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The Role of Information Pattern in Approximation of Control Systems

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We consider a control system

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x^0 \in \mathbf{R}^n, \qquad t \in [0,T], \tag{1}$$

where $u \in \mathcal{U} \subset L_1([0,T] \mapsto U)$, U is a convex compact subset of $\mathbf{R}^r$. For $u \in \mathcal{U}$ denote by $x[u]$ the solution of (1) that corresponds to u (assuming existence and uniqueness). Control theory and the set-membership estimation theory raise two main approximation problems related to (1): (i) approximate the set of trajectories, $\mathcal{X} = \{x[u] : u \in \mathcal{U}\}$, of (1); (ii) approximate the reachable set, $R = \{x[u](T) : u \in \mathcal{U}\}$, of (1).

Since the set of admissible controls $\mathcal{U}$ contains rather irregular functions[1] it is natural to split the approximation problems of (1) into two parts:

(P1) Replace the set of admissible controls $\mathcal{U}$ by a finitely parameterized subset $\mathcal{V}_N$ consisting only of functions u for which (1) can be discretized efficiently;
(P2) Apply a discretization scheme for solving (1) for $u \in \mathcal{V}_N$.

The requirement that $\mathcal{V}_N$ is a finitely parameterizable set (say, with a "degree of freedom" proportional to N) is needed to make the approximation "computable". Moreover, for each $u \in \mathcal{V}_N$ equation (1) should be well discretizable by single step methods, that is, the restrictions of the functions from $\mathcal{V}_N$ to each interval $[t_k, t_{k+1}]$ are sufficiently regular (for example, polynomial functions of a fixed degree and with uniformly bounded coefficients).[2] Then the error analysis of the discretization can be carried out in the usual way as for differential equations. Therefore we focus on the error analysis for the problem (P1), where one needs to estimate the *uniform error*

$$H_C(\mathcal{X}, \mathcal{X}_N) = \sup_{u \in \mathcal{U}} \inf_{v \in \mathcal{V}_N} \|x[v] - x[u]\|_{C[0,T]}$$

and the *terminal error*

$$H(R, R_N) = \sup_{u \in \mathcal{U}} \inf_{v \in \mathcal{V}_N} |x[v](T) - x[u](T)|,$$

where $\mathcal{X}_N$ and R_N are the set of trajectories and the reachable set corresponding to the set $\mathcal{V}_N$ of admissible controls.

[1] We mention that the reachable set R is usually not generated by "nice" controls (differentiable, Lipschitz, continuous). Even more, control functions of unbounded variation or non-integrable in Riemann sense may generate points of R that are not reachable by other controls, as in Fuller's phenomenon or as in [10]. This creates the main difficulty of approximating (1) by discrete schemes.

[2] Of course, there is a trade-off in choosing $\mathcal{V}_N$: the larger is $\mathcal{V}_N$, the better is the approximation to $\mathcal{X}$ and R by controls from $\mathcal{V}_N$; on the other hand, the lower is the accuracy of discretization.

Let us assume that the mapping $u \longrightarrow x[u]$ is continuous in L_1 and $\mathcal{V}_N$ is compact in the same space, hence the infimum in v is achieved. (These assumptions are normally fulfilled, as it will be the case in all considerations below.) Then there exists a mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_N$ such that

$$\sup_{u \in \mathcal{U}} \|x[\pi_N(u)] - x[u]\|_{C[0,T]} = H_C(\mathcal{X}, \mathcal{X}_N),$$

or

$$\sup_{u \in \mathcal{U}} |x[\pi_N(u)](T) - x[u](T)| = H(R, R_N),$$

respectively (the mapping π_N needs not be the same in the two equalities). The formulation has an advantage: one can study the information pattern of the mapping π_N that provides the best approximations in (2) or (3), or at least some approximations with a given order of accuracy with respect to N. Namely, we can distinguish the following cases of a mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_N$:

Definition 1 *(i) The mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_N$ is called "local" if for every $k =$ $0, \ldots, N-1$, and for every u', $u'' \in \mathcal{U}$ with $u'(t) = u''(t)$ on $[t_k, t_{k+1}]$ it holds that $\pi_N(u')(t) = \pi_N(u'')(t)$ on $[t_k, t_{k+1}]$;*
(ii) The mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_N$ is called "non-anticipative" if for every $k = 1, \ldots, N$, and for every u', $u'' \in \mathcal{U}$ with $u'(t) = u''(t)$ on $[0, t_k]$ it holds that $\pi_N(u')(t) =$ $\pi_N(u'')(t)$ on $[0, t_k]$;
(iii) The mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_N$ is called "anticipative" if it is not non-anticipative.

We shall see that for the same approximating set of inputs $\mathcal{V}_N$ it may happen that a certain order of approximation can be achieved by anticipative (non-anticipative) *approximating mappings* π_N but cannot be achieved by non-anticipative (resp. local) mappings π_N. That is, the information pattern of the approximating mapping may play a role for the order of the accuracy.

It is to be stressed that in different problems related to the control system (1) one may need to restrict the choice of the approximation mapping to a prescribed information pattern: local or non-anticipative. This is the case, for example, if one has to simulate a real system modeled by (1) only knowing the current, or the past information about the input u. For other problems, say for an optimal open-loop control problem one can freely employ anticipative approximation mappings to pass directly to mathematical programming.

The concept of information pattern in the approximation theory for control systems (or systems with "deterministic" uncertainties) opens a new filed of research in this area. The aim of this note is to revisit some known results in the area from this perspective.

First we recall a few often used approximation sets $\mathcal{V}_N$. For any natural number N denote $h = T/N$, $t_i = ih$, $i = 0, \ldots, N$.

Example 1.

$$\mathcal{V}_N = \mathcal{V}_{N,0} := \{u \in \mathcal{U} : \ u(t) \text{ is constant on each } (t_{i-1}, t_i))\}.$$

Example 2.

$$\mathcal{V}_N = \mathcal{V}_{N,1} := \{u \in \mathcal{U} : \ u(t) \text{ is p.-w. constant with at most 1 jump in each } (t_{i-1}, t_i)\}.$$

Example 3. Denote by U^{extr} the set of all extreme points of U and define

$$\mathcal{V}_N = \mathcal{V}_{N,0}^{\text{extr}} := \{u : [0,T] \mapsto U^{\text{extr}} : \ u(t) \text{ is constant on each } (t_{i-1}, t_i))\}.$$

Clearly $\mathcal{V}_{N,0}^{\text{extr}} \subset \mathcal{V}_{N,0} \subset \mathcal{V}_{N,1}$.

Below we reformulate some known results in a way which exhibits the information pattern of the approximation mapping implicitly involved in the proofs, and comment the validity of the results if a more stringent information pattern has to be used.

One commonly used approximation mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_{N,0}$ is defined as

$$\pi_N(u)(t) = \frac{1}{h} \int_{t_{k-1}}^{t_k} u(s)\,\mathrm{d}s \text{ for } t \in (t_{k-1}, t_k). \tag{4}$$

Obviously it is local and even more, it is independent of the specific form of the equation (1).

Let us consider first a liner control system, where $f(x,u) = Ax + Bu$ in (1). From a general result in [3] it follows that there exists a *local* approximation mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_{N,0}$ (namely, defined by (4)) such that

$$\|x[\pi_N(u)] - x[u]\|_{C[0,1]} \le ch \qquad \forall u \in \mathcal{U}. \tag{5}$$

In the same time the results in [13, 1] imply that there exists an *anticipative* approximation mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_{N,0}$ (which is not explicitly defined in these papers) such that

$$|x[\pi_N(u)](T) - x[u](T)| \le ch^2 \qquad \forall u \in \mathcal{U}. \tag{6}$$

We mention that the result holds for an arbitrary convex and compact set U, therefore it applies also to the "pathological" examples in [10] mentioned in footnote 2. A second order [illegible] above cannot be achieved by using local approximation mappings.

An important [illegible] in [9]: there exists an *anticipative* approximation mapping π_N : [illegible] simultaneously (5) and (6). This non-trivial result opens [illegible] for non-linear systems by local linearization. Of course, in [illegible] this simultaneous approximation cannot be achieved in [illegible] approximation mappings.

We recall also two results, again for a linear system as above, for the class $\mathcal{V}_{N,1}$ of approximating controls, restricted to polyhedral sets U. From the results in [5] it can be deduced that there exists a *local* approximation mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_{N,1}$ for which an estimation such as in 6 holds. On the other hand, the results in [1] show that there exists an *anticipative* $\pi_N : \mathcal{U} \mapsto \mathcal{V}_{N,1}$ such that

$$|x[\pi_N(u)](T) - x[u](T)| \leq ch^3 \qquad \forall u \in \mathcal{U}.$$

Third order accuracy cannot be achieved in the class $\mathcal{V}_{N,1}$ by local approximation mappings.

Now we turn to the non-linear case, assuming that the right-hand side in (1) has the form

$$f(x,u) = g_0(x) + \sum_{i=1}^{m} g_i(x)u_i, \qquad u = (u_1, \ldots, u_m) \in U.$$

We assume that U is convex and compact and that g_i are sufficiently smooth. There are only a few results obtaining higher than first order approximations in the nonlinear case.[3] The first is that in [12], where a second order approximation of $\mathcal{X}$ is proved (or an approximation of order 3/2 for a more general form of f than the above one) assuming, however, that $f(x, U)$ is uniformly strongly convex, which is rather strong for many applications. The implicitly involved approximation mapping π_N is *local.* Another group of results concern the case of commutative affine systems, i.e. such that the Lie brackets $[g_i, g_j]$ are all zero for $i, j \geq 1$. A rather general indirect (variational) estimation of $H(R, R_N)$ in the class $\mathcal{V}_{N,0}$ is obtained in [14]. It allows to obtain a second order estimation of $H(R, R_N)$ provided that R and R_N have the so called *external ball property.* This property is present if R and R_N are convex, for example. The approximation mappings π_N implicitly involve are *anticipative.* The drawback is that the external ball property does not hold, in general, and is difficult for verification.

A more advanced result for commutative systems is proved in a paper in preparation by the author and M. Krastanov. Namely, there exists a constant C such that for every N there exists an (*anticipative*) approximation mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_{N,0}$ such that

$$\|x[\pi_N(u)] - x[u]\|_{C[0,T]} \leq Ch, \tag{7}$$

$$|x[\pi_N(u)](T) - x[u](T)| \leq Ch^{1.5}. \tag{8}$$

It is an open question if the estimation is sharp, but in any case the second estimation cannot be achieved by using local approximation mappings only. Whether the result

[3] Higher than first order approximations to optimal control problems are known. However, most of these results are based on a priori assumption that the optimal control is sufficiently regular (i.e. is Lipschitz with derivative having bounded variation), see e.g. [4, 7]. The results recalled in the present paper are applicable in the optimal control context without such assumptions.

holds for a non-local but non-anticipative approximation mappings is another open question.

Higher than first order approximations are particularly difficult for non-commutative systems. An estimation like in (8) for class of non-commutative bilinear systems is proved in [9]. The class approximating controls is again $\mathcal{V}_{N,0}$ and the employed approximation mapping π_N is *anticipative*. A different result for non-commutative systems without drift term ($g_0 = 0$) and with $U = [0,1] \times [0,1] \subset \mathbf{R}^2$ is presented in [8]. The error estimate in terms of the trajectories is like in (7) but with h^2 in the right-hand side (that is, of second order) and it is achieved by *local* approximation mappings. However, the approximating control set $\mathcal{V}_N$ is much reacher even than $\mathcal{V}_{N,1}$, which creates a substantial problem in the numerical realization. The main contribution in this paper is, in fact, how to cope with this problem.

The last issue we briefly address is that of approximations using the class $\mathcal{V}_{N,0}^{\text{extr}}$ of controls. This issue is of substantial importance for numerical treatment of optimal control problems for switching systems, see e.g. [11]. The following estimation is proved, essentially, in [2, 6]: for the approximating class of controls $\mathcal{V}_{N,0}^{\text{extr}}$,

$$H_C(\mathcal{X}, \mathcal{X}_N) \le Ch^{1/2}. \tag{9}$$

This estimation is proved for the general system (1) under Lipschitz continuity of f. In [15] the author of the present paper conjectured that a first order estimation holds in (9) and proved this in several particular classes of systems. The paper [9] also contains a small contribution in this direction. A substantial progress in proving the conjecture is done in [11], where however, U is assumed to be a polyhedral set. In all the above contributions the (implicitly or explicitly) involved approximation mapping π_N is *non-local* and *non-anticipative*. Also, it is quite clear that local approximation mappings cannot provide even (9). In general, the problem of first order approximation is still open.

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