Fluid mechanics of lubrication I: fundamental aspects of a rigorous theory

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MINILUBES AC²Tion Day
8 June 2011
Main objectives

- existing gap in tribological literature: lubrication represented ‘unsatisfactorily accurate’ ⇒
- describing lube flows by adopting first principles of continuum mechanics: asymptotic theory of hydromechanical lubrication

Why is this expedient?

- rational estimate of methodical error
- rational extension of classical theory to include e.g. EHD, inertia, micro-scale effects (cavitation, surface roughness)
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Overview

1. Phenomenon of lubrication

2. Basic assumptions

3. Classical theory
   - First principles
   - Problem formulation
   - Asymptotic theory

4. Validation of tribo-systems

5. Further outlook
Phenomenon of lubrication

Pressurised counter-sliding (tilted) solid contacts: Striebeck curve

\[ \mu = \frac{\tilde{\tau}}{\tilde{\rho}} = \Pi(\text{Str}, \alpha, \ldots) \]

Str \gg 1:

\[ \frac{\tilde{\tau} \tilde{C}}{\tilde{\eta} \tilde{U}} \sim \text{const} \]

\[ \alpha = 0 \]

Fixed \( \tilde{\rho} > 0 \)

Relative motion \( d\tilde{\rho} < 0 \)

\[ \text{boundary} \quad \text{mixed-film} \quad \text{laminar hydrodynamic} \]

\[ \text{lubrication} \]

B. Scheichl (AC²T, VUT)

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Phenomenon of lubrication

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Basic (realistic) assumptions

lubricant flow

- ‘simple’ fluid
  excludes multi-phase flow (binary mixture lubricant–air):
  2 (intensive) state variables define local thermodynamic equilibrium
- Newtonian fluid
  lube oils, ionic liquids (vapour pressure very low), H$_2$O, many gases:
  at normal conditions, even for high pressures & shear rates, not for low
  temperatures
- laminar
- volume forces (gravity) neglected

bearing geometry

- clearance slender
  compared to typical macro-length (e.g. journal radius)
- perfectly hydrodynamic operation
  ‘hydraulically smooth’ surfaces:
  macroscopic flow description unaffected by mean asperities
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Outline

3 Classical theory
   • First principles
   • Problem formulation
   • Asymptotic theory
Governing eqs in Eulerian representation

any reference frame \( \tilde{x}, \tilde{t} \)

\[
\tilde{D}_t := \partial_{\tilde{t}} + \tilde{u} \cdot \nabla (\tilde{x})
\]

continuity

\[
\tilde{D}_t \tilde{\rho} + \tilde{\rho} \nabla \cdot \tilde{u} = 0
\]

momentum

\[
\tilde{p}\left(\tilde{x}_{\text{ref}} + 2\tilde{\Omega}_{\text{ref}} \times \tilde{u} + \tilde{D}_t \tilde{u}\right) = \nabla \cdot \tilde{\Sigma}, \quad \tilde{\Sigma} = -\tilde{\rho} I + \tilde{\Delta}, \quad \tilde{\Delta} = \tilde{\Delta}^{tr}
\]

thermal energy, 1st & 2nd law of thermodynamics

\[
\tilde{p} \tilde{c}_p \tilde{D}_t \tilde{T} = \beta \tilde{T} \tilde{D}_t \tilde{\rho} + \tilde{\Phi} - \nabla \cdot \tilde{\dot{q}}, \quad \tilde{\Phi} = \tilde{\Delta} \cdot \nabla \tilde{u} > 0
\]

constitutive laws for deviatoric & bulk stresses & heat flux

Newtonian fluid

\[
\tilde{\Delta} = \tilde{\eta} \left[ \nabla \tilde{u} + (\nabla \tilde{u})^\text{tr} \right] + (\tilde{\eta}' - \frac{2}{3} \tilde{\eta}) (\nabla \cdot \tilde{u}) I
\]

Fourier's law

\[
\tilde{\dot{q}} = -\tilde{\lambda} \nabla \tilde{T}
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\[ \tilde{\rho} \left( \ddot{\tilde{x}}_{\text{ref}} + 2 \tilde{\Omega}_{\text{ref}} \times \tilde{u} + \tilde{D}_t \tilde{u} \right) = \nabla \cdot \tilde{\Sigma} , \quad \tilde{\Sigma} = -\tilde{\rho} I + \tilde{\Delta} , \quad \tilde{\Delta} = \tilde{\Delta}^{\text{tr}} \]

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\[ \text{shear} \quad \text{bulk} \quad \text{viscosity} \]

Fourier's law
\[ \tilde{\dot{q}} = -\tilde{\lambda} \nabla \tilde{T} \]
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any reference frame \( \mathbf{x}, \mathbf{t} \)
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continuity
\[
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shear

bulk

viscosity

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constitutive laws for deviatoric & bulk stresses & heat flux

Newtonian fluid

\[ \tilde{\Delta} = \eta [\nabla \tilde{u} + (\nabla \tilde{u})^T] + (\eta' - \frac{2}{3} \eta) (\nabla \cdot \tilde{u}) I \]

shear, bulk, viscosity

Fourier's law

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shear \quad bulk \quad \text{viscosity}

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shear bulk viscosity

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constitutive laws for deviatoric & bulk stresses & heat flux

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Thermodynamic properties of ‘simple’ fluid

caloric eq of state

\[ \tilde{h} = \tilde{h}(\tilde{p}, \tilde{T}) \]
\[ \tilde{c}_p := \left( \frac{\partial \tilde{h}}{\partial \tilde{T}} \right)_{\tilde{p}} \left[ \frac{J}{kg \, K} \right] \]
\[ \tilde{\beta} \tilde{T} = 1 - \tilde{\rho} \left( \frac{\partial \tilde{h}}{\partial \tilde{p}} \right)_{\tilde{T}} \]

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\[ \tilde{\eta} = \tilde{\eta}(\tilde{p}, \tilde{T}) \left[ Pa \, s \right] \]
\[ \tilde{\lambda} = \tilde{\lambda}(\tilde{p}, \tilde{T}) \left[ W/(m \, K) \right] \]

2nd law of thermodynamics

\( \tilde{\eta}, \tilde{\lambda}, \tilde{\beta}, \tilde{c}_p > 0 \), seldom \( \tilde{\beta} < 0 \) (H2O l)
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2nd law of thermodynamics

\[ \tilde{\eta}, \tilde{\lambda}, \tilde{\beta}, \tilde{c}_p > 0, \quad \text{seldom } \tilde{\beta} < 0 \quad (\text{H}_2\text{O})! \]
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2nd law of thermodynamics

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\[ \tilde{\rho} = \tilde{\rho}(\tilde{\rho}, \tilde{T}) \]

\[ \tilde{\beta} := -\frac{1}{\tilde{\rho}} \left( \frac{\partial \tilde{\rho}}{\partial \tilde{T}} \right)_{\tilde{\rho}} \left[ \frac{1}{K} \right] \]

\[ \tilde{\eta} = \tilde{\eta}(\tilde{\rho}, \tilde{T}) \quad [Pa \, s] \]

\[ \tilde{\lambda} = \tilde{\lambda}(\tilde{\rho}, \tilde{T}) \quad [W/(m \, K)] \]

2nd law of thermodynamics

\[ \tilde{\eta}, \tilde{\lambda}, \tilde{\beta}, \tilde{c}_p > 0, \quad \text{seldom } \tilde{\beta} < 0 \quad (H_2O !) \]
Outline

3 Classical theory
  - First principles
  - Problem formulation
  - Asymptotic theory
Non-dimensional quantities

kinematic quantities
\[ t = \frac{\tilde{t}}{U/L}, \quad x = \frac{\tilde{x}}{L}, \quad c = \frac{\tilde{c}}{C}, \quad u = \frac{\tilde{u}}{\bar{U}} \]

reference state
\[ p = \frac{\tilde{p}}{p_r}, \quad \theta = \frac{(\tilde{T} - \tilde{T_a})}{T_r}, \quad \rho = \frac{\tilde{\rho}}{\rho_r}, \quad (\eta, \eta') = \frac{\tilde{\eta}, \tilde{\eta}'}{\eta_r}, \quad \lambda = \frac{\tilde{\lambda}}{\lambda_r}, \quad \beta = \frac{\tilde{\beta} T_a}{\bar{C} p}, \quad c_p = \frac{\tilde{c}_p}{\bar{c}_{p,r}} \]

key groups

- clearance slenderness: \( \epsilon := \frac{\tilde{C}}{L} \)
- temperature ratio: \( \gamma := \frac{T_r}{T_a} \)
Non-dimensional quantities

kinematic quantities
\[ t = \tilde{t} \tilde{U}/\tilde{L} , \quad x = \tilde{x}/\tilde{L} , \quad c = \tilde{c}/\tilde{C} , \quad u = \tilde{u}/\tilde{U} \]

reference state
\[ p = \tilde{p}/\tilde{p}_r , \quad \theta = (\tilde{T} - \tilde{T}_a)/\tilde{T}_r \]
\[ \rho = \tilde{\rho}/\tilde{\rho}_r , \quad (\eta, \eta') = (\tilde{\eta}, \tilde{\eta}')/\tilde{\eta}_r , \quad \lambda = \tilde{\lambda}/\tilde{\lambda}_r , \quad \beta = \tilde{\beta}\tilde{T}_a , \quad c_p = \tilde{c}_p/\tilde{c}_{p,r} \]

key groups

clearance slenderness \[ \epsilon := \tilde{C}/\tilde{L} \]
temperature ratio \[ \gamma := \tilde{T}_r/\tilde{T}_a \]
Reynolds number \[ Re := \tilde{U}\tilde{L}/\tilde{\eta} \]
Prandtl number \[ Pr := \tilde{c}_p/\tilde{\lambda} \]
Péclet number \[ Pe := Re Pr \]
Non-dimensional quantities

kinematic quantities
\[ t = \tilde{t} \tilde{U} / \tilde{L}, \quad x = \tilde{x} / \tilde{L}, \quad c = \tilde{c} / \tilde{C}, \quad u = \tilde{u} / \tilde{U} \]

reference state
\[ p = \tilde{p} / \tilde{p}_r, \quad \theta = (\tilde{T} - \tilde{T}_a) / \tilde{T}_r \]
\[ \rho = \tilde{\rho} / \tilde{\rho}_r, \quad (\eta, \eta') = (\tilde{\eta}, \tilde{\eta}') / \tilde{\eta}_r, \quad \lambda = \tilde{\lambda} / \tilde{\lambda}_r, \quad \beta = \beta \tilde{T}_a, \quad c_p = \tilde{c}_p / \tilde{c}_{p,r} \]

key groups

- clearance slenderness \( \epsilon := \tilde{C} / \tilde{L} \)
- temperature ratio \( \gamma := \tilde{T}_r / \tilde{T}_a \)
- Reynolds number \( Re := \tilde{U} \tilde{L} \tilde{p}_r / \tilde{\eta}_r \)
- Prandtl number \( Pr := \tilde{c}_{p,r} \tilde{\eta}_r / \tilde{\lambda}_r \)
- Péclet number \( Pe := Re Pr \)
Non-dimensional quantities

kinematic quantities
\[ t = \tilde{t} \frac{\bar{U}}{\bar{L}} , \quad x = \tilde{x} \frac{\bar{L}}{\bar{L}} , \quad c = \tilde{c} \frac{\bar{C}}{\bar{C}} , \quad u = \tilde{u} \frac{\bar{U}}{\bar{U}} \]

reference state
\[ p = \tilde{p} \frac{\bar{p}_r}{\bar{p}_r} , \quad \theta = (\tilde{T} - \tilde{T}_a) / \bar{T}_r \]
\[ \rho = \tilde{\rho} \frac{\bar{\rho}_r}{\bar{\rho}_r} , \quad (\eta, \eta') = (\tilde{\eta}, \tilde{\eta}') / \tilde{\eta}_r , \quad \lambda = \tilde{\lambda} / \bar{\lambda}_r , \quad \beta = \tilde{\beta} \bar{T}_a , \quad c_p = \tilde{c}_p / \tilde{c}_p r \]

key groups

- clearance slenderness \( \epsilon := \tilde{C} / \bar{L} \)
- temperature ratio \( \gamma := \tilde{T}_r / \tilde{T}_a \)
- Reynolds number \( Re := \tilde{U} \bar{L} \bar{p}_r / \tilde{\eta}_r \)
- Prandtl number \( Pr := \tilde{c}_p r \tilde{\eta}_r / \bar{\lambda}_r \)
- Péclet number \( Pe := Re Pr \)
Non-dimensional quantities

kinematic quantities

\[ t = \tilde{t} \frac{\tilde{U}}{\tilde{L}}, \quad x = \tilde{x} \frac{\tilde{L}}{\tilde{L}}, \quad c = \tilde{c} \frac{\tilde{C}}{\tilde{C}}, \quad u = \tilde{u} \frac{\tilde{U}}{\tilde{L}} \]

reference state

\[ p = \tilde{p} \frac{\tilde{p}_{r}}{\tilde{p}_{r}}, \quad \theta = \left( \tilde{T} - \tilde{T}_{a} \right) \frac{\tilde{T}_{r}}{\tilde{T}_{r}} \]

\[ \rho = \tilde{\rho} \frac{\tilde{\rho}_{r}}{\tilde{\rho}_{r}}, \quad (\eta, \eta') = (\tilde{\eta}, \tilde{\eta}') \frac{\tilde{\eta}_{r}}{\tilde{\eta}_{r}}, \quad \lambda = \tilde{\lambda} \frac{\tilde{\lambda}_{r}}{\tilde{\lambda}_{r}}, \quad \beta = \tilde{\beta} \frac{\tilde{T}_{a}}{\tilde{T}_{a}}, \quad c_p = \tilde{c}_p \frac{\tilde{c}_{p,r}}{\tilde{c}_{p,r}} \]

key groups

 clearance slenderness \[ \epsilon := \frac{\tilde{C}}{\tilde{L}} \]

 temperature ratio \[ \gamma := \frac{\tilde{T}_r}{\tilde{T}_a} \]

 Reynolds number \[ Re := \frac{\tilde{U} \tilde{L} \tilde{p}_r}{\tilde{n}_r} \]

 Prandtl number \[ Pr := \frac{\tilde{c}_{p,r} \tilde{n}_r}{\tilde{\lambda}_r} \]

 Péclet number \[ Pe := Re \cdot Pr \]

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Non-dimensional quantities

kinematic quantities

\[ t = \tilde{t} \tilde{U} / \tilde{L}, \quad x = \tilde{x} / \tilde{L}, \quad c = \tilde{c} / \tilde{C}, \quad u = \tilde{u} / \tilde{U} \]

reference state

\[ p = \tilde{p} / \tilde{p}_r, \quad \theta = (\tilde{T} - \tilde{T}_a) / \tilde{T}_r \]
\[ \rho = \tilde{\rho} / \tilde{\rho}_r, \quad (\eta, \eta') = (\tilde{\eta}, \tilde{\eta}') / \tilde{\eta}_r, \quad \lambda = \tilde{\lambda} / \tilde{\lambda}_r, \quad \beta = \tilde{\beta} \tilde{T}_a, \quad c_p = \tilde{c}_p / \tilde{c}_{p,r} \]

key groups

- clearance slenderness \( \epsilon := \tilde{C} / \tilde{L} \)
- temperature ratio \( \gamma := \tilde{T}_r / \tilde{T}_a \)
- Reynolds number \( Re := \tilde{U} \tilde{L} \tilde{\rho}_r / \tilde{\eta}_r \)
- Prandtl number \( Pr := \tilde{c}_{p,r} \tilde{\eta}_r / \tilde{\lambda}_r \)
- Péclet number \( Pe := Re Pr \)
Non-dimensional quantities, cont’d

natural metric

\[ x = x_\parallel + \epsilon e_n n, \quad u = u_\parallel + \epsilon e_n w, \quad u_\parallel = u_\parallel e_\parallel \]
\[ e_\parallel \cdot e_n = 0, \quad \partial_n e_\parallel = \partial_n e_n = 0 \]

\[ \nabla = \tilde{\nabla} = \nabla_\parallel + \epsilon^{-1} e_n \partial_n \]

\[ \nabla \cdot (\rho u) = \nabla_\parallel \cdot (\rho u_\parallel) + \underbrace{e_n \cdot \partial_n (\rho u_\parallel)}_{O(\epsilon)} + \underbrace{\epsilon \nabla_\parallel \cdot (\rho e_n w)}_{\rho w \nabla_\parallel \cdot e_n} + \underbrace{e_n \cdot \partial_n (\rho e_n w)}_{\partial_n (\rho w)} \]

\[ D_t = (\tilde{L}/\tilde{U}) \tilde{D}_t = \partial_t + u \cdot \nabla = u_\parallel \cdot \nabla_\parallel + w \partial_n \]
non-dimensional quantities, cont’d

\[ x = x_\parallel + \epsilon e_n n , \quad u = u_\parallel + \epsilon e_n w , \quad u_\parallel = u_\parallel e_\parallel \]
\[ e_\parallel \cdot e_n = 0 , \quad \partial_n e_\parallel = \partial_n e_n = 0 \]

\[ \nabla = \tilde{L} \tilde{\nabla} = \nabla_\parallel + \epsilon^{-1} e_n \partial_n \]
\[ \nabla \cdot (\rho u) = \nabla_\parallel \cdot (\rho u_\parallel) + e_n \cdot \partial_n (\rho u_\parallel) + \epsilon \nabla_\parallel \cdot (\rho e_n w) + e_n \cdot \partial_n (\rho e_n w) \]
\[ e_n \cdot e_\parallel \partial_n (\rho u_\parallel) = 0 \]

\[ \rho w \nabla_{\parallel} \cdot e_n \quad \partial_n (\rho w) \]
\[ D_t = (\tilde{L} / \tilde{U}) \ddot{D}_t = \partial_t + u \cdot \nabla = u_\parallel \cdot \nabla_\parallel + w \partial_n \]

natural metric
Non-dimensional quantities, cont’d

\[ x = x_\parallel + \epsilon e_n n, \quad u = u_\parallel + \epsilon e_n w, \quad u_\parallel = u_\parallel e_\parallel \]

\[ e_\parallel \cdot e_n = 0, \quad \partial_n e_\parallel = \partial_n e_n = 0 \]

\[ \nabla = \nabla_\parallel + \epsilon^{-1} e_n \partial_n \]

\[ \nabla \cdot (\rho u) = \nabla_\parallel \cdot (\rho u_\parallel) + e_n \cdot \partial_n (\rho u_\parallel) + \epsilon \nabla_\parallel \cdot (\rho e_n w) + e_n \cdot \partial_n (\rho e_n w) \]

\[ e_n \cdot e_\parallel \partial_n (\rho u_\parallel) = 0 \]

\[ D_t = (\nabla_\parallel + \eta_\parallel) = \partial_t + u \cdot \nabla = u_\parallel \cdot \nabla_\parallel + w \partial_n \]
Non-dimensional quantities, cont’d

natural metric

\[
x = x_\parallel + \epsilon e_n n, \quad u = u_\parallel + \epsilon e_n w, \quad u_\parallel = u_\parallel e_\parallel \\
e_\parallel \cdot e_n = 0, \quad \partial_n e_\parallel = \partial_n e_n = 0
\]

\[
\nabla = \tilde{\nabla} = \nabla_\parallel + \epsilon^{-1} e_n \partial_n \\
\nabla \cdot (\rho u) = \nabla_\parallel \cdot (\rho u_\parallel) + e_n \cdot \partial_n (\rho u_\parallel) + \epsilon \nabla_\parallel \cdot (\rho e_n w) + e_n \cdot \partial_n (\rho e_n w) \\
e_n \cdot e_\parallel \partial_n (\rho u_\parallel) = 0, \quad \rho w \nabla_\parallel \cdot e_n, \quad \partial_n (\rho w)
\]

\[
D_t = (\tilde{L}/\tilde{U}) \tilde{D}_t = \partial_t + u \cdot \nabla = u_\parallel \cdot \nabla_\parallel + w \partial_n
\]
Navier–Stokes eqs

\[ \tilde{p}_r := \tilde{\eta}_r \tilde{U} \tilde{L} / \tilde{C}^2, \quad \tilde{T}_r := \tilde{\eta}_r \tilde{U}^2 / \tilde{\lambda}_r \]

state

\[ q = q(\rho, 1 + \gamma \theta), \quad q = \rho, \eta, \lambda, c_p \Rightarrow \tilde{p}_r \]

continuity

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) \equiv \partial_t \rho + \nabla_{||} \cdot (\rho \mathbf{u}_{||}) + \epsilon \rho w \nabla_{||} \cdot \mathbf{e}_n + \partial_n (\rho w) = 0 \]

momentum

\[ Re \epsilon^2 \rho (\ddot{x}_{rel} + 2 \Omega_{rel} \times \mathbf{u} + D_t \mathbf{u}) + \nabla p = \epsilon^2 \nabla \cdot \Delta \]

\[ \Delta = \eta [\nabla \mathbf{u} + (\nabla \mathbf{u})^{tr}] + (\eta' - \frac{2}{3} \eta)(\nabla \cdot \mathbf{u}) I \]

energy

\[ Pe \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t p + \epsilon^2 [\Phi + \nabla \cdot (\lambda \nabla \theta)] \]

\[ \Phi = \Delta \cdot \nabla \mathbf{u}, \quad \gamma := \tilde{T}_r / \tilde{T}_a \]

\[ \epsilon \ll 1, \quad \nabla \sim \epsilon^{-1} \mathbf{e}_n \partial_n \]

momentum

\[ 0 \sim -\nabla_{||} p + \partial_n (\eta \partial_n u_{||}), \quad 0 \sim \epsilon^{-1} \partial_n p \]
Navier–Stokes eqs

\[ \tilde{p}_r := \tilde{\eta}_r \tilde{UL}/\tilde{C}^2 , \quad \tilde{T}_r := \tilde{\eta}_r \tilde{U}^2/\tilde{\lambda}_r \]

**state**

\[ q = q(\rho, 1 + \gamma \theta), \quad q = \rho, \eta, \lambda, c_p \quad \Rightarrow \quad \tilde{p}_r \]

**continuity**

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) + \epsilon \rho \omega \nabla \cdot \mathbf{e}_n + \partial_n (\rho \omega) = 0 \]

**momentum**

\[ \text{Re} \epsilon^2 \rho (\ddot{x}_{\text{ref}} + 2 \Omega_{\text{ref}} \times \mathbf{u} + D_t \mathbf{u}) + \nabla p = \epsilon^2 \nabla \cdot \Delta \]

\[ \Delta = \eta [\nabla \mathbf{u} + (\nabla \mathbf{u})^\text{tr}] + (\eta' - \frac{2}{3} \eta) (\nabla \cdot \mathbf{u}) I \]

**energy**

\[ \text{Pe} \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t p + \epsilon^2 [\Phi + \nabla \cdot (\lambda \nabla \theta)] \]

\[ \Phi = \Delta \cdot \nabla \mathbf{u}, \quad \gamma := \tilde{T}_r/\tilde{T}_a \]

\[ \epsilon \ll 1, \quad \nabla \sim \epsilon^{-1} \mathbf{e}_n \partial_n \]

**momentum**

\[ 0 \sim -\nabla \parallel \rho + \partial_n (\eta \partial_n \mathbf{u}) \parallel, \quad 0 \sim \epsilon^{-1} \partial_n \rho \]
Navier–Stokes eqs

\[ \tilde{p}_r := \tilde{\eta}_r \tilde{U}L / \tilde{C}^2 , \quad \tilde{T}_r := \tilde{\eta}_r \tilde{U}^2 / \tilde{\lambda}_r \]

state \[ q = q(\rho, 1 + \gamma \theta) , \quad q = \rho , \eta , \lambda , c_p \quad \Rightarrow \quad \tilde{p}_r \]
continuity \[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) \equiv \partial_t \rho + \nabla_{||} \cdot (\rho \mathbf{u}_{||}) + \epsilon \rho w \nabla_{||} \cdot \mathbf{e}_n + \partial_n (\rho w) = 0 \]
momentum \[ Re \epsilon^2 \rho (\ddot{x}_{ref} + 2 \Omega_{ref} \times \mathbf{u} + D_t \mathbf{u}) + \nabla p = \epsilon^2 \nabla \cdot \Delta \]
\[ \Delta = \eta [\nabla \mathbf{u} + (\nabla \mathbf{u})^t] + (\eta' - \frac{2}{3} \eta) (\nabla \cdot \mathbf{u}) I \]
energy \[ Pe \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t \rho + \epsilon^2 [\Phi + \nabla \cdot (\lambda \nabla \theta)] \]
\[ \Phi = \Delta \cdot \nabla \mathbf{u} , \quad \gamma := \tilde{T}_r / \tilde{T}_a \]

\[ \epsilon \ll 1 , \quad \nabla \sim \epsilon^{-1} \mathbf{e}_n \partial_n \]
momentum \[ 0 \sim -\nabla_{||} \rho + \partial_n (\eta \partial_n \mathbf{u}_{||}) , \quad 0 \sim \epsilon^{-1} \partial_n \rho \]
Navier–Stokes eqs

\[ \tilde{\rho}_r := \tilde{\eta}_r \tilde{U} \tilde{L} / \tilde{C}^2, \quad \tilde{T}_r := \tilde{\eta}_r \tilde{U}^2 / \tilde{\lambda}_r \]

**state**

\[ q = q(\rho, 1 + \gamma \theta), \quad q = \rho, \eta, \lambda, c_p \Rightarrow \tilde{\rho}_r \]

**continuity**

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) \equiv \partial_t \rho + \nabla_{\parallel} \cdot (\rho \mathbf{u}_{\parallel}) + \epsilon \rho w \nabla_{\parallel} \cdot \mathbf{e}_n + \partial_n (\rho w) = 0 \]

**momentum**

\[ \text{Re} \epsilon^2 \rho (\ddot{x}_{\text{ref}} + 2 \dot{\Omega}_{\text{ref}} \times \mathbf{u} + D_t \mathbf{u}) + \nabla p = \epsilon^2 \nabla \cdot \Delta \]

\[ \Delta = \eta \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^\text{tr} \right] + (\eta' - \frac{2}{3} \eta)(\nabla \cdot \mathbf{u}) I \]

**energy**

\[ \text{Pe} \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t p + \epsilon^2 \left[ \Phi + \nabla \cdot (\lambda \nabla \theta) \right] \]

\[ \Phi = \Delta \cdot \nabla \mathbf{u}, \quad \gamma := \tilde{T}_r / \tilde{T}_a \]

\[ \epsilon \ll 1, \quad \nabla \sim \epsilon^{-1} \mathbf{e}_n \partial_n \]

**momentum**

\[ 0 \sim -\nabla_{\parallel} p + \partial_n (\eta \partial_n \mathbf{u}_{\parallel}), \quad 0 \sim \epsilon^{-1} \partial_n p \]
Navier–Stokes eqs

\[ \tilde{\rho}_r := \tilde{\eta} \tilde{U} \tilde{L} / \tilde{C}^2, \quad \tilde{T}_r := \tilde{\eta} \tilde{U}^2 / \tilde{\lambda}_r \]

state

\[ q = q(\rho, 1 + \gamma \theta), \quad q = \rho, \eta, \lambda, c_p \quad \Rightarrow \quad \tilde{\rho}_r \]

continuity

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) \equiv \partial_t \rho + \nabla || \cdot (\rho \mathbf{u} ||) + \epsilon \rho w \nabla || \cdot \mathbf{e}_n + \partial_n (\rho w) = 0 \]

momentum

\[ \text{Re} \epsilon^2 \rho (\ddot{x}_{\text{ref}} + 2 \Omega_{\text{ref}} \times \mathbf{u} + D_t \mathbf{u}) + \nabla p = \epsilon^2 \nabla \cdot \Delta \]

\[ \Delta = \eta \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^\text{tr} \right] + (\eta' - \frac{2}{3} \eta)(\nabla \cdot \mathbf{u}) \mathbf{I} \]

energy

\[ \text{Pe} \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t p + \epsilon^2 \left[ \Phi + \nabla \cdot (\lambda \nabla \theta) \right] \]

\[ \Phi = \Delta \cdot \nabla \mathbf{u}, \quad \gamma := \tilde{T}_r / \tilde{T}_a \]

\[ \epsilon \ll 1, \quad \nabla \sim \epsilon^{-1} \mathbf{e}_n \partial_n \]

momentum

\[ 0 \sim -\nabla || \rho + \partial_n (\eta \partial_n \mathbf{u} ||), \quad 0 \sim \epsilon^{-1} \partial_n \rho \]
Navier–Stokes eqs

\[ \tilde{\rho}_r := \tilde{\eta}_r \tilde{U} \tilde{L} / \tilde{C}^2, \quad \tilde{T}_r := \tilde{\eta}_r \tilde{U}^2 / \tilde{\lambda}_r \]

state \quad q = q(\rho, 1 + \gamma \theta), \quad q = \rho, \eta, \lambda, c_p \quad \Rightarrow \quad \tilde{\rho}_r

continuity \quad \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) \equiv \partial_t \rho + \nabla|| \cdot (\rho \mathbf{u}||) + \epsilon \rho w \nabla|| \cdot \mathbf{e}_n + \partial_n (\rho w) = 0

momentum \quad Re \epsilon^2 \rho (\ddot{x}_{ref} + 2 \Omega_{ref} \times \mathbf{u} + D_t \mathbf{u}) + \nabla p = \epsilon^2 \nabla \cdot \Delta

\Delta = \eta \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^{tr} \right] + (\eta' - \frac{2}{3} \eta)(\nabla \cdot \mathbf{u}) I

energy \quad Pe \epsilon^2 \rho c_p D_t \theta = \beta (1 + \gamma \theta) D_t p + \epsilon^2 [\Phi + \nabla \cdot (\lambda \nabla \theta)]

\Phi = \Delta \cdot \nabla \mathbf{u}, \quad \gamma := \tilde{T}_r / \tilde{T}_a

\epsilon \ll 1, \quad \nabla \sim \epsilon^{-1} \mathbf{e}_n \partial_n

momentum \quad 0 \sim -\nabla|| \rho + \partial_n (\eta \partial_n \mathbf{u}||), \quad 0 \sim \epsilon^{-1} \partial_n \rho
Outline

3. Classical theory
   - First principles
   - Problem formulation
   - Asymptotic theory
Limit process

classical lubrication approximation

thin film $\epsilon \ll 1$

quasi-isothermal $\gamma \ll 1$

inertia neglected $Re \epsilon^2 \ll 1$, laminar flow: $Re \lesssim 10^5$

typical values $\epsilon \lesssim 10^{-3}$, $Pr_{oil} \approx 70 \ldots 10^3 \Rightarrow Pe \lesssim 10^8$, $Pe \epsilon^2 \lesssim 10^2$

$\nabla \cdot (\rho \mathbf{u}) \sim \nabla_\parallel \cdot (\rho \mathbf{u}_\parallel) + \partial_n (\rho \mathbf{w}) + O(\epsilon)$

$\rho (p, 1 + \gamma \theta) \sim \rho (p, 1) + O(\gamma)$

expansions

$\nabla_\parallel \sim \nabla_\parallel^0 + O(\epsilon)$

$\nabla_\parallel^0 = \nabla_\parallel$ for $n = 0$

$[\mathbf{u}_\parallel, w, p, \rho, \theta, \eta, \ldots] (x_\parallel, n, t; \epsilon, Re, \gamma, \ldots) \sim [\mathbf{U}, W, P, Q, \Theta, N] (x_\parallel, n, t) + \ldots$

$c \sim C(x_\parallel, t) + O(\epsilon)$ journal bearing

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Limit process

classical lubrication approximation

thin film \( \epsilon \ll 1 \)

quasi-isothermal \( \gamma \ll 1 \)

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\[ \nabla \cdot (\rho u) \sim \nabla_{||} \cdot (\rho u_{||}) + \partial_n (\rho w) + O(\epsilon) \]

\[ \rho (p, 1 + \gamma \theta) \sim \rho (p, 1) + O(\gamma) \]

expansions

\[ \nabla_{||} \sim \nabla_{||}^0 + O(\epsilon) \quad \nabla_{||}^0 = \nabla_{||} \quad \text{for} \quad n = 0 \]

\[ [u_{||}, w, p, \rho, \theta, \eta, \ldots] (x_{||}, n, t; \epsilon, Re, \gamma, \ldots) \sim [U, W, P, Q, \Theta, N](x_{||}, n, t) + \cdots \]

\[ c \sim C(x_{||}, t) + O(\epsilon) \quad \text{journal bearing}! \]
Limit process

classical lubrication approximation

<table>
<thead>
<tr>
<th>Condition</th>
<th>Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>thin film</td>
<td>$\epsilon \ll 1$</td>
</tr>
<tr>
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</tr>
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<td>inertia neglected</td>
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<td>laminar flow</td>
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<td></td>
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</tr>
</tbody>
</table>

\[
\nabla \cdot (\rho \mathbf{u}) \sim \nabla_{||} \cdot (\rho \mathbf{u}_{||}) + \partial_n (\rho \mathbf{w}) + O(\epsilon)
\]

\[
\rho(p, 1 + \gamma \theta) \sim \rho(p, 1) + O(\gamma)
\]

expansions

\[
\nabla_{||} \sim \nabla^0_{||} + O(\epsilon) \quad \nabla^0_{||} = \nabla_{||} \quad \text{for} \quad n = 0
\]

\[
[u_{||}, w, p, \rho, \theta, \eta, \ldots](x_{||}, n, t; \epsilon, Re, \gamma, \ldots) \sim [U, W, P, Q, \Theta, \mathcal{N}](x_{||}, n, t) + \ldots
\]

\[
c \sim C(x_{||}, t) + O(\epsilon) \quad \text{journal bearing}
\]
Limit process

classical lubrication approximation

thin film $\epsilon \ll 1$

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\[
\nabla \cdot (\rho \boldsymbol{u}) \sim \nabla_{||} \cdot (\rho \boldsymbol{u}_{||}) + \partial_n (\rho \boldsymbol{w}) + O(\epsilon)
\]
\[
\rho (p, 1 + \gamma \theta) \sim \rho (p, 1) + O(\gamma)
\]

expansions

$\nabla_{||} \sim \nabla_{||}^0 + O(\epsilon)$ \hspace{1cm} $\nabla_{||}^0 = \nabla_{||}$ for $n = 0$

$\begin{bmatrix} \boldsymbol{u}_{||}, w, p, \rho, \theta, \eta, \ldots \end{bmatrix}(\boldsymbol{x}_{||}, n, t; \epsilon, Re, \gamma, \ldots) \sim [\boldsymbol{U}, W, P, Q, \Theta, \mathcal{N}](\boldsymbol{x}_{||}, n, t) + \cdots$

$c \sim C(\boldsymbol{x}_{||}, t) + O(\epsilon)$ journal bearing
Limit process

classical lubrication approximation

thin film \( \epsilon \ll 1 \)

quasi-isothermal \( \gamma \ll 1 \)

inertia neglected \( Re \epsilon^2 \ll 1 \), laminar flow: \( Re \lesssim 10^5 \)

typical values \( \epsilon \lesssim 10^{-3} \), \( Pr_{\text{oil}} \approx 70 \ldots 10^3 \) \( 100 \ldots 20^\circ C \Rightarrow Pe \lesssim 10^8 \), \( Pe \epsilon^2 \lesssim 10^2 \)!

\[
\nabla \cdot (\rho \mathbf{u}) \sim \nabla || \cdot (\rho \mathbf{u} ||) + \partial_n (\rho w) + O(\epsilon)
\]

\[
\rho (p, 1 + \gamma \theta) \sim \rho (p, 1) + O(\gamma)
\]

expansions

\[
\nabla || \sim \nabla^0 || + O(\epsilon) \quad \nabla^0 || = \nabla || \quad \text{for} \quad n = 0
\]

\[
[u ||, w, p, \rho, \theta, \eta, \ldots](x ||, n, t; \epsilon, Re, \gamma, \ldots) \sim [U, W, P, Q, \Theta, N](x ||, n, t) + \cdots
\]

\[
c \sim C(x ||, t) + O(\epsilon) \quad \text{journal bearing!}
\]
Leading-order eqs

state & energy \[ Q = Q(P, 1), \quad Q = Q, \ \mathcal{N} \]

continuity \[ \partial_t Q + \nabla^0 \cdot (Q U) + \partial_N (Q W) = 0 \quad (1) \]

momentum \[ \nabla^0 P = \partial_n (\mathcal{N} \partial_n U), \quad \partial_n P = 0 \quad \Rightarrow \quad \partial_n Q = \partial_n \mathcal{N} = 0 \quad (2) \]

kinematic BCs

\( n = 0 : \quad U = U_1(x|_n, t), \quad W = W_{p,1}(x|_n, t) \quad (3) \)

\( n = C(x|_n, t) : \quad U = U_2(x|_n, t), \quad W = \partial_t C + U_2 \cdot \nabla^0 C + W_{p,2}(x|_n, t) \quad (4) \)

(1), (3), (4) \[ \Rightarrow \quad \partial_t (QC) + \nabla^0 \cdot \left( Q \int_0^C U \, dn \right) + Q(W_{p,2} - W_{p,1}) = 0 \]

(2), (3), (4) \[ \Rightarrow \quad U = \frac{\nabla^0 P}{2 \mathcal{N}(P)} n(n - C) + \frac{n}{C}(U_2 - U_1) \]

\( \frac{Hagen–Poisseuille}{Couette} \)
Leading-order eqs

state & energy \[ Q = Q(P, 1), \quad Q = Q, \quad N \]

continuity \[ \partial_t Q + \nabla_\parallel \cdot (Q U) + \partial_N(Q W) = 0 \quad (1) \]

momentum \[ \nabla_\parallel P = \partial_n(N \partial_n U), \quad \partial_n P = 0 \quad \Rightarrow \quad \partial_n Q = \partial_n N = 0 \quad (2) \]

kinematic BCs

\( n = 0 : \quad U = U_1(x_\parallel, t), \quad W = W_{p,1}(x_\parallel, t) \quad (3) \)

\( n = C(x_\parallel, t) : \quad U = U_2(x_\parallel, t), \quad W = \partial_t C + U_2 \cdot \nabla_\parallel C + W_{p,2}(x_\parallel, t) \quad (4) \)

\( (1), (3), (4) \quad \Rightarrow \quad \partial_t(QC) + \nabla_\parallel \cdot \left(Q \int_0^C U \, dn\right) + Q(W_{p,2} - W_{p,1}) = 0 \)

\( (2), (3), (4) \quad \Rightarrow \quad U = \frac{\nabla_\parallel P}{2N(P)} n(n - C) + \frac{n}{C} (U_2 - U_1) \)

Hagen–Poiseuille

Couette
Leading-order eqs

state & energy \[ Q = Q(P, 1), \quad Q = Q, \quad \mathcal{N} \]

continuity \[ \partial_t Q + \nabla_\parallel \cdot (Q \mathbf{U}) + \partial_N (Q \mathbf{W}) = 0 \quad (1) \]

momentum \[ \nabla_\parallel^0 P = \partial_n (\mathcal{N} \partial_n \mathbf{U}), \quad \partial_n P = 0 \quad \Rightarrow \quad \partial_n Q = \partial_n \mathcal{N} = 0 \quad (2) \]

kinematic BCs

\[ n = 0 : \quad \mathbf{U} = \mathbf{U}_1(x_\parallel, t), \quad \mathbf{W} = \mathbf{W}_{p,1}(x_\parallel, t) \quad (3) \]

\[ n = C(x_\parallel, t) : \quad \mathbf{U} = \mathbf{U}_2(x_\parallel, t), \quad \mathbf{W} = \partial_t C + \mathbf{U}_2 \cdot \nabla_\parallel^0 C + \mathbf{W}_{p,2}(x_\parallel, t) \quad (4) \]

\[ (1), (3), (4) \quad \Rightarrow \quad \partial_t (Q C) + \nabla_\parallel \cdot \left( Q \int_0^C \mathbf{U} \, dn \right) + Q (\mathbf{W}_{p,2} - \mathbf{W}_{p,1}) = 0 \]

\[ (2), (3), (4) \quad \Rightarrow \quad \mathbf{U} = \frac{\nabla_\parallel^0 P}{2 \mathcal{N}(P)} n(n - C) + \frac{n}{C} (\mathbf{U}_2 - \mathbf{U}_1) \]

\[ \text{Hagen–Poiseuille} \quad \text{Couette} \]
Leading-order eqs

state & energy \[ Q = Q(P, 1), \quad Q = Q, \quad N \]

continuity \[ \partial_t Q + \nabla^0 \cdot (QU) + \partial_N (QW) = 0 \quad (1) \]

momentum \[ \nabla^0 P = \partial_n (N \partial_n U), \quad \partial_n P = 0 \quad \Rightarrow \quad \partial_n Q = \partial_n N = 0 \quad (2) \]

kinematic BCs

\[ n = 0 : \quad U = U_1(x_\|, t), \quad W = W_{p,1}(x_\|, t) \quad (3) \]

\[ n = C(x_\|, t) : \quad U = U_2(x_\|, t), \quad W = \partial_t C + U_2 \cdot \nabla^0 C + W_{p,2}(x_\|, t) \quad (4) \]

(1), (3), (4) \[ \Rightarrow \quad \partial_t (Q C) + \nabla^0 \cdot \left( Q \int_0^C U \, dn \right) + Q (W_{p,2} - W_{p,1}) = 0 \]

(2), (3), (4) \[ \Rightarrow \quad U = \frac{\nabla^0 P}{2N(P)} n(n - C) + \frac{n}{C} (U_2 - U_1) + U_1 \]

Hagen–Poiseuille

Couette

sliding
Integral mass balance

\[ \int_0^C U \, dn = Q + C \, U_m, \quad Q := -\frac{C^3 \nabla^0 P}{12 \mathcal{N}}, \quad U_m := \frac{U_1 + U_2}{2} \]

Reynolds eq

O. Reynolds (1886), A. Sommerfeld (1904), L. Prandtl (1937)

\[ \nabla^0 \cdot (-Q \partial_t + U_m \cdot \nabla^0) (Q C) + Q C \nabla^0 \cdot U_m + Q (W_{p,2} - W_{p,1}) \]

\(
\) permeability

\[ Q = Q(P), \quad \mathcal{N} = \mathcal{N}(P) \]

elliptic 2nd-order PDE for \( P(x_\parallel, t) \) and given \( C(x_\parallel, t), \ U_m(x_\parallel, t) \)

kinematic wave operator \( \partial_t + U_m \cdot \nabla^0 \) most relevant for gas bearings

linear for incompressible lubricant with constant properties \( (Q \equiv \mathcal{N} \equiv 1) \)

in general to be solved numerically
Integral mass balance

\[ \int_{0}^{C} U \, dn = Q + C \, U_{m}, \quad Q := - \frac{C^{3} \nabla_{\parallel}^{0} P}{12 \, N}, \quad U_{m} := \frac{U_{1} + U_{2}}{2} \]

Reynolds eq

\[ \nabla_{\parallel}^{0} (Q \dot{Q}) = \left( \partial_{t} + U_{m} \cdot \nabla_{\parallel}^{0} \right) (Q \dot{C}) + QC \nabla_{\parallel}^{0} \cdot U_{m} + Q \left( W_{p,2} - W_{p,1} \right) \]

‘squeeze’ Couette + sliding = ‘wedge’

\[ Q = Q(P), \quad N = N(P) \]

elliptic 2nd-order PDE for \( P(x_{\parallel}, t) \) and given \( C(x_{\parallel}, t), \, U_{m}(x_{\parallel}, t) \)

kinematic wave operator \( \partial_{t} + U_{m} \cdot \nabla_{\parallel}^{0} \) most relevant for gas bearings

linear for incompressible lubricant with constant properties \( (Q \equiv N \equiv 1) \)

in general to be solved numerically
Reynolds eq – some important properties

\[ Q = -\frac{C^3 \nabla_1^0 P}{12N} \quad \text{,} \quad \boldsymbol{U}_m = \frac{\boldsymbol{U}_1 + \boldsymbol{U}_2}{2} \]

\[ \nabla_1^0 \cdot (-QQ) = (\partial_t + \boldsymbol{U}_m \cdot \nabla_1^0)(QC) + QC \nabla_1^0 \cdot \boldsymbol{U}_m + Q(W_{p,2} - W_{p,1}) \]

\[ Q = Q(P) \quad , \quad N = N(P) \]

rigid contacts, no Navier slip

\[ \nabla_1^0 \cdot [\boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_m] = 0 \]

Galilean transformation

\[ [\boldsymbol{x}_1, \ t] = [\boldsymbol{x}'_1 + \boldsymbol{S}(t'), \ t'] \]

\[ [\nabla_1^0, \partial_t] = [\nabla_1^0, \partial_{t'} - \dot{\boldsymbol{S}} \nabla_1^0] \]

\[ [C, P, \boldsymbol{U}_{1,2}](\boldsymbol{x}_1, \ t) = [C', P', \boldsymbol{U}'_{1,2}](\boldsymbol{x}'_1, \ t'), \quad [Q, N](P) = [Q', N'](P') \]

\[ [\boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_m] \rightarrow [\boldsymbol{U}'_1, \boldsymbol{U}'_2, \boldsymbol{U}'_m] - \dot{\boldsymbol{S}} \quad \text{ invariance against sliding motion } \boldsymbol{S} \]
Reynolds eq – some important properties

\[ Q = -\frac{C^3}{12N} \nabla_0^0 P, \quad U_m = \frac{U_1 + U_2}{2} \]

\[ \nabla_0^0 \cdot (-QQ) = (\partial_t + U_m \cdot \nabla_0^0)(QC) + QC \nabla_0^0 \cdot U_m + Q(W_{p,2} - W_{p,1}) \]

\[ Q = Q(P), \quad N = N(P) \]

rigid contacts, no Navier slip

\[ \nabla_0^0 \cdot [U_1, U_2, U_m] = 0 \]

Galilean transformation

\[ [x, t] = [x' + S(t'), t'] \]

\[ [\nabla_0^0, \partial_t] = [\nabla_0^0', \partial_{t'} - \dot{S} \nabla_0^0'] \]

\[ [C, P, U_{1,2}](x, t) = [C', P', U_{1,2}'](x', t'), \quad [Q, N](P) = [Q', N'](P') \]

\[ [U_1, U_2, U_m] \rightarrow [U'_1, U'_2, U'_m] - \dot{S} \]

invariance against sliding motion \( S \)
Reynolds eq – some important properties

\[ Q = -\frac{C^3 \nabla_0^0 P}{12 \mathcal{N}} , \quad U_m = \frac{U_1 + U_2}{2} \]

\[ \nabla_0^0 \cdot (-QQ) = (\partial_t + U_m \cdot \nabla_0^0)(QC) + QC \nabla_0^0 \cdot U_m + Q(W_{p,2} - W_{p,1}) \]

\[ Q = Q(P) , \quad \mathcal{N} = \mathcal{N}(P) \]

rigid contacts, no Navier slip

\[ \nabla_0^0 \cdot [U_1, U_2, U_m] = 0 \]

Galilean transformation

\[ [x_\parallel, t] = [x'_\parallel + S(t'), t'] \]

\[ [\nabla_0^0, \partial_t] = [\nabla_0^0', \partial_{t'} - \dot{S} \nabla_0^0'] \]

\[ [C, P, U_{1,2}](x_\parallel, t) = [C', P', U'_{1,2}](x'_\parallel, t') , \quad [Q, \mathcal{N}](P) = [Q', \mathcal{N}'](P') \]

\[ [U_1, U_2, U_m] \to [U'_1, U'_2, U'_m] - \dot{S} \quad \text{invariance against sliding motion } S \]
Reynolds eq – some important properties

\[ Q = -\frac{C^3 \nabla_0^0 P}{12N}, \quad U_m = \frac{U_1 + U_2}{2} \]

\[ \nabla_0^0 \cdot (-Q Q) = (\partial_t + U_m \cdot \nabla_0^0)(QC) + QC \nabla_0^0 \cdot U_m + Q(W_{p,2} - W_{p,1}) \]

\[ Q = Q(P), \quad N = N(P) \]

rigid contacts, no Navier slip \[ \nabla_0^0 \cdot [U_1, U_2, U_m] = 0 \]

Galilean transformation \[ [x_||, t] = [x'_|| + S(t'), t'] \]

\[ [\nabla_0^0, \partial_t] = [\nabla_0^0', \partial_{t'} - \dot{S} \nabla_0^0'] \]

\[ [C, P, U_{1,2}] (x_||, t) = [C', P', U'_{1,2}] (x'_||, t'), \quad [Q, N](P) = [Q', N'](P') \]

\[ [U_1, U_2, U_m] \mapsto [U'_1, U'_2, U'_m] - \dot{S} \quad \text{invariance against sliding motion } S \]
Validation of tribo-systems

typically find

- \( P(x_\parallel, t), \ x_\parallel \in \Omega \) subject to \( P(\partial \Omega, t) = P_a \)

- load-bearing capacity \( F(t) = \int_{\Omega} P e_n \, d\Omega \)

clearance \( C(x_\parallel, t) \) is

- prescribed
- found from fluid-structure interaction

machinery (e.g. shaft) dynamics \( \Rightarrow F = F(x_\parallel, t(C, C)) \)

EHL \( \Rightarrow P = P(C) \)
Validation of tribo-systems

typically find

- \( P(\mathbf{x}_||, t) \), \( \mathbf{x}_|| \in \Omega \) subject to \( P(\partial \Omega, t) = P_a \)
- load-bearing capacity \( \mathbf{F}(t) = \int_{\Omega} P\mathbf{e}_n \, d\Omega \)

clearance \( C(\mathbf{x}_||, t) \) is

- prescribed
- found from fluid–structure interaction machinery (e.g. shaft) dynamics \( \Rightarrow \mathbf{F} = F(\partial_{\mathbf{n}}C, \partial_{\mathbf{t}}C, C) \)
- EHL \( \Rightarrow P = P(C) \)
Validation of tribo-systems

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\[ P(x_{||}, t), \ x_{||} \in \Omega \quad \text{subject to} \quad P(\partial \Omega, t) = P_a \]

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machine (e.g. shaft) dynamics \( \Rightarrow F = F(\partial_{tt} C, \partial_t C, C) \)

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Validation of tribo-systems

typically find

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Validation of tribo-systems

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machinery (e.g. shaft) dynamics \[ F = F(\partial_{tt} C, \partial_t C, C) \]

EHL \[ P = P(C) \]
Classical application: journal bearing

\[
\tilde{C} \quad \theta \\
\tilde{\omega} \quad \tilde{\epsilon} \\
\text{liquid film} \\
\text{cavitation}
\]

reference quantities
\[
\tilde{U}_m = \tilde{\omega} \tilde{R}_i, \quad \tilde{p}_r = \tilde{\eta}_r \tilde{\omega} \tilde{R}_i^2 / \tilde{C}^2
\]

geometrical parameters
\[
\epsilon = \tilde{C} / \tilde{R}_i \ll 1, \quad \text{eccentricity} \quad \epsilon = \tilde{\epsilon} / (\tilde{R}_a - \tilde{R}_i)
\]

non-dimensional quantities
\[
C = 1 + \epsilon \cos \theta + O(\epsilon^2), \quad U_m (= N = Q) = 1
\]
Further outlook include

- EHL
- inertia \( (Re \epsilon^2 \sim 1, \text{start-up, high-speed rotors, rapid load cycles}) \)
- turbulence
- film rupture & cavitation (surface tension)
- effects acting on micro-scale \( \ll \epsilon \) (surface roughness, mixed friction)

Rational method: perturbation techniques

- multiple scales, matched asymptotic expansions
- numerical solution of reduced problem (simulation tools)
Further outlook

include

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Further outlook

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rational method: perturbation techniques

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Thank you for your attention!