Fluid mechanics of lubrication I: fundamental aspects of a rigorous theory

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Main objectives

- existing gap in tribological literature: lubrication represented ‘unsatisfactorily accurate’ ⇒
- describing lube flows by adopting first principles of continuum mechanics: asymptotic theory of hydromechanical lubrication

Why is this expedient?

- rational estimate of methodical error
- rational extension of classical theory to include e.g. EHD, inertia, micro-scale effects (cavitation, surface roughness)
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Overview

1. Phenomenon of lubrication

2. Basic assumptions

3. Classical theory
   - First principles
   - Problem formulation
   - Asymptotic theory

4. Validation of tribo-systems

5. Further outlook
Phenomenon of lubrication

pressurised counter-sliding (tilted) solid contacts: Striebeck curve

\[ \mu = \frac{\tilde{\tau}}{\tilde{p}} = \Pi(\text{Str}, \alpha, \ldots) \]

\[ \text{Str} \gg 1 : \frac{\tilde{\tau} C}{\tilde{\eta} \tilde{U}} \sim \text{const} \]

\[ \text{boundary} \quad \text{mixed-film} \quad \text{laminar hydrodynamic} \]

\[ \text{relative motion} \]

\[ \text{lubrication} \]

\[ \text{stiction} \]

Coulomb friction

\[ \alpha = 0 \]

\[ \tilde{L} \]

\[ \tilde{\rho} \]

\[ \alpha \]

\[ \tilde{C} \]

\[ \tilde{\eta} \]

\[ \tilde{u} \]

\[ \tilde{U} \]

\[ \text{fixed} \quad \tilde{\rho} > 0 \quad d\tilde{\rho} < 0 \]

\[ \text{Fluid mechanics of lubrication I} \]
Phenomenon of lubrication

pressurised counter-sliding (tilted) solid contacts: Striebeck curve

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Coulomb friction

\[ \alpha = 0 \]

\[ \alpha \]

boundary mixed-film laminar hydrodynamic lubrication

fixed \( \bar{p} > 0 \)

\( \bar{\rho} < 0 \)

relative motion

\[ \text{Str} = \frac{\bar{\eta} \bar{U}}{\bar{\rho} \bar{C}} \]
Basic (realistic) assumptions

lubricant flow

- ‘simple’ fluid
  excludes multi-phase flow (binary mixture lubricant–air):
  2 (intensive) state variables define local thermodynamic equilibrium

- Newtonian fluid
  lube oils, ionic liquids (vapour pressure very low), H$_2$O, many gases:
  at normal conditions, even for high pressures & shear rates, not for low temperatures

- laminar

- volume forces (gravity) neglected

bearing geometry

- clearance slender
  compared to typical macro-length (e.g. journal radius)

- perfectly hydrodynamic operation
  ‘hydraulically smooth’ surfaces:
  macroscopic flow description unaffected by mean asperities
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Outline

3 Classical theory
- First principles
- Problem formulation
- Asymptotic theory
Governing eqs in Eulerian representation

any reference frame \( \tilde{x}, \tilde{t} \)

\[ \tilde{D}_t := \partial_{\tilde{t}} + \tilde{u} \cdot \nabla (\tilde{x}) \]

continuity

\[ \tilde{D}_t \tilde{\rho} + \tilde{\rho} \nabla \cdot \tilde{u} = 0 \]

momentum

\[ \tilde{\rho} \left( \ddot{\tilde{x}}_{\text{ref}} + 2 \tilde{\Omega}_{\text{ref}} \times \tilde{u} + \tilde{D}_t \tilde{u} \right) = \nabla \cdot \tilde{\Sigma}, \quad \tilde{\Sigma} = -\tilde{\rho} I + \tilde{\Delta}, \quad \tilde{\Delta} = \tilde{\Delta}^{\text{tr}} \]

thermal energy, 1st & 2nd law of thermodynamics

\[ \tilde{\rho} \tilde{c}_p \tilde{D}_t \tilde{T} = \beta \tilde{T} \tilde{D}_t \tilde{\rho} + \tilde{\Phi} - \nabla \cdot \tilde{q}, \quad \tilde{\Phi} = \tilde{\Delta} \cdot \nabla \tilde{u} > 0 \]

constitutive laws for deviatoric & bulk stresses & heat flux

Newtonian fluid

\[ \tilde{\Delta} = \tilde{\eta} \left[ \nabla \tilde{u} + (\nabla \tilde{u})^{\text{tr}} \right] + (\tilde{\eta}' - \frac{2}{3} \tilde{\eta}) \left( \nabla \cdot \tilde{u} \right) I \]

shear \hspace{1cm} \text{bulk} \hspace{1cm} \text{viscosity}

Fourier's law

\[ \tilde{\dot{q}} = -\tilde{\lambda} \nabla \tilde{T} \]
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\[ \tilde{\rho} \tilde{c}_p \tilde{D}_t \tilde{T} = \beta \tilde{T} \tilde{D}_t \tilde{\rho} + \tilde{\Phi} - \nabla \cdot \tilde{\dot{q}} \], \( \tilde{\Phi} = \tilde{\Delta} \cdot \nabla \tilde{u} > 0 \)

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\[ \tilde{\Delta} = \tilde{\eta} \left[ \nabla \tilde{\dot{u}} + (\nabla \tilde{\dot{u}})^\text{tr} \right] + (\tilde{\eta}' - \frac{2}{3} \tilde{\eta}) (\nabla \cdot \tilde{\dot{u}})I \]

\( \tilde{\eta} \) shear \( \tilde{\eta}' \) bulk \( \tilde{\eta} \) viscosity

Fourier's law

\[ \tilde{\dot{q}} = -\tilde{\lambda} \nabla \tilde{T} \]
Governing eqs in Eulerian representation

any reference frame $\tilde{x}, \tilde{t}$ \hspace{1cm} $\tilde{D}_t := \partial_{\tilde{t}} + \tilde{u} \cdot \nabla(\tilde{x})$

continuity

$\tilde{D}_t \tilde{\rho} + \tilde{\rho} \nabla \cdot \tilde{\boldsymbol{u}} = 0$

momentum

$\tilde{\rho}(\ddot{\tilde{x}}_{\text{ref}} + 2\tilde{\Omega}_{\text{ref}} \times \tilde{\boldsymbol{u}} + \tilde{D}_t \tilde{\boldsymbol{u}}) = \nabla \cdot \tilde{\Sigma}, \quad \tilde{\Sigma} = -\tilde{\rho} I + \tilde{\Delta}, \quad \tilde{\Delta} = \tilde{\Delta}^{\text{tr}}$

thermal energy, 1st & 2nd law of thermodynamics

$\tilde{\rho} \tilde{c}_p \tilde{D}_t \tilde{T} = \beta \tilde{T} \tilde{D}_t \tilde{\rho} + \tilde{\Phi} - \nabla \cdot \tilde{\dot{q}}, \quad \tilde{\Phi} = \tilde{\Delta} \cdot \nabla \tilde{\boldsymbol{u}} > 0$

constitutive laws for deviatoric & bulk stresses & heat flux

Newtonian fluid \hspace{1cm} $\tilde{\Delta} = \tilde{\eta} \left[ \nabla \tilde{\boldsymbol{u}} + (\nabla \tilde{\boldsymbol{u}})^T \right] + \left( \tilde{\eta}' - \frac{2}{3} \tilde{\eta} \right) (\nabla \cdot \tilde{\boldsymbol{u}}) I$

shear \hspace{1cm} bulk \hspace{1cm} viscosity

Fourier's law \hspace{1cm} $\tilde{\dot{q}} = -\lambda \nabla \tilde{T}$
Governing eqs in Eulerian representation

any reference frame \( \tilde{x}, \tilde{t} \) \( \tilde{D}_t := \partial_{\tilde{t}} + \tilde{u} \cdot \nabla (\tilde{x}) \)

continuity
\[ \tilde{D}_t \tilde{\rho} + \tilde{\rho} \tilde{\nabla} \cdot \tilde{u} = 0 \]

momentum
\[ \tilde{\rho} (\ddot{\tilde{x}}_{\text{ref}} + 2\tilde{\Omega}_{\text{ref}} \times \tilde{u} + \tilde{D}_t \tilde{u}) = \tilde{\nabla} \cdot \tilde{\Sigma}, \quad \tilde{\Sigma} = -\tilde{\rho} I + \tilde{\Delta}, \quad \tilde{\Delta} = \tilde{\Delta}^{tr} \]

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Newtonian fluid
\[ \tilde{\Delta} = \tilde{\eta} [ \tilde{\nabla} \tilde{u} + (\tilde{\nabla} \tilde{u})^t ] + (\tilde{\eta}' - \frac{2}{3} \tilde{\eta}) (\tilde{\nabla} \cdot \tilde{u}) I \]

shear \quad \text{bulk} \quad \text{viscosity}

Fourier's law
\[ \tilde{q} = -\tilde{\lambda} \tilde{\nabla} \tilde{T} \]
Governing eqs in Eulerian representation

any reference frame $\mathbf{x}, \mathbf{t}$

\[ \hat{D}_t := \partial_t + \mathbf{u} \cdot \nabla_{(\mathbf{x})} \]

continuity

\[ \hat{D}_t \hat{\rho} + \hat{\rho} \nabla \cdot \mathbf{u} = 0 \]

momentum

\[ \hat{\bar{\rho}} (\ddot{\mathbf{x}}_{\text{ref}} + 2 \hat{\Omega}_{\text{ref}} \times \mathbf{u} + \hat{D}_t \mathbf{u}) = \nabla \cdot \hat{\Sigma} , \quad \hat{\Sigma} = -\hat{\rho} \mathbf{I} + \hat{\Delta} , \quad \hat{\Delta} = \hat{\Delta}^{\text{tr}} \]

thermal energy, 1st & 2nd law of thermodynamics

\[ \hat{\bar{\rho}} \hat{c}_p \hat{D}_t \hat{T} = \hat{\beta} \hat{T} \hat{D}_t \hat{\rho} + \hat{\Phi} - \nabla \cdot \mathbf{\hat{q}} , \quad \hat{\Phi} = \hat{\Delta} \cdot \nabla \mathbf{u} > 0 \]

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\[ \hat{\Delta} = \hat{\eta} \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^{\text{tr}} \right] + \left( \hat{\eta}' - \frac{2}{3} \hat{\eta} \right) (\nabla \cdot \mathbf{u}) \mathbf{I} \]

shear

\[ \hat{\Delta} = \hat{\lambda} \nabla \cdot \nabla \mathbf{T} \]

bulk viscosity

\[ \hat{\Phi} = \hat{\Delta} \cdot \nabla \mathbf{u} > 0 \]

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any reference frame $\tilde{x}$, $\tilde{t}$ \hspace{1cm} $\tilde{D}_t := \partial_{\tilde{t}} + \tilde{u} \cdot \nabla (\tilde{x})$

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$\tilde{p}(\ddot{\tilde{x}}_{\text{ref}} + 2\tilde{\Omega}_{\text{ref}} \times \tilde{u} + \tilde{D}_t \tilde{u}) = \tilde{\nabla} \cdot \tilde{\Sigma} , \quad \tilde{\Sigma} = -\tilde{\rho} I + \tilde{\Delta} , \quad \tilde{\Delta} = \tilde{\Delta}^t$

thermal energy, 1st & 2nd law of thermodynamics

$\tilde{\rho} \tilde{c}_p \tilde{D}_t \tilde{T} = \tilde{\beta} \tilde{T} \tilde{A}_t \tilde{\rho} + \tilde{\Phi} - \tilde{\nabla} \cdot \tilde{q} , \quad \tilde{\Phi} = \tilde{\Delta} \cdot \nabla \tilde{u} > 0$

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Newtonian fluid \hspace{1cm} $\tilde{\Delta} = \tilde{\eta} [\tilde{\nabla} \tilde{u} + (\tilde{\nabla} \tilde{u})^t] + (\tilde{\eta} - \frac{2}{3} \tilde{\eta}) (\tilde{\nabla} \cdot \tilde{u}) I$

\hspace{1cm} \text{shear} \hspace{1cm} \text{bulk} \hspace{1cm} \text{viscosity}

Fourier's law \hspace{1cm} $\tilde{q} = -\tilde{\lambda} \tilde{\nabla} \tilde{T}$
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any reference frame \( \tilde{x}, \tilde{t} \)  
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constitutive laws for deviatoric & bulk stresses & heat flux

\begin{align*}
\text{Newtonian fluid} & \quad \tilde{\Delta} = \tilde{\eta} \left[ \tilde{\nabla} \tilde{u} + (\tilde{\nabla} \tilde{u})^\text{tr} \right] + \left( \tilde{\eta}' - \frac{2}{3} \tilde{\eta} \right) (\tilde{\nabla} \cdot \tilde{u}) I \\
\text{shear} & \quad \text{bulk} \quad \text{viscosity} \\
\text{Fourier's law} & \quad \tilde{\dot{q}} = -\tilde{\lambda} \tilde{\nabla} \tilde{T}
\end{align*}
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any reference frame \( \tilde{x}, \tilde{t} \) \( \tilde{D}_t := \partial_{\tilde{t}} + \tilde{u} \cdot \nabla_{(\tilde{x})} \)

continuity
\( \tilde{D}_t \tilde{\rho} + \tilde{\rho} \nabla \cdot \tilde{u} = 0 \)

momentum
\( \tilde{\rho} (\tilde{\ddot{x}}_{\text{ref}} + 2 \tilde{\Omega}_{\text{ref}} \times \tilde{u} + \tilde{D}_t \tilde{u}) = \nabla \cdot \tilde{\Sigma} \), \( \tilde{\Sigma} = -\tilde{\rho} I + \tilde{\Delta} \), \( \tilde{\Delta} = \tilde{\Delta}^{\text{tr}} \)

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\( \tilde{\rho} \tilde{c}_p \tilde{D}_t \tilde{T} = \tilde{\beta} \tilde{T} \tilde{D}_t \tilde{\rho} + \tilde{\Phi} - \nabla \cdot \tilde{\dot{q}} \), \( \tilde{\Phi} = \tilde{\Delta} \cdot \nabla \tilde{u} > 0 \)

constitutive laws for deviatoric & bulk stresses & heat flux

Newtonian fluid \( \tilde{\Delta} = \tilde{\eta} \left[ \nabla \tilde{u} + (\nabla \tilde{u})^{\text{tr}} \right] + (\tilde{\eta}' - \frac{2}{3} \tilde{\eta}) (\nabla \cdot \tilde{u}) I \)

\[ \text{shear} \quad \text{bulk} \quad \text{viscosity} \]

Fourier’s law \( \tilde{\dot{q}} = -\tilde{\lambda} \nabla \tilde{T} \)
Governing eqs in Eulerian representation

any reference frame $\tilde{x}, \tilde{t}$

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constitutive laws for deviatoric & bulk stresses & heat flux

**Newtonian fluid**

\[ \tilde{\Delta} = \tilde{\eta} \left[ \tilde{\nabla} \tilde{u} + (\tilde{\nabla} \tilde{u})^{tr} \right] + (\tilde{\eta}' - \frac{2}{3} \tilde{\eta}) (\tilde{\nabla} \cdot \tilde{u}) I \]

- shear
- bulk viscosity

**Fourier’s law**

\[ \tilde{\dot{q}} = -\tilde{\lambda} \tilde{\nabla} \tilde{T} \]
Thermodynamic properties of ‘simple’ fluid

caloric eq of state

\[ \tilde{h} = \tilde{h}(\tilde{p}, \tilde{T}) \]

\[ \tilde{c}_p := \left( \frac{\partial \tilde{h}}{\partial \tilde{T}} \right)_\tilde{p} \left[ \frac{\text{J}}{\text{kg K}} \right] \]

\[ \tilde{\beta} \tilde{T} = 1 - \tilde{\rho} \left( \frac{\partial \tilde{h}}{\partial \tilde{p}} \right)_{\tilde{T}} \]

thermal eq of state

\[ \tilde{\rho} = \tilde{\rho}(\tilde{\rho}, \tilde{T}) \]

\[ \tilde{\beta} := -\frac{1}{\tilde{\rho}} \left( \frac{\partial \tilde{\rho}}{\partial \tilde{T}} \right)_\tilde{\rho} \left[ \frac{1}{\text{K}} \right] \]

\[ \tilde{\eta} = \tilde{\eta}(\tilde{\rho}, \tilde{T}) \quad [\text{Pa s}] \]

\[ \tilde{\lambda} = \tilde{\lambda}(\tilde{\rho}, \tilde{T}) \quad [\text{W/(m K)}] \]

2nd law of thermodynamics

\[ \tilde{\eta}, \tilde{\lambda}, \tilde{\beta}, \tilde{c}_p > 0, \quad \text{seldom } \tilde{\beta} < 0 \quad (\text{H}_2\text{O} \text{l}) \]
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\[ \tilde{\eta} = \tilde{\eta}(\tilde{\rho}, \tilde{T}) \quad \text{[Pa s]} \]
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thermal eq of state
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\[ \tilde{\eta} = \tilde{\eta}(\tilde{\rho}, \tilde{T}) \left[ \text{Pa s} \right] \]
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\[ \tilde{\lambda} = \tilde{\lambda}(\tilde{p}, \tilde{T}) \text{ [W/(m K)]} \]

2nd law of thermodynamics

\[ \tilde{\eta}, \tilde{\lambda}, \tilde{\beta}, \tilde{c}_p > 0, \text{ seldom } \tilde{\beta} < 0 \text{ (H}_2\text{O}!) \]
Thermodynamic properties of ‘simple’ fluid

caloric eq of state
\[ \tilde{h} = \tilde{h}(\tilde{\rho}, \tilde{T}) \quad \tilde{c}_p := \left( \frac{\partial \tilde{h}}{\partial \tilde{T}} \right)_{\tilde{\rho}} \left[ \frac{J}{kg \, K} \right], \quad \tilde{\beta} \tilde{T} = 1 - \tilde{\rho} \left( \frac{\partial \tilde{h}}{\partial \tilde{\rho}} \right)_{\tilde{T}} \]

thermal eq of state
\[ \tilde{\rho} = \tilde{\rho}(\tilde{\rho}, \tilde{T}) \quad \tilde{\beta} := -\frac{1}{\tilde{\rho}} \left( \frac{\partial \tilde{\rho}}{\partial \tilde{T}} \right)_{\tilde{\rho}} \left[ \frac{1}{K} \right] \]
\[ \tilde{\eta} = \tilde{\eta}(\tilde{\rho}, \tilde{T}) \quad [Pa \, s] \]
\[ \tilde{\lambda} = \tilde{\lambda}(\tilde{\rho}, \tilde{T}) \quad [W/(m \, K)] \]

2nd law of thermodynamics
\[ \tilde{\eta}, \tilde{\lambda}, \tilde{\beta}, \tilde{c}_p > 0, \quad \text{seldom} \quad \tilde{\beta} < 0 \quad (H_2O !) \]
Outline

3. Classical theory
   - First principles
   - Problem formulation
   - Asymptotic theory
Non-dimensional quantities

kinematic quantities
\[ t = \tilde{t} \frac{\tilde{U}}{\tilde{L}} , \quad x = \tilde{x} / \tilde{L} , \quad c = \tilde{c} / \tilde{C} , \quad u = \tilde{u} / \tilde{U} \]

reference state
\[ p = \tilde{p} / \tilde{p}_r , \quad \theta = (\tilde{T} - \tilde{T}_a) / \tilde{T}_r \]
\[ \rho = \tilde{\rho} / \tilde{\rho}_r , \quad (\eta, \eta') = (\tilde{\eta}, \tilde{\eta}') / \tilde{\eta}_r , \quad \lambda = \tilde{\lambda} / \tilde{\lambda}_r , \quad \beta = \tilde{\beta} \tilde{T}_a , \quad c_p = \tilde{c}_p / \tilde{c}_{p,r} \]

key groups

clearance slenderness \[ \epsilon := \tilde{C} / \tilde{L} \]
temperature ratio \[ \gamma := \tilde{T}_r / \tilde{T}_a \]
Non-dimensional quantities

kinematic quantities

\[ t = \tilde{t} \tilde{U}/\tilde{L}, \quad \mathbf{x} = \tilde{x}/\tilde{L}, \quad c = \tilde{c}/\tilde{C}, \quad \mathbf{u} = \tilde{u}/\tilde{U} \]

reference state

\[ p = \tilde{p}/\tilde{p}_r, \quad \theta = (\tilde{T} - \tilde{T}_a)/\tilde{T}_r \]
\[ \rho = \tilde{\rho}/\tilde{\rho}_r, \quad (\eta, \eta') = (\tilde{\eta}, \tilde{\eta}')/\tilde{\eta}_r, \quad \lambda = \tilde{\lambda}/\tilde{\lambda}_r, \quad \beta = \tilde{\beta}\tilde{T}_a, \quad c_p = \tilde{c}_p/\tilde{c}_{p,r} \]

key groups

- clearance slenderness \[ \epsilon := \tilde{C}/\tilde{L} \]
- temperature ratio \[ \gamma := \tilde{T}_r/\tilde{T}_a \]
- Reynolds number \[ \text{Re} := \tilde{U}\tilde{L}/\tilde{\eta} \]
- Prandtl number \[ Pr := \tilde{c}_p/\tilde{\eta} \]
- Péclet number \[ Pe := \text{Re Pr} \]
Non-dimensional quantities

kinematic quantities

\[ t = \frac{\tilde{t} \tilde{U}}{\tilde{L}}, \quad x = \frac{\tilde{x}}{\tilde{L}}, \quad c = \frac{\tilde{c}}{\tilde{C}}, \quad u = \frac{\tilde{u}}{\tilde{U}} \]

reference state

\[ p = \frac{\tilde{p}}{\tilde{p}_r}, \quad \theta = \frac{(\tilde{T} - \tilde{T}_a)}{\tilde{T}_r} \]

\[ \rho = \frac{\tilde{\rho}}{\tilde{\rho}_r}, \quad (\eta, \eta') = \frac{(\tilde{\eta}, \tilde{\eta}')}{\tilde{\eta}_r}, \quad \lambda = \frac{\tilde{\lambda}}{\tilde{\lambda}_r}, \quad \beta = \frac{\tilde{\beta} \tilde{T}_a}{\tilde{\lambda}_r}, \quad c_p = \frac{\tilde{c}_p}{\tilde{c}_{p,r}} \]

key groups

- clearance slenderness: \( \epsilon := \frac{\tilde{C}}{\tilde{L}} \)
- temperature ratio: \( \gamma := \frac{\tilde{T}_r}{\tilde{T}_a} \)
- Reynolds number: \( \text{Re} := \frac{\tilde{U} \tilde{L} \tilde{p}_r}{\tilde{\eta}_r} \)
- Prandtl number: \( \text{Pr} := \frac{\tilde{c}_{p,r} \tilde{\eta}_r}{\tilde{\lambda}_r} \)
- Péclet number: \( \text{Pe} := \text{Re} \text{Pr} \)
Non-dimensional quantities

kinematic quantities

t = \tilde{t} \tilde{U}/\tilde{L}, \quad x = \tilde{x}/\tilde{L}, \quad c = \tilde{c}/\tilde{C}, \quad u = \tilde{u}/\tilde{U}

reference state

\rho = \tilde{\rho}/\tilde{\rho}_r, \quad \theta = (\tilde{T} - \tilde{T}_a)/\tilde{T}_r
\rho = \tilde{\rho}/\tilde{\rho}_r, \quad (\eta, \eta') = (\tilde{\eta}, \tilde{\eta}')/\tilde{\eta}_r, \quad \lambda = \tilde{\lambda}/\tilde{\lambda}_r, \quad \beta = \tilde{\beta}\tilde{T}_a, \quad c_p = \tilde{c}_p/\tilde{c}_{p,r}

key groups

clearance slenderness \quad \epsilon := \tilde{C}/\tilde{L}
temperature ratio \quad \gamma := \tilde{T}_r/\tilde{T}_a
Reynolds number \quad Re := \tilde{U}\tilde{L}\tilde{\rho}_r/\tilde{\eta}_r
Prandtl number \quad Pr := \tilde{c}_{p,r}\tilde{\eta}_r/\tilde{\lambda}_r
Péclet number \quad Pe := Re Pr
Non-dimensional quantities

**Kinematic quantities**

\[ t = \tilde{t} \tilde{U}/\tilde{L}, \quad x = \tilde{x}/\tilde{L}, \quad c = \tilde{c}/\tilde{C}, \quad u = \tilde{u}/\tilde{U} \]

**Reference state**

\[ \rho = \tilde{\rho}/\tilde{\rho}_r, \quad \theta = (\tilde{T} - \tilde{T}_a)/\tilde{T}_r \]

\[ \rho = \tilde{\rho}/\tilde{\rho}_r, \quad (\eta, \eta') = (\tilde{\eta}, \tilde{\eta}')/\tilde{\eta}_r, \quad \lambda = \tilde{\lambda}/\tilde{\lambda}_r, \quad \beta = \tilde{\beta} \tilde{T}_a, \quad c_p = \tilde{c}_p/\tilde{c}_{p,r} \]

**Key groups**

- Clearance slenderness: \( \epsilon := \tilde{C}/\tilde{L} \)
- Temperature ratio: \( \gamma := \tilde{T}_r/\tilde{T}_a \)
- Reynolds number: \( Re := \tilde{U}\tilde{L}\tilde{\rho}_r/\tilde{\eta}_r \)
- Prandtl number: \( Pr := \tilde{c}_{p,r}\tilde{\eta}_r/\tilde{\lambda}_r \)
- Péclet number: \( Pe := Re Pr \)
Non-dimensional quantities

**kinematic quantities**
\[ t = \tilde{t} \frac{\tilde{U}}{\tilde{L}} , \quad x = \tilde{x} / \tilde{L} , \quad c = \tilde{c} / \tilde{C} , \quad u = \tilde{u} / \tilde{U} \]

**reference state**
\[ \rho = \frac{\tilde{\rho}}{\rho_r} , \quad \theta = \frac{\tilde{T} - \tilde{T}_a}{\tilde{T}_r} \]
\[ \rho = \frac{\tilde{\rho}}{\rho_r} , \quad (\eta, \eta') = (\tilde{\eta}, \tilde{\eta}') / \tilde{\eta}_r , \quad \lambda = \frac{\tilde{\lambda}}{\lambda_r} , \quad \beta = \beta \tilde{T}_a , \quad c_p = \frac{\tilde{c}_p}{\tilde{c}_{p,r}} \]

**key groups**

- **clearance slenderness** \( \epsilon := \tilde{C} / \tilde{L} \)
- **temperature ratio** \( \gamma := \frac{\tilde{T}_r}{\tilde{T}_a} \)
- **Reynolds number** \( Re := \frac{\tilde{U} \tilde{L} \tilde{\rho}_r}{\tilde{\eta}_r} \)
- **Prandtl number** \( Pr := \frac{\tilde{c}_{p,r} \tilde{\eta}_r}{\tilde{\lambda}_r} \)
- **Péclet number** \( Pe := Re Pr \)
Non-dimensional quantities, cont’d

natural metric

\[ x = x_{||} + \epsilon e_n n, \quad u = u_{||} + \epsilon e_n w, \quad u_{||} = u_{||} e_{||} \]

\[ e_{||} \cdot e_n = 0, \quad \partial_n e_{||} = \partial_n e_n = 0 \]

\[ \nabla = \tilde{\nabla} = \nabla_{||} + \epsilon^{-1} e_n \partial_n \]

\[ \nabla \cdot (\rho u) = \nabla_{||} \cdot (\rho u_{||}) + e_n \cdot \partial_n (\rho u_{||}) + \epsilon \nabla_{||} \cdot (\rho e_n w) + e_n \cdot \partial_n (\rho e_n w) \]

\[ D_t = (\tilde{L}/\tilde{U}) \tilde{D}_t = \partial_t + u \cdot \nabla = u_{||} \cdot \nabla_{||} + w \partial_n \]
Non-dimensional quantities, cont’d

\[ \mathbf{x} = \mathbf{x}_\parallel + \epsilon \mathbf{e}_n \mathbf{n}, \quad \mathbf{u} = \mathbf{u}_\parallel + \epsilon \mathbf{e}_n \mathbf{w}, \quad \mathbf{u}_\parallel = \mathbf{u}_\parallel \mathbf{e}_\parallel \]

\[ \mathbf{e}_\parallel \cdot \mathbf{e}_n = 0, \quad \partial_n \mathbf{e}_\parallel = \partial_n \mathbf{e}_n = 0 \]

\[ \nabla = \tilde{\nabla} = \nabla_\parallel + \epsilon^{-1} \mathbf{e}_n \partial_n \]

\[ \nabla \cdot (\rho \mathbf{u}) = \nabla_\parallel \cdot (\rho \mathbf{u}_\parallel) + \frac{\mathbf{e}_n \cdot \partial_n (\rho \mathbf{u}_\parallel)}{\epsilon} + \frac{\epsilon \nabla_\parallel \cdot (\rho \mathbf{e}_n \mathbf{w})}{\rho w \nabla_\parallel \cdot \mathbf{e}_n} + \frac{\mathbf{e}_n \cdot \partial_n (\rho \mathbf{e}_n \mathbf{w})}{\partial_n (\rho w)} \]

\[ D_t = (\tilde{\nabla} / \tilde{U}) \tilde{D}_t = \partial_t + \mathbf{u} \cdot \nabla = \mathbf{u}_\parallel \cdot \nabla_\parallel + \mathbf{w} \partial_n \]
Non-dimensional quantities, cont’d

\[ x = x_\parallel + \epsilon e_n n, \quad u = u_\parallel + \epsilon e_n w, \quad u_\parallel = u_\parallel e_\parallel \]

\[ e_\parallel \cdot e_n = 0, \quad \partial_n e_\parallel = \partial_n e_n = 0 \]

\[ \nabla = \tilde{\nabla} = \nabla_\parallel + \epsilon^{-1} e_n \partial_n \]

\[ \nabla \cdot (\rho u) = \nabla_\parallel \cdot (\rho u_\parallel) + e_n \cdot \partial_n (\rho u_\parallel) + \epsilon \nabla_\parallel \cdot (\rho e_n w) + e_n \cdot \partial_n (\rho e_n w) \]

\[ e_n \cdot e_\parallel \partial_n (\rho u_\parallel) = 0 \]

\[ O(\epsilon) \]

\[ \partial_n (\rho w) \]

\[ D_t = (\tilde{\nabla}/\tilde{U}) \tilde{D}_t = \partial_t + u \cdot \nabla = u_\parallel \cdot \nabla_\parallel + w \partial_n \]
Non-dimensional quantities, cont’d

\[ x = x_\parallel + \epsilon e_n n, \quad u = u_\parallel + \epsilon e_n w, \quad u_\parallel = u_\parallel e_\parallel \]

\[ e_\parallel \cdot e_n = 0, \quad \partial_n e_\parallel = \partial_n e_n = 0 \]

\[ \nabla = \tilde{L} \tilde{\nabla} = \nabla_\parallel + \epsilon^{-1} e_n \partial_n \]

\[ \nabla \cdot (\rho u) = \nabla_\parallel \cdot (\rho u_\parallel) + \underbrace{e_n \cdot \partial_n (\rho u_\parallel)} + \underbrace{\epsilon \nabla_\parallel \cdot (\rho e_n w)} + \underbrace{e_n \cdot \partial_n (\rho e_n w)} \]

\[ \partial_n (\rho w) \]

\[ D_t = (\tilde{L} / \tilde{\nabla}) \tilde{D}_t = \partial_t + u \cdot \nabla = u_\parallel \cdot \nabla_\parallel + w \partial_n \]
Navier–Stokes eqs

\[ \tilde{\rho}_r := \tilde{\eta}_r \tilde{U} \tilde{L} / \tilde{C}^2, \quad \tilde{T}_r := \tilde{\eta}_r \tilde{U}^2 / \tilde{\lambda}_r \]

state \[ q = q(\rho, 1 + \gamma \theta), \quad q = \rho, \eta, \lambda, c_p \Rightarrow \tilde{\rho}_r \]

continuity \[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = \partial_t \rho + \nabla_{\parallel} \cdot (\rho \mathbf{u}_{\parallel}) + \epsilon \rho w \nabla \cdot \mathbf{e}_n + \partial_n (\rho w) = 0 \]

momentum \[ \text{Re} \epsilon^2 \rho (\ddot{x}_{\text{ref}} + 2 \Omega_{\text{ref}} \times \mathbf{u} + D_t \mathbf{u}) + \nabla \rho = \epsilon^2 \nabla \cdot \Delta \]
\[ \Delta = \eta [\nabla \mathbf{u} + (\nabla \mathbf{u})^\text{tr}] + (\eta' - \frac{2}{3} \eta) (\nabla \cdot \mathbf{u}) I \]

energy \[ \text{Pe} \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t p + \epsilon^2 [\Phi + \nabla \cdot (\lambda \nabla \theta)] \]
\[ \Phi = \Delta \cdot \nabla \mathbf{u}, \quad \gamma := \tilde{T}_r / \tilde{T}_a \]

\[ \epsilon \ll 1, \quad \nabla \ll \epsilon^{-1} \partial_n \partial_n \]

momentum \[ 0 \sim -\nabla_{\parallel} p + \partial_n (\eta \partial_n \mathbf{u}_{\parallel}) , \quad 0 \sim \epsilon^{-1} \partial_n p \]
Navier–Stokes eqs

\[ \tilde{\rho}_r := \tilde{\eta}_r \tilde{U} \tilde{L} / \tilde{C}^2, \quad \tilde{T}_r := \tilde{\eta}_r \tilde{U}^2 / \tilde{\lambda}_r \]

state

\[ q = q(\rho, 1 + \gamma \theta), \quad q = \rho, \eta, \lambda, c_p \Rightarrow \tilde{\rho}_r \]

continuity

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = \partial_t \rho + \nabla_{\parallel} \cdot (\rho \mathbf{u}_{\parallel}) + \epsilon \rho w \nabla_{\parallel} \cdot \mathbf{e}_n + \partial_n (\rho w) = 0 \]

momentum

\[ \text{Re} \epsilon^2 \rho (\ddot{x}_{\text{ref}} + 2 \Omega_{\text{ref}} \times \mathbf{u} + D_t \mathbf{u}) + \nabla p = \epsilon^2 \nabla \cdot \Delta \]

\[ \Delta = \eta [\nabla \mathbf{u} + (\nabla \mathbf{u})^{\text{tr}}] + (\eta' - \frac{2}{3} \eta)(\nabla \cdot \mathbf{u}) I \]

energy

\[ \text{Pe} \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t \rho + \epsilon^2 [\Phi + \nabla \cdot (\lambda \nabla \theta)] \]

\[ \Phi = \Delta \cdot \nabla \mathbf{u}, \quad \gamma := \tilde{T}_r / \tilde{T}_a \]

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\[ 0 \sim -\nabla_{\parallel} \rho + \partial_n (\eta \partial_n \mathbf{u}_{\parallel}), \quad 0 \sim \epsilon^{-1} \partial_n \rho \]
Navier–Stokes eqs

\[ \tilde{\rho}_r := \tilde{\eta}_r \tilde{U} \tilde{L} / \tilde{C}^2 , \quad \tilde{T}_r := \tilde{\eta}_r \tilde{U}^2 / \tilde{\lambda}_r \]

state
\[ q = q(\rho, 1 + \gamma \theta) , \quad q = \rho , \eta , \lambda , c_p \quad \Rightarrow \quad \tilde{\rho}_r \]

continuity
\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) \equiv \partial_t \rho + \nabla_\| \cdot (\rho \mathbf{u}_\|) + \epsilon \rho \mathbf{w} \nabla_\| \cdot \mathbf{e}_n + \partial_n (\rho \mathbf{w}) = 0 \]

momentum
\[ \text{Re} \epsilon^2 \rho (\ddot{x}_{\text{ref}} + 2 \Omega_{\text{ref}} \times \mathbf{u} + D_t \mathbf{u}) + \nabla p = \epsilon^2 \nabla \cdot \Delta \]
\[ \Delta = \eta [\nabla \mathbf{u} + (\nabla \mathbf{u})^\text{tr}] + (\eta' - \frac{2}{3} \eta)(\nabla \cdot \mathbf{u}) \mathbf{l} \]

energy
\[ \text{Pe} \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t p + \epsilon^2 [\Phi + \nabla \cdot (\lambda \nabla \theta)] \]
\[ \Phi = \Delta \cdot \nabla \mathbf{u} , \quad \gamma := \tilde{T}_r / \tilde{T}_a \]

\[ \epsilon \ll 1 , \quad \nabla \sim \epsilon^{-1} \mathbf{e}_n \partial_n \]

momentum
\[ 0 \sim -\nabla_\| p + \partial_n (\eta \partial_n \mathbf{u}_\|) , \quad 0 \sim \epsilon^{-1} \partial_n p \]
Navier–Stokes eqs

\[ \tilde{p}_r := \tilde{\eta}_r \tilde{U} L / \tilde{C}^2 , \quad \tilde{T}_r := \tilde{\eta}_r \tilde{U}^2 / \tilde{\lambda}_r \]

state

\[ q = q(\rho, 1 + \gamma \theta) , \quad q = \rho , \eta , \lambda , c_p \quad \Rightarrow \quad \tilde{p}_r \]

continuity

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) \equiv \partial_t \rho + \nabla_{||} \cdot (\rho \mathbf{u}_{||}) + \epsilon \rho \mathbf{w} \nabla_{||} \cdot \mathbf{e}_n + \partial_n (\rho \mathbf{w}) = 0 \]

momentum

\[ \text{Re} \epsilon^2 \rho (\ddot{x}_{\text{ref}} + 2 \Omega_{\text{ref}} \times \mathbf{u} + D_t \mathbf{u}) + \nabla p = \epsilon^2 \nabla \cdot \Delta \]

\[ \Delta = \eta \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^\text{tr} \right] + (\eta' - \frac{2}{3} \eta)(\nabla \cdot \mathbf{u}) \mathbf{l} \]

energy

\[ \text{Pe} \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t p + \epsilon^2 \left[ \Phi + \nabla \cdot (\lambda \nabla \theta) \right] \]

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momentum

\[ 0 \sim -\nabla_{||} p + \partial_n (\eta \partial_n \mathbf{u}_{||}) , \quad 0 \sim \epsilon^{-1} \partial_n p \]
Navier–Stokes eqs

\[ \tilde{\rho}_r := \tilde{\eta}_r \tilde{U}L/\tilde{C}^2 , \quad \tilde{T}_r := \tilde{\eta}_r \tilde{U}^2/\tilde{\lambda}_r \]

**state**

\[ q = q(\rho, 1 + \gamma \theta) , \quad q = \rho , \eta , \lambda , c_p \quad \Rightarrow \quad \tilde{\rho}_r \]

**continuity**

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) \equiv \partial_t \rho + \nabla_{||} \cdot (\rho \mathbf{u}_{||}) + \epsilon \rho w \nabla_{||} \cdot \mathbf{e}_n + \partial_n (\rho w) = 0 \]

**momentum**

\[ Re \epsilon^2 \rho \left( \ddot{x}_{\text{ref}} + 2\Omega_{\text{ref}} \times \mathbf{u} + D_t \mathbf{u} \right) + \nabla p = \epsilon^2 \nabla \cdot \Delta \]

\[ \Delta = \eta \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^{\text{tr}} \right] + (\eta' - \frac{2}{3} \eta) (\nabla \cdot \mathbf{u}) \mathbf{l} \]

**energy**

\[ Pe \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t p + \epsilon^2 \left[ \Phi + \nabla \cdot (\lambda \nabla \theta) \right] \]

\[ \Phi = \Delta \cdot \nabla \mathbf{u} , \quad \gamma := \tilde{T}_r / \tilde{T}_a \]

\[ \epsilon \ll 1 , \quad \nabla \sim \epsilon^{-1} \mathbf{e}_n \partial_n \]

**momentum**

\[ 0 \sim -\nabla_{||} \rho + \partial_n (\eta \partial_n \mathbf{u}_{||}) , \quad 0 \sim \epsilon^{-1} \partial_n \rho \]

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Navier–Stokes eqs

\[ \tilde{\rho}_r := \tilde{\eta}_r \tilde{U} \tilde{L} / \tilde{C}^2, \quad \tilde{T}_r := \tilde{\eta}_r \tilde{U}^2 / \tilde{\lambda}_r \]

state \[ q = q(\rho, 1 + \gamma \theta), \quad q = \rho, \eta, \lambda, c_p \quad \Rightarrow \quad \tilde{\rho}_r \]

continuity \[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) \equiv \partial_t \rho + \nabla_\parallel \cdot (\rho \mathbf{u}_\parallel) + \epsilon \rho \mathbf{w} \nabla_\parallel \cdot \mathbf{e}_n + \partial_n (\rho \mathbf{w}) = 0 \]

momentum \[ \text{Re} \epsilon^2 \rho (\ddot{x}_{\text{ref}} + 2 \Omega_{\text{ref}} \times \mathbf{u} + D_t \mathbf{u}) + \nabla p = \epsilon^2 \nabla \cdot \Delta \]
\[ \Delta = \eta \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^\text{tr} \right] + (\eta' - \frac{2}{3} \eta) (\nabla \cdot \mathbf{u}) \mathbf{l} \]

energy \[ \text{Pe} \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t p + \epsilon^2 \left[ \Phi + \nabla \cdot (\lambda \nabla \theta) \right] \]
\[ \Phi = \Delta \cdot \nabla \mathbf{u}, \quad \gamma := \tilde{T}_r / \tilde{T}_a \]

\[ \epsilon \ll 1, \quad \nabla \sim \epsilon^{-1} \mathbf{e}_n \partial_n \]

momentum \[ 0 \sim -\nabla_\parallel \rho + \partial_n (\eta \partial_n \mathbf{u}_\parallel), \quad 0 \sim \epsilon^{-1} \partial_n \rho \]
Outline

3 Classical theory
   • First principles
   • Problem formulation
   • Asymptotic theory
Limit process

classical lubrication approximation

thin film \( \epsilon \ll 1 \)

quasi-isothermal \( \gamma \ll 1 \)

inertia neglected \( Re \epsilon^2 \ll 1 \), laminar flow: \( Re \lesssim 10^5 \)

typical values \( \epsilon \lesssim 10^{-3}, \quad Pr_{oil} \approx 70 \ldots 10^3 \quad \Rightarrow \quad Pe \lesssim 10^8, \quad Pe \epsilon^2 \lesssim 10^2 ! \)

\[
\nabla \cdot (\rho \mathbf{u}) \sim \nabla_{||} \cdot (\rho \mathbf{u}_{||}) + \partial_n (\rho \mathbf{w}) + O(\epsilon)
\]

\[
\rho(p, 1 + \gamma \theta) \sim \rho(p, 1) + O(\gamma)
\]

expansions

\[
\nabla_{||} \sim \nabla_{||}^0 + O(\epsilon) \quad \nabla_{||}^0 = \nabla_{||} \quad \text{for} \quad n = 0
\]

\[
[u_{||}, w, p, \rho, \theta, \eta, \ldots](x_{||}, n, t; \epsilon, Re, \gamma, \ldots) \sim [U, W, P, Q, \Theta, \mathcal{N}](x_{||}, n, t) + \ldots
\]

\[
c \sim C(x_{||}, t) + O(\epsilon) \quad \text{journal bearing !}
\]
Limit process

classical lubrication approximation

- thin film \( \epsilon \ll 1 \)
- quasi-isothermal \( \gamma \ll 1 \)
- inertia neglected \( Re \epsilon^2 \ll 1 \), laminar flow: \( Re \lesssim 10^5 \)

- typical values \( \epsilon \lesssim 10^{-3} \), \( Pr_{oil} \approx 70 \ldots 10^3 \) \( \Rightarrow Pe \lesssim 10^8 \), \( Pe \epsilon^2 \lesssim 10^2 \)

\[
\nabla \cdot (\rho \mathbf{u}) \sim \nabla_{||} \cdot (\rho \mathbf{u}_{||}) + \partial_n (\rho w) + O(\epsilon)
\]
\[
\rho(p, 1 + \gamma \theta) \sim \rho(p, 1) + O(\gamma)
\]

expansions

\[
\nabla_{||} \sim \nabla_{||}^0 + O(\epsilon) \quad \nabla_{||}^0 = \nabla_{||} \quad \text{for} \quad n = 0
\]
\[
[u_{||}, w, \rho, \rho, \theta, \eta, \ldots](x_{||}, n, t; \epsilon, Re, \gamma, \ldots) \sim [U, W, P, Q, \Theta, N](x_{||}, n, t) + \ldots
\]
\[
c \sim C(x_{||}, t) + O(\epsilon) \quad \text{journal bearing}
\]
Limit process

classical lubrication approximation

thin film $\epsilon \ll 1$

quasi-isothermal $\gamma \ll 1$

inertia neglected $Re \epsilon^2 \ll 1$, laminar flow: $Re \lesssim 10^5$

typical values $\epsilon \lesssim 10^{-3}$, $Pr_{oil} \approx 70 \ldots 10^3 \Rightarrow Pe \lesssim 10^8$, $Pe \epsilon^2 \lesssim 10^2$

$$\nabla \cdot (\rho \mathbf{u}) \sim \nabla_{||} \cdot (\rho \mathbf{u}_{||}) + \partial_n (\rho \mathbf{w}) + O(\epsilon)$$

$$\rho(p, 1 + \gamma \theta) \sim \rho(p, 1) + O(\gamma)$$

expansions

$$\nabla_{||} \sim \nabla^0_{||} + O(\epsilon) \quad \nabla^0_{||} = \nabla_{||} \quad \text{for} \quad n = 0$$

$$[\mathbf{u}_{||}, w, p, \rho, \theta, \eta, \ldots](x_{||}, n, t; \epsilon, Re, \gamma, \ldots) \sim [\mathbf{U}, W, P, Q, \Theta, N](x_{||}, n, t) + \ldots$$

$$c \sim C(x_{||}, t) + O(\epsilon) \quad \text{journal bearing}!$$

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Limit process

classical lubrication approximation

thin film \( \epsilon \ll 1 \)

quasi-isothermal \( \gamma \ll 1 \)

inertia neglected \( Re \epsilon^2 \ll 1 \), laminar flow: \( Re \lesssim 10^5 \)

typical values \( \epsilon \lesssim 10^{-3}, Pr_{oil} \approx 70 \ldots 10^3 \) \( 100 \ldots 20^\circ C \Rightarrow Pe \lesssim 10^8, Pe \epsilon^2 \lesssim 10^2 \)

\[ \nabla \cdot (\rho u) \sim \nabla_\parallel \cdot (\rho u_\parallel) + \partial_n (\rho w) + O(\epsilon) \]

\[ \rho (p, 1 + \gamma \theta) \sim \rho (p, 1) + O(\gamma) \]

expansions

\[ \nabla_\parallel \sim \nabla_\parallel^0 + O(\epsilon) \quad \nabla_\parallel^0 = \nabla_\parallel \quad \text{for} \quad n = 0 \]

\[ [u_\parallel, w, p, \rho, \theta, \eta, \ldots] (x_\parallel, n, t; \epsilon, Re, \gamma, \ldots) \sim [U, W, P, Q, \Theta, \mathcal{N}] (x_\parallel, n, t) + \cdots \]

\[ c \sim C(x_\parallel, t) + O(\epsilon) \quad \text{journal bearing} \]
Limit process

classical lubrication approximation

thin film $\epsilon \ll 1$

quasi-isothermal $\gamma \ll 1$

inertia neglected $Re \epsilon^2 \ll 1$, laminar flow: $Re \lesssim 10^5$

typical values $\epsilon \lesssim 10^{-3}$, $Pr_{oil} \approx 70 \ldots 10^3$ \Rightarrow $Pe \lesssim 10^8$, $Pe \epsilon^2 \lesssim 10^2$

$\nabla \cdot (\rho \mathbf{u}) \sim \nabla_{||} \cdot (\rho \mathbf{u}_{||}) + \partial_n (\rho w) + O(\epsilon)$

$\rho (p, 1 + \gamma \theta) \sim \rho (p, 1) + O(\gamma)$

expansions

$\nabla_{||} \sim \nabla^0_{||} + O(\epsilon)$ \quad $\nabla^0_{||} = \nabla_{||}$ for $n = 0$

$[\mathbf{u}_{||}, w, p, \rho, \theta, \eta, \ldots](x_{||}, n, t; \epsilon, Re, \gamma, \ldots) \sim [\mathbf{U}, W, P, Q, \Theta, \mathcal{N}](x_{||}, n, t) + \cdots$

c $\sim C(x_{||}, t) + O(\epsilon)$ \quad journal bearing!
Leading-order eqs

state & energy \[ Q = Q(P, 1), \quad Q = Q, \quad N \]

continuity \[ \partial_t Q + \nabla^0 \cdot (Q U) + \partial_N (Q W) = 0 \] (1)

momentum \[ \nabla^0 P = \partial_n (N \partial_n U), \quad \partial_n P = 0 \quad \Rightarrow \quad \partial_n Q = \partial_n N = 0 \] (2)

kinematic BCs

\( n = 0 : \quad U = U_1(x_\parallel, t), \quad W = W_{p,1}(x_\parallel, t) \) (3)

\( n = C(x_\parallel, t) : \quad U = U_2(x_\parallel, t), \quad W = \partial_t C + U_2 \cdot \nabla^0 C + W_{p,2}(x_\parallel, t) \) (4)

(1), (3), (4) \[ \Rightarrow \quad \partial_t (Q C) + \nabla^0 \cdot \left( Q \int_0^C U \, dn \right) + Q(W_{p,2} - W_{p,1}) = 0 \]

(2), (3), (4) \[ U = \frac{\nabla^0 P}{2N(P)} n(n - C) + \frac{n}{C}(U_2 - U_1) \]

Hagen–Poisseuille Couette
Leading-order eqs

state & energy \( Q = Q(P,1) , \quad Q = Q, \mathcal{N} \)

continuity \( \partial_t Q + \nabla_\parallel^0 \cdot (QU) + \partial_N (QW) = 0 \) \hspace{1cm} (1)

momentum \( \nabla_\parallel^0 P = \partial_n (\mathcal{N} \partial_n U) , \quad \partial_n P = 0 \quad \Rightarrow \quad \partial_n Q = \partial_n \mathcal{N} = 0 \) \hspace{1cm} (2)

kinematic BCs

\( n = 0 : \quad U = U_1(x_\parallel, t) , \quad W = W_{p,1}(x_\parallel, t) \) \hspace{1cm} (3)

\( n = C(x_\parallel, t) : \quad U = U_2(x_\parallel, t) , \quad W = \partial_t C + U_2 \cdot \nabla_\parallel^0 C + W_{p,2}(x_\parallel, t) \) \hspace{1cm} (4)

(1), (3), (4) \quad \Rightarrow \quad \partial_t (QC) + \nabla_\parallel^0 \cdot \left( Q \int_0^C U \, dn \right) + Q(W_{p,2} - W_{p,1}) = 0

(2), (3), (4) \quad \Rightarrow \quad U = \frac{\nabla_\parallel^0 P}{2N(P)} n(n - C) + \frac{n}{C} (U_2 - U_1)

\begin{align*}
\text{Hagen–Poiseuille} & \quad \text{Couette} \\
\end{align*}
Leading-order eqs

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kinematic BCs

\( n = 0: \) \[ \mathbf{U} = U_1(x_{||}, t), \quad \mathbf{W} = W_{p,1}(x_{||}, t) \] (3)

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(1), (3), (4) \[ \Rightarrow \quad \partial_t (QC) + \nabla^0 \cdot \left( Q \int_0^C \mathbf{U} \, dn \right) + Q \left( W_{p,2} - W_{p,1} \right) = 0 \]

(2), (3), (4) \[ \Rightarrow \quad \mathbf{U} = \frac{\nabla^0 P}{2N(P)} n(n - C) + \frac{n}{C} (U_2 - U_1) \]
Leading-order eqs

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\[(1), (3), (4) \quad \Rightarrow \quad \partial_t (QC) + \nabla_\parallel^0 \cdot \left( Q \int_0^C U \, dn \right) + Q \left( W_{p,2} - W_{p,1} \right) = 0 \]

\[(2), (3), (4) \quad \Rightarrow \quad U = \frac{\nabla_\parallel^0 P}{2 \mathcal{N}(P)} n(n - C) + \frac{n}{C} (U_2 - U_1) + U_1 \]

Hagen–Poisseuille \quad Couette \quad sliding
Integral mass balance

\[
\int_0^C U \, dn = Q + C \, U_m, \quad Q := -\frac{C^3 \nabla^0_\parallel P}{12 \, \mathcal{N}}, \quad U_m := \frac{U_1 + U_2}{2}
\]

Reynolds eq

O. Reynolds (1886), A. Sommerfeld (1904), L. Prandtl (1937)

\[
\nabla^0_\parallel (-Q \, Q) = (\partial_t + U_m \cdot \nabla^0_\parallel)(Q \, C) + Q \, C \, \nabla^0_\parallel \cdot U_m + Q \, (W_{p,2} - W_{p,1})
\]

\[
Q = Q(P), \quad \mathcal{N} = \mathcal{N}(P)
\]

elliptic 2nd-order PDE for \( P(x_\parallel, t) \) and given \( C(x_\parallel, t), \, U_m(x_\parallel, t) \)

kinematic wave operator \( \partial_t + U_m \cdot \nabla^0_\parallel \) most relevant for gas bearings

linear for incompressible lubricant with constant properties \( (Q \equiv \mathcal{N} \equiv 1) \)

in general to be solved numerically
Integral mass balance

\[ \int_{0}^{C} \mathbf{U} \, dn = Q + C \, \mathbf{U}_m, \quad Q := -\frac{C^3 \nabla^0 P}{12 \mathcal{N}}, \quad \mathbf{U}_m := \frac{\mathbf{U}_1 + \mathbf{U}_2}{2} \]

Reynolds eq

O. Reynolds (1886), A. Sommerfeld (1904), L. Prandtl (1937)

\[ \nabla^0_\parallel (\nabla^0_\parallel - Q \mathbf{Q}) = \left( \partial_t \right) + \left( \mathbf{U}_m \cdot \nabla^0_\parallel \right) (Q \mathbf{C}) + QC \nabla^0_\parallel \cdot \mathbf{U}_m + Q \left( W_{p,2} - W_{p,1} \right) \]

permeability

\[ Q = Q(P), \quad \mathcal{N} = \mathcal{N}(P) \]

elliptic 2nd-order PDE for \( P(\mathbf{x}_\parallel, t) \) and given \( C(\mathbf{x}_\parallel, t), \mathbf{U}_m(\mathbf{x}_\parallel, t) \)

kinematic wave operator \( \partial_t + \mathbf{U}_m \cdot \nabla^0_\parallel \) most relevant for gas bearings

linear for incompressible lubricant with constant properties \( (Q \equiv \mathcal{N} \equiv 1) \)

in general to be solved numerically
Reynolds eq – some important properties

\[ Q = -\frac{C^3 \nabla^0 P}{12 N}, \quad U_m = \frac{U_1 + U_2}{2} \]

\[ \nabla^0 \cdot (-QQ) = (\partial_t + U_m \cdot \nabla^0)(QC) + QC \nabla^0 \cdot U_m + Q(W_{p,2} - W_{p,1}) \]

\[ Q = Q(P), \quad N = N(P) \]

rigid contacts, no Navier slip

\[ \nabla^0 \cdot [U_1, U_2, U_m] = 0 \]

Galilean transformation

\[ [x, t] = [x', S(t'), t'] \]

\[ [\nabla^0, \partial_t] = [\nabla^0, \partial_t] - \dot{\mathbf{S}} \nabla^0 \]

\[ [C, P, U_{1,2}](x, t) = [C', P', U'_{1,2}](x', t'), \quad [Q, N](P) = [Q', N'](P') \]

\[ [U_1, U_2, U_m] \rightarrow [U'_1, U'_2, U'_m] - \dot{\mathbf{S}} \quad \text{invariance against sliding motion } \mathbf{S} \]
Reynolds eq – some important properties

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\[ [C, P, U_{1,2}](x_\parallel, t) = [C', P', U'_{1,2}](x'_\parallel, t'), \quad [Q, \mathcal{N}](P) = [Q', \mathcal{N}'](P') \]

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\[ [C, P, U_{1,2}](x_||, t) = [C', P', U'_{1,2}](x'_||, t'), \quad [Q, \mathcal{N}](P) = [Q', \mathcal{N}'](P') \]

\[ [U_1, U_2, U_m] \mapsto [U'_1, U'_2, U'_m] - \dot{S} \quad \text{invariance against sliding motion } S \]
Validation of tribo-systems

typically find

- \( P(x_{||}, t), \ x_{||} \in \Omega \) subject to \( P(\partial \Omega, t) = P_a \)

- load-bearing capacity \( F(t) = \int_{\Omega} P e_n \, d\Omega \)

clearance \( C(x_{||}, t) \) is

- prescribed

- found from fluid-structure interaction machinery (e.g., shaft) dynamics \( \Rightarrow F = F(t) \) and \( t(C, C) \)

EHL \( \Rightarrow P = P(C) \)
Validation of tribo-systems

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  machinery (e.g. shaft) dynamics \( \Rightarrow F = F(\partial_{\mathbf{n}} C, \partial_t C, C) \)
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Validation of tribo-systems

typically find

\begin{itemize}
  \item \( P(x_{||}, t) \), \( x_{||} \in \Omega \) subject to \( P(\partial \Omega, t) = P_a \)
  \item load-bearing capacity \( F(t) = \int_{\Omega} P e_n \, d\Omega \)
\end{itemize}

clearance \( C(x_{||}, t) \) is

\begin{itemize}
  \item prescribed
  \item found from fluid–structure interaction
  \item machinery (e.g. shaft) dynamics \( \Rightarrow F = F(\partial_{tt} C, \partial_t C, C) \)
  \item EHL \( \Rightarrow P = P(C) \)
\end{itemize}
Classical application: journal bearing

reference quantities

\[ \tilde{U}_m = \tilde{\omega} \tilde{R}_i , \quad \tilde{p}_r = \tilde{\eta}_r \tilde{\omega} \tilde{R}_i^2 / \tilde{C}^2 \]

geometrical parameters

\[ \epsilon = \tilde{C} / \tilde{R}_i \ll 1 , \quad \text{eccentricity} \quad \epsilon = \tilde{e} / (\tilde{R}_a - \tilde{R}_i) \]

non-dimensional quantities

\[ C = 1 + \epsilon \cos \theta + O(\epsilon^2) , \quad U_m (= \mathcal{N} = \mathcal{Q}) = 1 \]
Further outlook

include

- EHL
- inertia ($Re\epsilon^2 \sim 1$, start-up, high-speed rotors, rapid load cycles)
- turbulence
- film rupture & cavitation (surface tension)
- effects acting on micro-scale $\ll \epsilon$ (surface roughness, mixed friction)

rational method: perturbation techniques

- multiple scales, matched asymptotic expansions
- numerical solution of reduced problem (simulation tools)
Further outlook

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Thank you for your attention!