High-Reynolds-Number-Asymptotics of Turbulent Boundary Layers: Novel Aspects with Emphasis on Flow Separation

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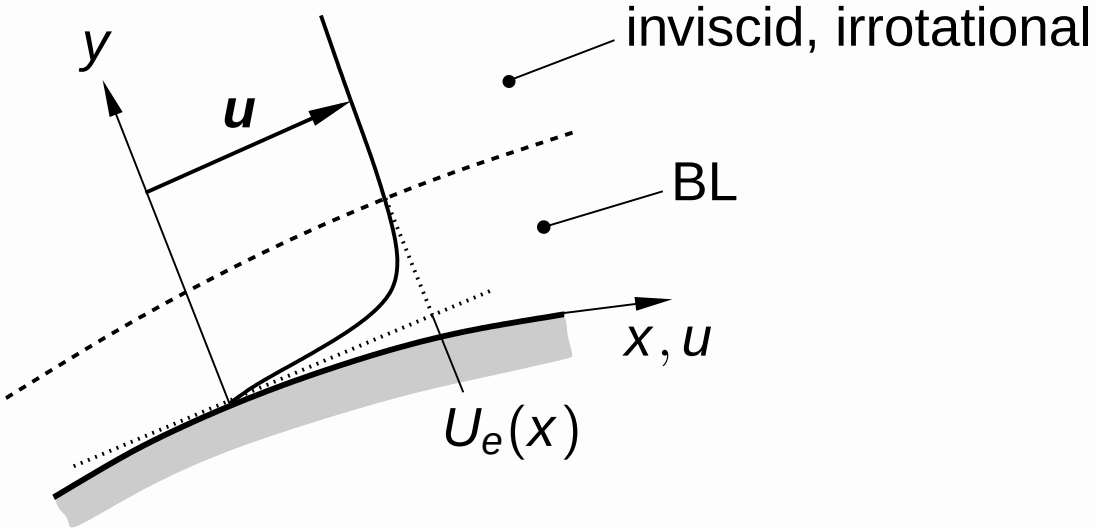
Overview

Wall-bounded high-Reynolds-number flows : $Re := \tilde{U}\tilde{L}/\tilde{\nu} \gg 1$



Navier–Stokes eqs ($\rho = const$)

$$\nabla \cdot \boldsymbol{u} = 0\,, \quad \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \frac{1}{Re}\nabla^2 \boldsymbol{u}\,, \quad \boldsymbol{u}|_{y=0} = \boldsymbol{0}$$

Reynolds-averaging, ergod hypothesis (nominally steady 2D flow)

$$\boldsymbol{Q}(\boldsymbol{x}, t; Re) = \langle \boldsymbol{Q} \rangle (x, y; Re) + \boldsymbol{Q}'(\boldsymbol{x}, t; Re)$$

$$\langle \boldsymbol{Q} \rangle := \lim_{t_{av} \to \infty} \frac{1}{t_{av}} \int_{-t_{av}/2}^{t_{av}/2} \boldsymbol{Q}(\boldsymbol{x}, t + t'; Re)\, \mathrm{d}t'$$

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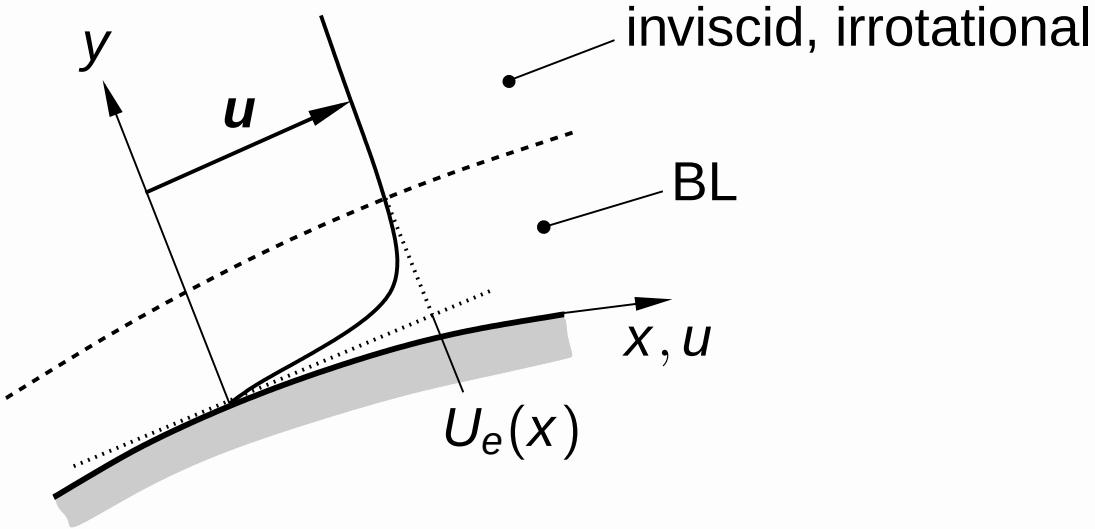

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Outline

Numerically-based methods

- DNS : resolution of all scales $\Rightarrow$ no modelling needed but at present restricted to moderately large values of Re
- LES : modelling of short scales $\Rightarrow$ reduction of computational efforts, which, however, are still too massive to be useful for the solution of engineering problems
- RANS : modelling of all scales $\Rightarrow$ engineering problems can be solved with an acceptable amount of computational efforts, which, however, increase with increasing values of Re

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1. Turbulent wall-bounded flows
   - Numerically-based methods
   - Analytically-based methods

Analytically-based methods

- full NS eqs : starting efforts, but still no complete theory exists
  Deriat & Guiraud (J Mec Theor Appl, 1986), Walker (AIAA J, 1989, CISM, 1998), Scheichl & Kluwick (JFS, 2008)

- non-dimensional Reynolds-averaged NS eqs
  (nominally 2D, curvature effects on BL flow of higher order):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} - \frac{\partial \langle u'^2 \rangle}{\partial x} - \frac{\partial \overbrace{\langle u'v' \rangle}^{-\tau}}{\partial y} + \frac{1}{Re}\nabla^2 u$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} - \frac{\partial \langle u'v' \rangle}{\partial x} - \frac{\partial \langle v'^2 \rangle}{\partial y} + \frac{1}{Re}\nabla^2 v$$

$$y = 0: \quad u = v = u' = v' = 0, \quad y \sim \delta(x; Re): \quad u \sim U_e(x), \quad \tau \sim 0$$

$\Rightarrow$ asymptotic theory faces closure problem for τ

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Classical theory of turbulent small-defect BLs : $u_\tau := \tau_w^{1/2} \to 0$

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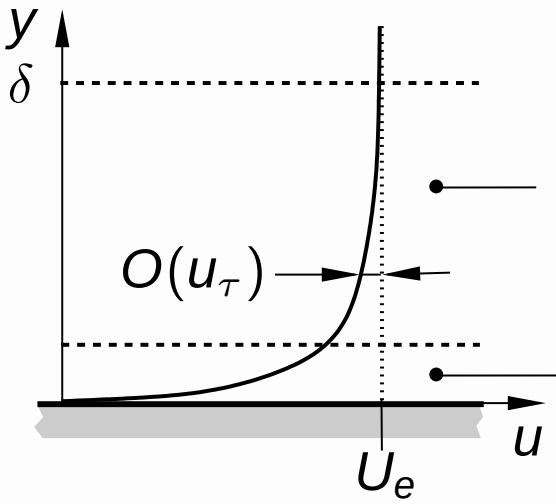

outer, predominately inviscid region

viscous wall layer

overlap region : $y^+ := y u_\tau Re \to \infty$

$$\frac{u}{u_\tau} \sim \frac{1}{\kappa} \ln y^+ + C_i$$

$\tau_w = Re^{-1} (\partial u / \partial y)_{y=0}$

assumptions — Walker (CISM, 1998), Scheichl, Kluwick & Alletto (Acta Mech, 2008)

(a) locally isotropic turbulence $\Rightarrow$ $\langle u'^2 \rangle$, $\langle u'v' \rangle$, $\langle v'^2 \rangle$ of same magnitude

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(c) direct match with outer fully turbulent region $\Rightarrow$ two-tiered BL

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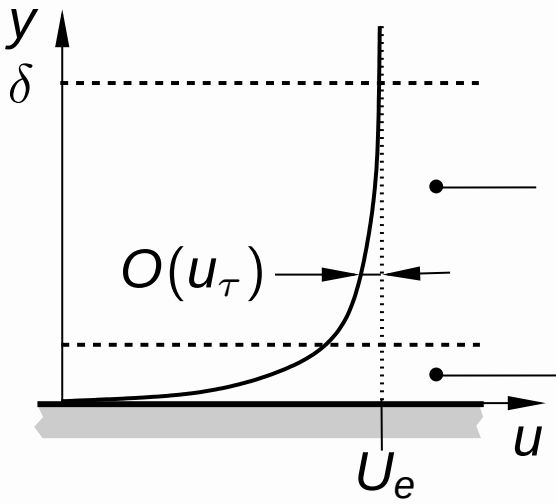

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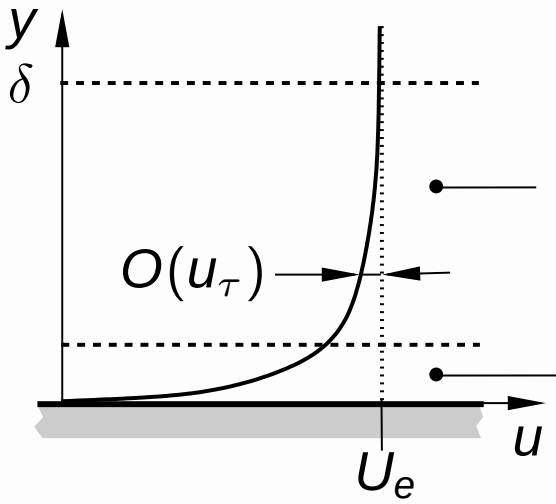

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Distinguished limit $\gamma := u_\tau / U_e \to 0\,,\ \ Re \to \infty$

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$$\frac{u}{U_e} \sim \gamma(x, Re)\, u^+(x, y^+) + \cdots$$

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$$p \sim p_0(x) + \cdots$$

outer defect layer: $\eta = y/\delta$

$$\frac{u}{U_e} \sim 1 - \gamma \frac{\partial F_1(x,\eta)}{\partial \eta} + O(\gamma^2)$$

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experimental observation / result from first principles (?)

Österlund et al. (Phys Fluids, 2000) Walker (AIAA J, 1989, CISM, 1998)

$$u^+(y^+) \sim \kappa^{-1} \ln y^+ + C_i\,, \quad y^+ \to \infty\,, \quad \kappa \approx 0.384\,,\ \ C_i \approx 4.1$$

provides expansion in outer layer and matching with wall layer

$$\eta \to 0\ : \quad \partial F_1/\partial \eta \sim -\kappa^{-1} \ln \eta + C_o(x)\,, \quad T_1 \to 1\,; \qquad p_0(x) = p_e(x)$$

skin-friction law $\kappa/\gamma \sim \ln(Re\, \gamma \delta U_e) + \kappa(C_i + C_o) \ \sim \ln Re$

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Substitution into Reynolds-averaged NS eqs

viscous wall layer: $\mathrm{d}u^+/\mathrm{d}y^+ + t^+ = 1$ ... constant total shear stress

outer defect layer: $\delta \sim \gamma\,\Delta_1(x) + O(\gamma^2)$

leading-order equation

$$(E + 2\beta_0)\eta F_1' - E\,F_1 - \Delta_1 F_{1,e}\,\partial F_1/\partial x = F_{1,e}\,(T_1 - 1)\,, \quad F_{1,e} = F_1(x,1)$$

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layer thickness ratio exponentially small

$$\frac{\delta^+}{\delta} = \frac{(u_\tau Re)^{-1}}{\delta} \sim \frac{1}{Re\,\gamma^2\Delta_1 U_e} \sim \frac{1}{\Delta_1 U_e}\,\frac{\exp(-\kappa/\gamma)}{\gamma^2}$$

consequences

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Self-similar flows : $\beta_0 = const$

- require $\partial[F_1, T_1]/\partial x \equiv 0 \;\Leftrightarrow\; [F_1, T_1] = [F_1, T_1](\eta; \beta_0)$
- Rotta–Clauser parameter $\beta := -\dfrac{\delta^* U_e U_{ex}}{\tau_w} \sim \beta_0 + O(\gamma)$

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$$(1 + 2\beta_0)\eta F_1' - F_1 = F_{1,e}\,(T_1 - 1)\,, \quad \beta_0 = -\frac{\Delta_1 F_{1,e} U_{ex}}{U_e}$$

$$U_e \propto (x - x_v)^m\,, \quad m = -\beta_0/(1 + 3\beta_0)\,, \quad \Delta_1 F_{1,e} = (1 + 3\beta_0)(x - x_v)$$

large values of $\beta_0 \;\Leftrightarrow\; m \to -1/3_+$ Mellor & Gibson (JFM, 1966)

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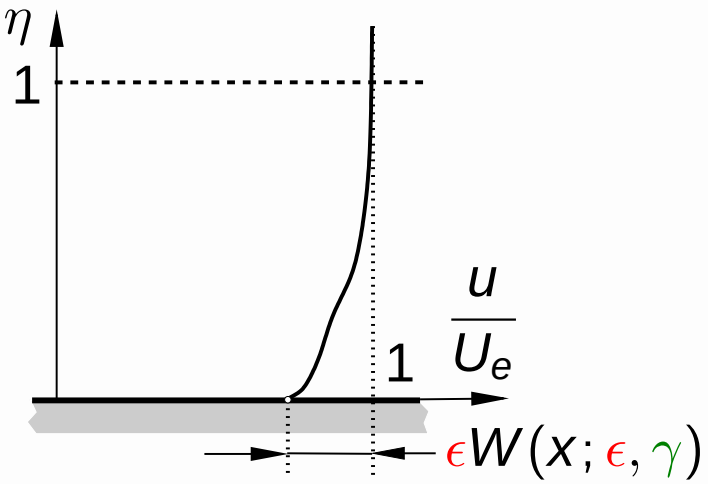
Outline

Moderately large velocity defect : $\gamma \ll \epsilon \ll 1$

Mellor & Gibson (JFM, 1966), Mellor (JFM, 1966), Perry & Schofield (Phys Fluids, 1973), Head (JFM, 1976), Kader & Yaglom (JFM, 1978), Schofield (JFM, 1981), Skote & Henningson (Flow Turbul Combust, 1998), Schlichting & Gersten (2000)

Outer main layer

$$\frac{u}{U_e} = 1 - \epsilon F'(x, \eta; \epsilon, \gamma)\,, \quad \frac{\tau}{U_e^2} = \epsilon^2 T(x, \eta; \epsilon, \gamma)\,, \quad \eta = \frac{y}{\delta}$$



$$\underbrace{u\,\frac{\partial u}{\partial x} + \cdots}_{O(\epsilon)} \sim \underbrace{\frac{\partial \tau}{\partial y}}_{O(\epsilon^2/\delta)} \quad \Rightarrow \quad \delta = \epsilon \Delta(x; \epsilon, \gamma)$$

flow behaviour as $\eta \to 0$: $\quad \tau_w / U_e^2 \sim \gamma^2 \sim \epsilon^2 T(x, 0; \ldots)$

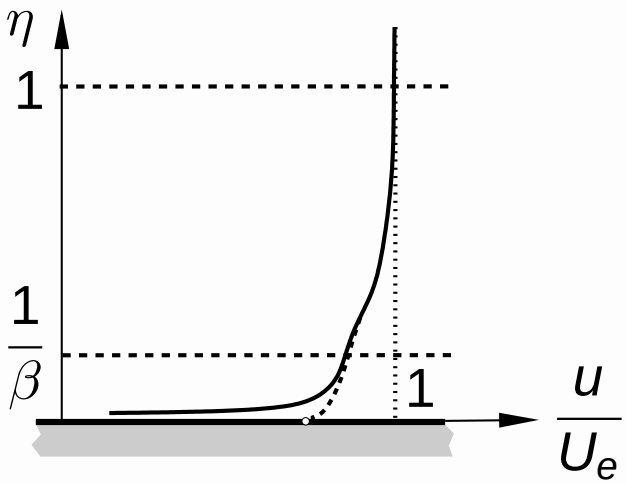
$$T(x, 0; 0, 0) = 0$$

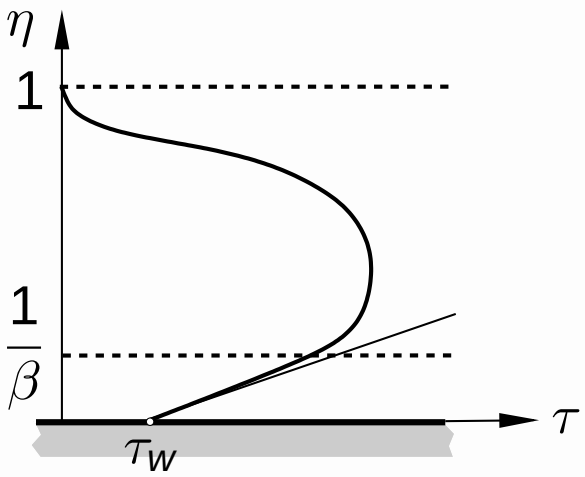
Intermediate layer

$$\frac{u}{U_e} \sim 1 - \epsilon W - \gamma \hat{u} + \cdots$$

estimate of layer thickness δ_i and velocity defect ϵ

$$u\,\frac{\partial u}{\partial x} + \cdots \sim -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} \quad \Rightarrow \quad \tau \sim \tau_w - \epsilon y \frac{\partial (U_e^2 W)}{\partial x}$$





- $\dfrac{\tau_w}{-U_e U_{ex}} \sim \delta_i\,\epsilon \quad \Rightarrow \quad \delta_i \sim \dfrac{\tau_w}{\epsilon(-U_e U_{ex})} \quad \Rightarrow \quad \dfrac{\delta_i}{\delta} \sim \dfrac{\tau_w}{\delta^*(-U_e U_{ex})} = \dfrac{1}{\beta}$
- $\dfrac{\tau_w}{U_e^2} = \gamma^2 \sim \epsilon \dfrac{\epsilon}{\beta} \quad \Rightarrow \quad \epsilon := \gamma \beta^{1/2}$

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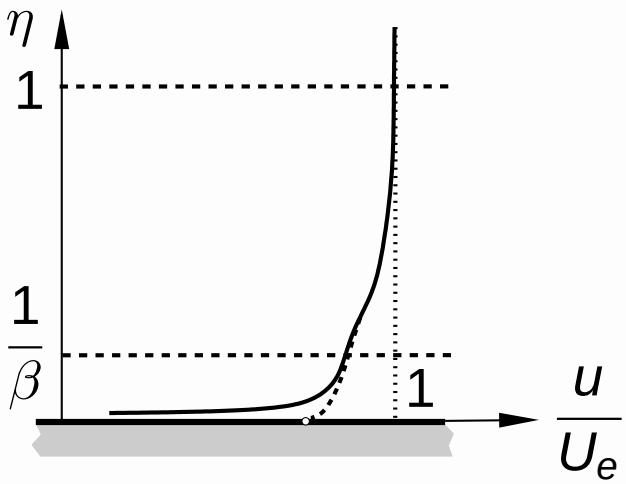

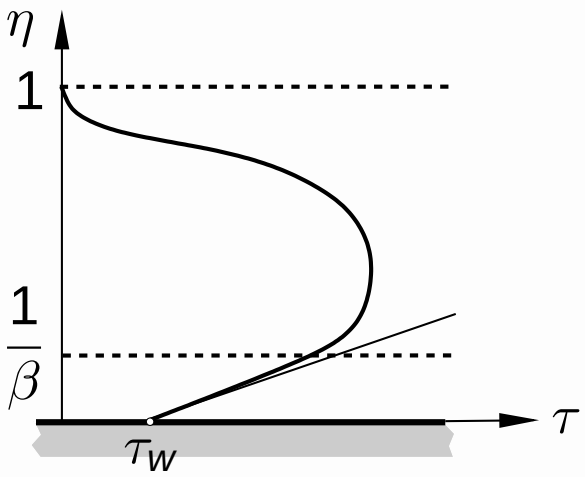

- $\dfrac{\tau_w}{-U_e U_{ex}} \sim \delta_i\,\epsilon \quad \Rightarrow \quad \delta_i \sim \dfrac{\tau_w}{\epsilon(-U_e U_{ex})} \quad \Rightarrow \quad \dfrac{\delta_i}{\delta} \sim \dfrac{\tau_w}{\delta^*(-U_e U_{ex})} = \dfrac{1}{\beta}$
- $\dfrac{\tau_w}{U_e^2} = \gamma^2 \sim \epsilon \dfrac{\epsilon}{\beta} \quad \Rightarrow \quad \epsilon := \gamma \beta^{1/2}$

formal expansions

$$\frac{u}{U_e} \sim 1 - \epsilon W(x;\epsilon,\gamma) - \gamma \hat{u}(x,\zeta) + \cdots, \quad \zeta = \frac{y}{\Delta}\frac{\epsilon}{\gamma^2} \sim \frac{y\beta}{\delta} \sim \frac{y}{\delta_i}$$

$$\frac{\tau}{U_e^2\gamma^2} \sim T_1(x,\zeta;\epsilon,\gamma) = 1 + \lambda\zeta, \quad \lambda = -\frac{\Delta}{U_e^2}\frac{\mathrm{d}(U_e^2 W)}{\mathrm{d}x}$$

matching with adjacent layers $\Rightarrow$ mixing length $\ell \sim \kappa y$

$$T_1 = (\kappa\zeta)^2(\partial\hat{u}/\partial\zeta)^2 = 1 + \lambda\zeta \;\Rightarrow$$

$$\kappa\hat{u} = -\ln\zeta + 2\ln\left[(1+\lambda\zeta)^{1/2} + 1\right] - 2(1+\lambda\zeta)^{1/2}$$

$\zeta \to \infty$: $\quad \kappa\hat{u} \sim -2(\lambda\zeta)^{1/2} + \cdots$ square-root law Townsend (JFM, 1961)

$\zeta \to 0$: $\quad \kappa\hat{u} \sim -\ln(\lambda\zeta/4) - 2 + \cdots$ log law

matching with classical wall layer $\Rightarrow$ generalised skin-friction law

$$\frac{\kappa}{\gamma} \sim \ln\left[\frac{Re\,\gamma^2 U_e^3}{\beta_0^{1/2}}\right] + \beta_0^{1/2}\kappa W + O(\gamma\beta_0) \sim \ln Re$$

Outer layer — cont'd

2nd-order analysis includes surface curvature $k(x)$

$$\eta = y/\delta(x;\epsilon,\gamma)$$

$$p = p_0(x) + U_e^2\left[k(x)\,y + \epsilon^2 P(x,\eta;\ldots)\right], \quad \mathrm{d}p_0/\mathrm{d}x = -U_e\,\mathrm{d}U_e/\mathrm{d}x$$

$$\frac{\psi}{U_e} = y - k(x)\frac{y^2}{2} - \delta\epsilon F(x,\eta;\ldots), \quad -\frac{[\langle u'^2\rangle,\langle v'^2\rangle,\langle u'v\rangle]}{U_e^2} = \epsilon^2[R,S,T](x,\eta;\ldots)$$

expansions

$$Q = Q_1 + \epsilon Q_2 + \cdots, \quad Q = F,P,R,S,T,W, \quad \delta/\epsilon = \Delta_1 + \epsilon\delta_2 + \cdots$$

$$\beta/\beta_0(x_0) = B_1(x) + \epsilon B_2(x) + \cdots, \quad B_1(x_0) = 1, \quad B_2(x_0) = \cdots = 0$$

1st-order problem

$$\frac{1}{U_e}\frac{\mathrm{d}(U_e\Delta_1)}{\mathrm{d}x}\eta F_1' - \frac{1}{U_e^3}\frac{\partial(U_e^3\Delta_1 F_1)}{\partial x} = T_1, \quad F_1(x,0) = F_1'(x,0) = T_1(x,1) = 0$$

$$\eta \to 0: \quad T_1 \sim (\kappa\eta F_1'')^2, \quad F_1' \sim W_1(x) - (2/\kappa)(\lambda\eta)^{1/2}$$

Outline

Quasi-equilibrium flows : self-similar up to 2nd order

classical theory : $\beta \sim \beta_0 = O(1)$

$B_1 \equiv 1\,, \quad \Delta_1 F_{1,e} = (1+3\beta_0)(x-x_v)\,, \quad U_e \propto (x-x_v)^m$

$m = -\beta_0/(1+3\beta_0)$ with $-1/3 < m < \infty$

limit of large β_0 : $m \sim -1/3 + 1/(9\beta_0) + \cdots$

1st-order problem : $\partial F_1/\partial x \equiv \partial T_1/\partial x \equiv 0\,, \quad \beta_0 \to \infty \;\Leftrightarrow\; m \to -1/3_+$

$U_e = \hat{U}(x)\,, \quad F_1 = \hat{F}(\eta)\,, \quad T_1 = \hat{T}(\eta)\,, \quad \Delta_1 = \hat{\Delta}(x)$

$B_1 \equiv 1\,, \quad \hat{\Delta}\hat{F}_e = 3(x-x_v)\,, \quad \hat{U} = (C/3)^{1/3}\,(x-x_v)^{-1/3}$

$2\eta\hat{F}' = \hat{F}_e\hat{T}\,, \quad \hat{F}'(1) = \hat{F}''(1) = \hat{T}(1) = \hat{F}(0) = \hat{T}(0) = 0$

mixing length model : $\hat{T} = l(\eta)^2\,\hat{F}''(\eta)^2$

$$\hat{F}'(\eta) = \frac{1}{2\hat{F}e}\left(\int_\eta^1 \frac{z^{1/2}}{l(z)}\,\mathrm{d}z\right)^2, \quad \hat{F}_e = \left[\frac{1}{2}\int_0^1\left(\int_\eta^1 \frac{z^{1/2}}{l(z)}\,\mathrm{d}z\right)^2 \mathrm{d}\eta\right]^{1/2}$$

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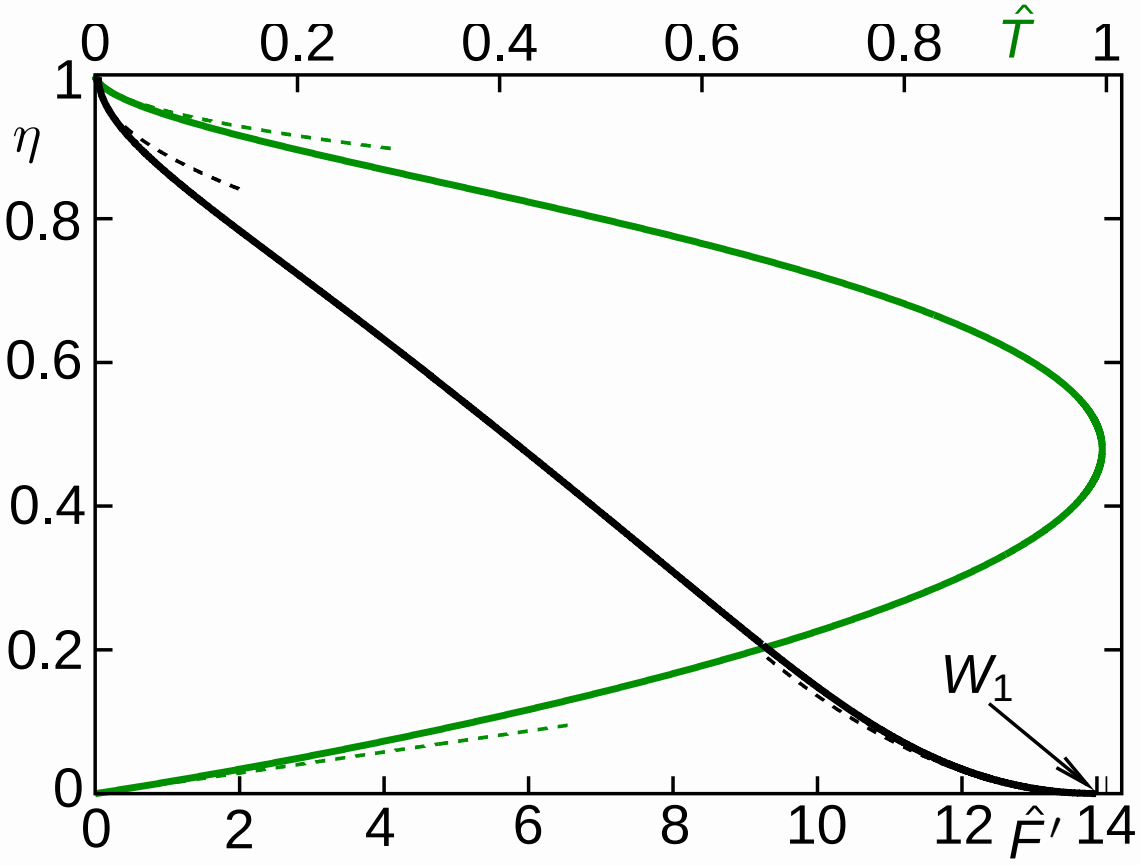
mixing length closure

Michel, Quémard & Durant (Proc Stanford Conf, 1969)

$$\ell(\eta) = c_\ell\, I(\eta)^{1/2} \tanh(\kappa\eta/c_\ell)\,, \quad c_\ell = 0.085\,, \quad I(\eta) = 1/(1+5.5\,\eta^6)$$

$$W_1(x) = \hat{F}'(0) \doteq 13.868\,, \quad \hat{F}(1) = 5.682\,, \quad \mathrm{d}\Delta_1/\mathrm{d}x \doteq 0.528$$



2nd-order problem: $\partial F_2/\partial x \equiv \partial T_2/\partial x \equiv 0\,, \quad \beta_0 \to \infty \quad \Leftrightarrow \quad m \to -1/3_+$

least-degenerate limit

$$\left.\begin{array}{l} \tau_w/U_e^2 \equiv \gamma^2 = \epsilon^3 T_2(0) \quad \Rightarrow \quad T_2(0) = 1/\Gamma\,, \quad \epsilon^3 = \Gamma\gamma^2 \\ \epsilon \equiv \gamma\,\beta_0^{1/2} \end{array}\right\} \quad \Gamma := \epsilon\,\beta_0 = O(1)$$

consistency with 1st-order results requires

$$\hat{U}(x) = (C/3)^{1/3}(x - x_v)^m\,, \quad m \sim -1/3 + \gamma^{2/3}\mu_1 + \cdots$$

solvability condition

$$9\Gamma^{2/3}\mu_1 = 1 + r\Gamma$$

$$r = \frac{1}{\hat{F}_e}\int_0^1 (\hat{F}'^2 + \hat{S} - \hat{R})(\eta)\,\mathrm{d}\eta - 12\frac{\hat{K}}{\hat{F}_e^2}\int_0^1 \eta\hat{F}'(\eta)\,\mathrm{d}\eta\,, \quad k = \frac{\hat{K}}{x - x_v}$$

canonical form for plane or concave surface: $\hat{K} \le 0 \;\Rightarrow\; r > 0$

$$9\hat{D}^{2/3}\hat{\mu} = 1 + \hat{D}^3 \qquad \hat{D} = r^{1/3}\Gamma^{1/3}\,, \quad \hat{\mu} = r^{-2/3}\mu_1$$

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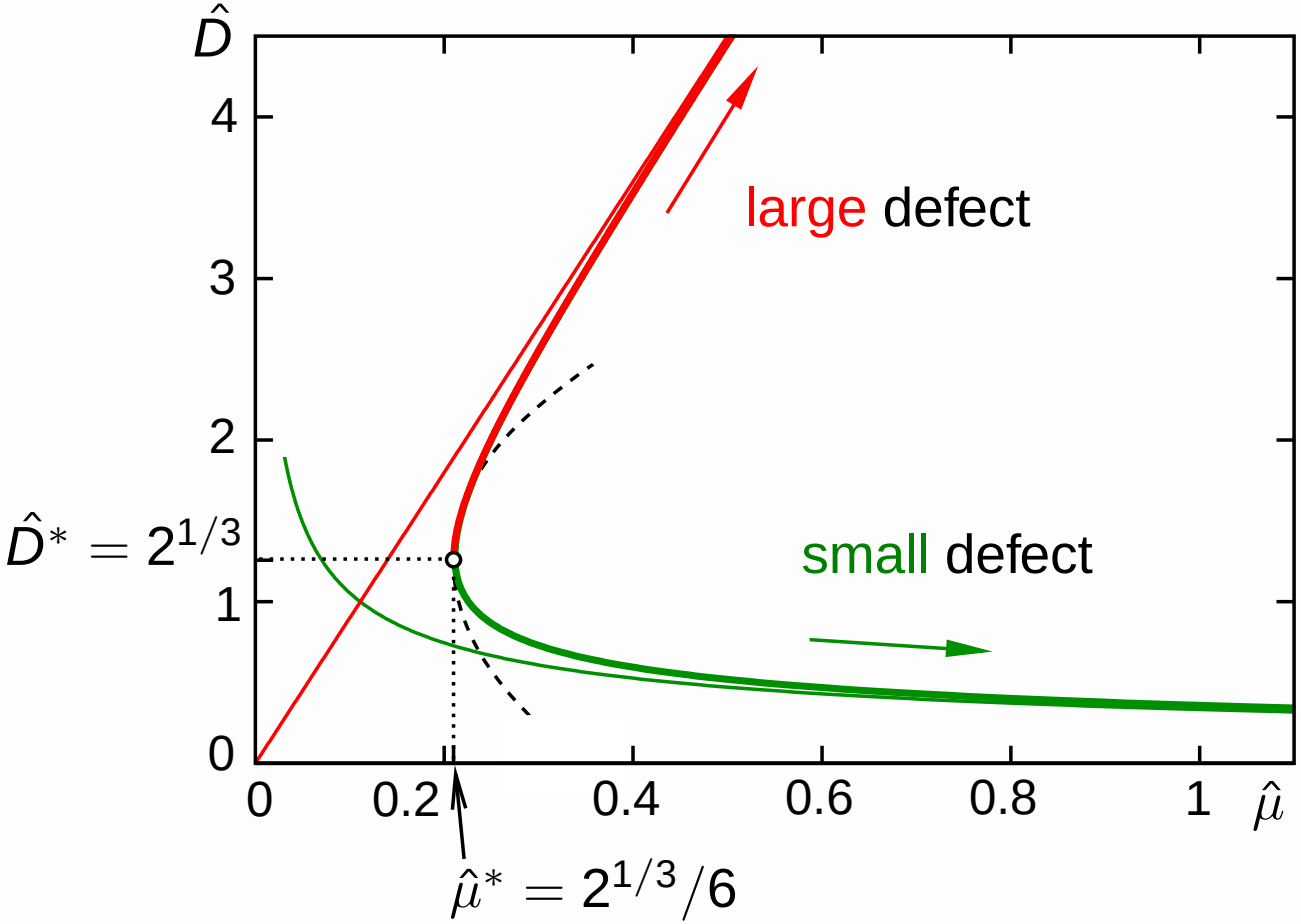
non-uniqueness supported experimentally

Clauser (Adv Appl Mech, 1956)



Outline

Large velocity defect : $1 - u/U_e = O(1)$

nonlinear momentum balance

$$u\frac{\partial u}{\partial x} + \cdots \sim \frac{\partial \tau}{\partial y} = O(1) \quad \Leftrightarrow \quad \tau = -\langle u'v' \rangle = O(\delta) \quad \Leftrightarrow \quad u', v' = O(\delta^{1/2})$$

consequences — Scheichl, Kluwick & Alletto (Acta Mech, 2008)

- $\delta \ll 1 \quad \Rightarrow \quad u', v' \ll 1$
- u_τ inappropriate turbulent velocity scale

question : why is BL still slender (lacking a small turbulent velocity scale) ?

answer : outer part of BL essentially behaves as turbulent free shear layer (wake), which is (empirically) known to be slender

strategy : combine asymptotic theory of turbulent free shear layers where slenderness is enforced by a slenderness (entrainment) parameter $\alpha \ll 1$ — Schneider (ZFW, 1991) with asymptotic description of turbulent wall-bounded flows

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Outer wake region: $\bar{y} := y/\alpha = O(1)$ Scheichl & Kluwick (AIAA J, 2007)

expansions

$$p \sim p_0(x) + O(\alpha)\,, \quad q \sim \alpha q_0(x,\bar{y}) + O(\alpha^2)\,, \quad q = \delta,\, \psi,\, \tau,\, \sigma_x,\, \sigma_y$$

leading-order problem

$$\frac{\partial\psi_0}{\partial\bar{y}}\frac{\partial^2\psi_0}{\partial\bar{y}\partial x} - \frac{\partial\psi_0}{\partial x}\frac{\partial^2\psi_0}{\partial\bar{y}^2} = -\frac{\partial p_0}{\partial x} + \frac{\partial\tau_0}{\partial\bar{y}}\,, \quad \frac{\partial p}{\partial x} = -U_e\frac{\partial U_e}{\partial x}\,, \quad \frac{\partial p_0}{\partial\bar{y}} = 0$$

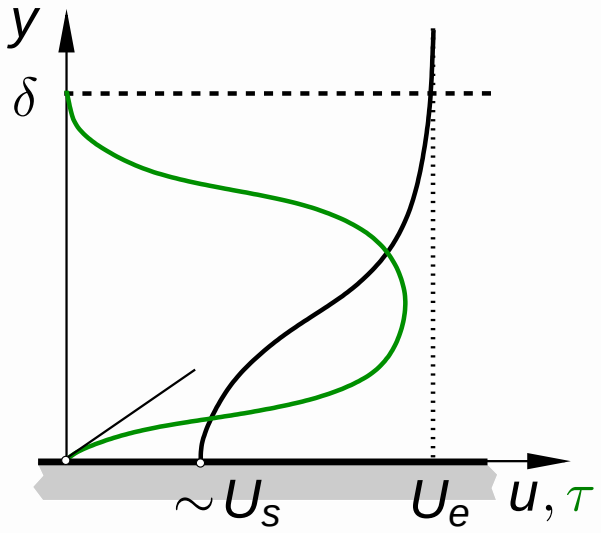
$$\bar{y} = 0: \quad \psi_0 = \tau_0 = 0\,, \qquad \bar{y} = \delta_0(x): \quad \partial\psi_0/\partial\bar{y} = U_e\,, \quad \tau_0 = 0$$

flow behaviour as $\bar{y} \to 0$

$$\partial\psi_0/\partial\bar{y} \sim U_s(x) + O(\bar{y}^{3/2})$$

$$\tau_0 \sim \Lambda_0(x)\bar{y} + O(\bar{y}^{3/2})$$

$$\Lambda_0 := U_s U_{sx} - U_e U_{ex}$$



Interpretation of mixing length ℓ

Hinze (1975), Scheichl & Kluwick (JFS, 2008)

$$-\langle u'v'\rangle \equiv \ell^2 \frac{\partial u}{\partial y}\left|\frac{\partial u}{\partial y}\right| = O(\alpha)\,, \quad \frac{\partial u}{\partial y} = O\left(\frac{1}{\alpha}\right) \quad \Rightarrow \quad \delta \gg \ell = O(\alpha^{3/2})$$

ℓ = distance where outer wake flow "begins to feel" presence of confining wall

$$\tau \sim \alpha\tau_0 = \alpha\Lambda_0\bar{y} = O(\alpha^{3/2})\,, \quad u - U_s = O(\alpha^{3/4})$$

Inner wake region: $Y := y/\alpha^{3/2} = \bar{y}/\alpha^{1/2}$

Scheichl & Kluwick (IJCSM, 2007)

expansions

$$p \sim p_0(x) + O(\alpha)\,, \quad \psi \sim \alpha^{3/2}U_s(x) + \alpha^{9/4}\Psi(x,Y) + \cdots$$

$$[\tau,\ell] \sim \alpha^{3/2}[\mathrm{T},L](x,Y) + \cdots\,, \quad L_\infty := \lim_{Y\to\infty} L = O(1)$$

leading-order problem

$$\mathrm{T} = \Lambda_0 Y\,, \quad \mathrm{T}(x,0) = \Psi(x,0) = 0\,, \quad Y \to \infty: \quad \frac{\partial\Psi}{\partial Y} \sim \frac{2}{3}\frac{\Lambda_0^{1/2}}{L_\infty}Y^{3/2} + \cdots$$

Inner wake region — cont'd

solution of leading-order problem

$$\frac{\partial \Psi}{\partial Y} = \frac{2}{3}\frac{\Lambda_0^{1/2}}{L_\infty} Y^{3/2} - J(x,Y)\,, \quad J := \int_Y^\infty \left[\frac{1}{L(x,z)} - \frac{1}{L_\infty}\right](\Lambda_0 z)^{1/2}\,\mathrm{d}z$$

flow behaviour as $Y \to 0\ldots$

overlap with intermediate layer $\Rightarrow$ square-root law

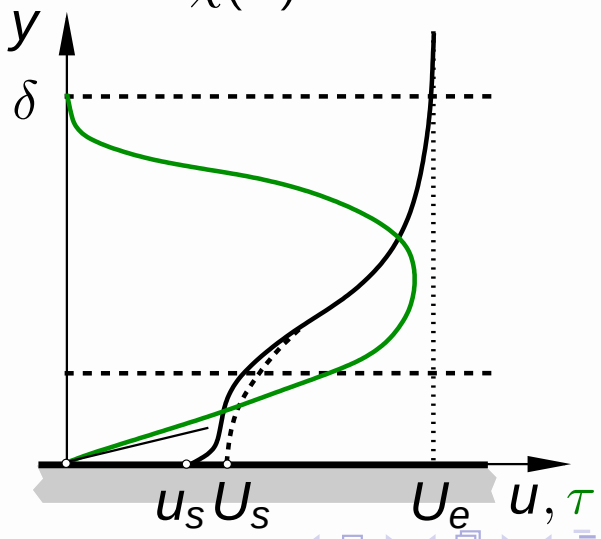
$$\frac{\partial u}{\partial y} \sim \frac{u_\tau}{u_\tau^2}\Pi\left(x, \frac{y}{u_\tau^2}\right) \quad \Rightarrow \quad \frac{\partial \Psi}{\partial Y} \sim \mathrm{U}_s(x) + 2\frac{(\Lambda_0 Y)^{1/2}}{\chi(x)} + \cdots\,, \quad L \sim \chi Y$$

slip velocity

$u_s \sim U_s(x) + \alpha^{3/4}\mathrm{U}_s(x) + \cdots$

$\mathrm{U}_s = -J(x,0) < 0$

…completes wake region analysis



Asymptotic investigation of shear stress closures

...confirms that α is independent of Re

specific algebraic closures — Schlichting & Gersten (2000)

- mixing length — Michel, Quémard & Durant (Proc Stanford Conf, 1969)

  $\alpha^{1/2} = c_\ell \approx 0.085$

- eddy viscosity — Baldwin & Lomax (AIAA P, 1978), Cebeci & Smith (1974)

  $\alpha \approx 0.0168$

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Outline

numerical solution of leading-order outer-wake problem

- algebraic closure : mixing length $\ell \sim \ell_0 + \cdots$

$$\tau_0 = \ell_0^2 \frac{\partial^2 \psi_0}{\partial \bar{y}^2} \left| \frac{\partial^2 \psi_0}{\partial \bar{y}^2} \right|, \quad \ell_0 = I(\xi)^{1/2} \Delta_0(x), \quad I(\xi) = \frac{1}{1 + 5.5\,\xi^6}, \quad \xi := \frac{\bar{y}}{\Delta_0(x)}$$

- 2-parameter family of retarded external flows

$$U_e(x; m_s, \sigma) = (1 + x)^{m(x; m_s, \sigma)}$$

$$\frac{m}{m_s} = 1 + \frac{\sigma}{1 - \sigma} H(2 - x) \left[1 - (1 - x)^2\right]^3, \quad m_s < 0, \quad 0 \le \sigma < 1$$

- similarity solutions : $\sigma = 0$, $m \equiv m_s > -1/3$

$$\psi_0 = \Delta_0(x) f(\xi), \quad \Delta_0 = b(1 + x)$$

specific choice for initial conditions at $x = 0$:

$$f'(0) = 0.95 \;\Rightarrow\; m_s \doteq -0.3292, \; b \doteq 0.3656$$

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Turbulent marginal separation

Scheichl & Kluwick (AIAA J, IJCSM, 2007)

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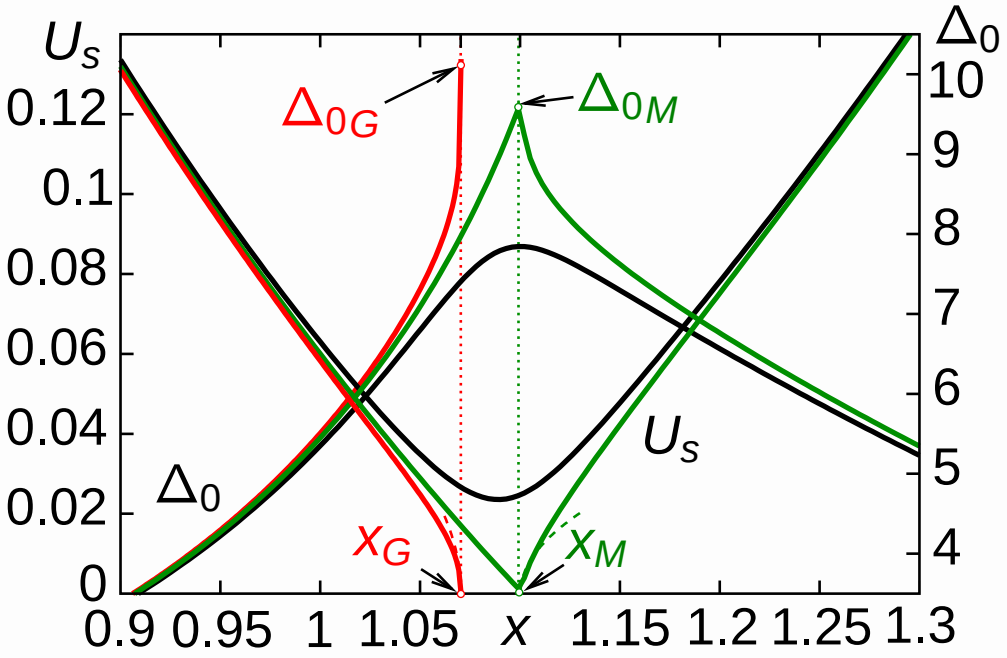


$$P_M = \partial p_0 / \partial x \quad \text{at} \quad x = x_M$$

critical conditions $\sigma = \sigma_M$

$s := x - x_M$

marginal separation

$s \to 0_- : \quad U_S / {P_M}^{1/2} \sim B\,(-s)$

$s \to 0_+ : \quad U_s / {P_M}^{1/2} \sim U_+ s^{1/2}$

$B > 0\,, \quad U_+ \doteq 1.1835$

supercritical conditions $\sigma > \sigma_M$

$s := x - x_G$

“turbulent” Goldstein singularity

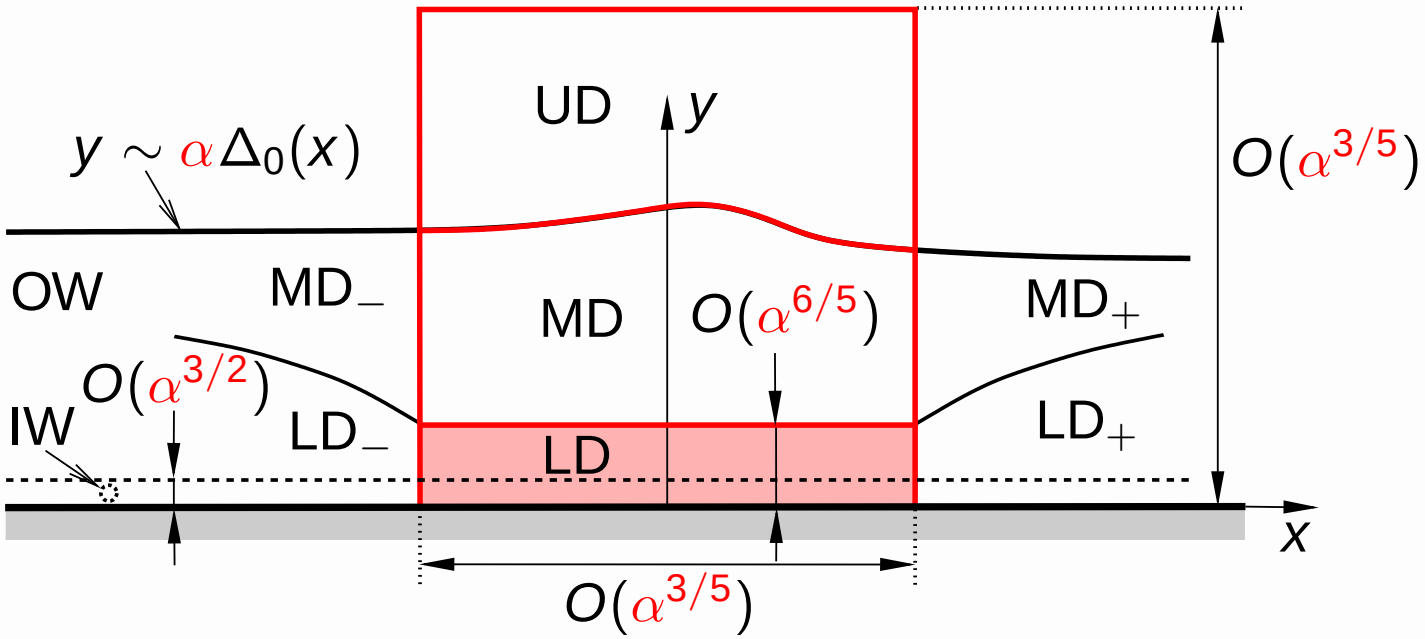
$s \to 0_- : \quad U_s / {P_M}^{1/2} \sim U_-(-s)^{1/2}$

marginal separation

breakdown of classical “hierarchical” BL theory in regions where BL thickness changes so rapidly that pressure response in external-flow region is large enough to affect the lowest-order BL approximation

$\Rightarrow$ local interaction strategy

“turbulent” (novel) triple-deck structure



“turbulent” triple-deck problem

similarity parameter $\hat{\Gamma}$

induced pressure gradient of strength $\hat{\Lambda}$

fundamental lower-deck problem

$$\frac{\partial\hat{\Psi}}{\partial\hat{Y}}\frac{\partial^2\hat{\Psi}}{\partial\hat{Y}\partial\hat{X}}-\frac{\partial\hat{\Psi}}{\partial\hat{X}}\frac{\partial^2\hat{\Psi}}{\partial\hat{Y}^2}=-1-\hat{\Lambda}(\hat{\Gamma})\hat{P}'(\hat{X})+\frac{\partial\hat{\mathsf{T}}}{\partial\hat{Y}}\,,\qquad \hat{\mathsf{T}}=\frac{\partial^2\hat{\Psi}}{\partial\hat{Y}^2}\left|\frac{\partial^2\hat{\Psi}}{\partial\hat{Y}^2}\right|$$

$$\hat{Y}=0:\quad \hat{\Psi}=\hat{\mathsf{T}}=0\,,\qquad \hat{Y}\to\infty:\quad \hat{\mathsf{T}}\to\hat{Y}+\hat{A}(\hat{X})$$

$$\hat{X}\to-\infty:\quad \hat{\Psi}\to(4/15)\hat{Y}^{5/2}+\hat{\Gamma}\hat{Y}\,,\qquad \hat{X}\to\infty:\quad \hat{\Psi}/\hat{X}^{5/6}\to\hat{\mathsf{F}}_+(\hat{Y}/\hat{X}^{1/3})$$

$$\hat{P}(X)=\frac{1}{\pi}\oint_{-\infty}^{\infty}\frac{\hat{A}'(\hat{S})}{\hat{X}-\hat{S}}\,\mathrm{d}\hat{S}$$

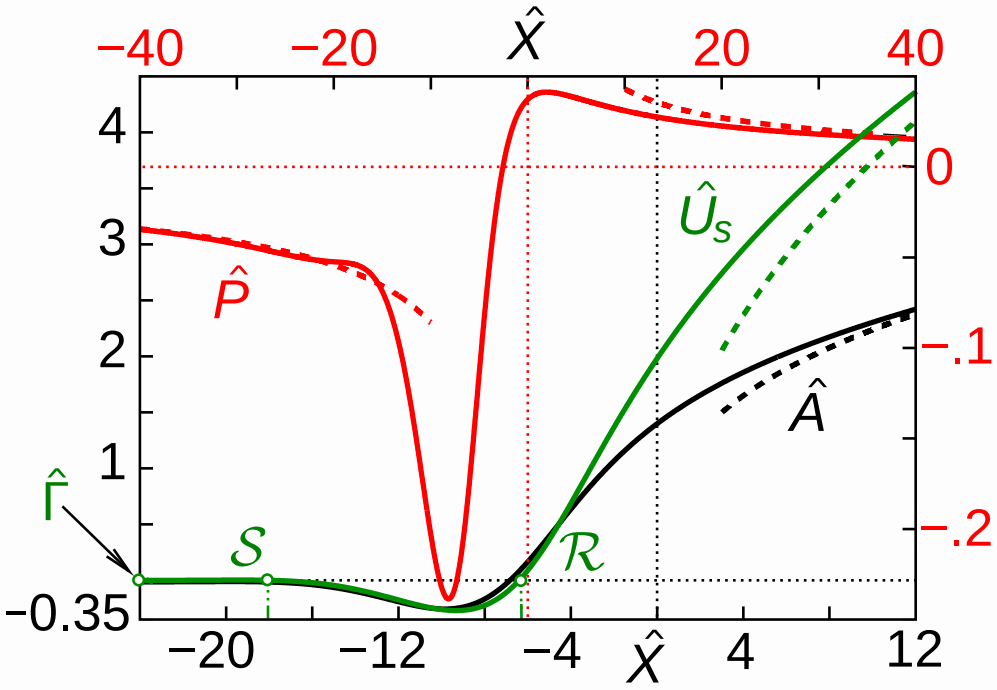
slip velocity

$\hat{U}_s = \partial\hat{\Psi}/\partial\hat{Y}$ at $Y = 0$

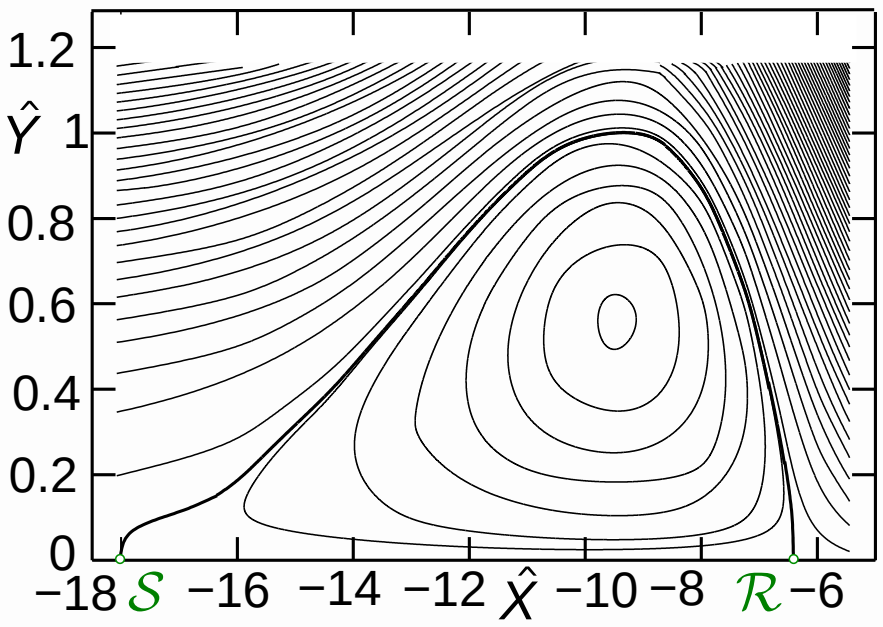
numerical solution

$\hat{\Gamma} = 0.019$

$\hat{\Lambda} = 3$



$\hat{\Psi} = const$



Outline

Sandborn & Liu (JFM, 1968)

"Turbulent boundary-layer separation is normally listed as one of the most important unsolved problems in fluid mechanics..."

Problem formulation — motivation

Local description of turbulent break-away separation . . .

. . . requires answers to three basic questions:

(1) Global topology of flow for ‘vanishing viscosity’?

(2) Characteristics of incident boundary layer flow?

(3) How do these issues interdepend?

How turbulent are the BL and the separated SL ?

Scheichl, Kluwick & Alletto (Acta Mech, AIAA P, 2008)

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basic assumptions

- flow incompressible, nominally steady, 2D
- free-stream turbulence disregarded
- $Re \simeq UL/\nu \to \infty$

canonical example: circular cylinder flow

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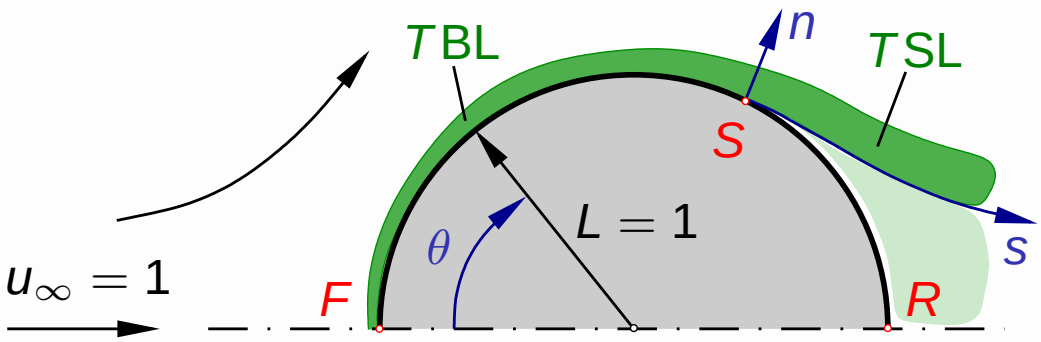

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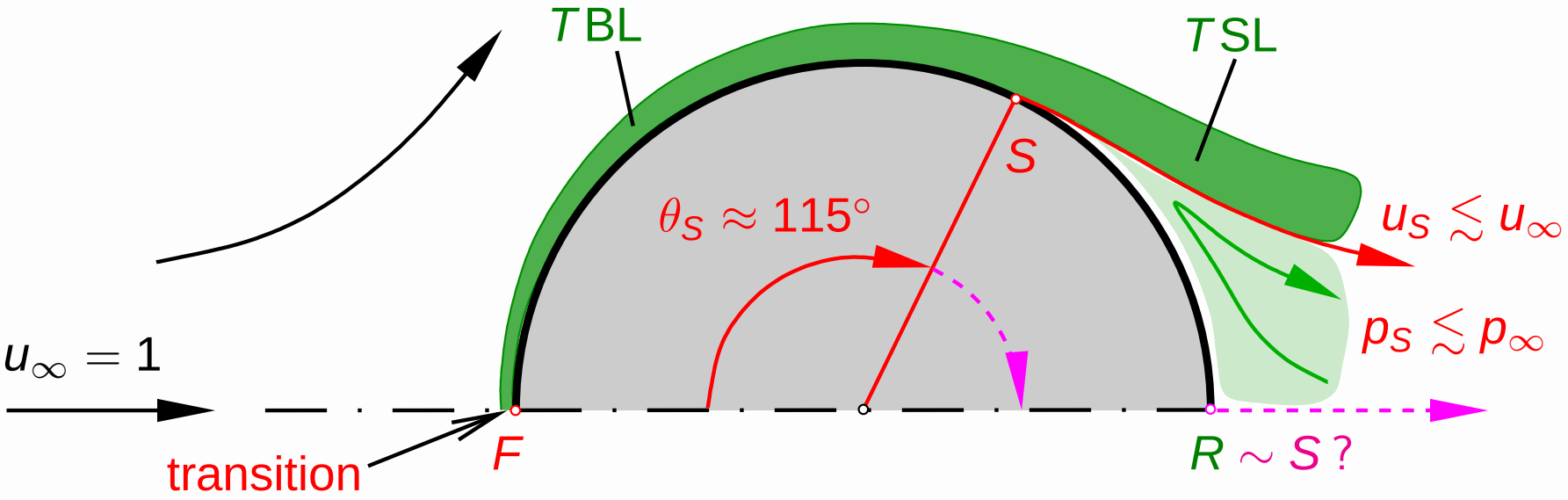
Outline

Global flow picture

$Re \gtrapprox 3 \times 10^6$: *postcritical* regime, $Re \to \infty$: transition point approximates F

$Re^{-1} = 0$: *ultimate* or T-state of flow, transition in F



Neish & Smith (JFM, 1992)

unlikely Scheichl & Kluwick (JFS, 2008)

corroborated experimentally up to $Re \approx 4 \times 10^7$

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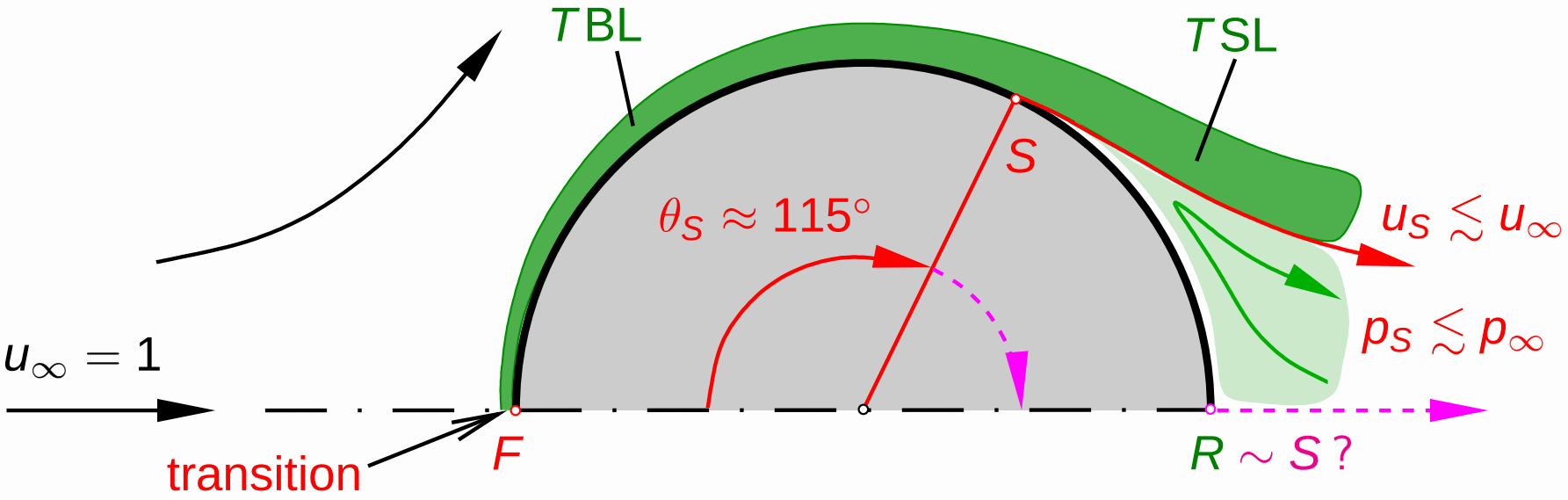

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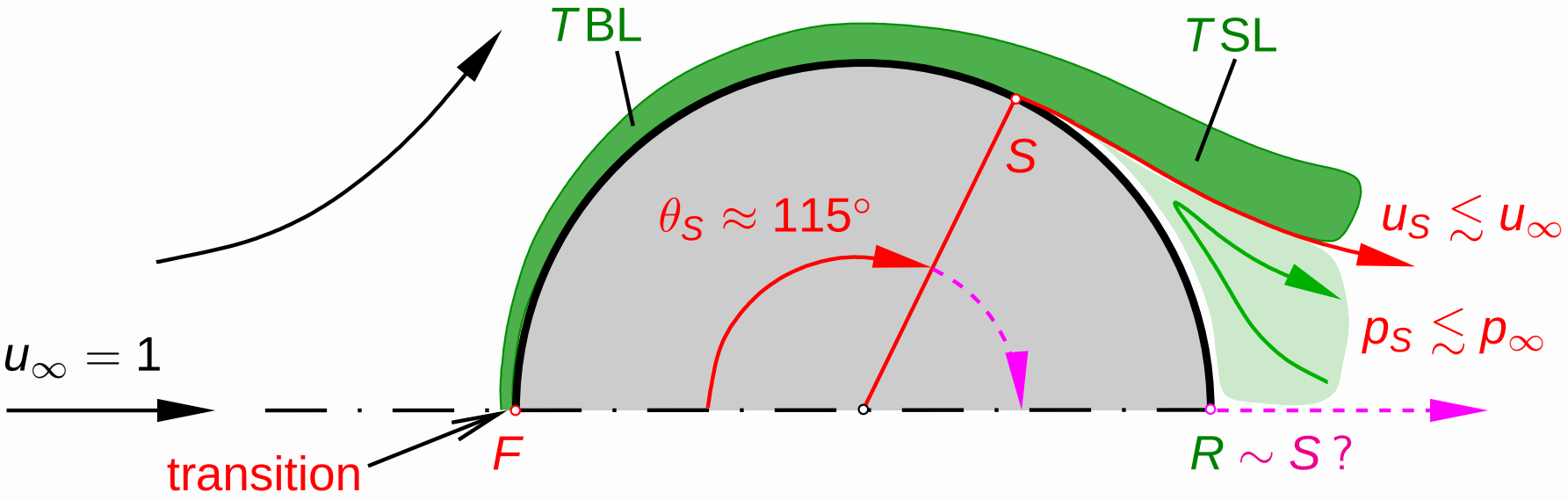

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Separation

Is there really a fully developed turbulent BL as $Re \to \infty$?

oil-flow measurements : $Re_C := \tilde{U}\tilde{C}/\tilde{\nu}$, $\tilde{C}$ = airfoil chord length

sub- *trans-* *super-* *post-*
-critical

by courtesy of G. Schewe (2001)

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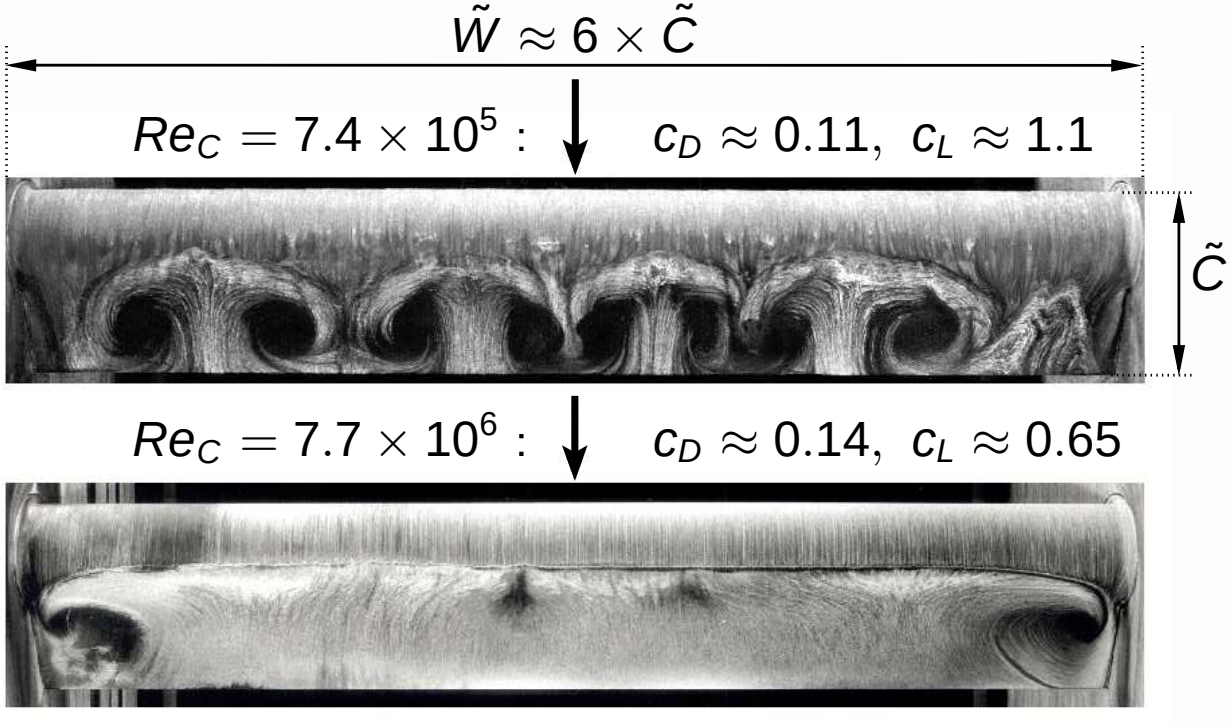

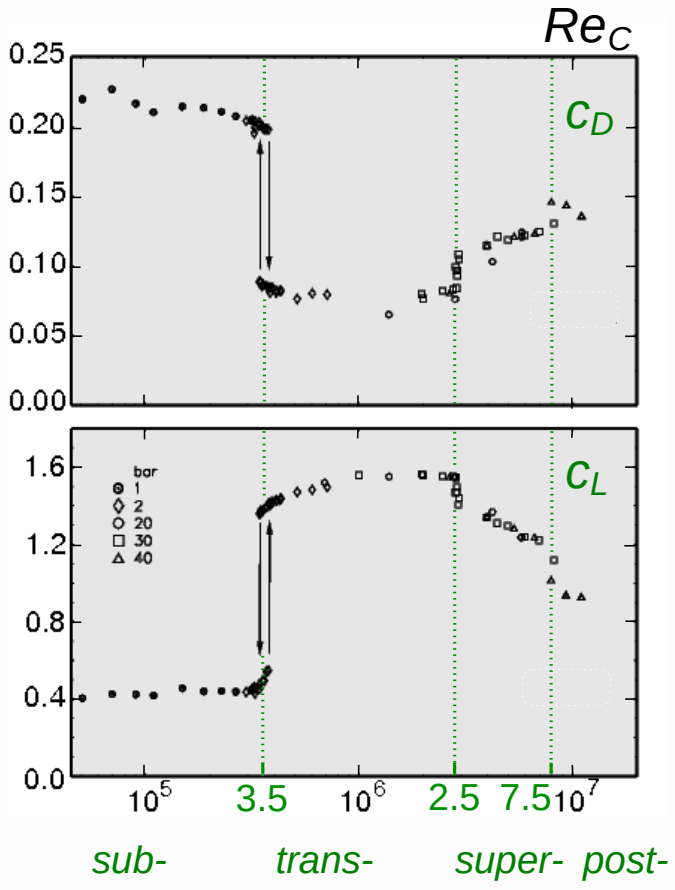

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Outline

6. Asymptotic structure of bluff-body flows
   - ‘Ideal-fluid limit’: Kirchhoff-type potential flow
   - Boundary layer

‘Ideal-fluid limit’

Hierarchy of external flow – shear layer: potential flow approached as $Re \to \infty$

Oseen (1927), Lamb (1932), Batchelor (1956), Prandtl (1961)

$Re^{-1} = 0$: external potential flows

Imai (J Phys Soc Japan, 1953),
Birkhoff & Zarantonello (1957), Gurevich (1966)

- free streamlines confine (open) dead-water region
- class of flows / θ_S controlled by
  Brillouin–Villat parameter $k \geq 0$, free-stream velocity $u_S \leq u_\infty$

$$u_e \begin{cases} \sim b(k)\theta + O(\theta^2)\,, & \theta \to 0_+ \\ \sim u_S\left[1 + 2k(-s)^{1/2} + O(-s)\right], & s \to 0_- \\ \equiv u_S\,, & s \geq 0 \end{cases}$$

$$\varkappa \begin{cases} = \varkappa_B(s)\,, & s < 0 \\ \sim -k\,s^{-1/2} + \varkappa_B(0) + O(s^{1/2})\,, & s \to 0_+ \end{cases}$$

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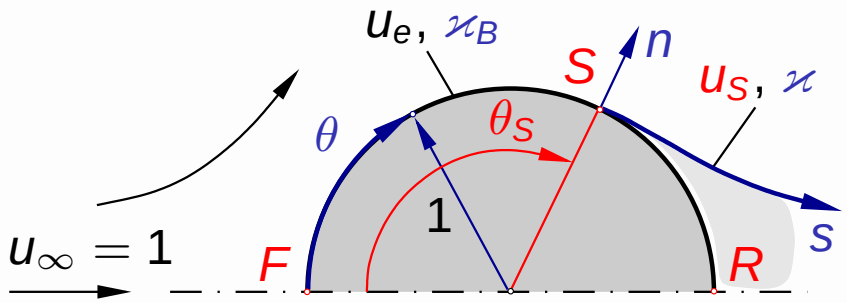

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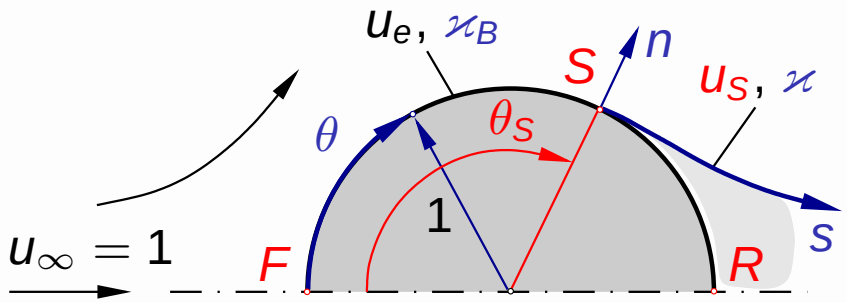

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Kirchhoff-type flows — numerical solutions

Open cavity: $u_S \equiv u_\infty = 1$

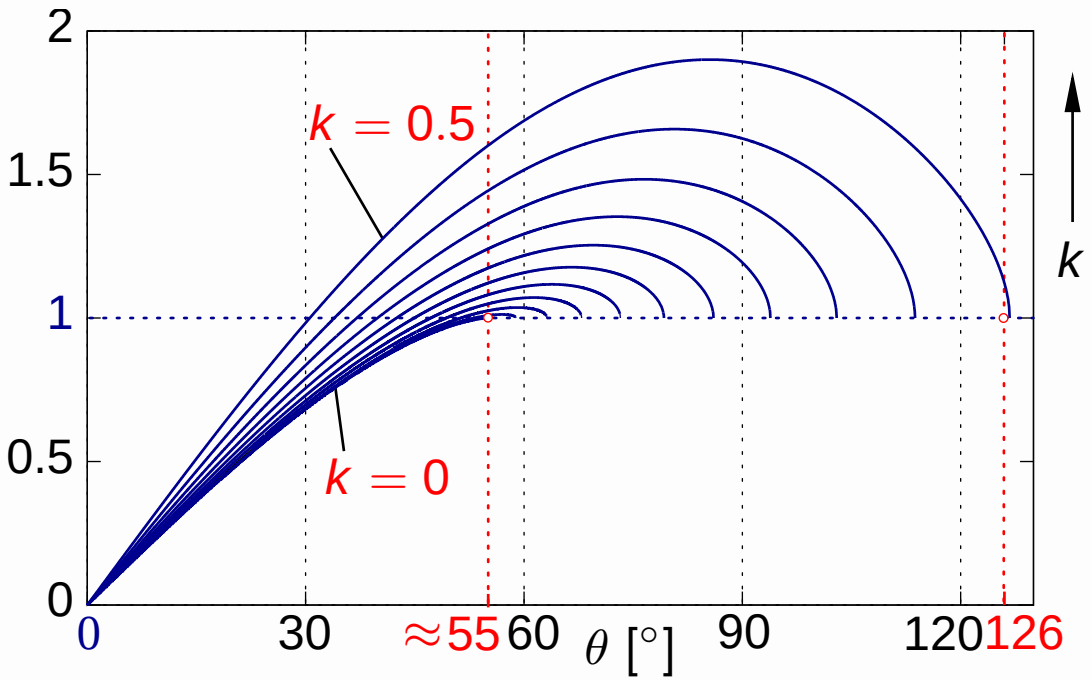
surface velocity $u_e(\theta, k)$

$k = 0,\ 0.05,\ 0.1,\ \ldots,\ 0.5$

$55^\circ\, 2'\, 30'' \ \leq\ \theta_S \ \lessapprox\ 126^\circ$

$k = 0$:

Brillouin–Villat condition



free streamlines

$126^\circ \lessapprox \theta_S \leq 180^\circ$

$0.5 \lessapprox k < \infty$

$1 \geq u_S \geq 0$?

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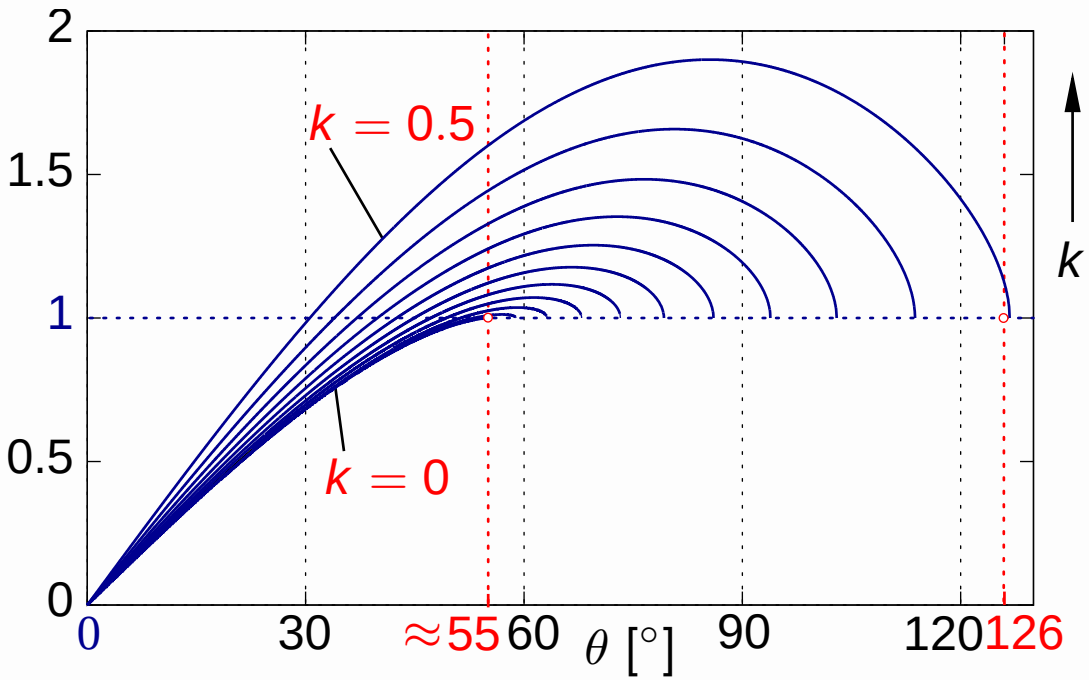

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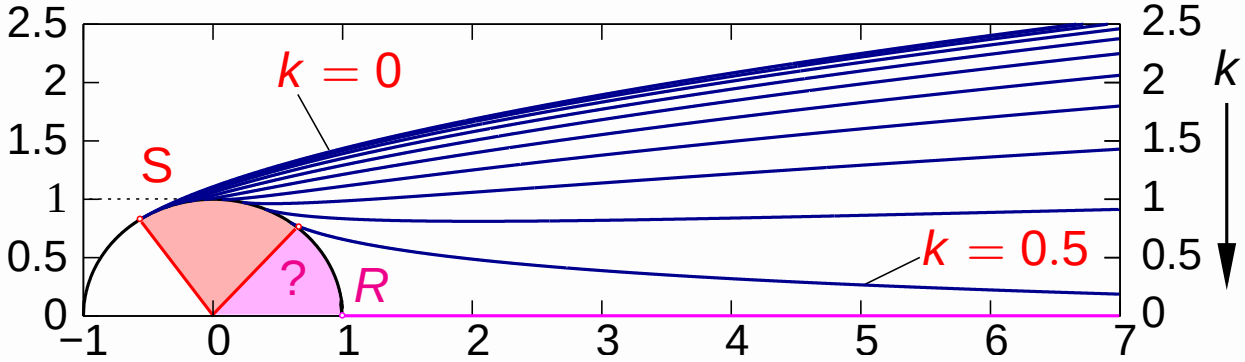

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Outline

Finite turbulence intensity level

Scheichl, Kluwick & Alletto (Acta Mech, AIAA P, 2008)

asymptotic analysis of Reynolds-averaged NS eqs

$\epsilon := Re^{-1/2} \to 0\,, \quad Tu := \sqrt{\langle u'^2 + v'^2 + w'^2\rangle/3} \ll 1$

hypothesis of *locally isotropic* turbulence

$[\tau,\, \sigma_s,\, \sigma_n,\, \sigma_z] := -\big[\langle u'v'\rangle,\, \langle u'^2\rangle,\, \langle v'^2\rangle,\, \langle w'^2\rangle\big] = O(Tu^2)$

classical inner expansions (Prandtl-type BL)

$p - p_F \sim -u_e^2(\theta; k)/2 + O(\epsilon)\,, \quad \mathrm{d}p_e/\mathrm{d}\theta = -u_e\,\mathrm{d}u_e/\mathrm{d}\theta$

$[\psi, r] \sim \epsilon\,[\psi_0, T r_0](\theta, N; k, T) + O(\epsilon^2)\,, \quad r := \tau,\, \sigma_s,\, \sigma_n,\, \sigma_z\,, \quad N := Re^{1/2} n$

time-mean flow governed by three parameters

Brillouin–Villat parameter k

Reynolds number Re

turbulence level gauge factor T $\sim O(0.1 \ldots 1.0)$

Neish & Smith (JFM, 1992)

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Finite turbulence intensity level — cont'd

$T = O(1)$

leading-order BIV problem

$$\frac{\partial\psi_0}{\partial N}\frac{\partial^2\psi_0}{\partial N\,\partial\theta} - \frac{\partial\psi_0}{\partial\theta}\frac{\partial^2\psi_0}{\partial N^2} = u_e\frac{\mathrm{d}u_e}{\mathrm{d}\theta} + \frac{\partial}{\partial N}\left[T\tau_0 + \frac{\partial^2\psi_0}{\partial N^2}\right]$$

$$N = 0: \quad \partial\psi_0/\partial N = \psi_0 = \tau_0 = 0$$

$$N \to \infty: \quad \partial\psi_0/\partial N \to u_e(\theta; k)\,, \quad \tau_0 \to 0$$

$$\theta \to 0_+: \quad u_e \sim b(k)\theta + O(\theta^2)\,, \quad \psi_0 \sim b^{1/2}\theta H(b^{1/2}N)\,, \quad r_0 = O(\theta^2)$$

flow laminar close to F

Hiemenz (1911)

$$H'^2 - HH'' = 1 + H'''\,, \qquad H(0) = H'(0) = 0\,, \quad H'(\infty) = 1$$

present study: BL in the 'turbulent limit' $T \to \infty$

$$T\tau_0 \gg \frac{\partial^2\psi_0}{\partial N^2} \quad \Rightarrow \quad \text{problem singularly perturbed} \quad \begin{cases} \theta \to 0\,, & N = O(1) \\ N \to 0\,, & \theta = O(1) \end{cases}$$

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$\theta \to 0_+$: $\quad u_e \sim b(k)\theta + O(\theta^2)\,, \quad \psi_0 \sim b^{1/2}\theta H(b^{1/2}N)\,, \quad r_0 = O(\theta^2)$

flow laminar close to F — Hiemenz (1911)

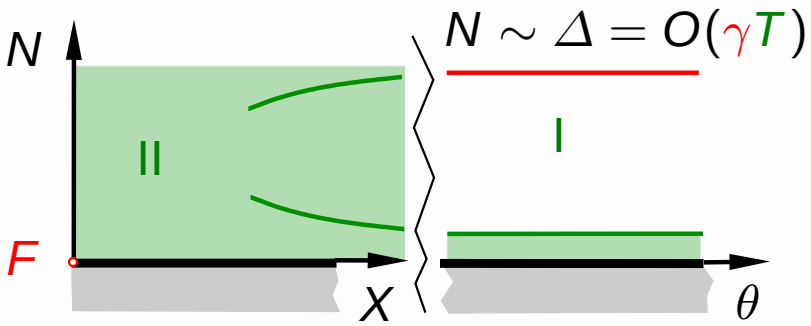
$$H'^2 - HH'' = 1 + H'''\,, \qquad H(0) = H'(0) = 0\,, \quad H'(\infty) = 1$$

present study: BL in the 'turbulent limit' $T \to \infty$

$$T\tau_0 \gg \frac{\partial^2\psi_0}{\partial N^2} \quad \Rightarrow \quad \text{problem singularly perturbed} \begin{cases} \theta \to 0\,, & N = O(1) \\ N \to 0\,, & \theta = O(1) \end{cases}$$

Outline

Large-T asymptotics



region I: two BL types possible

$$\left.\begin{array}{lll} \text{velocity scale } u_{\mathrm{I}}: & r_0 & = O(u_{\mathrm{I}}^2) \\ \text{velocity defect}: & u_e - u & = O(u_{\mathrm{I}}) \end{array}\right\} \quad Re \to \infty$$

$$[\psi_0, \tau_0] = [u_e \Delta f, \; T u_e^2 s] \qquad \zeta := N/\Delta(\theta; k, Re, T)$$

(1) $[f, s] \sim [f_0, s_0](\theta, \zeta; k) + \cdots, \quad s_0(\theta, 0; k) = 0, \qquad u_{\mathrm{I}}$ finite

(2) $[f, s] \sim [\zeta, 0] + [-\gamma f_1, \gamma^2 s_1](\theta, \zeta; k) + \cdots, \qquad \gamma := u_{\mathrm{I}}/u_e \to 0$

region II: transition $\quad X := T b(k)^{1/2}\theta = O(1) \quad \eta := b^{1/2} N \quad \rho := \zeta/\eta$

$$[u_e, \psi_0] \sim T^{-1} X [b^{1/2}, F(X, \eta)], \quad \tau_0 \sim T^{-2} X^2 b S(X, \eta)$$

$$\left.\begin{array}{l} (1) \quad [F, S] \sim [F_0, S_0](\zeta), \quad \rho \propto 1/X, \quad S_0(0) > 0 \\ (2) \quad [F, S] \sim [\gamma_F(X) F_1(\zeta)/\rho, \gamma_S(X) S_1(\zeta)], \quad \gamma_F \to 0 \end{array}\right\} \quad X \to \infty$$

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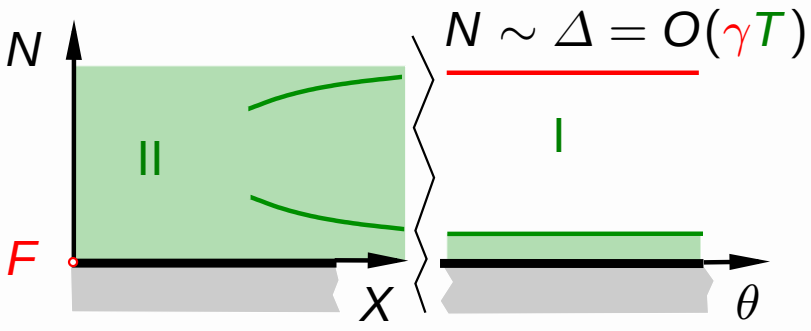

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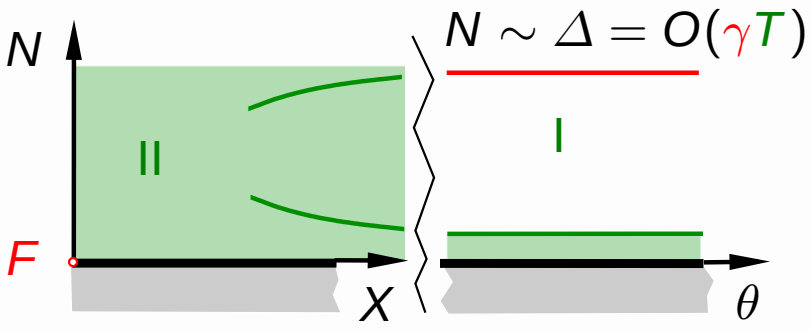

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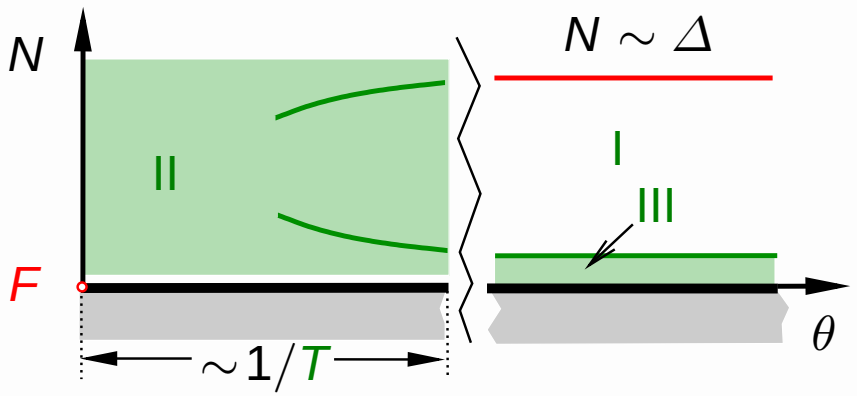
Outline

‘Underdeveloped’ turbulence



matching: regions I, II

(1) $s_0(\theta, 0, k) = 0\,,$ $\quad S_0(0) > 0 \quad \Rightarrow \quad s_0 \not\sim S_0$ no match

(2) $s_1(\theta \to 0, 0, k) > 0 \quad \Rightarrow \quad S_1(0) > 0 \quad \Rightarrow \quad [f_1, s_1] \sim [F_1, S_1]$

matching: outer region I, viscous wall layer III

main layer scale u_{I}: $\quad r_0 = O(u_{\mathrm{I}}^2) \quad u_e - u = O(u_{\mathrm{I}})$

wall layer scale u_{III}: $\quad r_0 = O(u_{\mathrm{III}}^2) \quad u = O(u_{\mathrm{III}})$

$\Big\}\ Re \to \infty$

(2) $s_1(\theta, 0, k) := 1 \quad u_t := u_{\mathrm{I}} = u_{\mathrm{III}} \quad \partial f_1/\partial\zeta \sim -\kappa^{-1}\ln\zeta + B_1(\theta)\,, \quad \zeta \to 0$

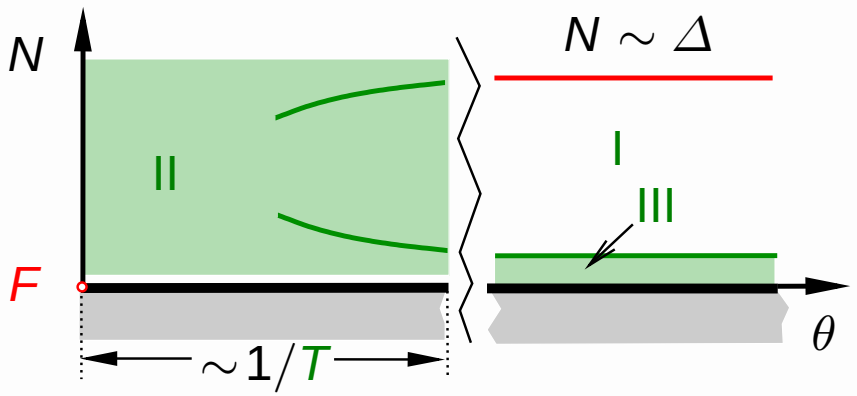
small-defect scaling $\quad \gamma = u_t/u_e \sim \kappa/(2\ln T)$

classical scaling $\quad \Rightarrow \quad T \sim Re^{1/2}$

flow structure near F and S $\quad \Rightarrow \quad T \ll Re^{1/2}$

Scheichl & Kluwick (AIAA P, 2008)

‘Underdeveloped’ turbulence



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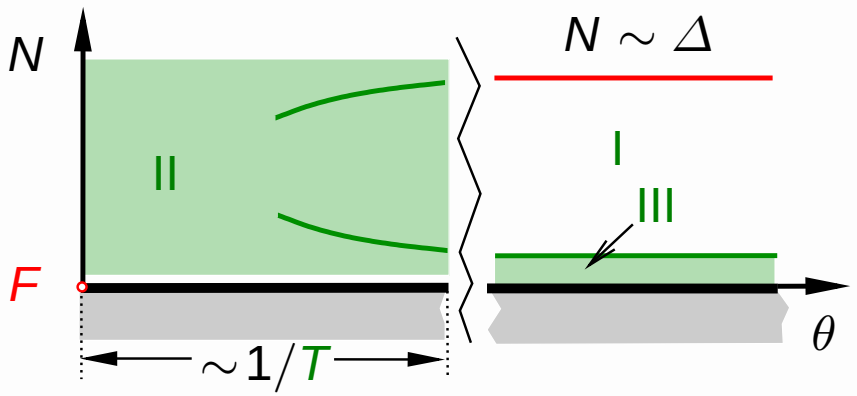

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Outline

Solution of BIV problem for transitional BL

$$[F, S](X, \eta) \qquad \eta := b^{1/2} N \qquad \rho(X) := \zeta/\eta \qquad X \to \infty : \quad [F', S] \sim [F_1', S_1](\zeta)$$

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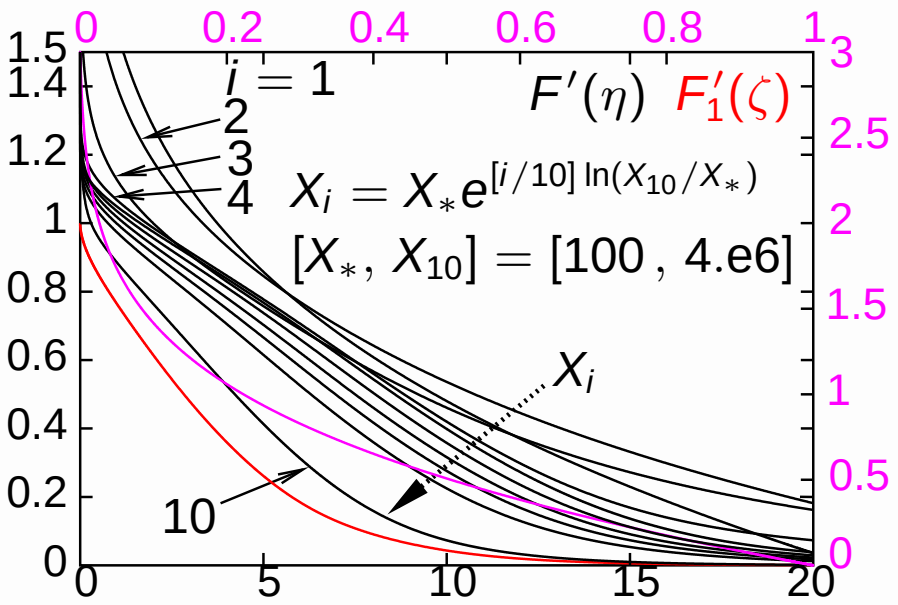

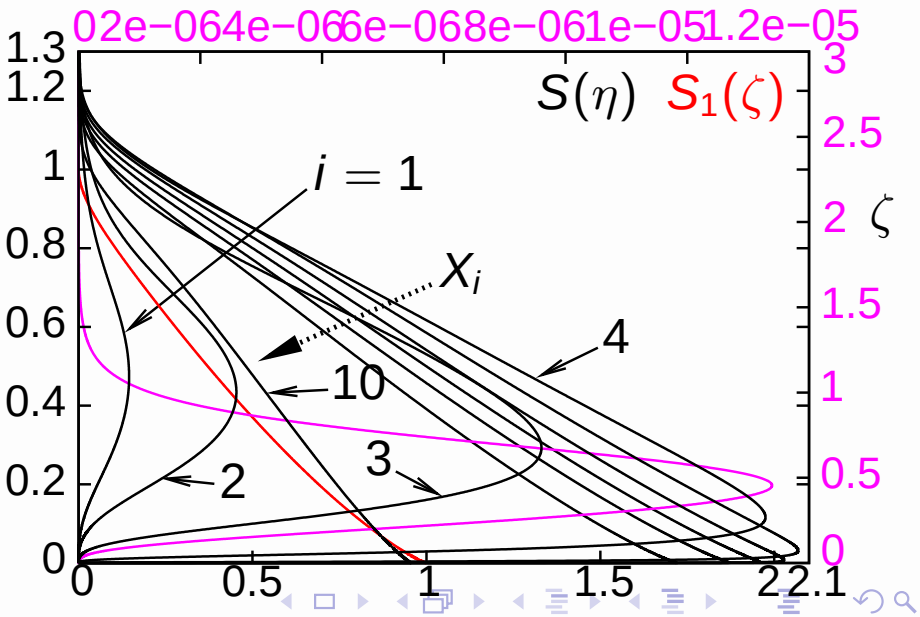

Conclusions and outlook

turbulent BLs on flat plates and related bodies

- small velocity defect
  $\Rightarrow$ unable to predict separation
- moderately large velocity defect
  $\Rightarrow$ loss of uniqueness (self-preserving flows)
- large velocity defect
  $\Rightarrow$ theory of marginal separation

turbulent BLs on bluff bodies

- stretch from front stagnation to separation point
  $\Rightarrow$ apparently never attain a large velocity defect
  $\Rightarrow$ apparently never reach a fully turbulent state
- modified small-defect theory
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Thank you for your attention !