Fluid mechanics of lubrication I: fundamental aspects of a rigorous theory

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Main objectives

- existing gap in tribological literature:
  lubrication represented ‘unsatisfactorily accurate’ ⇒
- describing lube flows by adopting first principles of continuum mechanics:
  asymptotic theory of hydromechanical lubrication

Why is this expedient?

- rational estimate of methodical error
- rational extension of classical theory to include e.g.
  EHD, inertia, micro-scale effects (cavitation, surface roughness)
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Overview

1. Phenomenon of lubrication

2. Basic assumptions

3. Classical theory
   - First principles
   - Problem formulation
   - Asymptotic theory

4. Validation of tribo-systems

5. Further outlook
Phenomenon of lubrication

Pressurised counter-sliding (tilted) solid contacts: Striebeck curve

\[ \mu = \frac{\tilde{\tau}}{\tilde{p}} = \Pi(\text{Str}, \alpha, \ldots) \]

Stiction

\[ \text{Str} \gg 1 : \frac{\tilde{\tau}\tilde{C}}{\tilde{\eta}\tilde{U}} \sim \text{const} \]

Coulomb friction

\[ \alpha = 0 \]

Fixed

Relative motion

\[ \tilde{\rho} > 0 \]

\[ d\tilde{\rho} < 0 \]

Boundary

Mixed-film

Laminar hydrodynamic

\[ \text{Str} = \frac{\tilde{\eta}\tilde{U}}{\tilde{\rho}\tilde{C}} \]

Fluid mechanics of lubrication I

B. Scheichl (AC²T, VUT)
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Lubrication

\[ \text{Str} = \frac{\eta \ddot{U}}{\dot{\rho} \dot{C}} \]
Basic (realistic) assumptions

lubricant flow

- ‘simple’ fluid
  excludes multi-phase flow (binary mixture lubricant–air):
  2 (intensive) state variables define local thermodynamic equilibrium

- Newtonian fluid
  lube oils, ionic liquids (vapour pressure very low), H₂O, many gases:
  at normal conditions, even for high pressures & shear rates, not for low temperatures

- laminar

- volume forces (gravity) neglected

bearing geometry

- clearance slender
  compared to typical macro-length (e.g. journal radius)

- perfectly hydrodynamic operation
  ‘hydraulically smooth’ surfaces:
  macroscopic flow description unaffected by mean asperities
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Outline

3 Classical theory
   - First principles
   - Problem formulation
   - Asymptotic theory
Governing eqs in Eulerian representation

any reference frame  \( \tilde{x}, \tilde{t} \)

\[ \tilde{D}_t := \partial_{\tilde{t}} + \tilde{u} \cdot \nabla (\tilde{x}) \]

continuity

\[ \tilde{D}_t \tilde{\rho} + \tilde{\rho} \nabla \cdot \tilde{\mathbf{u}} = 0 \]

momentum

\[ \tilde{\rho}(\ddot{\tilde{x}}_{\text{ref}} + 2\tilde{\Omega}_{\text{ref}} \times \tilde{\mathbf{u}} + \tilde{D}_t \tilde{\mathbf{u}}) = \nabla \cdot \tilde{\Sigma} , \quad \tilde{\Sigma} = -\tilde{\rho} I + \tilde{\Delta} , \quad \tilde{\Delta} = \tilde{\Delta}^{\text{tr}} \]

thermal energy, 1st & 2nd law of thermodynamics

\[ \tilde{\rho} \tilde{c}_p \tilde{D}_t \tilde{T} = \tilde{\beta} \tilde{T} \tilde{D}_t \tilde{\rho} + \tilde{\Phi} - \nabla \cdot \tilde{\mathbf{q}} , \quad \tilde{\Phi} = \tilde{\Delta} \cdot \nabla \tilde{\mathbf{u}} > 0 \]

constitutive laws for deviatoric & bulk stresses & heat flux

Newtonian fluid

\[ \tilde{\Delta} = \tilde{\eta} [\nabla \tilde{\mathbf{u}} + (\nabla \tilde{\mathbf{u}})^{\text{tr}}] + (\tilde{\eta}' - \frac{2}{3} \tilde{\eta}) (\nabla \cdot \tilde{\mathbf{u}}) I \]

\begin{align*}
\text{shear} & \quad \text{bulk} & \quad \text{viscosity} \\
\tilde{\mathbf{q}} &= -\tilde{\Lambda} \nabla \tilde{T} \\
\end{align*}

Fourier's law
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any reference frame $\tilde{x}, \tilde{t}$

$$\tilde{D}_t := \partial_{\tilde{t}} + \tilde{u} \cdot \nabla (\tilde{x})$$

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$$\tilde{\rho} \tilde{c}_p \tilde{D}_t \tilde{T} = \beta \tilde{T} \tilde{D}_t \tilde{\rho} + \tilde{\Phi} - \nabla \cdot \tilde{q} , \quad \tilde{\Phi} = \tilde{\Delta} \cdot \nabla \tilde{u} > 0$$

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Newtonian fluid

$$\tilde{\Delta} = \tilde{\eta} \left[ \nabla \tilde{u} + (\nabla \tilde{u})^T \right] + (\tilde{\eta}^{'} - \frac{2}{3} \tilde{\eta}) (\nabla \cdot \tilde{u}) I$$

shear

bulk viscosity

Fourier's law

$$\tilde{q} = -\tilde{\lambda} \nabla \tilde{T}$$
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any reference frame $\tilde{x}$, $\tilde{t}$

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shear\hspace{2cm} bulk\hspace{2cm} viscosity

Fourier's law

$\tilde{\dot{q}} = -\tilde{\lambda} \nabla \tilde{T}$
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any reference frame \( \tilde{x}, \tilde{t} \)

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\[ \tilde{\Delta} = \tilde{\eta} [\tilde{\nabla} \tilde{u} + (\tilde{\nabla} \tilde{u})^T] + (\tilde{\eta}' - \frac{2}{3} \tilde{\eta}) (\tilde{\nabla} \cdot \tilde{u}) I \]

shear \quad \eta \quad bulk \quad \eta' \quad \eta

Fourier's law

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shear

bulk viscosity

Fourier’s law $\tilde{\dot{q}} = -\tilde{\lambda} \tilde{\nabla} \tilde{T}$
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thermal energy, 1st & 2nd law of thermodynamics

$$\tilde{\rho} \tilde{c}_p \tilde{D}_t \tilde{T} = \beta \tilde{T} \tilde{D}_t \tilde{\rho} + \tilde{\Phi} - \nabla \cdot \tilde{q}, \quad \tilde{\Phi} = \tilde{\Delta} \cdot \nabla \tilde{u} > 0$$

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shear \quad bulk \quad \text{viscosity}

Fourier's law

$$\tilde{q} = -\lambda \nabla \tilde{T}$$
Governing eqs in Eulerian representation

any reference frame $\tilde{x}, \tilde{t} \quad \tilde{D}_t := \partial_{\tilde{t}} + \tilde{u} \cdot \nabla (\tilde{x})$

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<table>
<thead>
<tr>
<th>shear</th>
<th>bulk</th>
<th>viscosity</th>
</tr>
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</table>

**Fourier’s law**
$$\tilde{q} = -\tilde{\lambda} \nabla \tilde{T}$$
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any reference frame $\tilde{x}, \tilde{t}$  

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thermal energy, 1st & 2nd law of thermodynamics

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shear \hspace{2cm} \text{bulk viscosity}

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- shear
- bulk
- viscosity

**Fourier’s law**

$$\tilde{q} = -\tilde{\lambda} \tilde{\nabla} \tilde{T}$$
Thermodynamic properties of ‘simple’ fluid

caloric eq of state

\[ \tilde{h} = \tilde{h}(\tilde{\rho}, \tilde{T}) \]

\[ \tilde{c}_p := \left( \frac{\partial \tilde{h}}{\partial \tilde{T}} \right)_{\tilde{\rho}} \left[ \frac{\text{J}}{\text{kg K}} \right] \]

\[ \tilde{\beta} \tilde{T} = 1 - \tilde{\rho} \left( \frac{\partial \tilde{h}}{\partial \tilde{\rho}} \right)_{\tilde{T}} \]

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\[ \tilde{\eta} = \tilde{\eta}(\tilde{\rho}, \tilde{T}) \quad [\text{Pa s}] \]

\[ \tilde{\lambda} = \tilde{\lambda}(\tilde{\rho}, \tilde{T}) \quad [\text{W/(m K)}] \]

2nd law of thermodynamics

\[ \tilde{\eta}, \tilde{\lambda}, \tilde{\beta}, \tilde{c}_p > 0, \quad \text{seldom} \quad \tilde{\beta} < 0 \quad (\text{H}_2\text{O} \text{ l}) \]
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\[ \tilde{\rho} = \tilde{\rho}(\tilde{\rho}, \tilde{T}) \]

\[ \tilde{\beta} := -\frac{1}{\tilde{\rho}} \left( \frac{\partial \tilde{\rho}}{\partial \tilde{T}} \right)_{\tilde{\rho}} \left[ \frac{1}{K} \right] \]

\[ \tilde{\eta} = \tilde{\eta}(\tilde{\rho}, \tilde{T}) \left[ Pa\ s \right] \]

\[ \tilde{\lambda} = \tilde{\lambda}(\tilde{\rho}, \tilde{T}) \left[ W/(mK) \right] \]

2nd law of thermodynamics

\[ \tilde{\eta}, \tilde{\lambda}, \tilde{\beta}, \tilde{c}_p > 0 , \quad \text{seldom } \tilde{\beta} < 0 \ (H_2O!) \]
Outline

Classical theory
- First principles
- Problem formulation
- Asymptotic theory
Non-dimensional quantities

kinematic quantities
\[ t = \tilde{t} \tilde{U}/\tilde{L} , \quad x = \tilde{x}/\tilde{L} , \quad c = \tilde{c}/\tilde{C} , \quad u = \tilde{u}/\tilde{U} \]

reference state
\[ p = \tilde{p}/\tilde{p}_r , \quad \theta = (\tilde{T} - \tilde{T}_a)/\tilde{T}_r \]
\[ \rho = \tilde{\rho}/\tilde{\rho}_r , \quad (\eta, \eta') = (\tilde{\eta}, \tilde{\eta}')/\tilde{\eta}_r , \quad \lambda = \tilde{\lambda}/\tilde{\lambda}_r , \quad \beta = \tilde{\beta} \tilde{T}_a , \quad c_p = \tilde{c}_p/\tilde{c}_{p.r} \]

key groups

- clearance slenderness \( \epsilon := \tilde{C}/\tilde{L} \)
- temperature ratio \( \gamma := \tilde{T}_r/\tilde{T}_a \)
Non-dimensional quantities

**kinematic quantities**

\[ t = \frac{\tilde{t} \tilde{U}}{\tilde{L}}, \quad x = \frac{\tilde{x}}{\tilde{L}}, \quad c = \frac{\tilde{c}}{\tilde{C}}, \quad u = \frac{\tilde{u}}{\tilde{U}} \]

**reference state**

\[ p = \frac{\tilde{p}}{\tilde{p}_r}, \quad \theta = \frac{\tilde{T} - \tilde{T}_a}{\tilde{T}_r} \]

\[ \rho = \frac{\tilde{\rho}}{\tilde{\rho}_r}, \quad (\eta, \eta') = \frac{(\tilde{\eta}, \tilde{\eta}')}{\tilde{\eta}_r}, \quad \lambda = \frac{\tilde{\lambda}}{\tilde{\lambda}_r}, \quad \beta = \frac{\tilde{\beta}}{\tilde{T}_a}, \quad c_p = \frac{\tilde{c}_p}{\tilde{c}_{p,r}} \]

**key groups**

- Clearance slenderness: \[ \varepsilon := \frac{\tilde{C}}{\tilde{L}} \]
- Temperature ratio: \[ \gamma := \frac{\tilde{T}_r}{\tilde{T}_a} \]
- Reynolds number: \[ Re := \frac{\tilde{U} \tilde{L} \tilde{\rho}}{\tilde{\eta}_r} \]
- Prandtl number: \[ Pr := \frac{\tilde{c}_p}{\tilde{\eta}_r} \]
- Péclet number: \[ Pe := \frac{Re Pr}{\tilde{C}_p} \]
Non-dimensional quantities

**kinematic quantities**

\[ t = \tilde{t} \frac{\tilde{U}}{\tilde{L}} , \quad \mathbf{x} = \tilde{x} \frac{\tilde{L}}{\tilde{L}} , \quad c = \tilde{c} \frac{\tilde{C}}{\tilde{C}} , \quad u = \tilde{u} \frac{\tilde{U}}{\tilde{U}} \]

**reference state**

\[ p = \tilde{p} \frac{\tilde{p}_r}{\tilde{p}_r} , \quad \theta = (\tilde{T} - \tilde{T}_a) \frac{\tilde{T}_r}{\tilde{T}_r} \]

\[ \rho = \tilde{\rho} \frac{\tilde{\rho}_r}{\tilde{\rho}_r} , \quad (\eta, \eta') = (\tilde{\eta}, \tilde{\eta'}) \frac{\tilde{\eta}_r}{\tilde{\eta}_r} , \quad \lambda = \tilde{\lambda} \frac{\tilde{\lambda}_r}{\tilde{\lambda}_r} , \quad \beta = \tilde{\beta} \frac{\tilde{T}_a}{\tilde{T}_a} , \quad c_p = \tilde{c}_p \frac{\tilde{c}_{p,r}}{\tilde{c}_{p,r}} \]

**key groups**

- clearance slenderness \( \epsilon := \tilde{C} \frac{\tilde{L}}{\tilde{L}} \)
- temperature ratio \( \gamma := \tilde{T}_r \frac{\tilde{T}_a}{\tilde{T}_a} \)
- Reynolds number \( \text{Re} := \tilde{U} \frac{\tilde{L}}{\tilde{\rho}_r} \frac{\tilde{\eta}_r}{\tilde{p}_r} \)
- Prandtl number \( \text{Pr} := \tilde{c}_p \frac{\tilde{\eta}_r}{\tilde{\lambda}_r} \)
- Péclet number \( \text{Pe} := \text{Re} \frac{\text{Pr}}{\text{Pr}} \)
Non-dimensional quantities

kinematic quantities

\[ t = \tilde{t} \tilde{U} / \tilde{L}, \quad x = \tilde{x} / \tilde{L}, \quad c = \tilde{c} / \tilde{C}, \quad u = \tilde{u} / \tilde{U} \]

reference state

\[ p = \tilde{p} / \tilde{p}_r, \quad \theta = (\tilde{T} - \tilde{T}_a) / \tilde{T}_r \]
\[ \rho = \tilde{\rho} / \tilde{\rho}_r, \quad (\eta, \eta') = (\tilde{\eta}, \tilde{\eta}') / \tilde{\eta}_r, \quad \lambda = \tilde{\lambda} / \tilde{\lambda}_r, \quad \beta = \tilde{\beta} \tilde{T}_a, \quad c_p = \tilde{c}_p / \tilde{c}_{p,r} \]

key groups

clearance slenderness \( \epsilon := \tilde{C} / \tilde{L} \)

temperature ratio \( \gamma := \tilde{T}_r / \tilde{T}_a \)

Reynolds number \( Re := \tilde{U} \tilde{L} \tilde{p}_r / \tilde{\eta}_r \)

Prandtl number \( Pr := \tilde{c}_{p,r} \tilde{\eta}_r / \tilde{\lambda}_r \)

Péclet number \( Pe := Re Pr \)
Non-dimensional quantities

kinematic quantities
\[ t = \frac{\tilde{t}}{\tilde{U}}/\overline{L}, \quad x = \frac{\tilde{x}}{\overline{L}}, \quad c = \frac{\tilde{c}}{\overline{C}}, \quad u = \frac{\tilde{u}}{\overline{U}} \]

reference state
\[ p = \frac{\tilde{p}}{\tilde{p}_r}, \quad \theta = \frac{(\tilde{T} - \tilde{T}_a)}{\overline{T}_r} \]
\[ \rho = \frac{\tilde{\rho}}{\tilde{\rho}_r}, \quad (\eta, \eta') = \frac{(\tilde{\eta}, \tilde{\eta}')}{\tilde{\eta}_r}, \quad \lambda = \frac{\lambda}{\overline{\lambda}_r}, \quad \beta = \frac{\tilde{\beta}}{\overline{\beta}_a}, \quad c_p = \frac{\tilde{c}_p}{\tilde{c}_{p,r}} \]

key groups

- clearance slenderness \( \varepsilon := \frac{\tilde{C}}{\overline{L}} \)
- temperature ratio \( \gamma := \frac{\overline{T}_r}{\overline{T}_a} \)
- Reynolds number \( Re := \frac{\overline{U}\overline{L}\tilde{\rho}_r}{\tilde{\eta}_r} \)
- Prandtl number \( Pr := \frac{\tilde{c}_{p,r} \tilde{\eta}_r}{\overline{\lambda}_r} \)
- Péclet number \( Pe := \frac{Re}{Pr} \)
Non-dimensional quantities

kinematic quantities
\[ t = \tilde{t} \frac{\tilde{U}}{\tilde{L}}, \quad x = \tilde{x} \frac{\tilde{L}}{\tilde{L}}, \quad c = \tilde{c} \frac{\tilde{C}}{\tilde{C}}, \quad u = \tilde{u} \frac{\tilde{U}}{\tilde{U}} \]

reference state
\[ p = \tilde{p} \frac{\tilde{p}_r}{\tilde{p}_r}, \quad \theta = (\tilde{T} - \tilde{T}_a) \frac{\tilde{T}_r}{\tilde{T}_r} \]
\[ \rho = \tilde{\rho} \frac{\tilde{\rho}_r}{\tilde{\rho}_r}, \quad (\eta, \eta') = (\tilde{\eta}, \tilde{\eta'}) \frac{\tilde{\eta}_r}{\tilde{\eta}_r}, \quad \lambda = \tilde{\lambda} \frac{\tilde{\lambda}_r}{\tilde{\lambda}_r}, \quad \beta = \tilde{\beta} \frac{\tilde{T}_a}{\tilde{T}_a}, \quad c_p = \tilde{c}_p \frac{\tilde{c}_{p,r}}{\tilde{c}_{p,r}} \]

key groups

- clearance slenderness \( \epsilon := \frac{\tilde{C}}{\tilde{L}} \)
- temperature ratio \( \gamma := \frac{\tilde{T}_r}{\tilde{T}_a} \)
- Reynolds number \( Re := \frac{\tilde{U} \tilde{L} \tilde{p}_r}{\tilde{\eta}_r} \)
- Prandtl number \( Pr := \frac{\tilde{c}_{p,r} \tilde{\eta}_r}{\tilde{\lambda}_r} \)
- Péclet number \( Pe := Re \cdot Pr \)
Non-dimensional quantities, cont’d

natural metric

\[ x = x_\parallel + \epsilon e_n n, \quad u = u_\parallel + \epsilon e_n w, \quad u_\parallel = u_\parallel e_\parallel \]
\[ e_\parallel \cdot e_n = 0, \quad \partial_n e_\parallel = \partial_n e_n = 0 \]

\[ \nabla = \tilde{\nabla} = \nabla_\parallel + \epsilon^{-1} e_n \partial_n \]
\[ \nabla \cdot (\rho u) = \nabla_\parallel \cdot (\rho u_\parallel) + \underbrace{e_n \cdot \partial_n (\rho u_\parallel)} + \underbrace{\epsilon \nabla \parallel \cdot (\rho e_n w)} + \underbrace{e_n \cdot \partial_n (\rho e_n w)} \]
\[ e_n \cdot e_\parallel \partial_n (\rho u_\parallel) = 0 \]
\[ D_t = (\tilde{\nabla} / \tilde{U}) \tilde{D}_t = \partial_t + u \cdot \nabla = u_\parallel \cdot \nabla_\parallel + w \partial_n \]
Non-dimensional quantities, cont’d

natural metric

\[
x = x || + \epsilon e_n n, \quad u = u || + \epsilon e_n w, \quad u || = u|| e ||
\]

\[
e || \cdot e_n = 0, \quad \partial_n e || = \partial_n e_n = 0
\]

\[
\nabla = \tilde{L} \tilde{\nabla} = \nabla || + \epsilon^{-1} e_n \partial_n
\]

\[
\nabla \cdot (\rho u) = \nabla || \cdot (\rho u ||) + \underbrace{e_n \cdot \partial_n (\rho u ||)}_{O(\epsilon)} + \epsilon \underbrace{\nabla || \cdot (\rho e_n w)}_{\rho w \nabla || \cdot e_n} + \underbrace{e_n \cdot \partial_n (\rho e_n w)}_{\partial_n (\rho w)}
\]

\[
D_t = (\tilde{L} / \tilde{U}) \tilde{D}_t = \partial_t + u \cdot \nabla = u || \cdot \nabla || + w \partial_n
\]
Non-dimensional quantities, cont’d

\[ x = x_{||} + \epsilon e_n n, \quad u = u_{||} + \epsilon e_n w, \quad u_{||} = u_{||} e_{||} \]

\[ e_{||} \cdot e_n = 0, \quad \partial_n e_{||} = \partial_n e_n = 0 \]

\[ \nabla = \tilde{L} \tilde{\nabla} = \nabla_{||} + \epsilon^{-1} e_n \partial_n \]

\[ \nabla \cdot (\rho u) = \nabla_{||} \cdot (\rho u_{||}) + e_n \cdot \partial_n (\rho u_{||}) + \epsilon \nabla_{||} \cdot (\rho e_n w) + e_n \cdot \partial_n (\rho e_n w) \]

\[ O(\epsilon) \]

\[ e_n \cdot e_{||} \partial_n (\rho u_{||}) = 0 \]

\[ \rho w \nabla_{||} \cdot e_n \]

\[ \partial_n (\rho w) \]

\[ D_t = (\tilde{L}/\tilde{U}) \tilde{D}_t = \partial_t + u \cdot \nabla = u_{||} \cdot \nabla_{||} + w \partial_n \]

natural metric
Non-dimensional quantities, cont’d

natural metric

\[ \mathbf{x} = \mathbf{x}_\parallel + \epsilon \mathbf{e}_n \mathbf{n}, \quad \mathbf{u} = \mathbf{u}_\parallel + \epsilon \mathbf{e}_n \mathbf{w}, \quad \mathbf{u}_\parallel = \mathbf{u}_\parallel \mathbf{e}_\parallel \]
\[ \mathbf{e}_\parallel \cdot \mathbf{e}_n = 0, \quad \partial_n \mathbf{e}_\parallel = \partial_n \mathbf{e}_n = 0 \]

\[ \nabla = \tilde{\nabla} = \nabla_\parallel + \epsilon^{-1} \mathbf{e}_n \partial_n \]

\[ \nabla \cdot (\rho \mathbf{u}) = \nabla_\parallel \cdot (\rho \mathbf{u}_\parallel) + \mathbf{e}_n \cdot \partial_n (\rho \mathbf{u}_\parallel) + \epsilon \nabla_\parallel \cdot (\rho \mathbf{e}_n \mathbf{w}) + \mathbf{e}_n \cdot \partial_n (\rho \mathbf{e}_n \mathbf{w}) \]
\[ \mathbf{e}_n \cdot \mathbf{e}_\parallel \partial_n (\rho \mathbf{u}_\parallel) = 0 \]

\[ O(\epsilon) \]

\[ \partial_n (\rho \mathbf{w}) \]

\[ D_\parallel = (\tilde{\mathcal{L}} / \tilde{\mathcal{U}}) \tilde{D}_\parallel = \partial_t + \mathbf{u} \cdot \nabla = \mathbf{u}_\parallel \cdot \nabla_\parallel + \mathbf{w} \partial_n \]
Navier–Stokes eqs

\[ \tilde{p}_r := \tilde{\eta}_r \tilde{U}L/\tilde{C}^2, \quad \tilde{T}_r := \tilde{\eta}_r \tilde{U}^2/\tilde{\lambda}_r \]

state \[ q = q(\rho, 1 + \gamma \theta), \quad q = \rho, \eta, \lambda, c_p \Rightarrow \tilde{p}_r \]

continuity \[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) \equiv \partial_t \rho + \nabla_{||} \cdot (\rho \mathbf{u}_{||}) + \epsilon \rho w \nabla_{||} \cdot e_n + \partial_n (\rho w) = 0 \]

momentum \[ Re \epsilon^2 \rho (\ddot{x}_{ref} + 2 \Omega_{ref} \times \mathbf{u} + D_t \mathbf{u}) + \nabla p = \epsilon^2 \nabla \cdot \Delta \]

\[ \Delta = \eta [ \nabla \mathbf{u} + (\nabla \mathbf{u})^T] + (\eta' - \frac{2}{3} \eta) (\nabla \cdot \mathbf{u}) I \]

energy \[ Pe \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t p + \epsilon^2 [\Phi + \nabla \cdot (\lambda \nabla \theta)] \]

\[ \Phi = \Delta \cdot \nabla \mathbf{u}, \quad \gamma := \tilde{T}_r/\tilde{T}_a \]

momentum \[ 0 \sim -\nabla_{||} p + \partial_n (\eta \partial_n \mathbf{u}_{||}), \quad 0 \sim \epsilon^{-1} \partial_n p \]
Navier–Stokes eqs

\[ \tilde{\rho}_r := \tilde{\eta}_r \tilde{U}L / \tilde{C}^2, \quad \tilde{T}_r := \tilde{\eta}_r \tilde{U}^2 / \tilde{\lambda}_r \]

state \hspace{1cm} q = q(\rho, 1 + \gamma \theta), \quad q = \rho, \eta, \lambda, c_p \quad \Rightarrow \quad \tilde{\rho}_r

continuity \hspace{1cm} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) \equiv \partial_t \rho + \nabla_{\parallel} \cdot (\rho \mathbf{u}_{\parallel}) + \epsilon \rho \mathbf{w} \nabla_{\parallel} \cdot \mathbf{e}_n + \partial_n (\rho \mathbf{w}) = 0

momentum \hspace{1cm} \text{Re} \epsilon^2 \rho (\ddot{\mathbf{x}}_{\text{ref}} + 2 \Omega_{\text{ref}} \times \mathbf{u} + D_t \mathbf{u}) + \nabla \mathbf{p} = \epsilon^2 \nabla \cdot \Delta

\[ \Delta = \eta \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^{\text{tr}} \right] + \left( \eta' - \frac{2}{3} \eta \right)(\nabla \cdot \mathbf{u}) \mathbf{l} \]

energy \hspace{1cm} \text{Pe} \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t \rho + \epsilon^2 \left[ \Phi + \nabla \cdot (\lambda \nabla \theta) \right]

\[ \Phi = \Delta \cdot \nabla \mathbf{u}, \quad \gamma := \tilde{T}_r / \tilde{T}_a \]

\[ \epsilon \ll 1, \quad \nabla \sim \epsilon^{-1} \mathbf{e}_n \partial_n \]

momentum \hspace{1cm} 0 \sim -\nabla_{\parallel} \rho + \partial_n (\eta \partial_n \mathbf{u}_{\parallel}), \quad 0 \sim \epsilon^{-1} \partial_n \rho
Navier–Stokes eqs

\[ \tilde{\rho}_r := \tilde{\eta}_r \tilde{U} \tilde{L} / \tilde{C}^2, \quad \tilde{T}_r := \tilde{\eta}_r \tilde{U}^2 / \tilde{\lambda}_r \]

state

\[ q = q(\rho, 1 + \gamma \theta), \quad q = \rho, \ \eta, \ \lambda, \ c_p \quad \Rightarrow \quad \tilde{\rho}_r \]

continuity

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) \equiv \partial_t \rho + \nabla_{\parallel} \cdot (\rho \mathbf{u}_{\parallel}) + \epsilon \rho \mathbf{w} \nabla_{\parallel} \cdot \mathbf{e}_n + \partial_n (\rho \mathbf{w}) = 0 \]

momentum

\[ \text{Re} \epsilon^2 \rho(\ddot{x}_{\text{ref}} + 2\Omega_{\text{ref}} \times \mathbf{u} + D_t \mathbf{u}) + \nabla p = \epsilon^2 \nabla \cdot \Delta \]

\[ \Delta = \eta [\nabla \mathbf{u} + (\nabla \mathbf{u})^{\text{tr}}] + (\eta' - \frac{2}{3} \eta)(\nabla \cdot \mathbf{u})I \]

energy

\[ \text{Pe} \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t p + \epsilon^2 [\Phi + \nabla \cdot (\lambda \nabla \theta)] \]

\[ \Phi = \Delta \cdot \nabla \mathbf{u}, \quad \gamma := \tilde{T}_r / \tilde{T}_a \]

\[ \epsilon \ll 1, \quad \nabla \sim \epsilon^{-1} \mathbf{e}_n \partial_n \]

momentum

\[ 0 \sim -\nabla_{\parallel} \rho + \partial_n (\eta \partial_n \mathbf{u}_{\parallel}), \quad 0 \sim \epsilon^{-1} \partial_n \rho \]
Navier–Stokes eqs

\[ \mathcal{\tilde{r}} := \mathcal{\tilde{\eta}} \tilde{U} \bar{L} / \mathcal{\tilde{C}}^2 , \quad \mathcal{\tilde{T}} := \mathcal{\tilde{\eta}} \tilde{U}^2 / \mathcal{\tilde{\lambda}} \]

state \quad q = q(\rho, 1 + \gamma \theta) , \quad q = \rho , \ \eta , \ \lambda , \ c_p \quad \Rightarrow \quad \mathcal{\tilde{\rho}} \\
continuity \quad \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) \equiv \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) + \epsilon \rho \mathbf{w} \nabla \cdot \mathbf{e}_n + \partial_n (\rho \mathbf{w}) = 0 \\
momentum \quad Re \epsilon^2 \rho (\ddot{x}_{ref} + 2 \Omega_{ref} \times \mathbf{u} + D_t \mathbf{u}) + \nabla p = \epsilon^2 \nabla \cdot \Delta \\
\Delta = \eta \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^\text{tr} \right] + (\eta' - \frac{2}{3} \eta) (\nabla \cdot \mathbf{u}) \mathbf{l} \\
energy \quad Pe \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t \rho + \epsilon^2 [\Phi + \nabla \cdot (\lambda \nabla \theta)] \\
\Phi = \Delta \cdot \nabla \mathbf{u} , \quad \gamma := \mathcal{\tilde{T}} / \mathcal{\tilde{T}}_a \\

\epsilon \ll 1 , \quad \nabla \sim \epsilon^{-1} \mathbf{e}_n \partial_n \\
momentum \quad 0 \sim -\nabla \cdot p + \partial_n (\eta \partial_n u) , \quad 0 \sim \epsilon^{-1} \partial_n p
Navier–Stokes eqs

\[ \tilde{p}_r := \tilde{\eta}_r \tilde{U}L / \tilde{C}^2 , \quad \tilde{T}_r := \tilde{\eta}_r \tilde{U}^2 / \tilde{\lambda}_r \]

state
\[ q = q(\rho, 1 + \gamma \theta) , \quad q = \rho , \eta , \lambda , c_p \quad \Rightarrow \quad \tilde{p}_r \]

continuity
\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) \equiv \partial_t \rho + \nabla_{||} \cdot (\rho \mathbf{u}_{||}) + \epsilon \rho w \nabla_{||} \cdot \mathbf{e}_n + \partial_n (\rho w) = 0 \]

momentum
\[ Re \epsilon^2 \rho (\ddot{x}_{ref} + 2 \Omega_{ref} \times \mathbf{u} + D_t \mathbf{u}) + \nabla p = \epsilon^2 \nabla \cdot \Delta \]
\[ \Delta = \eta \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^{tr} \right] + (\eta' - \frac{2}{3} \eta) (\nabla \cdot \mathbf{u}) \mathbf{l} \]

energy
\[ Pe \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t p + \epsilon^2 [\Phi + \nabla \cdot (\lambda \nabla \theta)] \]
\[ \Phi = \Delta \cdot \nabla \mathbf{u} , \quad \gamma := \tilde{T}_r / \tilde{T}_a \]

\[ \epsilon \ll 1 , \quad \nabla \sim \epsilon^{-1} \mathbf{e}_n \partial_n \]

momentum
\[ 0 \sim - \nabla_{||} \rho + \partial_n (\eta \partial_n \mathbf{u}_{||}) , \quad 0 \sim \epsilon^{-1} \partial_n p \]
Navier–Stokes eqs

\[ \bar{\rho}_r := \bar{\eta}_r \bar{U} \bar{L} / \bar{C}^2, \quad \bar{T}_r := \bar{\eta}_r \bar{U}^2 / \bar{\lambda}_r \]

**state**

\[ q = q(\rho, 1 + \gamma \theta), \quad q = \rho, \eta, \lambda, c_p \quad \Rightarrow \quad \bar{\rho}_r \]

**continuity**

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) \equiv \partial_t \rho + \nabla_{||} \cdot (\rho \mathbf{u}_{||}) + \epsilon \rho w \nabla_{||} \cdot \mathbf{e}_n + \partial_n (\rho \mathbf{w}) = 0 \]

**momentum**

\[ \text{Re} \epsilon^2 \rho (\ddot{x}_{\text{ref}} + 2 \Omega_{\text{ref}} \times \mathbf{u} + D_t \mathbf{u}) + \nabla p = \epsilon^2 \nabla \cdot \Delta \]

\[ \Delta = \eta \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^\text{tr} \right] + (\eta' - \frac{2}{3} \eta)(\nabla \cdot \mathbf{u}) \mathbf{l} \]

**energy**

\[ \text{Pe} \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t p + \epsilon^2 \left[ \Phi + \nabla \cdot (\lambda \nabla \theta) \right] \]

\[ \Phi = \Delta \cdot \nabla \mathbf{u}, \quad \gamma := \bar{T}_r / \bar{T}_a \]

\[ \epsilon \ll 1, \quad \nabla \sim \epsilon^{-1} \mathbf{e}_n \partial_n \]

**momentum**

\[ 0 \sim -\nabla_{||} \rho + \partial_n (\eta \partial_n \mathbf{u}_{||}), \quad 0 \sim \epsilon^{-1} \partial_n \rho \]
Outline

Classical theory
- First principles
- Problem formulation
- Asymptotic theory
Limit process

classical lubrication approximation

thin film $\epsilon \ll 1$

quasi-isothermal $\gamma \ll 1$

inertia neglected $Re \epsilon^2 \ll 1$, laminar flow: $Re \lesssim 10^5$

typical values $\epsilon \lesssim 10^{-3}$, $Pr_{oil} \approx 70 \ldots 10^3 \Rightarrow Pe \lesssim 10^8$, $Pe \epsilon^2 \lesssim 10^2$

$$\nabla \cdot (\rho \mathbf{u}) \sim \nabla || \cdot (\rho \mathbf{u} ||) + \partial_n (\rho w) + O(\epsilon)$$

$$\rho (p, 1 + \gamma \theta) \sim \rho (p, 1) + O(\gamma)$$

expansions

$$\nabla || \sim \nabla_0 || + O(\epsilon), \quad \nabla_0 || = \nabla || \quad \text{for} \quad n = 0$$

$$[\mathbf{u} ||, w, p, \rho, \theta, \eta, \ldots](\mathbf{x} ||, n, t; \epsilon, Re, \gamma, \ldots) \sim [\mathbf{U}, W, P, Q, \Theta, N](\mathbf{x} ||, n, t) + \cdots$$

c $\sim C(\mathbf{x} ||, t) + O(\epsilon)$ journal bearing!
Limit process

classical lubrication approximation

thin film $\epsilon \ll 1$

quasi-isothermal $\gamma \ll 1$

inertia neglected $Re \epsilon^2 \ll 1$, laminar flow: $Re \lesssim 10^5$

typical values $\epsilon \lesssim 10^{-3}$, $Pr_{oil} \approx 70 \ldots 10^3$ $\Rightarrow$ $Pe \lesssim 10^8$, $Pe \epsilon^2 \lesssim 10^2$

\[ \nabla \cdot (\rho \mathbf{u}) \sim \nabla_\parallel \cdot (\rho \mathbf{u}_\parallel) + \partial_n (\rho w) + O(\epsilon) \]

\[ \rho(p, 1 + \gamma \theta) \sim \rho(p, 1) + O(\gamma) \]

expansions

\[ \nabla_\parallel \sim \nabla_\parallel^0 + O(\epsilon) \quad \nabla_\parallel^0 = \nabla_\parallel \quad \text{for} \quad n = 0 \]

\[ [\mathbf{u}_\parallel, w, p, \rho, \theta, \eta, \ldots](x_\parallel, n, t; \epsilon, Re, \gamma, \ldots) \sim [\mathbf{U}, W, P, Q, \Theta, N](x_\parallel, n, t) + \ldots \]

\[ c \sim C(x_\parallel, t) + O(\epsilon) \quad \text{journal bearing}! \]
Limit process

classical lubrication approximation

thin film \( \epsilon \ll 1 \)

quasi-isothermal \( \gamma \ll 1 \)

inertia neglected \( Re \epsilon^2 \ll 1 \), laminar flow: \( Re \lesssim 10^5 \)

typical values \( \epsilon \gtrsim 10^{-3}, \quad Pr_{oil} \approx 70 \ldots 10^3 \) \( 100 \ldots 20^\circ C \) \( \Rightarrow Pe \lesssim 10^8, \quad Pe \epsilon^2 \lesssim 10^2 ! \)

\[
\nabla \cdot (\rho \mathbf{u}) \sim \nabla_\parallel \cdot (\rho \mathbf{u}_\parallel) + \partial_n (\rho w) + O(\epsilon)
\]

\[
\rho(p, 1 + \gamma \theta) \sim \rho(p, 1) + O(\gamma)
\]

expansions

\[
\nabla_\parallel \sim \nabla_\parallel^0 + O(\epsilon) \quad \nabla_\parallel^0 = \nabla_\parallel \quad \text{for} \quad n = 0
\]

\[
[u_\parallel, \ w, \ p, \ \rho, \ \theta, \ \eta, \ldots](x_\parallel, n, t; \epsilon, Re, \gamma, \ldots) \sim [U, \ W, \ P, \ Q, \ \Theta, \ \mathcal{N}](x_\parallel, n, t) + \cdots
\]

\[
c \sim C(x_\parallel, t) + O(\epsilon) \quad \text{journal bearing !}
\]
Limit process

classical lubrication approximation

thin film $\epsilon \ll 1$

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typical values $\epsilon \lesssim 10^{-3}$, $Pr_{oil} \approx 70 \ldots 10^3$ \Rightarrow $Pe \lesssim 10^8$, $Pe \epsilon^2 \lesssim 10^2$

\[ \nabla \cdot (\rho u) \sim \nabla_{||} \cdot (\rho u_{||}) + \partial_n (\rho w) + O(\epsilon) \]

\[ \rho (p, 1 + \gamma \theta) \sim \rho (p, 1) + O(\gamma) \]

expansions

\[ \nabla_{||} \sim \nabla_{||}^0 + O(\epsilon) \quad \nabla_{||}^0 = \nabla_{||} \quad \text{for} \quad n = 0 \]

\[ [u_{||}, w, p, \rho, \theta, \eta, \ldots](x_{||}, n, t; \epsilon, Re, \gamma, \ldots) \sim [U, W, P, Q, \Theta, \mathcal{N}](x_{||}, n, t) + \ldots \]

\[ c \sim C(x_{||}, t) + O(\epsilon) \quad \text{journal bearing}! \]
Limit process

classical lubrication approximation

thin film \( \epsilon \ll 1 \)

quasi-isothermal \( \gamma \ll 1 \)

inertia neglected \( Re \epsilon^2 \ll 1 \), laminar flow: \( Re \lesssim 10^5 \)

typical values \( \epsilon \lesssim 10^{-3} \), \( Pr_{oil} \approx 70 \ldots 10^3 \) ⇒ \( Pe \lesssim 10^8 \), \( Pe \epsilon^2 \lesssim 10^2 \) !

\[ \nabla \cdot (\rho \mathbf{u}) \sim \nabla_\parallel \cdot (\rho \mathbf{u}_\parallel) + \partial_n (\rho w) + O(\epsilon) \]

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\[ c \sim C(\mathbf{x}_\parallel, t) + O(\epsilon) \] journal bearing!
Leading-order eqs

state & energy
\[ Q = Q(P, 1), \quad Q = \mathcal{Q}, \ N \]

continuity
\[ \partial_t Q + \nabla_\parallel \cdot (Q \mathbf{U}) + \partial_N (Q \mathbf{W}) = 0 \] (1)

momentum
\[ \nabla_\parallel \mathbf{P} = \partial_n (\mathcal{N} \partial_n \mathbf{U}), \quad \partial_n \mathbf{P} = 0 \quad \Rightarrow \quad \partial_n Q = \partial_n \mathcal{N} = 0 \] (2)

kinematic BCs

\[ n = 0 : \quad \mathbf{U} = U_1(x_\parallel, t), \quad \mathbf{W} = W_{p, 1}(x_\parallel, t) \] (3)

\[ n = C(x_\parallel, t) : \quad \mathbf{U} = U_2(x_\parallel, t), \quad \mathbf{W} = \partial_t C + U_2 \cdot \nabla_\parallel C + W_{p, 2}(x_\parallel, t) \] (4)

\[ (1), (3), (4) \quad \Rightarrow \quad \partial_t (Q C) + \nabla_\parallel \cdot \left( Q \int_0^C \mathbf{U} \, dn \right) + Q (W_{p, 2} - W_{p, 1}) = 0 \]

\[ (2), (3), (4) \quad \Rightarrow \quad \mathbf{U} = \frac{\nabla_\parallel \mathbf{P}}{2 \mathcal{N}(P)} n(n - C) + \frac{n}{C} (U_2 - U_1) \]

Hagen–Poisseuille

Couette
Leading-order eqs

state & energy

\[ Q = Q(P, 1) , \quad Q = Q , \quad \mathcal{N} \]

continuity

\[ \partial_t Q + \nabla_\parallel^0 \cdot (Q U) + \partial_n (Q W) = 0 \]  \hspace{1cm} (1)

momentum

\[ \nabla_\parallel^0 P = \partial_n (N \partial_n U) , \quad \partial_n P = 0 \quad \Rightarrow \quad \partial_n Q = \partial_n \mathcal{N} = 0 \]  \hspace{1cm} (2)

kinematic BCs

\[ n = 0 : \quad U = U_1(x_\parallel, t) , \quad W = W_{p,1}(x_\parallel, t) \]  \hspace{1cm} (3)

\[ n = C(x_\parallel, t) : \quad U = U_2(x_\parallel, t) , \quad W = \partial_t C + U_2 \cdot \nabla_\parallel^0 C + W_{p,2}(x_\parallel, t) \]  \hspace{1cm} (4)

\[ (1), (3), (4) \quad \Rightarrow \quad \partial_t (QC) + \nabla_\parallel^0 \cdot \left( Q \int_0^C U \, dn \right) + Q(W_{p,2} - W_{p,1}) = 0 \]

\[ (2), (3), (4) \quad \Rightarrow \quad U = \frac{\nabla_\parallel^0 P}{2 \mathcal{N}(P)} n(n - C) + \frac{n}{C} (U_2 - U_1) \]

Hagen–Poiseuille \hspace{1cm} Couette
Leading-order eqs

state & energy \[ Q = Q(P, 1) , \quad Q = Q, \mathcal{N} \]

continuity \[ \partial_t Q + \nabla_\parallel \cdot (QU) + \partial_N (QW) = 0 \quad (1) \]

momentum \[ \nabla_\parallel^0 P = \partial_n (\mathcal{N} \partial_n U) , \quad \partial_n P = 0 \quad \Rightarrow \quad \partial_n Q = \partial_n \mathcal{N} = 0 \quad (2) \]

kinematic BCs

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Leading-order eqs

state & energy \[ Q = Q(P, 1), \quad Q = Q, \quad N \]

continuity
\[ \partial_t Q + \nabla_\parallel \cdot (Q U) + \partial_N(Q W) = 0 \tag{1} \]

momentum
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\[ (2), (3), (4) \quad \Rightarrow \quad U = \frac{\nabla_\parallel P}{2N(P)} n(n - C) + \frac{n}{C} (U_2 - U_1) + U_1 \]
Integral mass balance

\[ \int_{0}^{C} \mathbf{U} \, dn = Q + C \, U_m, \quad Q := -\frac{C^3 \, \nabla_\parallel^0 P}{12 \, N}, \quad U_m := \frac{U_1 + U_2}{2} \]

Reynolds eq

\[ \nabla_\parallel^0 \cdot (-Q \mathbf{Q}) = \left( \frac{\partial}{\partial t} + U_m \cdot \nabla_\parallel^0 \right)(Q \mathbf{C}) + Q \mathbf{C} \nabla_\parallel^0 \cdot U_m + Q \left( W_{p,2} - W_{p,1} \right) \]

\[ Q = Q(P) , \quad N = N(P) \]

elliptic 2nd-order PDE for \( P(x_\parallel, t) \) and given \( C(x_\parallel, t) , \ U_m(x_\parallel, t) \)

kinematic wave operator \( \partial_t + U_m \cdot \nabla_\parallel^0 \) most relevant for gas bearings

linear for incompressible lubricant with constant properties \( (Q \equiv N \equiv 1) \)

in general to be solved numerically
Integral mass balance

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‘squeeze’ Couette + sliding = ‘wedge’

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elliptic 2nd-order PDE for \( P(x_{||}, t) \) and given \( C(x_{||}, t) , \, U_m(x_{||}, t) \)

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Reynolds eq – some important properties

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\[ Q = Q(P), \quad N = N(P) \]

rigid contacts, no Navier slip

\[ \nabla_0^0 \cdot [U_1, U_2, U_m] = 0 \]

Galilean transformation

\[ [x_||, t] = [x_||' + S(t'), t'] \]

\[ [\nabla_0^0, \partial_t] = [\nabla_0^0, \partial_t] - \dot{S} \nabla_0^0 \]

\[ [C, P, U_{1,2}](x_||, t) = [C', P', U_{1,2}'](x_||', t'), \quad [Q, N](P) = [Q', N'](P') \]

\[ [U_1, U_2, U_m] \rightarrow [U_{1}', U_{2}', U_{m}'] - \dot{S} \quad \text{invariance against sliding motion } S \]
Reynolds eq – some important properties

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[C, P, U_{1,2}](x||, t) = [C', P', U'_{1,2}](x'||, t') \]

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Reynolds eq – some important properties

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rigid contacts, no Navier slip \[ \nabla^0 \cdot [\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_m] = 0 \]

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\[ [\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_m] \mapsto [\mathbf{U}'_1, \mathbf{U}'_2, \mathbf{U}'_m] - \dot{\mathbf{S}} \quad \text{invariance against sliding motion } \mathbf{S} \]
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Validation of tribo-systems

typically find

- $P(x_{||}, t), \ x_{||} \in \Omega$, subject to $P(\partial \Omega, t) = P_a$

- load-bearing capacity $F(t) = \int_{\Omega} P e_n \, d\Omega$

clearance $C(x_{||}, t)$ is

- prescribed

- found from fluid–structure interaction machinery (e.g. shaft) dynamics ⇒ $F = F(\text{mech}, t(C, C))$

EHL ⇒ $P = P(C)$
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  machinery (e.g. shaft) dynamics $\Rightarrow F = F(\partial_{\parallel}C, \partial_{\perp}C, C)$
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Validation of tribo-systems

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  machinery (e.g. shaft) dynamics \( \Rightarrow F = F(\partial_{tt}C, \partial_t C, C) \)
  EHL \( \Rightarrow P = P(C) \)
Classical application: journal bearing

Reference quantities

$$\tilde{U}_m = \tilde{\omega}\tilde{R}_i, \quad \tilde{p}_r = \tilde{\eta}_r\tilde{\omega}\tilde{R}_i^2/\tilde{C}^2$$

Geometrical parameters

$$\epsilon = \tilde{C}/\tilde{R}_i \ll 1, \quad \text{eccentricity} \quad \epsilon = \tilde{e}/(\tilde{R}_a - \tilde{R}_i)$$

Non-dimensional quantities

$$C = 1 + \epsilon \cos \theta + O(\epsilon^2), \quad U_m (= N = Q) = 1$$
Further outlook

include

- EHL
- inertia ($Re \epsilon^2 \sim 1$, start-up, high-speed rotors, rapid load cycles)
- turbulence
- film rupture & cavitation (surface tension)
- effects acting on micro-scale $\ll \epsilon$ (surface roughness, mixed friction)

rational method: perturbation techniques

- multiple scales, matched asymptotic expansions
- numerical solution of reduced problem (simulation tools)
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Thank you for your attention!