{P} Q)(A \bar{B})$ are

$$
\begin{align*}
& \int\left(+\bar{Y}_{+ \pm+} Y_{+ \pm}\right)\left(-Y_{+\mp-} \bar{Y}_{+\mp}\right) \mathrm{d} \Omega^{2}=\frac{1}{3 \pi}  \tag{C.4}\\
& \int\left(\bar{Y}_{+ \pm+},\right.  \tag{C.5}\\
& \int\left(Y_{+ \pm}\right)\left(-Y_{+ \pm-} \bar{Y}_{+ \pm}\right) \mathrm{d} \Omega^{2}=\frac{1}{6 \pi},  \tag{C.6}\\
& \left.\int Y_{+\mp}\right)\left(-Y_{+ \pm-} \bar{Y}_{+\mp}\right) \mathrm{d} \Omega^{2}=-\frac{1}{6 \pi} .
\end{align*}
$$

Terms of type $(\bar{A} B)(P \bar{Q})$ immediately follow from complex conjugation. Thus on gets

$$
\begin{align*}
& \int \bar{P} Q P \bar{Q} \Phi^{2} \mathrm{~d} \Omega^{2} \omega_{g^{(2)}}=\frac{1}{3 \pi} \int\left(\left(\bar{P}_{+ \pm} Q_{+ \pm} P_{+ \pm} \bar{Q}_{+ \pm}\right)\right. \\
& \left.\quad+\frac{1}{2}\left(\bar{P}_{+ \pm} Q_{+ \pm} P_{+\mp} \bar{Q}_{+\mp}+\bar{P}_{+ \pm} Q_{+\mp} P_{+ \pm} \bar{Q}_{+\mp}\right)\right) \Phi^{2} \omega_{g}  \tag{C.7}\\
& \int \bar{A} B A \bar{B} \Phi^{2} \mathrm{~d} \Omega^{2} \omega_{g^{(2)}}=\frac{1}{3 \pi} \int\left(\left(\bar{A}_{+ \pm} B_{+ \pm} A_{+ \pm} \bar{B}_{+ \pm}\right)\right. \\
& \left.\quad+\frac{1}{2}\left(\bar{A}_{+ \pm} B_{+ \pm} A_{+\mp} \bar{B}_{+\mp}+\bar{A}_{+ \pm} B_{+\mp} A_{+ \pm} \bar{B}_{+\mp}\right)\right) \Phi^{2} \omega_{g} \tag{C.8}
\end{align*}
$$

For the mixed ones one finds

$$
\begin{align*}
& \int \bar{P} Q A \bar{B} \Phi^{2} \mathrm{~d} \Omega^{2} \omega_{g^{(2)}}=\frac{1}{3 \pi} \int\left(\left(\bar{P}_{+ \pm} Q_{+ \pm} A_{+\mp} \bar{B}_{+\mp}\right)\right. \\
& \left.\quad+\frac{1}{2}\left(\bar{P}_{+ \pm} Q_{+ \pm} A_{+ \pm} \bar{B}_{+ \pm}-\bar{P}_{+ \pm} Q_{+\mp} A_{+ \pm} \bar{B}_{+\mp}\right)\right) \Phi^{2} \omega_{g} \tag{C.9}
\end{align*}
$$

together with its complex conjugate.
The terms of the inner products can be written in terms of two-spinors

$$
\begin{array}{ll}
\bar{\Psi}_{j m}^{I} P_{+} \Psi_{j m}^{I}=\bar{B}_{j m} A_{j m}, & \\
\bar{\Psi}_{j m}^{I} P_{-} \Psi_{j m}^{I}=\bar{A}_{j m} B_{j m}  \tag{C.11}\\
\bar{\Psi}_{j m}^{I I} P_{+} \Psi_{j m}^{I}=-\bar{P}_{j m} Q_{j m}, & \\
\bar{\Psi}_{j m}^{I} P_{-} \Psi_{j m}^{I I}=-\bar{Q}_{j m} P_{j m}
\end{array}
$$

Hence we find

$$
\begin{align*}
& \int(\bar{A} B A \bar{B}+\bar{P} Q P \bar{Q}) \Phi^{2} \mathrm{~d} \Omega^{2} \omega_{g}= \\
& \frac{1}{3 \pi} \sum_{m= \pm \frac{1}{2}} \int\left(\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I, I} P_{+} \Psi_{\frac{1}{2} \pm m}^{I, I I}\right)\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I, I I} P_{-} \Psi_{\frac{1}{2} \pm m}^{I, I I}\right)\right. \\
&+\frac{1}{2}\left(\bar{\Psi}_{\frac{1}{2} \mp m}^{I, I I} P_{+} \Psi_{\frac{1}{2} \mp m}^{I, I I}\right)\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I, I I} P_{-} \Psi_{\frac{1}{2} \pm m}^{I, I I}\right) \\
&\left.+\frac{1}{2}\left(\bar{\Psi}_{\frac{1}{2} \mp m}^{I, I I} P_{+} \Psi_{\frac{1}{2} \pm m}^{I, I I}\right)\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I, I I} P_{-} \Psi_{\frac{1}{2} \mp m}^{I, I}\right)\right) \Phi^{2} \omega_{g} \tag{C.12}
\end{align*}
$$

where the first and second term of the left-hand side is obtained if $\Psi^{I}$ or $\Psi^{I I}$ is considered respectively.

For the mixed terms we find

$$
\begin{align*}
& \int(\bar{P} Q A \bar{B}+\bar{Q} P B \bar{A}) \Phi^{2} \mathrm{~d} \Omega^{2} \omega_{g}= \\
&-\frac{1}{3 \pi} \sum_{m= \pm \frac{1}{2}} \int\left(\left(\bar{\Psi}_{\frac{1}{2} \mp m}^{I} P_{ \pm} \Psi_{\frac{1}{2} \mp m}^{I}\right)\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I I} P_{ \pm} \Psi_{\frac{1}{2} \pm m}^{I I}\right)\right. \\
&+\frac{1}{2}\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I} P_{ \pm} \Psi_{\frac{1}{2} \pm m}^{I}\right)\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I I} P_{ \pm} \Psi_{\frac{1}{2} \pm m}^{I I}\right) \\
&\left.-\frac{1}{2}\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I} P_{ \pm} \Psi_{\frac{1}{2} \mp m}^{I}\right)\left(\bar{\Psi}_{\frac{1}{2} \pm m}^{I I} P_{ \pm} \Psi_{\frac{1}{2} \mp m}^{I I}\right)\right) \Phi^{2} \omega_{g} \tag{C.13}
\end{align*}
$$

where the upper sign of the projector corresponds to the first term of the left-hand side and the lower to the second.

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[^0]:    *e-mail: hbalasin@tph.tuwien.ac.at
    ${ }^{\dagger}$ e-mail: boehmer@hep.itp.tuwien.ac.at
    $\ddagger$ e-mail: grumil@hep.itp.tuwien.ac.at

[^1]:    ${ }^{1}$ It is impossible to present a complete list of references regarding spherical symmetry, because ever since Coulomb we estimate that ca. $10^{5}$ publications appeared in this context. However, whenever a certain technical detail is used of course some of the original literature, or at least reviews for further orientation, will be provided.

[^2]:    ${ }^{2}$ Because from the context it will always be clear whether we mean a Lie-group or its associated algebra we do not discriminate notationally between them.

[^3]:    ${ }^{3}$ Alternatively, one can try to eliminate the dilaton by means of its EOM, thus obtaining an action which depends non-linearly on curvature. Reviews on this approach are [24] and [25].
    ${ }^{4}$ It should be noted that even in the absence of torsion in $d=4$ the connection $\omega^{a}{ }_{b}$ in 3.32) is torsion free if and only if $\mathcal{V}$ depends on $X$ only.

[^4]:    ${ }^{5}$ One can describe a patch in which $X^{+}=0=X^{-}$at a certain point e.g. by a coordinate system similar to the one introduced by Israel [28] or by Kruskal like coordinates.

[^5]:    ${ }^{6}$ By "all" in the parlance of [37] it is meant that the "good" function $\tilde{w}$ can attain any form. The "muggy" function $\tilde{I}$, however, cannot be chosen independently.

[^6]:    ${ }^{7}$ We note that we follow here the usage of most of the literature, taking Latin indices for abstract indices, whereas they were used for anholonomic indices in the previous section.

[^7]:    ${ }^{8}$ If the spinor $\psi_{A}$ is labelled by spinorial indices $A, A^{\prime}, \ldots$ then $\bar{\psi}_{\mathcal{A}^{\prime}}$ denotes the complex conjugate of the spinor $\psi_{A}$. In all other cases the quantity $\bar{\Psi}$ is the Dirac conjugate of the spinor $\Psi$.

[^8]:    ${ }^{9}$ We stick with the notation of [51].

