Properties of Zero-Free Spectral Matrices

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***Abstract*—In factor analysis, which is used for example in econometrics, by definition the number of latent variables has to exceed the number of factor variables. The associated transfer function matrix has more rows than columns, and when the factor variables are independent zero mean white noise sequences and the transfer function matrix is stable, then the output spectrum is singular. While a related paper focusses on the properties of such a nonsquare transfer function matrix, in this paper, we explore a number of properties of the spectral matrix and associated covariance sequence. In particular, a zero free minimum degree spectral factor can be computed with a finite number of rational calculations from the spectrum (in contrast to typical spectral factor calculations), assuming the spectrum fulfills a generic condition. Application of the result to Kalman filtering is indicated, and presentation of the results is also achieved using finite block Toeplitz matrices with entries obtained from the covariance of the latent variable vector.**

***Index Terms*—Kalman filtering, spectral factorization, stochastic systems, system identification.**

I. Introduction

A. The Problem of Interest

THE purpose of this paper is to present a number of new results concerning spectral matrices, their spectral factorization, and associated Kalman filtering problems, when these spectral matrices arise as the output spectrum of a finite-dimensional, linear, time-invariant, stable system excited by white noise *with more outputs than inputs*. Such spectra arise in factor analysis, with the number of latent variables exceeding the number of factor variables. In particular, in recent times, in econometrics, so-called generalized dynamic factor models [1], [2] have been developed to deal with such problems and the system theory of such transfer functions and their spectra may be useful in this field. In the applications domain, questions arise both of modelling and prediction; the work reported here will underpin consideration of both types of question.

The paper does not however focus on the applications problem, but rather the underlying system theoretic issues; the authors became aware of these issues when directly addressing the applications problem. For an introduction to the application of these ideas in generalized dynamic factor modelling, see [3].

By way of a one paragraph summary, we note that a generalized dynamic factor model envisages a finite number of uncorrelated stationary white noise processes (termed the factors), q say, entering a stable linear time-invariant system of state dimension n say, to generate a very large number of scalar outputs, N say, termed the latent variables. The latent variables are corrupted by zero mean stationary noise; when the scalar noise processes are independent, the model is termed a linear dynamic factor model, but when there can be a limited amount of dependence, it is termed a generalized linear dynamic factor model. Unusually for signal processing or filtering problems, one now contemplates N tending to infinity (or becoming very large in relation to q) in the sense that more and more noisy measurements of the common state are obtained, as is common in econometric time-series. With several further reasonable structural assumptions, e.g. to ensure that no factor affects only a bounded number of output variables as N grows, it is possible to determine the output covariance or power spectrum uncorrupted by the noise signals, if not perfectly, at least to a high degree of accuracy. It is such a spectrum which we take as the starting point for this paper; its rank evidently will be bounded by q, the number of factors, even though it is $N \times N$.

While this paper is self-contained, we do quote several results from a closely related paper [4], treating properties of tall rational transfer function matrices, i.e. transfer function matrices of linear, finite-dimensional systems with more outputs than inputs.

The most obvious property of a spectral matrix of the output of a system with fewer inputs than outputs is that it is necessarily singular. However, a less obvious but certainly useful property is that for generic values of the parameters in the underlying linear system, that underlying linear system has no zeros, which can be shown to imply a zero-free property for the spectral matrix also.

The zero-free property has been explored in detail in [4] for transfer function matrices, and key conclusions of that examination are summarized in this paper, particularly an extension to a result of Moylan [5] on the invertibility of systems with no zeros. This invertibility property is exploited in this paper in studying the problem of *spectral factorization* [6]–[8]; this is the task of passing from a spectral matrix to a linear system which when driven by white noise provides an output process with spectral matrix equal to the prescribed spectral matrix. Typically, interest centers around finding a canonical spectral factor, namely a stable, minimum phase transfer function. We show that for the class of spectral matrices of interest in this paper, such a canonical factor can be computed from the spectral matrix *using a finite number of rational calculations*. Moreover, among the spectral factors of least McMillan degree, or equivalently, least

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state dimension, all spectral factors are canonical, differing from one another in a trivial way. We argue that, as a consequence, a Kalman filter will provide error-free state estimates after a finite number of time instants.

The structure of this paper is as follows. Section II is largely devoted to review: we recall the notion of transfer function zeros and spectral factorization. In the process however, properties of zero-free tall rational transfer function matrices are also recalled, and a characterization of the zero-free property in terms of the associated spectral matrix is also provided. In Section III, we show that a Kalman filtering state-estimation problem linked to the spectral matrix has a solution providing an error-free state estimate after a finite interval, and in Section IV, we present an algorithm using a finite number of rational calculations to obtain a zero-free (and minimum phase) stable spectral factor of a prescribed zero-free spectral matrix. The calculations in Sections III and IV are all executed using state-variable descriptions of the spectral matrix and the associated spectral factor. In Sections V and VI, we adopt a different description of the spectral matrix, by working with a finite number of lagged covariances, arranged in a finite block Toeplitz matrix. We exhibit an algorithm for checking the zero-free property and for obtaining a zero-free spectral factor through manipulation of this block Toeplitz matrix. Section VII contains concluding remarks.

II. Spectral Factorization and System Zeros

In this section, we first review results from the closely related paper [4] dealing with zero-free transfer functions. Following this, we review the standard result on spectral factorization of a rational spectral matrix, and then we make connections between the zeros of the spectrum and of the spectral factor, with attention also being given to the zero-free case.

A. Transfer Function Zeros

Suppose that $W(z)$ is a $p \times m$ rational transfer function matrix with minimal realization $\{A, B, C, D\}$ of dimension n. The associated state variable equations are

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k. \end{aligned} \tag{1}$$

Zeros of $W(z)$ or equivalently any minimal realization of $W(z)$ are defined as follows.

Definition 1: The finite zeros of the transfer function matrix $W(z)$ with minimal realization $\{A, B, C, D\}$ are defined to be the finite values of z for which the rank of the following matrix falls below its normal rank, i.e. the rank of the matrix for generic values of z:

$$M(z) = \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix}. \tag{2}$$

Further, $W(z)$ is said to have an infinite zero precisely when $n + \mathrm{rank} D$ is less than the normal rank of M, or equivalently when rank $D < \text{normal rank } W$.

The above definition of zeros is consistent with treatments in the literature, see e.g. [9], [10]. In particular, for finite zeros, with $W(z)$ expressed as a coprime polynomial matrix fraction, $A^{-1}(z)B(z)$ or $B(z)A^{-1}(z)$ say, the zeros are those values of z causing the numerator matrix $B(z)$ to have rank less than its normal rank, and in a Smith-McMillan decomposition of $W(z)$, the zeros of $W(z)$ are given by the zeros of the numerator polynomials of the diagonal matrix of the decomposition. To study infinite zeros, one can define $\tilde{W}(q) = W((aq+b)/(q+c))$ for some real a, b, c such that a is not a pole of $W(z)$. Then $W(z)$ has an infinite zero if and only if $\tilde{W}(q)$ has a zero at $q = -c$.

B. Zero-Free Transfer Function Matrices

Rational transfer function matrices which are zero-free are of interest in this paper. They can occur frequently; the following result is from [4].

Proposition 1: Consider a rational transfer function matrix $W(z)$ with minimal realization $\{A, B, C, D\}$ of dimension n in which B, C have m columns and p rows, respectively, with $p > m$. Let $M(z)$ be as in (2). If the entries of A, B, C, D assume generic values, then $W(z)$ has no finite or infinite zeros.

By contrast, a nonconstant square transfer function matrix necessarily has zeros; they may all be at $z = \infty$ of course, or indeed all at $z = 0$. Zero-free discrete-time transfer function matrices, and indeed those that are almost zero-free as indicated below, have an important invertibility property, summed up in the theorem below, which amalgamates results from [4]. These results represent minor extensions of the key result of [5]. First we require:

Definition 2: For all $k \geq 0$ let u_k^1 and u_k^2 be any two inputs at time k to the system (1) with initial conditions x_0^1 and x_0^2 and let y_k^1 and y_k^2 be the corresponding outputs at time k. Then the system is said to be left invertible with unknown initial state if $y_k^1 = y_k^2$ for all k implies that $u_k^1 = u_k^2$ for all k and that $x_0^1 = x_0^2$.

The key result now follows.

Theorem 1: (a) Consider the system (1) with input, state and output dimensions m, n and p, with $p \geq m$, and with $[B^T\ D^T]^T$ of full column rank. Then it is left invertible with unknown initial state if and only if rank $M(z) = n + m$ for all finite $z \in \mathcal{C}$, i.e., $W(z)$ has no finite zeros. In particular, under this condition, there exists an integer $L \leq n$ such that the state and input at an arbitrary time k are computable from the L measurements $y_k, y_{k+1}, \ldots, y_{k+L-1}$. (b) Assume the same hypothesis as for part (a), save that now $p > m$. Then there exists an integer $L \leq n$ such that the state at time j is computable for $k \leq j \leq k+L$ and the input at time j is computable for $k \leq j \leq k+L-1$ from the L measurements $y_k, y_{k+1}, \ldots, y_{k+L-1}$ if and only if $W(z)$ has no finite or infinite zeros. (c) Assume the same hypothesis as for part (a). If $W(z)$ has no finite zero except at $z = 0$, there exists a positive integer $J \leq n$ such that for some integer L with $J \leq L \leq n + Jm$ and arbitrary k the state at time $k + J$ is computable from $y_k, y_{k+1}, \ldots, y_{k+L-1}$. Moreover, if $W(z)$ has no finite zero except at $z = 0$ and further has no infinite zero, the state at time j is computable for $k + J \leq j \leq k + L$, and the input at time j is computable for $k + J \leq j \leq k + L - 1$ from the same data.

Whether or not there is a zero at 0 affects whether or not the state at the start of the measurement interval is computable, and whether or not there is a zero at ∞ affects whether or not the state at the finish of the measurement interval is computable. In all cases, the measurement interval needs to be of adequate length. The issue of how to determine this length, and more generally how to execute the system inversion, is set out in [4]. The results hold for $p \geq m$ except if the system has no finite or infinite zeros, one must have $p > m$.

The zero-free property can also be characterized using the language of coprime matrix fraction descriptions.

Proposition 2: Let $W(z)$ be a rational $p \times m$ transfer function of rank m which has no finite zeros. Then there exists a $m \times p$ polynomial left inverse for $W(z)$, call it $W^{L1}(z)$. If $W(z)$ has no zeros in $\mathcal{C} \cup \infty$, except possibly at $z = 0$, then there exists a $m \times p$ rational left inverse for $W(z)$, call it $W^{L2}(z)$, all of whose poles are at the origin, which guarantees that $W^{L2}(z)$ is proper, i.e. is finite at $z = \infty$.

Proof: Let $A^{-1}(z)B(z)$ be a polynomial left coprime fractional representation of $W(z)$. Because $W(z)$ has no finite zeros, there exists a polynomial $C(z)$ such that $C(z)B(z) = I_m$. Then it is immediately seen that a $W^{L1}(z)$ is given by the polynomial matrix $C(z)A(z)$. For the second part, define $\tilde{W}(q) = W(q^{-1})$. If $W(z)$ has no zeros except possibly at $z = 0$, then $\tilde{W}(q)$, which is necessarily rational, has no finite zeros. There will then exist a polynomial left inverse, call it $\tilde{D}(q)$, for $\tilde{W}(q)$. Let $W^{L2}(z) = \tilde{D}(z^{-1})$. Observe that $W^{L2}(z)$ will be rational in z with all poles at $z = 0$ and there will hold $W^{L2}(z)W(z) = I$. ■

Both the left inverses identified in the above proposition define systems which when driven by the output sequence of (1) produce the input sequence. A polynomial left inverse produces u_k using $y_k, y_{k+1}, \ldots, y_{k+L}$ for some L. A proper left inverse with all poles at the origin produces u_{k+L} using $y_k, y_{k+1}, \ldots, y_{k+L}$ for some L. This is consistent with Theorem 1.

C. Spectral Factorization

A *real rational spectral matrix* is a square real rational matrix function $\Phi(z)$ where $\Phi(z) = \Phi^T(z^{-1})$ with $\Phi(e^{j\omega})$ bounded and nonnegative definite for $\omega \in [-\pi, \pi]$. Based on a factorization of such matrices in which the unit circle is replaced by the imaginary axis (via a simple bilinear transformation), the following result is well known, see [6] for an expression of the result in continuous-time, i.e. with a Laplace transform variable s, instead of z. The substitution $z = (s+1)(s-1)^{-1}$ allows one result to be obtained from the other. A theorem equivalent to the following and expressed in the z^{-1}-domain can be found in [11], with amendment in [12].

Theorem 2: Let $\Phi(z)$ be a real rational spectral matrix of size $p \times p$ and with normal rank m. Then there exists a $p \times m$ real rational spectral factor, call it $\bar{W}(z)$, such that

$$\Phi(z) = \bar{W}(z)\bar{W}^T(z^{-1}) \tag{3}$$

and $\bar{W}(z)$ has no poles in $|z| \geq 1$, including $z = \infty$, and rank m in $|z| > 1$ including $z = \infty$. Further, such a $\bar{W}(z)$ is unique up to multiplication on the right by a constant $m \times m$ orthogonal matrix. If $\Phi(z)$ has rank m on $|z| = 1$ then $\bar{W}(z)$ has constant rank m in $|z| \geq 1$.

The transfer function matrix $\bar{W}(z)$ is sometimes known as the stable minimum phase spectral factor, or sometimes just minimum phase spectral factor, in the light of the pole and rank restriction. (Stability of a transfer function matrix corresponds to all poles of all entries being confined to $|z| < 1$). Two other properties are worth noting. First, the McMillan degree of $\bar{W}(z)$ is one half that of $\Phi(z)$ [The McMillan degree of $\Phi(z)$ when $\Phi(z)$ has no pole at infinity is defined in the standard way, regarding $\Phi(z)$ simply as a rational function of z. When $\Phi(z)$ has a pole at infinity, its McMillan degree is the same as that of $\tilde{\Phi}(q) = \Phi((aq+b)/(q+c))$ where a, b, c are such that a is not a pole of $\Phi(z)$.] Second, all stable rational spectral factors $W(z)$ of $\Phi(z)$, i.e. all stable $W(z)$ satisfying $\Phi(z) = W(z)W^T(z^{-1})$, can be written as $W(z) = \bar{W}(z)V(z)$ for some stable rational $V(z)$, not necessarily square, obeying $V(z)V^T(z^{-1}) = I_m$. Conversely, any such $V(z)$ defines a stable rational spectral factor.

Further, if $\Phi(z)$ is expressed as

$$\Phi(z) = R + C(zI - A)^{-1}K + K^T(z^{-1}I - A^T)^{-1}C^T \tag{4}$$

for some quadruple $\{A, K, C, R\}$ with $[A, K]$ reachable and $[A, C]$ observable, then $\bar{W}(z)$ and indeed all spectral factors with the same McMillan degree as $\bar{W}(z)$ have a minimal state-variable realization of the form $\{A, B, C, D\}$, i.e., with the same matrices A, C as $\Phi(z)$. See e.g. [7], [8]. Calculation of the stable minimum phase spectral factor is not necessarily trivial, as set out in the references. In particular, typically either polynomial factorization is required, or the steady state solution of a discrete-time Riccati difference equation with constant coefficients must be computed, or operations equivalent to these must be executed.

To further understand the essence of the spectral factorization problem from a state-variable point of view, consider first how $\Phi(z)$ expressed in the form of (4) might be determined, given a transfer function description of the form (1). Let us assume that the input sequence u_k is zero mean white noise, with $E[u_k u_k^T] = I$, and that the input has been applied from time $-\infty$. Further, suppose that all eigenvalues of A lie in the interior of the unit circle. Then the state covariance of the system (1) achieves a steady state value $E[x_k x_k^T] = P$ satisfying

$$P - APA^T = BB^T. \tag{5}$$

Now make the definitions

$$\begin{aligned} R &= CPC^T + DD^T \\ K &= APC^T + BD^T. \end{aligned} \tag{6}$$

Then straightforward calculation shows that the spectrum of the output y_k of (1) is precisely the spectral matrix $\Phi(z)$ defined in (4). One way to think about the spectral factorization is to regard it as the task of solving (5) and (6) for B, D, P with P nonnegative, given A, K, C, R.

D. Spectral Factor Zeros, Spectral Matrix Zeros and the Zero-Free Property

The minimum phase property described in Theorem 2 as being one requiring constant rank of $\bar{W}(z)$ in $|z| > 1$ or in its strengthened form $|z| \geq 1$ is equivalent to requiring $\bar{W}(z)$ to be zero-free in this region (including $z = \infty$).

Our interest however is with the situation where there is a completely zero-free spectral factor. The following result connects zero-free spectral factors and minimum phase spectral factors.

Theorem 3: Let $\Phi(z)$ be a real rational spectral matrix of size $p \times p$ and with normal rank m. If it has a stable spectral factor of size $p \times m$ with no zeros in $\mathcal{C} \cup \infty$, this spectral factor, call it $\bar{W}(z)$, is necessarily minimum phase. If it has a stable spectral factor of size $p \times m$ with no zeros in $\mathcal{C}$ but with a zero at infinity, then this spectral factor is necessarily of the form $\bar{W}(z)V(z)$ where $\bar{W}(z)$ has no zeros in $\mathcal{C} \cup \infty \setminus \{0\}$, $V(z)$ is square with $V(z)V^T(z^{-1}) = I$, and $V(z)$ is not constant but has all poles at $z = 0$.

Proof: Any stable spectral factor $\bar{W}(z)$ of size $p \times m$ with no zeros in $\mathcal{C} \cup \infty$ meets the conditions set out in the hypothesis of Theorem 2, and accordingly is minimum phase. Suppose now $W(z)$ is a stable spectral factor of size $p \times m$ with no zero except at $z = \infty$. Then with $\bar{W}(z)$ as in Theorem 2, it follows that $W(z) = \bar{W}(z)V(z)$ for some square stable $V(z)$ with $V(z)V^T(z^{-1}) = I$. Suppose to obtain a contradiction that $V(z)$ has a finite zero. Since $W(z)$ has no finite zeros, this zero must cancel a pole of $\bar{W}(z)$, and thus be stable. But then it follows easily that $V(z)$ would have an unstable pole at the reciprocal point, which contradicts the fact that it is stable. Hence $V(z)$ has no finite zeros, and consequently $V^T(z^{-1})$ is not infinite for any finite value of z, i.e. $V^T(z)$ is not infinite for any value of z except possibly $z = 0$. Equivalently, $V(z)$ has no poles apart possibly from $z = 0$. Since $W(z)$ has a zero at infinity, it is not minimum phase and so $V(z)$ cannot be constant; hence $V(z)$ must have a pole or poles, but only at $z = 0$. Since $W(z)$ has no finite zeros, any finite zero of $\bar{W}(z)$ must be cancelled by a pole of $V(z)$. Since the only pole of $V(z)$ can be at $z = 0$, the only finite zero of $\bar{W}(z)$ is at $z = 0$. Since $\bar{W}(z)$ is minimum phase, it has no infinite zero. ■

Apparently, to check whether there is a zero-free spectral factor associated with a spectral matrix, given only the spectral matrix initially, one would have to perform a spectral factorizaton and then check the minimum phase spectral factor's zeros. Actually, it is possible to test the spectrum itself.

We need a concept of zeros for the spectral matrix, and yet we have no expression for it in the *standard* state-variable form and it may be improper, i.e. have a pole at $z = \infty$. Introduce temporarily a continuous-time power spectrum $\tilde{\Phi}(s) = \Phi((s+1)/(s-1))$ for which there will hold $\tilde{\Phi}(s) = \tilde{\Phi}^T(-s)$. Evidently, $\tilde{\Phi}(s)$ is rational, and is proper. Accordingly, it has a state-variable equation, and our earlier definition of zeros can be applied. When this is translated back to the z-domain for $\Phi(z)$, the following characterizations of finite zeros and zeros at $0, \infty$ result.

The finite zeros of $\Phi(z)$, other than possibly at zero, are given by the values of z at which the following matrix has rank less than its normal rank of $2n + m$:

$$M_\Phi(z) = \begin{bmatrix} zI - A & 0 & -K \\ 0 & z^{-1}I - A^T & -C^T \\ C & K^T & R \end{bmatrix} \tag{7}$$

Further, $\Phi(z)$ has a zero at infinity and at zero (all zeros occurring in reciprocal pairs) if and only if the matrix

$$M_0 = \begin{bmatrix} -A & -K \\ C & R \end{bmatrix} \tag{8}$$

has rank less than $n + m$.

Now we can relate the occurrence of zeros in a spectral factor to zeros in the spectral matrix, and conversely.

Theorem 4: Consider a $p \times m$ transfer function $W(z)$ of normal rank m defined by (1) with minimal realization $\{A, B, C, D\}$. Suppose further that all eigenvalues of A lie in the interior of the unit circle, and that u_k is a zero mean, white, unit covariance process applied from time $-\infty$. Let $\Phi(z)$ be the associated power spectrum, given by (4) using (5) and (6). Then (a) the system (1) has no zeros in $\mathcal{C} \setminus 0$ if and only if $\Phi(z)$ as given by (4) has no zeros in $\mathcal{C} \setminus 0$; (b) the system (1) has no zeros at $z = 0$ or $z = \infty$ if and only if $\Phi(z)$ as given by (4) has no zeros at $z = 0$ or $z = \infty$.

Proof: Observe first that the rank constraint on $W(z)$ implies that the matrix $[B^T \ D^T]$ has full row rank m. Now observe the following identity

$$\begin{aligned} &\begin{bmatrix} zI - A & 0 & -K \\ 0 & z^{-1}I - A^T & -C^T \\ C & K^T & R \end{bmatrix} \\ &= \begin{bmatrix} I & AP & 0 \\ 0 & I & 0 \\ 0 & -CP & I \end{bmatrix} \begin{bmatrix} zI - A & 0 & -B \\ 0 & I & 0 \\ C & 0 & D \end{bmatrix} \\ &\quad \times \begin{bmatrix} zI & 0 & 0 \\ 0 & z^{-1}I - A^T & -C^T \\ 0 & B^T & D^T \end{bmatrix} \begin{bmatrix} I & z^{-1}P & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \end{aligned} \tag{9}$$

which follows by direct verification. Observe that all the matrices in this equation are well-defined for $z \neq 0, \infty$. The first and last matrix on the right side of the equation are obviously nonsingular. The second and third matrices on the right lose rank just at the zeros of $W(z)$ (if there are any zeros) or of $W(z^{-1})$, and then have less than full column and full row rank respectively. Accordingly, their product will lose rank. Hence any zero of $W(z)$ in $z \neq 0, \infty$ will be a zero of $\Phi(z)$. Conversely, suppose that $\Phi(z)$ has a finite zero other than at $z = 0$. Then either the second matrix on the right loses full column rank or the third matrix loses full row rank, and this corresponds to either $W(z)$ having a zero or $W(z^{-1})$ having a zero. This establishes part (a) of the theorem. To establish part (b), consider the identity

$$\begin{aligned} \begin{bmatrix} -A & K \\ C & R \end{bmatrix} &= \begin{bmatrix} -A & -APC^T - BD^T \\ C & CPC^T + DD^T \end{bmatrix} \\ &= \begin{bmatrix} -A & -B \\ C & D \end{bmatrix} \begin{bmatrix} I & PC^T \\ 0 & D^T \end{bmatrix}. \end{aligned} \tag{10}$$

Observe that the matrix M_0 appears on the left of this equation and this has rank less than $n + m$ if and only if $\Phi(z)$ has a zero at $z = 0$ and thus also $z = \infty$. The matrix $M(0)$ appearing as the first matrix on the right side has rank less than $n + m$ if and only if $W(z)$ has a zero at $z = 0$. The second matrix on the right side has rank less than $n + m$ if and only if D has rank less than m, i.e. $W(z)$ has a zero at infinity. Clearly, M_0 on the left has rank less than $n + m$ if and only if one or both of the two matrices on the right has rank less than $n+m$. Therefore, $\Phi(z)$ has no zero at 0 or ∞ if and only if $W(z)$ has no zero at 0 or ∞. ■

By way of example, observe that both $W_1(z) = (z+.5)^{-1}$ and $W_2(z) = z(z+.5)^{-1}$ obey $W_1(z)W_1(z^{-1}) = W_2(z)W_2(z^{-1}) = \Phi(z) = z(.5z^2+1.25z+.5)^{-1}$. The spectrum has a zero at $z=0$ and thus $z=\infty$; one of the spectral factors has a zero at $z=0$ but not at $z=\infty$, and the other has no zero at $z=0$ but a zero at $z=\infty$.

III. Kalman Filter Result for a System With No Zeros

In this section, we shall consider the linear time-invariant $p \times m$ system defined by

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k. \end{aligned} \tag{11}$$

We shall suppose throughout this and the next section unless otherwise indicated that $p > m$ and the system is minimal and has no zeros, finite or infinite. Equivalently, with $M(z)$ as given in (2), there holds rank $M(z) = n+m$ for all finite z, and the matrix D has full column rank. We also assume that all eigenvalues of A lie in the interior of the unit circle.

We shall further suppose that u_k is a zero mean white sequence, such that $E[u(k)u^T(k)] = I$, and that the input has been applied from time $-\infty$. This will result in a steady state covariance $E[x(k)x^T(k)] = P$ satisfying (5).

Suppose that the measurements y_k are collected starting only at time 0, and suppose a Kalman filter to estimate x_k in $k \geq 0$ is constructed using these measurements. Let us denote the associated error covariance by

$$\Sigma_{k/k-1} = E\left[(x_k - \hat{x}_{k/k-1})(x_k - \hat{x}_{k/k-1})^T\right] \tag{12}$$

where of course $\hat{x}_{k/k-1} = E[x_k|y_0, y_1, \ldots, y_{k-1}]$.

As is well known [13], the error covariance satisfies the following discrete time Riccati equation:

$$\begin{aligned} \Sigma_{k+1/k} = {} & A\Sigma_{k/k-1}A^T - (A\Sigma_{k/k-1}C^T + BD^T) \\ & \times (DD^T + C\Sigma_{k/k-1}C^T)^\sharp (A\Sigma_{k/k-1}C^T + BD^T)^T \\ & + BB^T. \end{aligned} \tag{13}$$

The initialization is $\Sigma_{0/-1} = P$, and the symbol $\sharp$ denotes pseudo inverse.

In the previous section, we saw in Theorem 1 that it is possible to uniquely determine x_{k+L} and u_{k+L}, and therefore x_{k+L+1}, from $y_k, y_{k+1}, \ldots, y_{k+L}$ for some L with $L \leq n$. Since this estimation problem can be performed with zero error, the error covariance resulting from the Kalman filter equations, which is optimal, must also evaluate as zero when we take $k = L$, and thereafter it stays at zero. Actually, Theorem 1 indicates it is possible to mildly relax the assumption that $p > m$ and that there are no zeros anywhere, permitting $p \geq m$ and zeros to occur at $z = 0$.

Accordingly, we have proved the following result.

Theorem 5: Consider the system (1) with input, state and output dimensions m, n and p, with $p > m$, and with $[B^T \; D^T]^T$ of full column rank. Assume the system is minimal and has no finite or infinite zeros (which implies in fact that D has full column rank), that all eigenvalues of A lie in the interior of the unit circle, and that u_k is a zero mean, white, unit covariance process applied from time $-\infty$. Suppose that a Kalman filter is applied commencing at time zero. Then after at most n time steps, the Kalman filter error covariance (13) will be zero, and remain at that value thereafter. Further, the same conclusion holds if the condition $p > m$ is relaxed to $p \geq m$ and the system is permitted to have a zero at $z = 0$ but nowhere else.

This result is evidently straightforward to obtain. Further, it is hardly surprising and is rooted in earlier examinations of transients in solving Riccati equations with constant coefficients. Beginning in the 1970's, a concept termed 'invariant' or 'constant' directions of Riccati equations was formulated, see e.g. [14] and [15] for two important early contributions. In a Kalman filtering problem, an invariant direction, a say, is a vector with the property that $\Sigma_{k+1/k}a$ is invariant with k and with nonnegative $\Sigma_{0/-1}$ for all k exceeding some finite integer. The intuition behind the notion was that for some systems, differentiation in continuous time or differencing in discrete time of output measurements could result in more accurate, even perfect, information about part of the state vector; indeed, [15] notes that a constant direction is equivalent to a filter direction in which the best one-step predictor is a fixed linear combination of the last d observations for $n \geq d$. More sophisticated investigations again of these concepts, and their relation to spectral zeros and splitting subspaces, were undertaken by Lindquist and colleagues, [16] and [17]. It is possible to demonstrate that the result above, which says effectively that all directions are constant directions if and only if a certain zero property holds, is indeed a special case of these more general results. From a pedagogical point of view and for the purposes of this paper, the argument presented here leading to the theorem appears to us more attractive. The distinction between $p > m$ and $p \geq m$ in the theorem is also helpful, and not easy to draw out from the other work.

Purely autoregressive models will lead to a finite time convergence of the Riccati equation. That is not exactly what we have here, due to the nonsquareness of the transfer function. Nevertheless, a zero free transfer function is analagous to a standard autoregressive model, and indeed one can represent such a system using a variant on a standard autoregressive model.

IV. Spectral Factorization Result

In this section, we switch focus onto the key problem of computing a spectral factor from the spectral matrix, using a finite number of rational calculations. In econometric applications, the transfer function in question is that linking the factors to the latent variables.

The set-up we contemplate is the following. Suppose we know there is a system of the form of (1) giving rise to a stationary covariance through excitation of the system by a zero mean white noise process of unit covariance. We do *not* assume we know the matrices defining the system, though we must assume that the A matrix has all eigenvalues inside the unit circle; we do assume that the system has no zeros. We suppose that the output power spectrum is known, in minimal state-variable form (see below), and we wish to recover the system.

In the light of the material to this point, we can understand that this is a problem of spectral factorization [6]–[8]. Of particular interest to us is the obtaining of the (minimum phase) spectral factor by a finite number of rational calculations through some exploitation of the zero-free property. This section will present one method, based on a Riccati equation. Other approaches will come later.

A. Going From the System to the Power Spectrum

Consider the system (1); suppose that the A matrix has all eigenvalues inside the unit circle, and under the assumption of zero mean white unit covariance noise sequence u_k applied from time $-\infty$, let the state covariance be P. Then P is given by (5), repeated as

$$P = APA^T + BB^T. \tag{14}$$

The spectrum of the y_k process is given by (4), repeated as

$$\Phi(z) = R + C(zI - A)^{-1}K + K(z^{-1}I - A^T)^{-1}C^T \tag{15}$$

where, repeating (6)

$$\begin{aligned} R &= CPC^T + DD^T \\ K &= APC^T + BD^T. \end{aligned} \tag{16}$$

B. Essence of the Spectral Factorization Problem

We now suppose our data is the quadruple $\{A, K, C, R\}$. We regard the key task of spectral factorization as one of finding the state covariance matrix P; for once it is found, the matrices D and B are determinable up to multiplication on the right by a constant orthogonal matrix using the following consequence of (14) and (16)

$$\begin{bmatrix} P - APA^T & K - APC^T \\ K^T - CPA^T & R - CPC^T \end{bmatrix} = \begin{bmatrix} B \\ D \end{bmatrix} [B^T \ D^T]. \tag{17}$$

Determination of P is evidently a task of finding a symmetric nonnegative matrix which makes the left side of (17) nonnegative. If the spectral factor is to have rank m, we further want the matrix on the left of (17) to be of rank m. Generally too, we will be interested in finding the P which defines the minimum phase spectral factor. With no special properties of the underlying system, obtaining the minimum phase spectral factor can be achieved by a number of methods, one [8] of which is now set out.

Consider the following discrete Riccati equation:

$$T_{k+1} = AT_kA^T + (AT_kC^T - K)(R - CT_kC^T)^\sharp (AT_kC^T - K)^T \tag{18}$$

initialized by $T_0 = 0$. Then

$$lim_{k\to\infty} T_k = P. \tag{19}$$

Of course, use of this equation apparently requires an infinite number of calculations (although one would hope that convergence in practical terms could occur after a finite number).

C. Finite Time Convergence of the Discrete Riccati Equation

Recall from the previous section that the discrete Riccati equation associated with a Kalman filtering equation converged in a finite number of steps. We will demonstrate the same property for the spectral factorization Riccati equation (18). In fact, we have the following easy result; for all k, there holds

$$P = T_k + \Sigma_{k/k-1}. \tag{20}$$

We indicate how to prove this equation immediately below. Observe though that it yields the finite convergence property of T_k precisely because of the finite time convergence property of $\Sigma_{k/k-1}$.

To establish (20) for all k, observe first that the equation holds at $k = 0$ by virtue of the initializations of the two Riccati equations. Suppose equality holds for $k = 0, 1, \ldots, j$. Then it is straightforward to verify using (16) that

$$\begin{aligned} R - CT_jC^T &= DD^T + C\Sigma_{j/j-1}C^T \\ AT_jC^T - K &= -A\Sigma_{j/j-1}C^T - BD^T. \end{aligned} \tag{21}$$

Using these equations and the two Riccati equations for T_{j+1} and $\Sigma_{j+1/j}$, equality in (20) is established for $k = j + 1$. In more detail, setting $k = j$ in (18), we obtain

$$\begin{aligned} T_{j+1} &= AT_jA^T + (A\Sigma_{j/j-1}C^T + BD^T) \\ &\quad \times (DD^T + C\Sigma_{j/j-1}C^T)^\sharp (A\Sigma_{j/j-1}C^T + BD^T)^T \\ &= AT_jA^T - \Sigma_{j+1/j} + A\Sigma_{j/j-1}A^T + BB^T \end{aligned} \tag{22}$$

where the second equality follows on using (13). By the inductive hypothesis, we have $T_j + \Sigma_{j/j-1} = P$ and so there results

$$T_{j+1} + \Sigma_{j+1/j} = APA^T + BB^T = P \tag{23}$$

from which the inductive step is established. To sum up, we have proved the following result:

Theorem 6: Suppose that the power spectrum (15) is generated by the system (1) under the hypotheses that the system is minimal, has no finite or infinite zeros and has more outputs than inputs, that all eigenvalues of A lie in the interior of the unit circle, and that u_k is a zero mean, white, unit covariance process applied from time $-\infty$. Then there is a unique matrix P computable in a finite number of rational calculations from $\{A, K, C, R\}$, so that the system (1) can be determined, save that D and B are only determined to within multiplication on the right by a constant orthogonal matrix. The result extends to the case when $p \geq m$ and the system is permitted to have a zero at $z = 0$, but nowhere else.

The following corollary is immediate.

Corollary 1: Let $\Phi(z)$ be a rational spectrum given by (15), with $\{A, K, C\}$ minimal, and with no finite or infinite zeros. Then there exists a spectral factor $\{A, B, C, D\}$ with D of full column rank that is unique up to right multiplication by an orthogonal matrix. This spectral factor has no zeros in $\mathcal{C} \cup \infty$ and is computable in a finite number of rational calculations. If the zero-free property of $\Phi(z)$ is relaxed to permit a zero at $z = 0$, then a stable minimum phase spectral factor free of zeros except at $z = 0$ can be computed by a finite number of rational calculations.

V. Working Directly With Covariance Data

In this section, we consider the question of how we can work with a finite number of lagged covariances to determine a state estimate from an output, and, if desired, also to find a state variable description of the spectral factor knowing a state variable description of the spectrum matrix. This material makes contact with the earlier section providing an alternative approach to the proof of the important result concerning absence of zeros.

So our underlying assumptions are that the signal model with transfer function $W(z)$ is defined by a quadruple $\{A, B, C, D\}$ which is unknown, and that the spectrum matrix is defined by the known minimal quadruple $\{A, K, C, R\}$, as per (4).

From the minimality of the quadruple $\{A, K, C, R\}$, it is easy to argue the minimality of the quadruple $\{A, B, C, D\}$. Of course, as before, it is assumed that all eigenvalues of A lie in $|z| < 1$. As we know, from Theorem 1, if the matrix $M(z)$ of (2) defined using the transfer function has no finite zeros, then x_k, u_k can be uniquely determined from $y_k, y_{k+1}, \ldots, y_{k+L-1}$. If the matrix $M(z)$ has no finite or infinite zero, we can (uniquely) reconstruct $x_k, x_{k+1}, \ldots, x_{k+L}$ and $u_k, u_{k+1}, \ldots, u_{k+L-1}$.

The derivation of the key result of this section uses the following equation, which is an easy consequence of (1), and is independent of the dimension of x_k, u_k, y_k

$$\begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ \vdots \\ y_{k+L-1} \end{bmatrix} = \begin{bmatrix} C & D & 0 & \cdots & 0 \\ CA & CB & D & \cdots & 0 \\ CA^2 & CAB & CB & \cdots & 0 \\ \vdots & & & & \vdots \\ CA^{L-1} & CA^{L-2}B & CA^{L-3}B & \cdots & D \end{bmatrix} \times \begin{bmatrix} x_k \\ u_k \\ u_{k+1} \\ \vdots \\ u_{k+L-1} \end{bmatrix}. \quad (24)$$

Let N_L denote the matrix appearing on the right side of (24). We are first going to prove a result concerning N_L and its connection to the zero-free property of the associated transfer function. Then we shall carry over this result to one involving a Toeplitz matrix of covariance data with connection to the zero-free property of the associated power spectrum.

Lemma 1: Consider the system (1) with input, state and output dimensions m, n and p, with $p \geq m$, and with $[B^T \; D^T]^T$ of full column rank. Let $M(z)$ be as defined in (2) and let N_L be the matrix appearing on the right side of (24). Then the following conditions are equivalent.

1) The matrix $M(z)$ has rank $n + m$ for all finite z.
2) The vector x_k is uniquely determinable from $y_k, y_{k+1}, \ldots, y_{k+L-1}$ for some L.
3) For some L, the row echelon form [18] of the matrix N_L has its first n rows as $[I_n \; 0]$.
4) For some L, there exists a nonsingular matrix $\bar{S}$ satisfying
$$\bar{S}[C \; CA \; CA^2 \; \ldots \; CA^{L-1}]^T = [I_n \; 0]^T \quad (25)$$
with $\bar{S}N_L$ of the form
$$\bar{S}N_L = \begin{bmatrix} I_n & 0 \\ 0 & X \end{bmatrix} \quad (26)$$
for some X.

Moreover, if an L exists satisfying any of conditions 2, 3 or 4, it is necessarily as large as the observability index of $[A, C]$, a satisfying value exists overbounded by n, and all larger values of L also satisfy the conditions.

Proof: The equivalence of the first two conditions was established in Theorem 1, with indeed the restriction that $L \leq n$. Obviously, if condition 2 holds for any L, it holds for all greater values. Now we argue the equivalence of conditions 2 and 3. Let S be a nonsingular matrix such that SN_L has row echelon form. (Thus all zero rows are at the bottom of the matrix, the first nonzero entry of each nonzero row after the first occurs to the right of the first nonzero entry of the previous row, the first nonzero entry in any nonzero row is 1, and all entries in the column above and below a first nonzero entry of a row are zero.) In order that x_k be uniquely determinable from the output data, it is necessary and sufficient that the kernel of N_L and equivalently SN_L contain no vector with other than zeros in the first n entries. The only way this can occur is if the echelon form has the structure in the Lemma statement--else a contradiction is immediate. Note also that it is immediate that if L is less than the observability index of $[A, C]$, such a vector with other than zeros in its first n entries will exist in the kernel of N_L. This establishes the lower bound on L. The matrix S actually encodes the procedure for computing x_k: to see this, suppose there holds for some X of full row rank whose entries are immaterial

$$SN_L = \begin{bmatrix} I_n & 0 \\ 0 & X \\ 0 & 0 \end{bmatrix} \quad (27)$$

(with the bottom zero blocks possibly not being present), then it is trivial to see using this equation and (24) that

$$[I_n \; 0]S \left[y_k^T \; y_{k+1}^T \; \ldots \; y_{k+L-1}^T\right]^T = x_k.$$

It now remains to prove that condition 4 on $\bar{S}N_L$ implies and is implied by the echelon form condition 3. Assume the echelon form condition. Then it is trivial that the S of the echelon form condition works as the matrix $\bar{S}$. Conversely, assume $\bar{S}$ is available with the stated property of condition 4. Let $\bar{S}_1$ be a nonsingular matrix such that $\bar{S}_1 X = X_1$ is in echelon form. Then it is immediate that

$$\begin{bmatrix} I_n & 0 \\ 0 & \bar{S}_1 \end{bmatrix} \bar{S}N_L = \begin{bmatrix} I_n & 0 \\ 0 & X_1 \end{bmatrix}$$

is itself in echelon form. Hence an S exists converting N_L to the echelon form of condition 3.

Finally, suppose, to obtain a contradiction, that while some L exists satisfying any of conditions 2, 3 or 4, no L exists with $L \leq n$. Then $M(z)$ could not have rank $n+m$ for all finite z, by Theorem 1. Suppose that $M(z_0)$ has rank less than $n + m$. Let $[x_k^T \; u_k^T]^T \neq 0$ be in the kernel of $M(z_0)$. Because $[B^T \; D^T]^T$ has full column rank, it is easily seen that $x_k \neq 0$. It is trivial to check that the input sequence $u_j = z_0^{j-k} u_k$ for $j \geq k$ will lead to $y_j = 0$ for all $j \geq k$. Accordingly, $x_k \neq 0$ will be indistinguishable on the basis of output measurements from the zero state, contradicting condition 2. If L exists satisfying condition 2, all larger values obviously also ensure satisfaction of the condition. ∎

In the main result of this section, we will need a Toeplitz matrix formulated from the the covariance data. The data are of course available from the power spectrum $\Phi(z)$. Indeed, in the light of (4), there holds under white noise input conditions and in steady state $E[y_k y_k^T] = R$, $E[y_{k+1} y_k^T] = CK$, $E[y_{k+2} y_k^T] = CAK$, etc. Let us define for $L = 1, 2, \ldots$ the Toeplitz matrix

$$\mathcal{T}_L = \begin{bmatrix} R & (CK)^T & (CAK)^T & \ldots & (CA^{L-2}K)^T \\ CK & R & (CK)^T & \ldots & (CA^{L-3}K)^T \\ CAK & CK & R & \ldots & (CA^{L-4}K)^T \\ \vdots & & & & \vdots \\ CA^{L-2}K & CA^{L-3}K & CA^{L-4}K & \ldots & R \end{bmatrix}. \tag{28}$$

It is easy to verify the following connection between N_L and $\mathcal{T}_L$. This relation is crucial for establishing the connection between $\mathcal{T}_L$ and the zero-free property.

Lemma 2: Consider the system (1) with realization $\{A, B, C, D\}$, with input, state and output dimensions m, n and p, and where all eigenvalues of A lie in the interior of the unit circle. Let $\Phi(z)$ be the associated power spectrum, given by (15) and (16), that would result if u_k is taken as a zero mean unit covariance white noise process from $t = -\infty$. Let N_L be as defined in (24) and $\mathcal{T}_L$ as in (28). Then with $E[x_k x_k^T] = P$ as the state covariance, there holds

$$N_L diag[P, I_m, \ldots, I_m] N_L^T = \mathcal{T}_L. \tag{29}$$

Proof: Multiply each side of (24) by its transpose and take the expectation. Equation (29) is immediate. ∎

Theorem 7: Adopt the same hypothesis as Lemma 2, suppose additionally that $p > m$ and that D has full column rank m, and let $\bar{S}$ be defined as in Lemma 1. Suppose that the state-variable quadruple $\{A, K, C, R\}$ is minimal. Then the following conditions involving the matrices $\{A, K, C, R\}$ are equivalent:

1) the spectral matrix has no zeros in $\mathcal{C} \cup \infty$;
2) There exists a value of L such that $\mathcal{T}_L$ has rank $n + Lm$;
3) Let $\bar{S}$ be as defined in (25). Then there holds for some nonnegative symmetric Q of rank Lm

$$\bar{S}\mathcal{T}_L\bar{S}^T = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \tag{30}$$

4) There exists a nonsingular matrix $\tilde{S}$ satisfying (25) with $\bar{S}$ replaced by $\tilde{S}$ and such that for some nonsingular matrix $\tilde{P}$ and some nonnegative symmetric $\tilde{Q}$ of rank Lm

$$\tilde{S}\mathcal{T}_L\bar{S}^T = \begin{bmatrix} \tilde{P} & 0 \\ 0 & \tilde{Q} \end{bmatrix}. \tag{31}$$

If an L exists satisfying any of conditions 2, 3 or 4, it is necessarily as large as the observability index of $[A, C]$, a satisfying value exists overbounded by n, and all larger values of L also satisfy the conditions. The only value of $\tilde{P}$ for which (31) holds is $\tilde{P} = P$ with P the state covariance of the underlying minimum phase, zero-free spectral factor.

Proof: Observe first that because the state-variable quadruple $\{A, K, C, R\}$ is minimal, a straightforward argument will show that the pair $[A, B]$ is completely controllable and accordingly the state covariance matrix P is nonsingular.

Assume condition 1. Since the spectral matrix is zero free, the associated minimum phase spectral factor is zero free. Further, because the spectral factor has no zero in $\mathcal{C} \cup \infty$, by Theorem 1, (24) is uniquely solvable for some minimum value of L at least as large as the observability index of $[A, C]$ and overbounded by n, and indeed all larger values of L. Thus N_L has full column rank, viz. $n + Lm$. By (29), it follows that $\mathcal{T}_L$ has rank $n + Lm$, i.e., condition 2 holds. Conversely, if $\mathcal{T}_L$ has rank $n + Lm$, then N_L must have full column rank. Then (24) is uniquely solvable and by Theorem 1, the underlying system has no zeros. Accordingly, condition 1 holds.

If condition 1 holds, the underlying system is zero-free and condition 1 of Lemma 1 holds. Accordingly, by condition 4 of Lemma 1 there exists $\bar{S}$ defined as in (25) with also $\bar{S}N_L$ satisfying (26). Condition 3 is then an immediate consequence of (29). Since condition 3 presupposes that $\bar{S}$ exists satisfying condition 4 of Lemma 1, it is trivial that condition 1 of the Lemma holds, and then, because also D has full column rank, $\Phi(z)$ is zero free in $\mathcal{C} \cup \infty$, i.e. condition 1 holds.

Condition 3 trivially implies condition 4, by taking $\tilde{S} = \bar{S}$. For the converse, suppose, to obtain a contradiction, that for some nonzero $X = [X_1^T \; X_2^T]^T$ with Lm columns there holds

$$\tilde{S}N_L = \begin{bmatrix} I_n & X_1 \\ 0 & X_2 \end{bmatrix}.$$

(The fact that the first n columns are as shown is a consequence of the definition of $\tilde{S}$). Now by using (29), we have

$$\begin{aligned} \bar{S}\mathcal{T}_L\bar{S}^T &= \begin{bmatrix} I_n & X_1 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I_{Lm} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ X_1^T & X_2^T \end{bmatrix} \\ &= \begin{bmatrix} P + X_1X_1^T & X_1X_2^T \\ X_2X_1^T & X_2X_2^T \end{bmatrix}. \end{aligned} \tag{32}$$

Evidently from (31), we have $X_1X_2^T = 0$, while the rank constraint on $\mathcal{T}_L$ ensures that X_2 has full column rank. Therefore $X_1 = 0$ and it is evident that $\tilde{S}$ meets the conditions on $\bar{S}$ of Lemma 1. Further, since $\tilde{P} = P + X_1X_1^T$ and $X_1 = 0$, there holds $\tilde{P} = P$. ∎

We make several summarizing remarks. If one has a realization of a spectral factor, $\{A, B, C, D\}$, the absence of zeros can be tested for by working with a matrix $\bar{S}$ computed from the observability matrix of $[A, C]$ (condition 3 of Lemma 1). If one has a realization instead of the associated spectrum, call it $\{A, K, C, R\}$ then by working with the matrix $\tilde{S}$ one can test for the absence of zeros (condition 4 of the theorem). In the latter case, there is a cheap way to perform spectral factorization, due to the state covariance matrix being delivered via the calculation of condition 4. Note that the calculation of $\tilde{S}$, if it exists, is straightforward. One finds a congruency transformation taking $\mathcal{T}_L$ to a diagonal matrix, and then adjusts the transformation to ensure satisfaction of (25) with $\tilde{S}$ replacing $\bar{S}$.

A further insight is that if the zero-free property is known to hold, and if covariance data is available but values are not known a priori for n and m, these values can in principle be obtained by examining the ranks of $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \ldots$; from some point on, there will hold rank $\mathcal{T}_j = n + jm$ and then n, m will be available from two successive rank values.

VI. Directly Factoring the Toeplitz Matrix

In this section, our main focus is to work to the greatest extent possible with the Toeplitz matrix. First, we shall make a connection between the standard Cholesky factorization of a nonnegative definite matrix, and the matrix sequence T_k introduced earlier.

The following is a standard extension of the Cholesky factorization for an arbitrary nonnegative definite symmetric matrix. Such connections go back a long way, see [19], the contents of which are also connected to ideas of the previous section. In the scalar case, Bauer [20] may have been the first to apply the idea to spectral factorization, while [21] recognizes the possibility of doing the same thing in the matrix case.

Proposition 3: Let X be a symmetric nonnegative definite matrix written in block form as

$$X = \begin{bmatrix} X_{11} & X_{21}^T & X_{31}^T & \ldots & X_{q1}^T \\ X_{21} & X_{22} & X_{32}^T & \ldots & X_{q2}^T \\ X_{31} & X_{32} & X_{33} & \ldots & X_{q3}^T \\ \vdots & & & & \vdots \\ X_{q1} & X_{q2} & X_{q3} & \ldots & X_{qq} \end{bmatrix} \tag{33}$$

with X_{ii} square and symmetric for each i. Then there exists a decomposition $X = YY^T$ where

$$Y = \begin{bmatrix} Y_{11} & 0 & 0 & \ldots & 0 \\ Y_{21} & Y_{22} & 0 & \ldots & 0 \\ Y_{31} & Y_{32} & Y_{33} & \ldots & 0 \\ \vdots & & & & \vdots \\ Y_{q1} & Y_{q2} & Y_{q3} & \ldots & Y_{qq} \end{bmatrix} \tag{34}$$

and for all i, Y_{ii} has full column rank. Moreover, the submatrices Y_{ij} can be determined recursively. One possible order is $Y_{11}, Y_{21}, Y_{31}, \ldots, Y_{q1}, Y_{22}, Y_{32}, \ldots, Y_{q2}, Y_{31}, Y_{32}, \ldots$; all Y_{ij} for $i \neq j$ are determinable by linear operations, and Y_{ii} for each i is determined by

$$Y_{ii}Y_{ii}^T = X_{ii} - \sum_{1}^{i-1} Y_{ij}Y_{ij}^T \tag{35}$$

The matrices Y_{ii} are unique up to right multiplication by an orthogonal matrix, and the block matrix factor Y is unique to within right multiplication by a diagonal block matrix of orthogonal matrices.

Notice that the existence of the Cholesky factor Y in the previous proposition is a simple consequence of the nonnegative symmetric nature of X. In particular, this ensures that the diagonal blocks Y_{ii} are well defined, or that the right side of (35) is nonnegative definite.

There is a straightforward interpretation of the Y_{ij} as coefficient matrices of the impulse response of a time-varying linear system which when excited by unit variance white noise (albeit at different times possibly of different vector dimension) produces an output sequence for which the covariance matches the X_{ij}. More precisely, let $v_1, v_2, v_3, \ldots$ denote an infinite sequence of independent zero mean gaussian random vectors with unit covariance, and with the dimension of v_i equal to the number of columns of Y_{ii}. Define a second sequence $y_1, y_2, y_3, \ldots$ by

$$y_i = \sum_{j=1}^{i} Y_{ij}v_j. \tag{36}$$

Then it is a trivial exercise to verify that for $i \geq j \geq 1$, there holds

$$E\left[y_i y_j^T\right] = X_{ij}. \tag{37}$$

In fact, there is a second calculation that we will shortly make use of. Observe that because the Y_{ii} have full column rank, knowledge of the sequence $y_1, y_2, y_3, \ldots$ allows recursive reconstruction of the sequence $v_1, v_2, v_3, \ldots$, with the consequence that for any $i > j$

$$E[y_i|y_1, y_2, \ldots, y_j] = E[y_i|v_1, v_2, \ldots, v_j] = \sum_{k=1}^{j} Y_{ik}v_k. \tag{38}$$

Consequently, the associated estimation error is given by

$$\tilde{y}_{i|j} := y_i - E[y_i|y_1, y_2, \ldots, y_j] = \sum_{k=j+1}^{i} Y_{ik}v_k \tag{39}$$

and the covariance of the error follows easily as

$$E\left[\tilde{y}_{i|j}\tilde{y}_{i|j}^T\right] = \sum_{k=j+1}^{i} Y_{ik}Y_{ik}^T. \tag{40}$$

We shall now apply the block Cholesky decomposition to a Toeplitz matrix associated with a rational power spectrum, and use it to characterize the error covariance associated with prediction. Initially, the power spectrum may have zeros.

Theorem 8: Consider a power spectrum $\Phi(z)$ defined by (4), in which $\{A, K, C, R\}$ is minimal, and all eigenvalues of A lie inside the unit circle. Let $\mathcal{T}_L$ for all $L = 1, 2, \ldots$, be the associated set of nonnegative definite matrices as defined in (28). Let the sequence T_k be defined by (18) with initial condition $T_0 = 0$. Then for all k, the matrix $R - CT_kC^T$ is nonnegative definite. Define further the matrices D_k, B_k by

$$D_kD_k^T = R - CT_kC^T \tag{41}$$

with D_k having a minimum number of columns, and

$$B_k = (K - AT_kC^T)D_k\left(D_k^TD_k\right)^{-1}. \tag{42}$$

Then with the definition

$$\mathcal{H}_L = \begin{bmatrix} D_0 & & & \ldots & 0 \\ CB_0 & D_1 & & \ldots & 0 \\ CAB_0 & CB_1 & D_2 & \ldots & 0 \\ CA^2B_0 & CAB_1 & CB_2 & \ldots & 0 \\ \vdots & & & & \vdots \\ CA^{L-2}B_0 & CA^{L-3}B_1 & CA^{L-4}B_2 & \ldots & D_{L-1} \end{bmatrix} \tag{43}$$

there holds

$$\mathcal{T}_L = \mathcal{H}_L\mathcal{H}_L^T. \tag{44}$$

Proof: Apply direct verification, using the various defining equations; the construction procedure is precisely that of the block Cholesky decomposition, and assures the nonnegativity of $R-CT_kC^T$. To this end, it is helpful to recognize the following three equalities:

$$T_{k+1} = AT_kA^T + B_kB_k^T \tag{45}$$
$$R = CT_kC^T + D_kD_k^T \tag{46}$$
$$K = AT_kC^T + B_kD_k^T \tag{47}$$

■

To this point, no assumption has been made about zeros. We have simply shown that the block Cholesky factorization is related to the construction, for an arbitrary stationary covariance sequence corresponding to a rational spectrum, of a time-varying system impulse response with the property that when the associated system is driven by white noise commencing at time $t=0$, as opposed to commencing in the infinite past, the output process has the prescribed covariance.

We are now going to introduce the zero-free property, in considering the prediction of future values of the y_i given values from time 1 up to a certain time, L say. For convenience of notation, let $\mathcal{Y}_{L+1}^{L+q}$ denote the column vector whose block entries are in order $y_{L+1}, y_{L+2}, \ldots, y_{L+q}$ and let $\tilde{\mathcal{Y}}_{L+q|L}$ denote the error in estimating $\mathcal{Y}_{L+1}^{L+q}$ using the measurements $y_1, y_2, \ldots, y_L$. We have described above how to obtain the covariance of this error (or submatrices of the whole covariance). We have the following evaluation for $1 \le L < L+q$:

$$E\left[\tilde{\mathcal{Y}}_{L+q|L}\tilde{\mathcal{Y}}_{L+q|L}^T\right] = \tilde{\mathcal{H}}_{L+q|L}\tilde{\mathcal{H}}_{L+q|L}^T \tag{48}$$

where $\tilde{\mathcal{H}}_{L+q|L}$ is a submatrix of $\mathcal{H}_{L+q}$ given by

$$\tilde{\mathcal{H}}_{L+q|L} = \begin{bmatrix} D_L & & & \cdots & 0 \\ CB_L & D_{L+1} & & \cdots & 0 \\ CAB_L & CB_{L+1} & D_{L+2} & \cdots & 0 \\ CA^2B_L & CAB_{L+1} & CB_{L+2} & \cdots & 0 \\ \vdots & & & & \vdots \\ CA^{q-2}B_L & CA^{q-3}B_{L+1} & CA^{q-4}B_{L+2} & \cdots & D_{L+q-1} \end{bmatrix}. \tag{49}$$

The key result is as follows:

Theorem 9: Assume the same hypothesis as for Theorem 8. Assume further that $\Phi(z)$ has no zeros. Then for some $L \le n$, there holds $T_L = T_{L+1} = T_{L+2} = \ldots$. Further, with block orthogonal multiplication on the right if required, $\tilde{\mathcal{H}}_{L+q|L} = \tilde{\mathcal{H}}_{L+q+1|L+1} = \tilde{\mathcal{H}}_{L+q+2|L+2} = \ldots$, implying that for all $K \ge L$, the matrices $\mathcal{H}_K$ with first L block rows and columns deleted are Toeplitz.

Proof: Beginning with the top left corner, it is not hard utilizing the equations (45), (46) and (47) to obtain, with $\mathcal{O}_q = [C^T\ (CA)^T\ (CA^2)^T\ \ldots\ (CA^{q-1})^T]^T$

$$E\left[\tilde{\mathcal{Y}}_{L+q|L}\tilde{\mathcal{Y}}_{L+q|L}^T\right] = \mathcal{T}_q - \mathcal{O}_qT_L\mathcal{O}_q^T. \tag{50}$$

Now in light of the zero-free property, we know for some finite L and all nonnegative q, the estimation of values of the y-process q time steps in the future requires at most L successive past values, in the sense that if $L+1, L+2, \ldots$ values were available, no improvement in the error covariance would result. [We have of course earlier argued that the no-zero property guarantees that Kalman filtering can occur with zero error using a finite number of past values of the output, and no knowledge of the initial state. This is equivalent to saying that in estimating present and future outputs, apart from uncertainty associated with the present and future inputs, all uncertainty associated with all past inputs can be eliminated with knowledge of a *finite interval* of past inputs.]

The property can be expressed as

$$E\left[\tilde{\mathcal{Y}}_{L+q|L}\tilde{\mathcal{Y}}_{L+q|L}^T\right] = E\left[\tilde{\mathcal{Y}}_{L+q+1|L+1}\tilde{\mathcal{Y}}_{L+q+1|L+1}^T\right] = \ldots \tag{51}$$

Then it is clear that this implies

$$\mathcal{O}_qT_L\mathcal{O}_q^T = \mathcal{O}_qT_{L+1}\mathcal{O}_q^T = \ldots \tag{52}$$

Since the pair $[A, C]$ is observable, this implies that $T_L = T_{L+1} = T_{L+2} = \ldots$. It is straightforward then to see using the defining equations for T_k, B_k, D_k that B_k, D_k for $k \ge L$ can be chosen to be independent of k, and then the remaining claim of the theorem follows.

Actually, if there holds $T_L = T_{L+1}$ or equivalently only the first equality of (51) is known to hold, or equivalently again, $\tilde{\mathcal{H}}_{L+q|L} = \tilde{\mathcal{H}}_{L+q+1|L+1}$ (which is an equality involving different block submatrices of $\mathcal{H}_{L+q+1}$), it is straightforward to see using the defining equations for T_k, B_k, D_k that $T_L = T_{L+1} = T_{L+2} = \ldots$. ■

In summary, starting with an infinite block Toeplitz matrix $\mathcal{T}$ corresponding to a covariance sequence for which a bound on the McMillan degree of the associated power spectral matrix is known, one performs a block Cholesky decomposition described, and if one finds that certain submatrices of the Cholesky factor become identical before the number of block rows factored has attained the McMillan degree bound, one knows that one has the no-zero property, along with an ability to estimate the current state perfectly and forecast the future state using a finite interval of past data.

VII. Conclusion

In this paper, we have considered the properties of the output spectral matrix obtained from a stable finite-dimensional linear system excited by white noise, under the assumption that the input dimension is less than the output dimension. A number of interesting properties flow from the fact that generically, the underlying systems has no finite zeros, and may have no zeros at infinity either. This property can be checked with spectral data, i.e., without knowing the system generating the spectrum, and the paper views this property using Kalman filtering ideas. Such systems have the property that the input and state at a given instant of time can be constructed from a finite interval of outputs. We have shown how the problem of spectral factorization can be solved in a finite number of rational calculations, using different viewpoints, including use of a Riccati difference equation, and with a Toeplitz matrix of covariance data.

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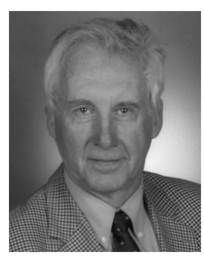

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