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AN ASYMPTOTIC ITERATION METHOD FOR THE NUMERICAL ANALYSIS OF NEAR-CRITICAL FREE-SURFACE FLOWS

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Dedicated to Prof. Dr.-Ing. Dr.techn. E.h. Dr.h.c. Jürgen ZIEREP on the Occasion of his 80th Birthday

ABSTRACT

Satisfying the boundary conditions at the free surface may impose severe difficulties to the computation of turbulent open-channel flows with finite-volume or finite-element methods, in particular, when the flow conditions are nearly critical. It is proposed to apply an iteration procedure that is based on an asymptotic expansion for large Reynolds numbers and Froude numbers close to the critical value 1.

The iteration procedure starts by prescribing a first approximation for the free surface as it is obtained from solving an ordinary differential equation that has been derived previously by means of an asymptotic expansion [1]. The numerical solution of the full equations of motion then gives a surface pressure distribution that differs from the constant value required by the dynamic boundary condition. To determine a correction to the elevation of the free surface we next solve an ordinary differential equation that is obtained from the asymptotic analysis of the flow with a prescribed pressure disturbance at the free surface. The full equations of motion are then solved for the corrected surface, and the procedure is repeated until criteria of accuracy for surface elevation and surface pressure, respectively, are satisfied.

The method is applied to an undular hydraulic jump, using the finite-volume code FLUENT to obtain the numerical solutions.

Keywords: asymptotic iteration method, critical flow, free-surface flow, free-boundary problem, turbulent flow, undular hydraulic jump

NOMENCLATURE

A	[-]	coupling parameter, Eq. (9)
c_f	[-]	friction coefficient
Fr	[-]	Froude number, Eq. (4)
G_1	[-]	auxiliary function, Eq. (15)
g	[m/s^2]	gravity acceleration
H	[-]	dimensionless surface elevation, Eq. (2)
h	[m]	elevation of surface above bottom
P	[-]	dimensionless pressure, Eq. (2)
p	[Pa]	pressure
r	[-]	relaxation factor, Eq. (18)
U, V	[-]	dimensionless velocity components in X and Y directions, respectively, Eq. (2)
U_τ	[-]	dimensionless friction velocity, Eq. (2)
u, v	[m/s]	velocity components in x and y directions, respectively
u_τ	[m/s]	friction velocity, Eq. (1)
$\dot{V}$	[m^2/s]	volume flow rate per unit width
X, Y	[-]	dimensionless x and y coordinates, respectively, Eq. (2)
x, y	[m]	longitudinal and transversal coordinates, respectively, Fig. 1
α	[-]	channel slope
β	[-]	damping parameter, Eq. (14)
δ	[-]	stretching parameter, Eq. (8)
Δ	[-]	surface correction, Eq. (19)
ε	[-]	perturbation parameter, Eq. (7)
ρ	[kg/m^3]	density of liquid

Subscripts

r	reference state
S	surface
1, 2, …	orders in asymptotic expansion

Superscripts

(1), (2), … , (n), (n+1)	number of iteration steps
(num)	numerical result
–	time-averaged quantity
'	turbulent fluctuation

1. INTRODUCTION

Presently available finite-element and finite-volume codes, e.g. FLUENT, allow the computation of turbulent shear flows with free surfaces. But finding the solution of a particular problem is not always an easy task. The system of equations is highly non-linear, in particular, when advanced turbulence models are incorporated. Furthermore, non-linear boundary conditions have to be satisfied at the a-priori unknown free surface. Thus it can be rather difficult, if not impossible, to come up with a first guess of the free surface that is sufficiently close to the correct solution to allow convergence. The numerical difficulties are particularly pronounced when the Froude number is close to the critical value 1; cf. [2] for an example. The near-critical flow field is very sensitive to perturbations, and, furthermore, the free surface may attain a wavy shape. In addition, the wavy surface may be associated with several transitions from super-critical to subcritical states, and vice versa. A typical example is the well-known undular hydraulic jump.

It appears that computational methods have, so far, not gained much from the progress that has been made in recent years in the asymptotic analysis of undular hydraulic jumps; cf. [1] and further references given below. Thus the present work has been undertaken with the aim of extending the analysis given in [1] such as to provide a method that is suitable for determining iteratively the free surface of turbulent flows with Froude numbers close to the critical value 1.

2. BASIC EQUATIONS

To be specific, open channel flow over a plane bottom with small, constant slope α is considered, cf. Fig. 1. The effects of surface tension will be neglected. The time-averaged flow is assumed to be steady and two-dimensional (plane).

As usual, mean values are characterized by a bar, while a prime refers to the fluctuation with respect to the mean. A Cartesian coordinate system is introduced, with the *x* axis in the plane of the bottom and the *y* axis pointing towards the surface. The components of the flow velocity in *x* and *y* direction are denoted by *u* and *v*, respectively. The mean elevation of the free surface above the bottom is $\bar{h}(x)$.

In determining $\bar{h}(x)$ by computational methods, the main difficulty originates from the boundary condition of constant pressure, *p*, at the surface. Without loss of generality, $p = 0$ may be used as a boundary condition at the free surface. However, for the purpose of providing a basis for an iteration procedure (as will be described below), a non-vanishing pressure perturbation $p_S(x)$ acting on the surface is prescribed, cf. Fig. 1, and it is the aim of the following asymptotic analysis to find an approximate solution for this case.

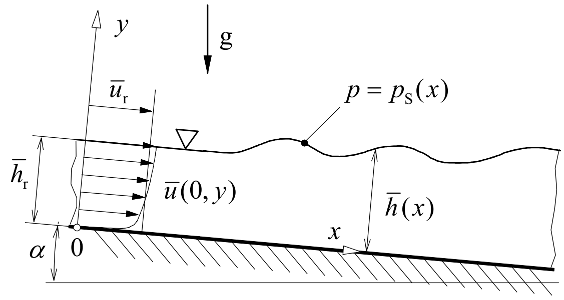

Figure 1. Open-channel flow under nearly critical conditions with prescribed pressure disturbance at the free surface

2.1. Non-dimensional quantities

To define non-dimensional quantities, the fully developed flow over the bottom with slope α is chosen as a reference state. The surface height in the reference state is denoted by $\bar{h}_r$, cf. Fig. 1. Thus a suitable reference pressure is the hydrostatic pressure at the bottom in the reference state, i.e. $g\rho\bar{h}_r$, where g is the gravity acceleration and ρ is the constant density of the liquid. The Reynolds stresses are referred to

$$u_{\tau,r}^2 = g\alpha\bar{h}_r , \qquad (1)$$

with $u_{\tau,r}$ being the friction velocity in the reference state. Note that Eq. (1) follows from the equilibrium between gravity and friction forces in fully developed flow. Finally it is convenient to define a reference velocity $\bar{u}_r$ in terms of the volume flow rate (per unit width of the channel), $\dot{V}$, which is conserved in steady flow and, besides, is accessible to routine measurement techniques. Thus $\bar{u}_r$ is defined as volumetric mean velocity, i.e. $\bar{u}_r = \dot{V} / \bar{h}_r$.

It is then quite natural to refer the lateral coordinate and the surface elevation to $\bar{h}_r$. Concerning the longitudinal coordinate, however, it is convenient for numerical computations – and essential for an asymptotic analysis – of near-critical flows to allow for a coordinate stretching in terms of a small parameter, say δ. In view of the equation of continuity, the lateral velocity component has to be stretched accordingly. Hence, non-dimensional variables are defined as follows:

$$
\begin{aligned}
&X = \delta x / \overline{h}_\mathrm{r}\,, \quad Y = y / \overline{h}_\mathrm{r}\,, \quad \overline{H} = \overline{h} / \overline{h}_\mathrm{r}\,,\\
&\overline{U} = \overline{u} / \overline{u}_\mathrm{r}\,, \quad \overline{V} = \delta^{-1} \overline{v} / \overline{u}_\mathrm{r}\,,\\
&\overline{P} = \overline{p} / \mathrm{g}\rho \overline{h}_\mathrm{r}\,, \quad P_\mathrm{S} = p_\mathrm{S} / \mathrm{g}\rho \overline{h}_\mathrm{r}\,,\\
&\overline{U'^2} = \overline{u'^2} / u_{\tau,\mathrm{r}}^2\,, \quad \overline{V'^2} = \overline{v'^2} / u_{\tau,\mathrm{r}}^2\,,\\
&\overline{U'V'} = \overline{u'v'} / u_{\tau,\mathrm{r}}^2\,, \quad U_\tau = u_\tau / u_{\tau,\mathrm{r}}\,.
\end{aligned}
\tag{2}
$$

2.2. Equations of motion

Written in terms of the non-dimensional quantities defined above, the equations of motion for the mean flow become

$$
\begin{aligned}
&\frac{\partial \overline{U}}{\partial X} + \frac{\partial \overline{V}}{\partial Y} = 0\,;\\
&\mathrm{Fr}^2 \left(\overline{U} \frac{\partial \overline{U}}{\partial X} + \overline{V} \frac{\partial \overline{U}}{\partial Y} \right)\\
&= -\frac{\partial \overline{P}}{\partial X} + \frac{\alpha}{\delta} - \frac{\alpha}{\delta} \left(\delta \frac{\partial \overline{U'^2}}{\partial X} + \frac{\partial \overline{U'V'}}{\partial Y} \right)\,;\\
&\delta^2\, \mathrm{Fr}^2 \left(\overline{U} \frac{\partial \overline{V}}{\partial X} + \overline{V} \frac{\partial \overline{V}}{\partial Y} \right)\\
&= -\frac{\partial \overline{P}}{\partial Y} - 1 - \alpha \left(\delta \frac{\partial \overline{U'V'}}{\partial X} + \frac{\partial \overline{V'^2}}{\partial Y} \right)\,,
\end{aligned}
\tag{3}
$$

where

$$
\mathrm{Fr} = \overline{u}_\mathrm{r} / \sqrt{\mathrm{g} \overline{h}_\mathrm{r}}
\tag{4}
$$

is the Froude number. Viscous stresses are neglected in Eq. (3). This requires the use of matching conditions as boundary conditions at the bottom.

2.3. Boundary conditions

Matching to the viscous wall layer is accomplished by satisfying the conventional boundary condition for the normal velocity component, i.e. $\overline{V}(X,0) = 0$, and by applying the logarithmic “law of the wall” (“overlap law”) [3].

If the time-averaged free surface is defined as a streamline in the time-averaged velocity field, the kinematic boundary condition simply reads

$$
\overline{V}(X, \overline{H}) = \overline{U}(X, \overline{H}) \frac{\mathrm{d}\overline{H}}{\mathrm{d}X}\ .
\tag{5}
$$

With surface tension and viscous stresses being neglected, the dynamic boundary conditions reduce to the conditions that, firstly, the tangential Reynolds stress component must vanish at the free surface, and, secondly, the normal component of the Reynolds stress must balance the pressure disturbance at the free surface. While it is rather easy to satisfy the first condition with presently available computational codes, e.g. by applying a local symmetry condition at each point of the free surface, it is the second condition that usually causes trouble in the computations, in particular if the Froude number is close to the critical value 1. An iteration procedure based on an asymptotic analysis can be of some help in such cases.

3. ASYMPTOTIC ANALYSIS

The basis of the iteration procedure will be an asymptotic solution for a prescribed surface pressure disturbance $P_\mathrm{S}(X)$. This gives rise to additional terms in the dynamic boundary conditions. To avoid unnecessary complexity, only the leading terms of an expansion in terms of δ are taken into account, implying a long-wave approximation. This gives

$$
\begin{aligned}
&\left[\overline{P}(X, \overline{H}) + \alpha \overline{U'^2}(X, \overline{H}) \right] \delta \frac{\mathrm{d}\overline{H}}{\mathrm{d}X}\\
&\quad - \alpha \overline{U'V'}(X, \overline{H}) = P_S(X) \delta \frac{\mathrm{d}\overline{H}}{\mathrm{dX}}\,;\\
&\left[\overline{P}(X, \overline{H}) + \alpha \overline{V'^2}(X, \overline{H}) \right]\\
&\quad - \alpha \overline{U'V'}(X, \overline{H}) \delta \frac{\mathrm{d}\overline{H}}{\mathrm{d}X} = P_S(X)\,.
\end{aligned}
\tag{6}
$$

3.1. Asymptotic expansions

Since it is assumed that the Froude number differs but little from the critical value 1, a perturbation parameter ε is defined according to the relationship

$$
\mathrm{Fr} = 1 + \tfrac{3}{2} \varepsilon\,,
\tag{7}
$$

where the coefficient $\frac{3}{2}$ is introduced in order to obtain the main result of the analysis in a convenient form. Note that the reference state is a supercritical flow for $\varepsilon > 0$, whereas it is subcritical in case of $\varepsilon < 0$.

Following [1], the non-dimensional variables are expanded in terms of powers of the perturbation parameter ε. There are, however, two other small parameters in the problem, i.e. the stretching parameter δ and the bottom slope α. As shown in [1], the asymptotic analysis of two-dimensional (rather than one-dimensional) near-critical flow requires that $\delta^2 = \mathrm{O}(\varepsilon)$. Introducing a coefficient 3 for later convenience gives

$$
\delta = 3\sqrt{|\varepsilon|}\ .
\tag{8}
$$

To fix the relative order of α and ε, a coupling parameter A is introduced with the relation

$$A = \alpha / \varepsilon^2 . \quad (9)$$

As shown in [1], $A = \mathrm{O}(1)$ ensures that the order of magnitude of the Reynolds stresses is sufficiently small to make an analysis without turbulence modelling feasible.

Finally it is assumed that, in the course of the iteration process, the actual surface differs but slightly from the correct one. On the basis of ref. [1] it can be anticipated that the equation for the first-order surface perturbation will be obtained from a solvability condition for the second-order equations. Thus the surface pressure disturbance is assumed to be as small as ε^2, i.e.

$$P_S = \varepsilon^2 P_{S2}(X) , \quad (10)$$

with $P_{S2}(X) = \mathrm{O}(1)$.

Hence, with the reference state (i.e. fully developed flow) taken as the basic state, the expansions are as follows:

$$\begin{aligned} \overline{H}(X) &= 1 + \varepsilon H_1(X) \\ &\quad + \varepsilon^2 H_2(X) + O(\varepsilon^3) ; \\ \overline{P}(X,Y) &= 1 - Y + \varepsilon P_1(X,Y) \\ &\quad + \varepsilon^2 P_2(X,Y) + O(\varepsilon^3) ; \\ \overline{U'V'}(X,Y) &= -1 + Y + \varepsilon (\overline{U'V'})_1(X,Y) \\ &\quad + \varepsilon^2 (\overline{U'V'})_2(X,Y) + O(\varepsilon^3) , \end{aligned} \quad (11)$$

and analogous equations hold for the other dependent variables.

3.2. First-order solution

Expanding the basic equations and the boundary conditions according to Eq. (11) up to first-order quantities gives differential equations that can easily be solved to obtain the following solutions [1]:

$$P_1 = H_1 , \ (\overline{U'V'})_1(X,0) = 2H_1 . \quad (12)$$

To determine $H_1(X)$, i.e. the first-order perturbation of the surface elevation, the second-order equations have to be inspected. The analysis, which requires a careful consideration of the velocity defect in the turbulent shear flow together with the logarithmic "law of the wall", follows ref. [1]. Without recourse to turbulence modelling, the following differential equation is obtained from a solvability condition:

$$\frac{\mathrm{d}^3 H_1}{\mathrm{d}X^3} + (H_1 - 1)\frac{\mathrm{d}H_1}{\mathrm{d}X} - \beta H_1 = -\frac{1}{3}\frac{\mathrm{d}P_{S2}}{\mathrm{d}X} , \quad (13)$$

with the "damping coefficient"

$$\beta = \tfrac{1}{3} A \sqrt{|\varepsilon|} = \tfrac{1}{3}\alpha |\varepsilon|^{-3/2} . \quad (14)$$

For constant surface pressure, the term on the right-hand side of Eq. (13) vanishes, and Eq. (13) reduces to the homogeneous equation derived previously in ref. [1].

Eq. (13) may be seen as an extension of the steady-state version of the Korteweg - de Vries equation. Equations of this type have been extensively investigated in the past, in particular by R. Grimshaw and colleagues, cf. [4], and also by H. Steinrück, cf. [5].

4. ITERATION PROCEDURE

The computation of turbulent open channel flows by numerical means, e.g. by finite-volume or finite element methods, requires to satisfy the dynamic boundary conditions at the a-priori unknown free surface. For dealing with this problem in the notoriously cumbersome case of Froude numbers close to the critical value 1, an iteration procedure based on Eq. (13) is proposed in what follows. Applying Eq. (13) directly is, however, not quite convenient, as it would require numerical differentiation of the surface pressure $P_S(X)$ that is obtained from the numerical solution of the full equations of motion. Thus it appears preferable to integrate Eq. (13) to obtain the following system of differential equations:

$$\begin{aligned} &\frac{\mathrm{d}^2 H_1}{\mathrm{d}X^2} + \tfrac{1}{2}H_1^2 - H_1 - \beta G_1 \\ &\quad = -\tfrac{1}{3}P_{S2} + \left.\frac{\mathrm{d}^2 H_1}{\mathrm{d}X^2}\right|_{X=0} , \\ &\frac{\mathrm{d}G_1}{\mathrm{d}X} = H_1 , \end{aligned} \quad (15)$$

with

$$H_1(0) = 0, \ \left.\frac{\mathrm{d}H_1}{\mathrm{d}X}\right|_{X=0} = 0, \ G_1(0) = 0. \quad (16)$$

$P_{S2}(0)$ has been dropped in the first equation of the system (15) on the basis of the assumption that the numerically obtained surface pressure disturbance vanishes for the reference state, i.e. a fully developed flow with $H_1 = 0$. The choice of a proper value of $\mathrm{d}^2 H_1/\mathrm{d}X^2$ at $X = 0$ depends on the particular problem; cf. Section 5 for an example.

The first step in the iteration procedure is to take the result of the (exact) asymptotic expansion for $\mathrm{Fr} \to 1$, i.e. $\varepsilon \to 0$ according to Eq. (7), as a first approximation for small, but finite values of ε; i.e. the first approximation for the elevation of the free surface is

$$\overline{H}^{(1)}(X) = 1 + \varepsilon H_1^{(1)}(X) . \qquad (17)$$

Surface tension and Reynolds normal stresses being neglected as discussed above, the dynamic boundary condition requires constant pressure at the free surface. Thus the equations to be solved in order to find $H_1^{(1)}(X)$ are Eqs. (15) and (16) with $P_{S2} \equiv 0$. According to Eq. (14), the damping parameter β is small compared to 1, which is also in accord with conditions of practical interest. Though for small values of β a solution in terms of slowly modulated Jacobian elliptical functions (cn functions) is available [5], it is recommended to solve the system of Eqs. (15) and (16) numerically, e.g. by a Runge-Kutta method, for further use in the (numerical) iteration procedure.

Prescribing this first approximation for the free surface as a fixed boundary, a numerical solution of the full equations of motion (i.e. Eq. (3) if the present non-dimensional variables are used) is determined. Of course, the system of equations has to be closed by applying a suitable turbulence model. Boundary conditions at the bottom may be taken into account in any appropriate manner. As boundary conditions at the surface, the kinematic boundary condition and the condition of vanishing Reynolds shear stress are prescribed. Since the first approximation of the surface elevation is, most likely, not good enough to satisfy the dynamic boundary condition for the normal stresses with sufficient accuracy, the non-vanishing pressure at the surface is considered as a perturbation, and a second approximation for the free surface is obtained from the results of the asymptotic analysis given above, i.e. Eqs. (15) and (16).

To start the iteration, the system of Eqs. (15) and (16) is solved with

$$P_{S2} = r P_{S2}^{(\mathrm{num})} = r \varepsilon^{-2} P_S^{(\mathrm{num})} , \qquad (18)$$

where $P_S^{(\mathrm{num})}$ is the surface pressure disturbance that is obtained from the numerical solution of the full equations of motion, and $0 < r < 1$ is a relaxation factor introduced to improve convergence. The result for H_1 differs, of course, from the solution for constant surface pressure, i.e. $H_1^{(1)}$. Since the numerical solution ought to satisfy the condition of constant surface pressure, the surface elevation is corrected by the amount

$$\Delta(X) = \varepsilon [H_1^{(1)}(X) - H_1(X)] \qquad (19)$$

to obtain the following second approximation:

$$\overline{H}^{(2)}(X) = \overline{H}^{(1)}(X) + \Delta(X) \; . \qquad (20)$$

The procedure is then repeated as follows. The system of Eqs. (15) and (16) is solved with the new numerical surface pressure disturbance to obtain a

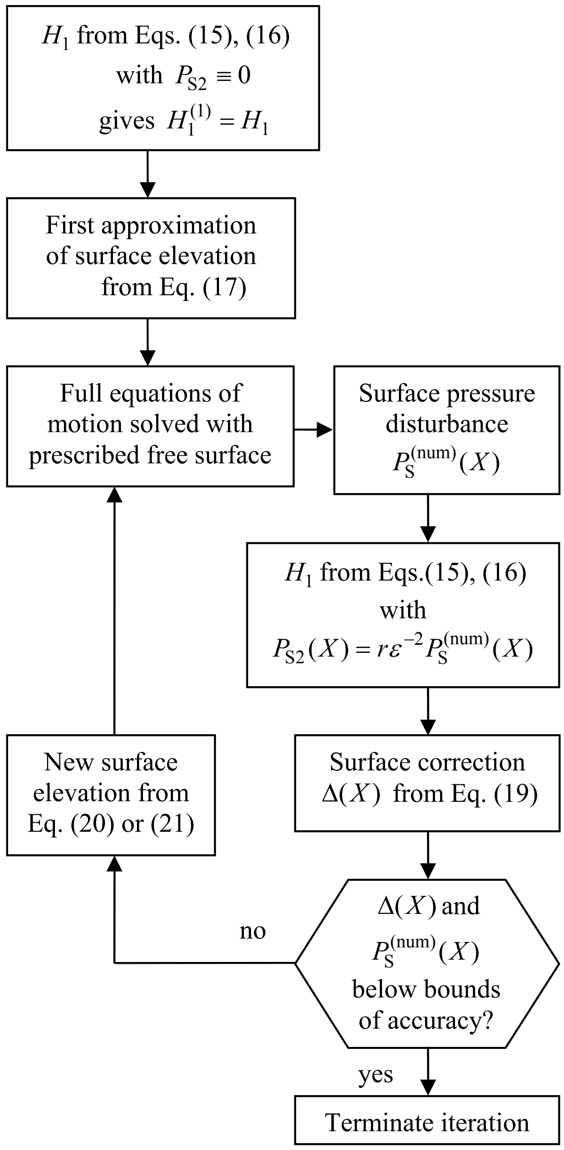

Figure 2. Flow chart of the iteration procedure

new function $H_1(X)$ and, with Eq. (19), a new surface correction $\Delta(X)$. The new surface elevation is then determined from the equation

$$\overline{H}^{(\mathrm{n}+1)}(X) = \overline{H}^{(\mathrm{n})}(X) + \Delta(X) , \quad (\mathrm{n} = 2, 3, 4, ...) . \qquad (21)$$

The iteration is continued until the surface corrections and the surface pressure disturbances are found to be below prescribed bounds of accuracy. A flow chart of the iteration procedure is given in Fig. 2.

5. APPLICATION: THE UNDULAR JUMP

An example of near-critical free-surface flow is the undular hydraulic jump. This is a peculiar version of the hydraulic jump that is observed when the upstream Froude number is slightly *larger* than 1, i.e. $\varepsilon > 0$ in the present notation. The undular jump, which is of relevance for applications in nature and engineering, has been investigated both analytically and experimentally, but the present authors were unable to solve the undular-jump problem by the standard numerical techniques provided by FLUENT 6.2.16. Thus the undular-jump problem was chosen as a test case for the proposed iteration procedure.

Though more general upstream conditions may be of interest, cf. [6], [7], it was assumed that the flow upstream of the undular jump, say at $X = 0$, is fully developed; cf. Fig. 1. This upstream state was, therefore, taken as the reference state, with $H_1(0) = 0$. A small initial disturbance of $\mathrm{d}^2 H_1/\mathrm{d}X^2$ at $X = 0$ in Eq. (15) generates a surface that resembles the undular jump that is to be computed. A value of 0.1 was chosen for the initial disturbance in the present case. Other (small) values were found to change mainly the location of the undular jump, but to have little effect on the shape.

The numerical solution of the full equations of motion was obtained by applying the finite-volume code FLUENT 6.2.16. At the bottom, FLUENT's standard wall functions were used. The kinematic boundary condition as well as the dynamic boundary condition for the tangential stress components was satisfied by applying local symmetry conditions at the surface, whereas the dynamic boundary condition for the normal stresses was satisfied iteratively as described above. To make the local symmetry conditions applicable, the hydrostatic pressure was split off according to the relation $\tilde{p} = p - p_{\mathrm{h}}$, with $p_{\mathrm{h}} = \mathrm{g}\rho(\overline{h}_{\mathrm{r}} - y)$. The FLUENT computations were then made with $\tilde{p}$ as the pressure, and the hydrostatic contribution to the pressure was added after each iteration step to obtain the surface pressure. FLUENT's 2D double precision pressure-based segregated solver with a first-order upwind discretisation scheme and standard criteria of convergence and residua, respectively, were applied in each iteration step. Using GAMBIT, the computational grid was recreated after each iteration step according to the change in geometry as obtained from Eq. (21), dividing a domain of approximately 0.1m height and 2.05m length into 60×1400 quadrilateral cells. The whole iteration process was controlled by a shell-script batch procedure.

Two different turbulence models, i.e. the standard k-epsilon model and the Reynolds stress model, respectively, were applied in order to get an idea about the influence of turbulence modelling on the numerical results. Using the k-epsilon model, the errors after 100 iteration steps with a relaxation factor of 0.1 and a computing time (as given in FLUENT's output files) of about 5 days on a single processor Intel Xenon 36 bit CPU machine were found to be bounded as follows:

$$\mathrm{Max}\{\Delta(\mathrm{X})\} = 4.3 \times 10^{-5},$$
$$\mathrm{Max}\left\{P_{\mathrm{S}}^{(\mathrm{num})}(\mathrm{X})\right\} = 1.4 \times 10^{-4}.$$

To save computing time, the computations with the Reynolds stress model were started with the final k-epsilon solution for the free surface. After 100 iteration steps with a relaxation factor of 0.1 and a computing time of about 2.7 days the errors were found to satisfy

$$\mathrm{Max}\{\Delta(\mathrm{X})\} = 1.2 \times 10^{-4},$$
$$\mathrm{Max}\left\{P_{\mathrm{S}}^{(\mathrm{num})}(\mathrm{X})\right\} = 3.6 \times 10^{-4}.$$

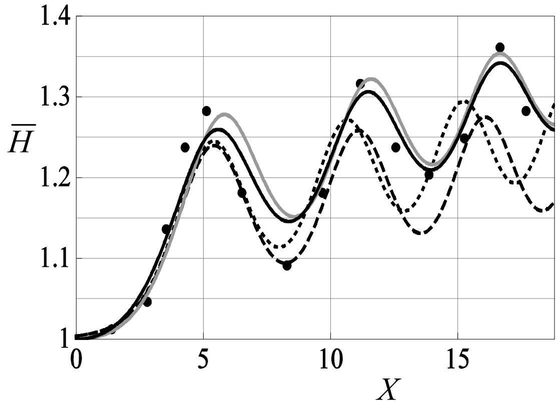

Figure 3. Non-dimensional surface elevation. Dots: experimental values [8]-[10]. $\mathrm{Fr} = 1.11$ **(i.e.** $\varepsilon = 0.076$**),** $\mathrm{Re} = 93{,}000$ **.**
Solid black line: FLUENT, Reynolds stress model.
Solid grey line: FLUENT, k-epsilon model.
Dashed line: asymptotic analysis with measured value of friction coefficient, $c_{\mathrm{f}} = 0.0038$ **.**
Dotted line: asymptotic analysis with measured value of bottom inclination angle, $\alpha = 1/282$ **.**

Figs. 3 and 4 show results for the mean surface elevation and the mean wall shear stress, respectively, as functions of the distance in main flow direction. All quantities are given in non-dimensional form. The parameters are chosen to allow comparison with the experimental data according to refs. [8] to [10]. To provide the required inflow velocity profile at $X = 0$ for the computations of the undular jump with FLUENT, the velocity profile of the *fully-developed* flow was computed on the basis of the measured values of the volume flow rate and the bottom slope, respectively.

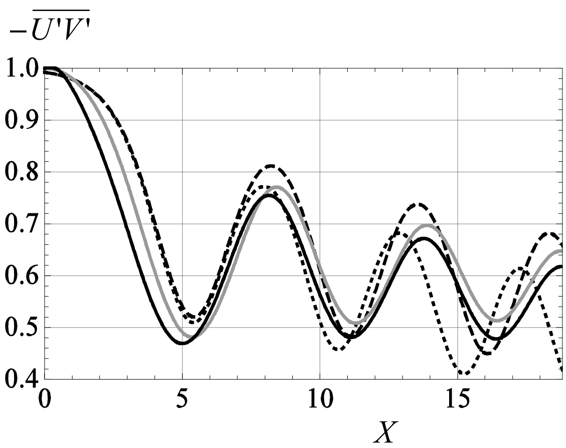

Figure 4. Non-dimensional wall shear stress. Parameters as in Fig. 3.
Solid black line: FLUENT, Reynolds stress model.
Solid grey line: FLUENT, k-epsilon model.
Dashed line: asymptotic analysis with measured value of friction coefficient, c_f .
Dotted line: asymptotic analysis with measured value of bottom inclination angle, α .

For the purpose of comparison, the results of the asymptotic analysis, which is free of turbulence modelling, are also given in Figs. 3 and 4. The latter results were obtained from solving Eq. (13) with $P_{S2} \equiv 0$ subject to initial conditions describing small perturbations of the fully-developed flow far upstream. There is, however, the problem that the flow upstream the undular jump was supposedly fully developed in the experiment chosen for the comparison, but there is a discrepancy between the experimentally determined value of the friction coefficient and the measured value of the slope α as given in refs. [8]-[10]. To comply with this problem, the results of the asymptotic analysis were evaluated for two different cases, i.e. based on the friction coefficient and the bottom slope, respectively. Thus $H_1(0)$ was chosen to be $H_1(0) = 0.05$, while the first and second derivatives, respectively, at the reference position $X = 0$ were prescribed in accordance with an expansion of Eq. (13) for small values of H_1. Using the measured values of the *bottom-inclination angle* α, the expansion gives $\left(\mathrm{d}H_1/\mathrm{d}X\right)_{X=0} = 0.051$ and $\left(\mathrm{d}^2H_1/\mathrm{d}X^2\right)_{X=0} = 0.053$, while the slightly different results $\left(\mathrm{d}H_1/\mathrm{d}X\right)_{X=0} = 0.051$ and $\left(\mathrm{d}^2H_1/\mathrm{d}X^2\right)_{X=0} = 0.052$ are obtained with the measured values of the *friction coefficient* c_f .

6. CONCLUSIONS

The proposed iteration procedure, which is based on an asymptotic analysis, appears suitable to lend essential support to the numerical solution of near-critical free-surface flows. As a test case, the undular-jump problem was successfully solved. As a by-product of the numerical solutions it became evident that the choice of the turbulence model has little effect on the results. This may be understood as a consequence of the fact that the undular jump is characterized by small damping of the amplitudes, implying that the flow in the undular jump is nearly inviscid.

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