Adaptive Boundary Element Methods Based on Accurate A Posteriori Error Estimation

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Abstract — *The boundary element method is one strategy to solve partial differential equations of elliptic type. As model problem, we consider the computation of the charge density ϕ and the capacitance C of a thin electrified plate. We introduce an intelligent algorithm based on certain error estimators. The mesh-refinement is steered automatically in the sense that the mesh is locally refined, where the error appears to be large. Numerical experiments show that the new method reveals the optimal order of convergence and is therefore significantly faster than a standard uniform approach. All mathematical results are valid in a quite general framework and thus apply to a large problem class, including, e.g., the Laplace problem, the Stokes system, and the Lamé equation.*

I. Introduction

Computer aided simulations – and therefore usually the solution of partial differential equations (PDEs) – have established themselves as cost and time efficient methods in research and development.

Certain PDEs of elliptic type can be solved by the so-called boundary element method (BEM). Compared to the more popular finite element method (FEM), the essential disadvantage of BEM is that it leads to large dense matrices. Mathematical strategies to overcome this disadvantage, e.g. fast multipole methods, have been developed and are usually employed by engineers nowadays. Besides the fact that one only has to discretize the boundary of the simulation domain, one key advantage of BEM is its generically higher order of convergence when compared to FEM. With respect to the mesh-size h of the discretization $\mathcal{T}_h$ of the simulation domain optimal convergence for lowest-order approximations reads:

- FEM: $\mathcal{O}(h)$ as $h \to 0$,
- BEM: $\mathcal{O}(h^{3/2})$ as $h \to 0$.

However, these convergence rates are only observed if the unknown solution is sufficiently smooth. This is generically not the case in practice. One possibility to recover the optimal order of convergence is to estimate the simulation error, and to refine the spatial discretization only locally, where the error appears to be large.

In the context of BEM, only few a posteriori error estimators and adaptive strategies have been proposed in the literature. All of them imply a significant implementational and computational overhead. In the following, we introduce a simple and efficient approach of practical relevance.

II. Model problem

We consider the Dirichlet screen problem

$$\begin{aligned} \Delta u &= 0 && \text{in} \quad \mathbb{R}^3 \setminus \Gamma, \\ u &= f && \text{on} \quad \Gamma \subseteq \mathbb{R}^2 \times \{0\}, \\ u &= \mathcal{O}(\|x\|^{-1}) && \text{as} \quad x \to \infty, \end{aligned} \tag{1}$$

which describes the potential u away from an electrified thin plate Γ loaded with potential f, see [1]. This problem is equivalent to Symm's integral equation

$$V\phi(x) := -\frac{1}{4\pi} \int_\Gamma \frac{1}{\|x-y\|} \phi(y)\, ds_y = f(x). \tag{2}$$

Then, ϕ is the charge density on the plate Γ which is known to show singularities along the edges. Of special interest in physics is the so-called capacitance $C = -\frac{1}{4\pi} \int_\Gamma \phi\, ds$, where the charge density ϕ now solves Symm's integral equation (2) with $f = 1$.

III. Main results

A. A posteriori error estimation

The h-$h/2$-strategy is a basic technique for the a posteriori error estimation, well-known from the context of ordinary differential equations. In [2], this approach is proposed in the context of BEM. Let ϕ denote the exact (but in general unknown) solution of (2). One then considers

$$\eta_H := \|\phi_h - \phi_{h/2}\| \tag{3}$$

to estimate the error $\|\phi - \phi_h\|$, where ϕ_h is the Galerkin solution with respect to a mesh $\mathcal{T}_h$ and $\phi_{h/2}$ is the Galerkin solution for a mesh $\mathcal{T}_{h/2}$ obtained from a uniform refinement of $\mathcal{T}_h$. We stress that η_H is always a lower bound

$$\eta_H \le \|\phi - \phi_h\|, \tag{4}$$

even with known constant 1. Under the saturation assumption $\|\phi - \phi_{h/2}\| \le \sigma \, \|\phi - \phi_h\|$ with some constant $\sigma \in (0,1)$, there holds

$$\|\phi - \phi_h\| \le \frac{1}{\sqrt{1-\sigma^2}}\, \eta_H. \tag{5}$$

This means, η_H also gives an upper bound for the error.

However, for boundary element methods, the energy norm $\|\cdot\|$ is non-local and thus the error estimator η_H does not provide information about the local error. Recent localization techniques from [3] allow to replace the energy norm by mesh-size weighted L^2-norms. For instance, the estimator μ_H defined by

$$\mu_H^2 = \sum_{T\in\mathcal{T}_h} \mu_{H,T}^2 := \sum_{T\in\mathcal{T}_h} h_T \|\phi_h - \phi_{h/2}\|_{L^2(T)}^2, \quad (6)$$

where h_T denotes the diameter of an element $T \in \mathcal{T}_h$, is equivalent to η_H, see [4].

B. Adaptive Algorithm

Based on the error estimators η_H and μ_H from the previous section, we now introduce an adaptive algorithm. With a fixed parameter $\theta \in (0,1)$ as well as an initial mesh $\mathcal{T}_h$, our strategy reads as follows: Until η_H is sufficiently small, do:

1. Refine $\mathcal{T}_h$ uniformly to obtain $\mathcal{T}_{h/2}$.
2. Compute discrete solutions ϕ_h and $\phi_{h/2}$ as well as corresponding error estimators η_H and μ_H.
3. Find minimal set $\mathcal{M} \subseteq \mathcal{T}_h$ such that

$$\theta\mu_H^2 = \theta \sum_{T\in\mathcal{T}_h} \mu_{H,T}^2 \le \sum_{T\in\mathcal{M}} \mu_{H,T}^2 \quad (7)$$

4. Refine $T \in \mathcal{M}$ to obtain new mesh $\mathcal{T}_h$.

IV. Numerical Experiments

Figure 1 shows the discrete charge density ϕ_h computed over an adaptively generated mesh $\mathcal{T}_h$ which discretizes an L-shaped plate. The solution shows strong singularities, i.e. peaks, at all edges and the convex corners of the simulation domain. Our algorithm leads to a well aligned mesh showing refinements towards the edges, thus resolving the singularities efficiently.

Figure 2 shows the true error $\|\phi - \phi_h\|$ as well as the error estimators η_H and μ_H in the uniform and adaptive case. All quantities are plotted over the number of boundary elements N. In the double logarithmic plot the convergence rate is the slope of a straight line. We stress that for uniform meshes the equivalence $h \sim N^{-1/2}$ implies that the optimal order of convergence is $\mathcal{O}(N^{-3/4})$. A uniform approach only reveals a convergence order of $\mathcal{O}(N^{-1/4})$ due to the singularities of the unknown solution ϕ. Our proposed adaptive algorithm, however, recovers the optimal order of convergence $\mathcal{O}(N^{-3/4})$.

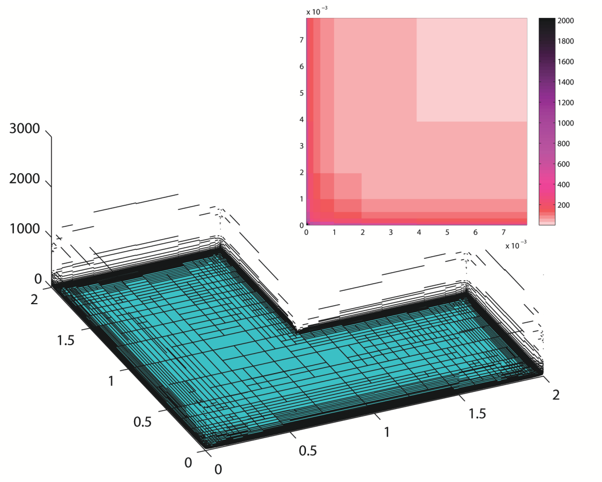

Figure 1: Discrete charge density ϕ_h on L-shaped plate and enlargement of the solution at the lower left corner.

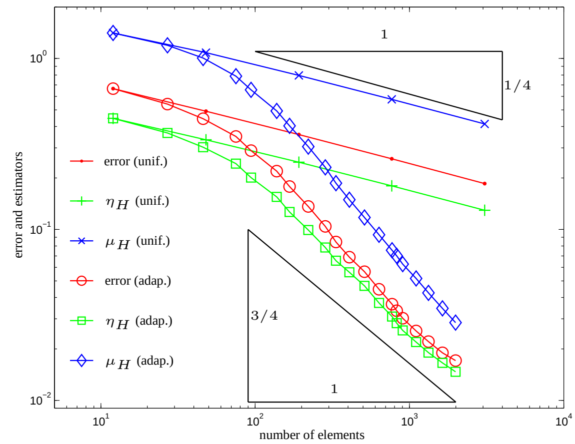

Figure 2: Error and error estimators plotted over the number of boundary elements used.

References

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