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Dephasing in two decoupled one-dimensional Bose-Einstein condensates and the subexponential decay of the interwell coherence

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Abstract. We provide a simple physical picture of the loss of coherence between two coherently split one-dimensional Bose-Einstein condensates. The source of the dephasing is identified with nonlinear corrections to the elementary excitation energies in either of the two independent condensates. We retrieve the result by Burkov, Lukin and Demler [Phys. Rev. Lett. **98**, 200404 (2007)] on the subexponential decay of the coherence $\propto \exp[-(t/t_0)^{2/3}]$ for the large time t, however, the scaling of t_0 differs.

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Effectively one-dimensional (1D) systems of ultracold atoms are a model systems to study the fundamental processes of the coherent dynamics and decoherence in interacting many body systems. In addition in the limit of zero temperature they are a primary example of the exactly integrable Lieb-Liniger model [1,2].

Recently experimental progress on both, optical lattices [3] and atom chips [4,5], allow to confine ultra cold atoms in strongly elongated traps with $\omega_r \gg \omega_z$ (ω_r, ω_z being the frequencies of the radial and longitudinal confinement, respectively). These traps are an ideal system for studying 1D physics as long as both the temperature T and chemical potential μ are small compared to the energy scale given by the transverse confinement: $\mu < \hbar\omega_r$, $k_BT < \hbar\omega_r$. Optical lattices enable study of global properties of ensembles of 1D systems down to very small atom numbers and into the strongly correlated regime, atom chips allow to study the properties and dynamics of single 1D systems.

Strong inhibition of thermalization, a signature of integrability, was observed with bosons deep in the 1D regime [6], and interference experiments on atom chips with pairs of weakly interacting Bose gases, easily fulfilling the above conditions for one dimensionality, allowed to study the dynamics of decoherence [7] and the interplay between thermal and quantum noise [8].

The special interest in the decoherence in such effectively one-dimensional ultracold atomic systems is rooted in the fact that they display decoherence and thermalization despite being at the first glance a prime example of the exactly integrable Lieb-Liniger model [1,2]. In a recent paper [9], we have shown that even if the temperature and chemical potential are well below the energy of the radial excitation ($\mu < \hbar\omega_r$, $k_BT < \hbar\omega_r$) radial modes can be excited virtually. These virtually excited radial modes give rise to effective three-body velocity-changing collisions which lead to thermalization and the break down of integrability. The typical thermalization scale for typical 1D atom chip experiments [7,8,10] is in the order of 100 ms even when thermalization due to two body collisions is completely suppressed. However the dephasing dynamics observed in such coherently split ultracold 1D atomic clouds [7,11] is even one order of magnitude faster.

In addition, the experiment on the time evolution of interference between two coherently split one-dimensional (1D) atomic Bose-Einstein condensates (BECs) [7] revealed a surprising sub-exponential decay of the inter-well coherence $\langle \hat{\psi}_R^\dagger \hat{\psi}_L \rangle$.

$$\langle \hat{\psi}_R^\dagger(x,t)\hat{\psi}_L(x,t)\rangle \propto \exp[-(t/t_0)^\alpha]. \qquad (1)$$

The measured decay exponent $\alpha \approx 2/3$ is in agreement with the theoretical calculations of Burkov, Lukin and Demler [12] which predicts $\alpha = 2/3$. In their theoretical approach the inter-well coherence decay is treated in the terms of the heat flow between symmetric, $\hat{\psi}_+ = (\hat{\psi}_L + \hat{\psi}_R)/\sqrt{2}$, and antisymmetric, $\hat{\psi}_- = (\hat{\psi}_L - \hat{\psi}_R)/\sqrt{2}$, modes. The two individual (fully split) condensates are designated as right (R) and left (L), $\hat{\psi}_{R,L}(x,t)$ being the

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corresponding atomic field annihilation operator in the coordinate representation.

In the present work we choose a different, more intuitive way of describing the system and consider *dephasing* of two independent integrable systems with intercorrelated initial conditions. Similarly to Burkov, Lukin and Demler [12], we consider the weak interaction limit, which, together with the finite size of the system, allows for a finite condensed fraction, in contrast to the Tonks limit or the case of an infinitely long quasicondensate. In what follows we use the system of units where Planck's and Boltzmann's constants are set to 1.

After the splitting, the system consists of two independent BECs, each being described by the Hamiltonian

$$\hat{\mathcal{H}} = \int dx \left[\frac{1}{2m}\left(\frac{\partial}{\partial x}\hat{\psi}_j^\dagger\right)\left(\frac{\partial}{\partial x}\hat{\psi}_j\right) + \frac{g_{1D}}{2}\hat{\psi}_j^\dagger\hat{\psi}_j^\dagger\hat{\psi}_j\hat{\psi}_j\right],\quad j = L, R, \tag{2}$$

where m is the atomic mass, g_{1D} is the 1D coupling constant. The main contribution to the coherence reduction stems from phase fluctuation, so that

$$\langle\hat{\psi}_R^\dagger(x,t)\hat{\psi}_L(x,t)\rangle \approx \bar{n}\exp\left(-\langle\hat{\phi}_-^2\rangle\right), \tag{3}$$

where the phase operators $\hat{\phi}_\pm = (\hat{\phi}_L \pm \hat{\phi}_R)/\sqrt{2}$ and their conjugate density fluctuations operators $\delta\hat{n}_\pm = (\delta\hat{n}_L \pm \delta\hat{n}_R)/\sqrt{2}$ are defined via $\hat{\psi}_j = (\bar{n} + \delta\hat{n}_j)^{1/2}\exp(i\hat{\phi}_j)$, $j = L, R$, and $\bar{n}$ being the average 1D BEC density. In what follows, we consider only classical (thermal-like) fluctuations and therefore omit operator notations, writing simply ϕ_j etc.

The local fluctuations can be expanded in plane waves,

$$\phi_\ell(x,t) = \frac{1}{\sqrt{\mathcal{L}}}\sum_k \tilde{\phi}_k^\ell(t)\exp(ikx),$$
$$\delta n_\ell(x,t) = \frac{1}{\sqrt{\mathcal{L}}}\sum_k \tilde{n}_k^\ell(t)\exp(ikx), \tag{4}$$

$\mathcal{L}$ being the quantization length, $\ell = +, -, L, R$. The linearization of equation (2) yields Bogoliubov spectrum $\omega_k = \sqrt{k^2/(2m)[k^2/(2m) + 2mc^2]}$, c being the speed of sound. In what follows we consider BEC at temperatures below the chemical potential, so that we assume phonon spectrum of excitations of the uniform 1D Bose gas at rest, $\omega_k = c|k|$.

The key idea of our treatment is to recall that the local density and velocity fields in BEC fluctuate, thus giving rise to the *random non-linear* corrections $\delta\omega_k^j$ to the phonon frequency $\Omega_k^j = \omega_k + \delta\omega_k^j$, $j = L, R$. Phonons propagating in the left and right condensates interact with different fluctuations, and this is the source of dephasing.

The phonon energy depends on the BEC local density $n_j = \bar{n} + \delta n_j$ via the speed of sound, which is proportional to the sqare root of density. Also the fluctuating local velocity $V_j = m^{-1}\partial\phi_j/\partial x$ in the BEC contributes to the random energy correction as the advective term, so that the total correction to the average phonon energy $c(\bar{n})|k|$ is

$$\delta\omega_k^j = \left[\frac{d}{d\bar{n}}c(\bar{n})\delta n_j + V_j\right]|k|$$
$$= c|k|\left(\frac{\delta n_j}{2\bar{n}} + \frac{V_j}{c}\right). \tag{5}$$

Fluctuations of the density and local velocity differ for the right and left BECs, because the splitting process is never completely adiabatic (the degree of nonadiabaticity has been recently quantified by Polkovnikov and Gritsev [13]), and initial quantum (zero-point) fluctuations of ϕ^- and n^- can be amplified to the values, comparable at low momenta to the initial thermal (classical) fluctuations of ϕ^+ and n^+. Still quantum fluctuations on their own can initiate the decay of the interwell coherence [14] (on a different time scale). Since the density and local velocity fluctuations in the right and left BECs are different, the random energy shift is also different for the excitations propagating in these two BECs:

$$\delta\omega_k^{L,R} = \delta\omega_k^+ \pm \delta\omega_k^-. \tag{6}$$

By definition, the upper sign (+) in the right-hand side of equation (6) corresponds to the left BEC, the lower sign (-) corresponds to the right BEC. Since $|\delta\omega_k^{L,R}| \ll c|k|$ and $|\frac{\partial}{\partial t}\delta\omega_k^{L,R}| \ll c^2k^2$, we can approximately describe the evolution of the phases in each of the two independent BECs on the time interval from t to $t+\Delta t$ by $\tilde{\phi}_k^j(t+\Delta t) = \tilde{\phi}_k^j(t)\cos\int_t^{t+\Delta t}dt'\,\Omega_k^j - [m\omega_k/(\bar{n}k^2)]\tilde{n}_k^j(t)\sin\int_t^{t+\Delta t}dt'\,\Omega_k^j$. Transformation from the basis of the $j = L, R$ modes to that of the symmetric and antisymmetric modes results in the following expression:

$$\begin{aligned}\tilde{\phi}_k^-(t+\Delta t) = {}& \Big[\tilde{\phi}_k^-(t)\cos\Phi_k(t,\,\Delta t) - \frac{m\omega_k}{\bar{n}k^2}\tilde{n}_k^-(t)\\ &\times\sin\Phi_k(t,\,\Delta t)\Big]\cos J_k(t,\,\Delta t) - \Big[\tilde{\phi}_k^+(t)\\ &\times\sin\Phi_k(t,\,\Delta t) + \frac{m\omega_k}{\bar{n}k^2}\tilde{n}_k^+(t)\cos\Phi_k(t,\,\Delta t)\Big]\\ &\times\sin J_k(t,\,\Delta t),\end{aligned} \tag{7}$$

where $\Phi_k(t,\,\Delta t) = c|k|\Delta t + \int_t^{t+\Delta t}dt'\,\delta\tilde{\omega}_k^+(t')$ and

$$J_k(t,\,\Delta t) = \int_t^{t+\Delta t}dt'\,\delta\omega_k^-(t'). \tag{8}$$

We assume that fluctuations in each k-mode at the time t are not correlated with the values of the random frequency corrections at later times. This is reasonable, because $\delta\omega_k^\pm$ is determined, as we shall see below, by a superposition of a large number of modes and, hence, does not bear any significant correlation with a particular k-mode. This enables us to decouple correlations in equation (7). Then we take into account the statistical properties $\langle\tilde{\phi}_k^\ell\tilde{n}_k^\ell\rangle = 0$ and $[m\omega_k/(\bar{n}k^2)]^2\langle(\tilde{n}_k^\ell)^2\rangle = \langle(\tilde{\phi}_k^\ell)^2\rangle$, $\ell = +, -$ (which mean that energy in each k-mode is equally distributed between the phase and density fluctuations, although initially there is no equilibrium between modes *with different momenta* or between the symmetric and antisymmetric modes with the same k), and obtain a finite-difference equation

$$\langle[\tilde{\phi}_k^-(t+\Delta t)]^2\rangle - \langle[\tilde{\phi}_k^-(t)]^2\rangle = \Big\{\langle[\tilde{\phi}_k^+(t)]^2\rangle - \langle[\tilde{\phi}_k^-(t)]^2\rangle\Big\}\left\langle\sin^2 J_k(t,\,\Delta t)\right\rangle. \quad (9)$$

If the time difference Δt is taken to be shorter than the typical time of the phase evolution (decoherence) in the system, equation (9) reduces to the differential equation

$$\frac{\partial}{\partial t}\langle(\tilde{\phi}_k^-)^2\rangle = \Gamma_k(t)\left[\langle(\tilde{\phi}_k^+)^2\rangle - \langle(\tilde{\phi}_k^-)^2\rangle\right], \quad (10)$$

where $\Gamma_k(t) = \lim_{\Delta t\to 0}\left\langle J_k^2(t,\,\Delta t)\right\rangle/\Delta t$. Moreover, since the system under consideration is closed and consists of two independent (L and R) integrable subsystems, the sum $\langle(\tilde{\phi}_k^L)^2\rangle + \langle(\tilde{\phi}_k^R)^2 = \langle(\tilde{\phi}_k^+)^2\rangle + \langle(\tilde{\phi}_k^-)^2\rangle$ is time-independent, and we obtain

$$\frac{\partial}{\partial t}\left[\langle(\tilde{\phi}_k^-)^2\rangle - \langle(\tilde{\phi}_k^+)^2\rangle\right] = -2\Gamma_k(t)\left[\langle(\tilde{\phi}_k^-)^2\rangle - \langle(\tilde{\phi}_k^+)^2\rangle\right]. \quad (11)$$

Since the correlation function for the random interwell frequency shift can be expressed as

$$\langle\delta\omega_k^-(t')\delta\omega_k^-(t'')\rangle = (\delta\omega_k^A)^2 f_k(t''-t'), \quad (12)$$

where $\delta\omega_k^A$ characterizes the amplitude of the frequency shift fluctuations for the mode with given k, and $f_k(\tau)$ is the correlation function in dimensionless form, equal to 1 at $\tau = 0$ and rapidly approaching 0 if its argument exceeds certain correlation time τ_k^c. Then equation (11) yields

$$\langle(\tilde{\phi}_k^-)^2\rangle - \langle(\tilde{\phi}_k^+)^2\rangle \propto \exp\left[-\Gamma_k(0)t^2/\tau_k^c\right], \quad t \ll \tau_k^c, \quad (13)$$

$$\langle(\tilde{\phi}_k^-)^2\rangle - \langle(\tilde{\phi}_k^+)^2\rangle \propto \exp\left[-2\int_0^t dt'\,\Gamma_k(t')\right], \quad t \gtrsim \tau_k^c, \quad (14)$$

where the dephasing rate is

$$\Gamma_k(t) \approx (\delta\omega_k^A)^2\tau_k^c. \quad (15)$$

Note that the validity of the assumptions resulting in equations (9–11) does not depend on the ratio between the time t, elapsed since the end of the BEC splitting process and the onset of independent evolution of the two separate BECs, and characteristic time scales such as τ_k^c, $1/T$ or $1/(mc^2)$.

Assuming that the condensate splitting is close to adiabatic and, hence,

$$\langle[\tilde{\phi}_k^-(0)]^2\rangle \ll \langle[\tilde{\phi}_k^+(0)]^2\rangle \quad (16)$$

and neglecting the change (approximately by a factor of 2) of $\langle(\tilde{\phi}_k^+)^2\rangle$ in the course of the system evolution (similar approximations are assumed in Ref. [12]), thus making a qualitatively good approximation $\Gamma_k \approx$ const., we obtain in the limit $t \gtrsim \tau_k^c$

$$\langle(\tilde{\phi}_k^-)^2\rangle = \langle(\tilde{\phi}_k^+)^2\rangle\,[1 - \exp(-2\Gamma_k t)]. \quad (17)$$

The remaining issue is to determine $\delta\omega_k^A$ and τ_k^c. To estimate $\delta\omega_k^A$ at least roughly, we apply the following method. Instead of plain waves, we consider a wave packet centered at the momentum k and having the momentum uncertainty $\Delta k \sim k$. Such a width allows us to choose the shape of the wave packet close to the minimum-uncertainty wave packet, that allows us to localize it on a length of the order of $2\pi/k$. Then we can identify the random energy shift experienced by such a wave packet with that of the phonon with momentum k.

In such a context, obviously, only the density and velocity fluctuations at wavelengths longer than $2\pi/k$ contribute to $\delta\omega_k^-$. Fluctuations at shorter wavelengths are effectively averaged out and cause no influence to the dynamics of a wave packet of a spatial extension $\sim 2\pi/k$. Taking into account that fluctuations at different momenta are not correlated and replacing sum over discrete states by integration over continuous spectrum, we obtain

$$\delta\omega_k^{A\,2} = c^2k^2\int_{-k}^{k}\frac{dk'}{2\pi}\left[\frac{\langle(\tilde{n}_{k'}^-)^2\rangle}{8\bar{n}^2} + \frac{k^2\langle(\tilde{\phi}_{k'}^-)^2\rangle}{2m^2c^2}\right]. \quad (18)$$

The remaining issue is to estimate the correlation time τ_k^c. Say, the wavepacket propagates along x in the positive direction. Half of the surrounding fluctuations propagates in the opposite direction, bringing about a short correlation time scale $\sim 1/(ck)$. However, half of the fluctuations co-propagate with the wave packet, in the first approximation at the same velocity. If there were no dephasing (with respect to each other) of fluctuations at different momenta, the corresponding correlation time would be infinite. However, longer-wavelength correlations dephase as well, at a rate similar to that of the wave packet under consideration. Averaging the dephasing rates over the ensemble of fluctuations restricted to $|k'| < |k|$, we obtain the correlation time of the fluctuations affecting the dynamics of phonons with momentum k to be

$$\tau_k^c \approx \Gamma_k^{-1}. \quad (19)$$

The correlation time cannot be longer than given by equation (19), because otherwise equation (13) holds instead of equation (14), and the dephasing is slowed down significantly. Equations (15, 19) result in

$$\Gamma_k \approx \delta\omega_k^A. \quad (20)$$

It is unlikely that deeper insight to the short-time dynamics of nonlierly interacting modes of a 1D BEC can enhance the dephasing rate further compared to equation (20), because short-time dephasing given by equation (13) is of the form $\exp(-\text{const.}\,t^2) \approx 1 - \text{const.}\,t^2$, and fast perturbation can cause only slowdown of the evolution (Quantum Zeno effect), but not speed up (anti-Zeno effect) [15]. Equations (17, 20) yield finally the following dephasing dynamics:

$$\langle(\tilde{\phi}_k^-)^2\rangle = \langle(\tilde{\phi}_k^+)^2\rangle\left[1-\exp(-\delta\omega_k^A t)\right] \qquad (21)$$

(the factor of 2 in the decrement is omitted, because of inexact nature of the estimations involved).

In the asymptotically long time limit we may estimate fluctuation amplitudes from the 1D Bogoliubov treatment for the phononic ensemble at temperature T as illustrated in [16] (the same assumption is taken in Ref. [12]):

$$\frac{1}{\bar{n}^2}\langle(\tilde{n}_k^-)^2\rangle = \left(\frac{k}{mc}\right)^2\langle(\tilde{\phi}_k^-)^2\rangle \approx \frac{T}{\mu\bar{n}}, \qquad (22)$$

where $\mu = mc^2$ is the zero-temperature approximation for the chemical potenial. Substituting equation (22) into equation (18), we obtain

$$\delta\omega_k^A \approx \sqrt{\frac{T}{2\pi\mu\bar{n}}}c|k|^{3/2}. \qquad (23)$$

Finally, equation (21) takes the form

$$\langle(\tilde{\phi}_k^-)^2\rangle = \frac{mT}{\bar{n}k^2}\left[1-\exp\left(-0.4\sqrt{\frac{T}{\mu\bar{n}}}c|k|^{3/2}t\right)\right], \qquad (24)$$

which is to a certain extent similar to equation (14) of reference [12]: in both cases the dephasing rate is proportional to $|k|^{3/2}$, but there is also an important difference: the relaxation rate predicted in reference [12] is by the factor of $\bar{n}/(mc)$ faster than our estimate. Equation (24) has a generic form

$$\langle(\tilde{\phi}_k^-)^2\rangle = \frac{b_1}{k^2}\mathcal{F}(b_2|k|^{3/2}t), \qquad (25)$$

where b_1, b_2 are certain constants and $\mathcal{F}(\Theta)$ is a function that is finite at $\Theta \to \infty$ and decreases faster than $\Theta^{2/3}$ if $\Theta \to 0$, to provide the convergence of the integral

$$\langle\phi_-^2\rangle = \int_{-k_{max}}^{k_{max}}\frac{dk}{2\pi}\langle(\tilde{\phi}_k^-)^2\rangle \qquad (26)$$

that appears in the coherence factor equation (3). The cut-off momentum k_{max} depends on the temperature and chemical potential and is of the order of mc if $T \sim \mu$. For $t \gg T^{-1}$, μ^{-1} we can substitute k_{max} by ∞. Changing the integration variable from momentum k to the dimensionless time $\Theta = b_2|k|^{3/2}t$ we obtain

$$\langle\phi_-^2\rangle = (b_2t)^{2/3}\frac{2b_1}{3\pi}\int_0^\infty d\Theta\,\Theta^{-5/3}\mathcal{F}(\Theta) \equiv (t/t_0)^{2/3}, \qquad (27)$$

that provides the subexponential decay equation (1) with $\alpha = 2/3$. Comparing equations (24) and (25) we find the scaling time t_0 of the subexponential decay equation (1):

$$t_0 \approx 3.2\frac{\mu}{T^2}\left(\frac{\bar{n}}{mc}\right)^2 = 3.2\frac{\bar{n}^2}{mT^2}. \qquad (28)$$

Note that the time limit $t \gg T^{-1}$, μ^{-1} is not sufficient for equation (27) to hold, since the latter is derived under assumption of significant thermalization of the antisymmetric mode that should happen at $t \sim t_0$. Surprisingly, the inter-well coherence decay experimentally measured in reference [7] is well described by equation (1) also on its initial stage. Our estimation of t_0 is by the factor of $0.24\,\bar{n}/(mc) > 1$ longer than that of reference [12] and the scaling with experimental parameters are also different. Recalling that $\mu = g_{1D}\bar{n}$ with g_{1D} proportional to the radial trapping frequency $\nu_\perp$ of the atomic waveguide and regarding it, the 1D number density and experimentally obtained t_0 as input parameters, we obtain $T \propto \bar{n}/t_0^{1/2}$, whereas reference [12] gives $T \sim \bar{n}^{3/4}\nu_\perp^{1/4}/t_0^{1/2}$.

Let us now compare our results to the findings of Burkov, Lukin and Demler [12] in more detail. In the theoretical approach of reference [12], the inter-well coherence decay is treated in the terms of the heat flow between symmetric, $\hat{\psi}_+ = (\hat{\psi}_L + \hat{\psi}_R)/\sqrt{2}$, and antisymmetric, $\hat{\psi}_- = (\hat{\psi}_L - \hat{\psi}_R)/\sqrt{2}$, modes. This point of view is counterintuitive for two completely split one-dimensional BECs which are two independent systems [1,2] close to exact integrability. The thermalization times calculated, even when including the virtual 3-body collisions, whose role in breaking down the integrability in quasi-1D BECs has been recently studied theoretically [9], are much longer then the observed decoherence. The termalization time scale should be the one at which the heat flow arguments should not apply.

The results of reference [12] seem to imply some unphysical consequences: if we consider $T \sim \mu$ and estimate the dephasing rate for phonons with the energy close to the chemical potential reference [12] gives a rate in the order of $\sqrt{\bar{n}/(mc)}\,\mu \gg \mu$ (since the experimentally accessible 1D BECs are characterized by $\bar{n}/(mc) \approx 30$), which is quite counterintuitive: the phonons with $k \sim mc$ become overdamped, and their dephasing rate exceeds any frequency scale available in the system with $T \sim \mu$. Under the same condition our theory predicts dephasing rate $\sim\mu/\sqrt{\bar{n}/(mc)} \ll \mu$.

The time scale t_0 as suggested by our estimates implies the reconsideration of the temperature estimations for the experimental data of reference [7]. Our model suggests that the actual final temperatures were higher by a factor of order 2 than it was concluded from [12]. This may easily happen, since the mechanisms of the heating of a BEC during the splitting (which is adaibatic only partially, taking into account its time scale ~10 ms) are not yet explored and understood.

This difference between the prediction of the timescales and the scaling with experimental parameters between our model and the calculations by Burkov, Lukin and Demler [12] demands both new more detailed experiments over a wider parameter range and a comprehensive numerical calculation of the splitting and coherence dynamics. New ways to measure temperature using the statistics of interference patterns [8], and more refined RF potentials for atom manipulation [17] will greatly extend the capability for experimental investigations.

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