KONINKLIJKE VLAAMSE ACADEMIE VAN BELGIE
VOOR WETENSCHAPPEN EN KUNSTEN

ACTUARIAL AND FINANCIAL MATHEMATICS CONFERENCE
Interplay between Finance and Insurance

February 4-5, 2010

Michèle Vanmaele, Griselda Deelstra, Ann De Schepper, Jan Dhaene, Wim Schoutens & Steven Vanduffel (Eds.)

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Actuarial and Financial Mathematics Conference
Interplay between finance and insurance

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PREFACE

On February 4 and 5, the contactforum "Actuarial and Financial Mathematics Conference" (AFMathConf2010) took place in the buildings of the Royal Flemish Academy of Belgium for Science and Arts in Brussels. The main goal of this conference is to strengthen the ties between researchers in actuarial and financial mathematics from Belgian universities and from abroad on the one side, and professionals of the banking and insurance business on the other side. The conference attracted 124 participants from 12 different countries, illustrating the large interest from academia as well as from practitioners.

For this 2010 edition, we have organized the first conference day as a thematic day on "Market Consistent Valuation in Insurance", which is a hot topic at the moment among academics and practitioners. During the first day, we welcomed four internationally esteemed invited speakers: *Eckhard Platen (University of Technology – Sydney, Australia), Philippe Artzner (Université de Strasbourg, France), Ragnar Norberg (London School of Economics, U.K.) and Antoon Pelsser (Maastricht University, the Netherlands)*. They all gave first-class lectures throwing more light on several aspects of Market Consistent Valuation. Together with *Guy Roelandt (CEO Dexia Insurance Services, Belgium)*, they afterwards participated in an animated panel discussion, moderated by *Steven Vanduffel (Vrije Universiteit Brussel, Belgium)*. The second day, the attendants had the opportunity to listen to four more invited speakers: *Hansjoerg Albrecher (Université de Lausanne, Switzerland), Maria de Lourdes Centeno (Technical University of Lisbon, Portugal), Dilip Madan (University of Maryland, USA) and Monique Jeanblanc (Université d'Evry Val d'Essonne, France)*, as well as to seven interesting contributions from *Donatien Hainaut (ESC Rennes, France), Daniel Alai (ETH Zurich, Switzerland), Katrien Antonio (University of Amsterdam, the Netherlands), Robert Salzmann (ETH Zurich, Switzerland), Marc Henrard (Dexia Bank, Belgium), Antonis Papapantoleon (QP Lab and TU Berlin, Germany) and Mateusz Maj (Vrije Universiteit Brussel, Belgium)*. In addition, nine researchers presented a poster during an appreciated poster session. We thank them all for their enthusiasm and their nice presentations which made the conference a great success.

The present proceedings give an overview of the activities at the conference. They contain comments on one of the invited talks, three papers corresponding to contributed talks, and five short communications of posters presented during the poster sessions on both conference days.

We are much indebted to the members of the scientific committee, *Freddy Delbaen (ETH Zurich, Switzerland), Rob Kaas (University of Amsterdam, the Netherlands), Ernst Eberlein (University of Freiburg, Germany), Michel Denuit (Université Catholique de Louvain, Belgium), Noel Veraverbeke (Universiteit Hasselt, Belgium) and Griselda Deelstra (Université Libre de Bruxelles & Vrije Universiteit Brussel, Belgium)*, for the excellent scientific support. We also thank *Wouter Dewolf (Ghent University, Belgium)*, for the administrative work.

We cannot forget our sponsors, who made it possible to organise this event in a very agreeable and inspiring environment. We are very grateful to the Royal Flemish Academy of Belgium for Science and Arts, the Research Foundation – Flanders (FWO), the Scientific Research Network (WOG) "Fundamental Methods and Techniques in Mathematics", le Fonds de la Recherche Scientific (FNRS), the KULeuven Fortis Chair in Financial and Actuarial Risk Management and the ESF program "Advanced Mathematical Methods in Finance" (AMaMeF).

The success of the meeting encourages us to go on with the organisation of this contactforum. We are sure that continuing this event will provide more opportunities to facilitate the exchange of ideas and results in our fascinating research field.

The editors:
Griselda Deelstra
Ann De Schepper
Jan Dhaene
Wim Schoutens
Steven Vanduffel
Michèle Vanmaele

The other members of the organising committee:
Pierre Devolder
Paul Van Goethem
David Vyncke

INVITED TALK

SUPERVISORY ACCOUNTING:

COMPARISON BETWEEN SOLVENCY II AND COHERENT RISK MEASURES

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Abstract

We examine the ingredients of Solvency II, namely its free capital, provision and solvency capital requirement. They are of course linked by the accounting equality but we claim that they should be more deeply related to each other since solvency naturally should require positivity of available capital. Taken in general, this condition indeed almost dictates a formula to derive provision from free capital. The derivation suggests the property of market consistency of provision and the definition of optimal replicating portfolio. This does not show up in actual building of Solvency II, while we show that coherent risk measures allow an integrated construction.

1. INTRODUCTION AND NOTATION

1.1. Introductory remarks

The following is essentially a sequence of comments on the actual presentation at the Actuarial and Financial Mathematics Conference, Brussels, Feb. 4-5, 2010, and it incorporates some feedback received during the conference. The presentation itself was based on the paper Artzner and Eisele (2010).

The paper intended to draw connections between academia, in particular the theory of risk measures, and industry in looking for a definition of "solvency", further exploring the reason and method for studying "provision" as well as the deduced requirement on available capital.

The current comments go one step further with respect to Solvency II by reviewing its ingredients (see Subsection 2.1) and by asking how related they are or should be. A first link is of course the accounting equality:

[1]Partial support from AERF/CKER, The Actuarial Foundation is gratefully acknowledged.

$$\text{free capital} + \text{provision} + \text{capital requirement} = \text{initial asset value}$$

where the free capital is chosen by the assessment procedure, like the 0.5% quantile of the net (cash-flow) position whose positivity is taken as definition of solvency. Solvency is usually rephrased as:

$$\text{available capital } (= \text{initial asset value} - \text{provision}) \geq \text{capital requirement}$$

which strongly suggests that available capital must be positive when solvency is reached.
This fact in turn puts a condition on the definition of provision via the positivity of free capital since market possibilities show up in this procedure: roughly speaking definition of the provision functional (on random obligation cash-flows) is mandated by the free capital functional (see Definition 2.2). This is a first, very powerful instrument to judge a solvency system.
Unfortunately, the construction of Solvency II by a VaR-operator reveals a number of serious difficulties, in particular the possibility of supervisory arbitrage. We show these lacks in some examples.
The much spoken about property of "market consistency" of obligation assessments is defined and shown to result from the definition of provision out of free capital.
Guided by these observations on Solvency II, we juxtapose the "top-down" approach provided by the test probabilities at the basis of coherent risk measurement (see Artzner and Eisele (2010)). It starts from the free capital *functional* (Section 3.1.1) then obtains the provision (Section 3.2) either by an infimum of the price of asset portfolios "covering" the company's obligations and providing positive free capital (the method of Definition 2.2), or by a characterization of market consistency in terms of the trading-risk neutral test probabilities. Finally it recovers the requirement on available capital from an asset-liability management approach as well as from the supervisory accounting equality.

1.2. Notations

1.2.1. Probabilistic framework

We deal with a one-period model with $t = 0$ and $t = 1$. Based on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a random variable X is interpreted as a "risky position" in date 1 money. By $Z_1 \geq 0$ we denote an **obligation cash-flow** at date 1 and by A_1 an **asset cash-flow** at date 1 in a financial market, described as follows:

1.2.2. Financial markets

For $i = 0, \ldots, d$, assets S^i are given as random variables: $S_1^i(\omega)$ is the cash-flow of date 1 money in state ω. S_0^i is the initial market price of S^i. We take $S_1^0 = r$ as the "numéraire", assuming $r > 0$ and $S_0^0 = 1$. Now, the asset cash flow A_1 has the form of a **traded** portfolio: $A_1 = \sum_{i=0}^d \alpha_i S_1^i$, $\alpha_i \in \mathbb{R}$, with initial market value $A_0 = \pi(A_1) := \sum_{i=0}^d \alpha_i S_0^i$. The pricing functional is denoted as π.
The set $\mathcal{M}$ of **"trading-risk neutral" probabilities** $\mathbb{Q}$ is

$$\mathcal{M} := \left\{ \mathbb{Q} \mid \mathbb{E}_{\mathbb{Q}}\left[S_1^i / r\right] = S_0^i \text{ for all } i = 1, \ldots, d \right\},$$

such that

$$\pi(A_1) = \mathbb{E}_{\mathbb{Q}}\left[A_1/r\right] \quad \text{for all } \mathbb{Q} \in \mathcal{M} \text{ and all tradeable } A_1. \tag{1}$$

The set of zero-cost portfolios is

$$N_{\mathcal{M}} := \{D \mid D \text{ is a traded portfolio with } \pi(D) = 0\}. \tag{2}$$

Thus, a self-financed **rebalancing** of the insurance's asset choice is possible at date 0 by passing from A_1 to $A_1 + D$ with $D \in N_{\mathcal{M}}$.

1.2.3. Business plan of an insurance company

A business plan (A_1, Z_1) of a company consists of an asset cash flow A_1 and an obligation cash-flow Z_1. The obligation Z_1 is exogenously given by the company's signed contracts and in general not tradeable, while the asset cash flow A_1 is tradeable and subject to the choice of the company's management. So $C_1 := A_1 - Z_1$ is the "net position". We call $D_1 := A_1 - r \cdot A_0$ the **"trading-risk exposure"**, where $A_0 = \pi(A_1)$.

2. SOLVENCY II REVISITED

2.1. Components of Solvency II

2.1.1. Free Capital

Solvency II is mainly based on a **quantile approach** since it requires the future net cash-flow $A_1 - Z_1$ to satisfy the following "solvency" condition:

$$\mathbb{P}\left[A_1 - Z_1 \geq 0\right] \geq 99.5\%. \tag{3}$$

We define the quantile $\widetilde{F}_0(A_1, Z_1) = q_{0.5\%}((A_1 - Z_1)/r)$. Relation (3) is a requirement on $\widetilde{F}_0$, namely $\widetilde{F}_0(A_1, Z_1) \geq 0$. We read $\widetilde{F}_0(A_1, Z_1)$ as the company's **"free capital"** since the company satisfies the solvency condition as long as $\widetilde{F}_0(A_1, Z_1)$ remains positive.
In principle the solvency condition says all about the supervisor's opinion on the risky position $A_1 - Z_1$. However, one main task of accounting in general and also of supervisory accounting is to distinguish between commitments with respect to third parties and those with respect to shareholders.

2.1.2. Provision

Provision (also called "liability") is thought of as an assessment of the obligation Z_1 contracted by the company with respect to the contingent creditor, namely the policyholder. At one point of the elaboration of Solvency II it was defined as:

$$\widetilde{L}_0(Z_1) \text{ is the 75\%- quantile of } Z_1/r: \quad \mathbb{P}\left[Z_1/r > \widetilde{L}_0(Z_1)\right] \leq 25\%. \tag{4}$$

We remark that $\widetilde{L}_0$ bears no formal relation to $\widetilde{F}_0$, and that it is independent of the financial market, except for r.

2.1.3. Solvency capital requirement

The definitions of the free capital $\widetilde{F}_0(A_1, Z_1)$ and the provision $\widetilde{L}_0(Z_1)$ give an implicit definition of the "solvency capital requirement" $\widetilde{M}_0(A_1, Z_1)$ via the **(supervisory) accounting equality**

$$\pi = \widetilde{L}_0 + \widetilde{M}_0 + \widetilde{F}_0. \tag{5}$$

This is an equality of functionals on the couple (A_1, Z_1).

Remark 2.1 *We denote $\widetilde{M}_0(A_1, Z_1)$ by the traditional "Solvency capital requirement" (SCR), though it is only the definition of an amount of capital, rather then a requirement. The requirement lies in the condition $\pi(A_1) \geq \widetilde{L}_0(Z_1) + \widetilde{M}_0(A_1, Z_1)$. Personally we would prefer for $\widetilde{M}_0(A_1, Z_1)$ the name "required solvency capital".*

Sometimes the solvency capital requirement is also defined using a sort of asset-liability management approach, by considering net asset values at date $t = 0$ and $t = 1$ and their difference. The quantile flavour of free capital is reflected in the definition of $\widetilde{M}_0$ as the (negative) quantile:

$$\mathbb{P}\left[\text{NAV}_1/r - \text{NAV}_0 \geq -\widetilde{M}_0(A_1, Z_1)\right] \geq 99.5\%. \tag{6}$$

In the one-period model, we have $\text{NAV}_1 = A_1 - Z_1$ and $\text{NAV}_0 = A_0 - \widetilde{L}_0(Z_1)$. Using (5) , it is easily seen that (6) is equivalent to the accounting equality definition of $\widetilde{M}_0$.
Rewriting (5) as $\pi - \widetilde{L}_0 = \widetilde{F}_0 + \widetilde{M}_0$ shows that the required positivity of both the free capital $\widetilde{F}_0(A_1, Z_1)$ and $\widetilde{M}_0(A_1, Z_1)$ implies the positivity of the **"available capital"** $A_0 - \widetilde{L}_0(Z_1)$. We shall see in Example 2.1 below that the converse is not true.

2.2. Positivity of the available capital

We now examine the interdependence between the roles of free capital and of provision. A very natural requirement on the provision as assessment of obligations is the fact that if solvency is granted by the supervision, then the market value of assets should be at least equal to the provision i.e. available capital, assets minus liabilities, should be positive. We show that this is not always satisfied by a quantile based definition of provision.
In the following we write random variables and probabilities on a finite probability space as row vectors.

Example 2.1 *Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ with probability vector $\mathbb{P} = (0.005, 0.245, 0.25, 0.5)$ and $S_1^0 = r = (1, 1, 1, 1)$. We assume that $\mathbb{P}$ is also a market-risk neutral probability. At date 1, the company's obligation will be $Z_1 = (60, 1000/49, 40, 0)$ and its asset value $A_1 = (0, 1000/49, 40, 0)$. Obviously the company satisfies the "solvency" condition (3), since its free capital $\widetilde{F}_0(A_1 - Z_1) =$*

0. *By (4), the provision is* $\widetilde{L}_0(Z_1) = 40$ *while the initial asset value is* $A_0 = 15$. *Thus, the company does not satisfy the requirement of the positivity of the available capital: both the available capital and the solvency capital requirement are equal to* -25.
Only for the sake of completeness, we mention that if we replace A_1 *by* $A'_1 = (0, 2000/49, 100, 10)$, *we get* $A'_0 = 40$ *such that the available capital* $A'_0 - \widetilde{L}_0(Z_1) = 0$ *is non-negative, while* $\mathbb{P}[A'_1 - Z_1 = (-60, 1000/49, 60, 10) \geq 10] = 99.5\%$ *shows that the solvency capital requirement* $\widetilde{M}_0(A'_1, Z_1) = -10$ *is negative.*

2.3. Attempting a systematic approach to supervisory provision

In existing solvency regulations and in the insurance industry there exists a great number of definitions of provision. In some of them it is sometimes not clear which part covers the firm's obligations with respect to the policyholder and which part serves as protection against other uncertainties, for example shareholders' risks.

Remark 2.2 *In this paper, as in Artzner and Eisele (2010), we study exclusively the notion of supervisory provision, in the following shortly called provision. But we emphasize the fact that supervisory provision has to be distinguished from other notions of provision, in particular those defined by a cost-of-capital method. This is a matter of ongoing research.*

We believe that any notion of provision should satisfy the following two conditions:

Conditions 2.1

1. *(**Independence**): Any two companies having the same obligation* Z_1 *with respect to the policyholders should have the same provision.*
2. *(**Positivity of available capital**): If an insurance company is acceptable to the supervisor, then its initial asset value should be at least as great as the provision.*

The quantile approach of Solvency II given in (3) and (4) does not satisfy the positivity of available capital condition above, as the Example 2.1 has shown.
Based on the requirements above, a general and natural derivation of provision from a solvency assessment would be as follows:
We suppose that Ψ is a functional, called free capital, on random variables such that a net position $A_1 - Z_1$ satisfies the solvency condition if and only if

$$\Psi(A_1 - Z_1) \geq 0.$$

Definition 2.2 (Derivation of supervisory provision from the free capital functional Ψ)
The provision $L_\Psi(Z_1)$ *of an obligation* Z_1 *deduced from the free capital functional* Ψ *is the minimal initial value* $A_0 = \pi(A_1)$ *of a tradeable portfolio* A_1 *such that* $A_1 - Z_1$ *satisfies the solvency condition:*

$$L_\Psi(Z_1) = \inf\left\{\pi(A_1) \mid A_1 \text{ tradeable with } \Psi(A_1 - Z_1) \geq 0\right\}. \tag{7}$$

As is evident from the formulation, L_Ψ depends on the whole external market structure (but is defined without reference to an asset portfolio). We shall even show below that it defines a market consistent assessment.

2.4. Market consistent assessment

At the Oberwolfach miniworkshop on the Mathematics of Solvency, Cheridito et al. (2008) presented a definition of a market consistent functional (see also Malamud et al. (2008)). In fact, it is a generalization of cash-invariance: $\Psi(X + c \cdot r) = \Psi(X) + c$.

Definition 2.3 (Market Consistency)
*An assessment Ψ is **market consistent** (MC) if and only if for each X and each traded U:*

$$\Psi(X + U) = \Psi(X) + \pi(U). \tag{8}$$

Proposition 2.1 *For any functional Ψ defining the solvency condition, the provision L_Ψ is market consistent:*

$$L_\Psi(Z_1 + U) = L_\Psi(Z_1) + \pi(U) \tag{9}$$

for all obligations Z_1 and all traded U.

Proof. If U is a traded position, then

$$\begin{aligned} L_\Psi(Z_1 + U) &= \inf\{\pi(A_1) \mid A_1 \text{ tradeable with } \Psi(A_1 - Z_1 - U) \geq 0\} \\ &= \inf\{\pi(A_1' + U) \mid A_1' \text{ tradeable with } \Psi(A_1' - Z_1) \geq 0\} = L_\Psi(Z_1) + \pi(U). \end{aligned}$$

■

Example 2.2 *We resume Example 2.1 with $Z_1 = (60, 1000/49, 40, 0)$, $S_1^0 = r = (1, 1, 1, 1)$, $S_1^1 = (0, 1000/49, 40, 1)$ and $S_1^2 = (300, 900/49, 38, 0)$ such that $S_0^1 = S_0^2 = 15.5$. It follows that $D = S_1^1 - S_1^2 = (-300, 100/49, 2, 1)$ is a zero-cost portfolio with a short position in S^2.*
Since $Z_1 - D_1 = (360, 900/49, 38, -1)$, we find

$$\widetilde{L}_0(Z_1 - D_1) = 38 \neq \widetilde{L}_0(Z_1) = 40.$$

This shows that $\widetilde{L}_0$ is not market consistent.

2.5. Supervisory arbitrage

We consider Example 2.1 once more.

Example 2.3 *Define Z_1 and D_1 as before. Now for any asset value $A_1 = (a_1, a_2, a_3, a_4)$, we get*

$$A_1 + \lambda D_1 - Z_1 = (a_1 - 300\lambda - 60, a_2 + 100/49\lambda - 1000/49, a_3 + 2\lambda - 40, a_4 + \lambda).$$

Therefore, a company with the arbitrary asset value A_1 can satisfy the solvency condition (3) by putting at zero cost a sufficiently large amount of the contract D into its portfolio:

$$\mathbb{P}[A_1 + \lambda D_1 - Z_1 \geq 0] \geq 99.5\%$$

for $\lambda > 0$ sufficiently large.

Remark 2.3 *The example shows that the solvency condition (3) allows supervisory arbitrage. This can be expressed as $L_{\widetilde{F}_0} \equiv -\infty$.*
One may argue that such "abstract" examples do not reflect the real situation of an insurance company. However, recent experiences show that astute people soon may find out these deficiencies and create special financial products (like D above), allowing companies having difficulties to pass the solvency condition, no matter how bad their situation is.
We shall find the same problem in the second part of this paper where we discuss supervisory accounting by a coherent risk-adjusted assessment. However, there the simple assumption 3.1 prevents the existence of supervisory arbitrage, while in the situation of Solvency II it seems very difficult, if not impossible, to avoid supervisory arbitrage.

3. SUPERVISION BY COHERENT RISK-ADJUSTED ASSESSMENT

The observations on Solvency II comfort us into the "top-down" approach provided by the test probabilities at the basis of coherent risk measurement. We shall start from the free capital *functional*, the negative of a *coherent* risk measure (Section 3.1.1). We then obtain the provision (Section 3.2) by an infimum of the price of asset portfolios "covering" the company's obligations and providing positive free capital (the method of Definition 2.2), and identify it explicitly as a market consistent functional. Finally we shall recover the requirement on available capital from an asset-liability management approach as well as from the supervisory accounting equality.

3.1. Coherent risk-adjusted assessment

3.1.1. Acceptability and free capital

For a (closed, convex) set $\mathcal{P}$ of **"test probabilities"** ("scenarios", "stress tests"), we define the risk-adjusted assessment[2] $\Phi_{\mathcal{P},r}$ for each random variable X by

$$\Phi_{\mathcal{P},r}(X) := \inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[X/r\right]. \tag{10}$$

Definition 3.1 *A risky position X is **acceptable** w.r.t. $(\mathcal{P}, r)$ if and only if*

$$\Phi_{\mathcal{P},r}(X) \geq 0.$$

According to Section 2.1.1, we get the free capital of a business plan:

Definition 3.2 *The **free capital** of (A_1, Z_1) is*

$$F_0(C_1) := \Phi_{\mathcal{P},r}(A_1 - Z_1) = \inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[(A_1 - Z_1)/r\right]. \tag{11}$$

[2] $-\Phi_{\mathcal{P},r}$ is called risk measure in Artzner et al. (1999).

3.1.2. No supervisory arbitrage

It is intuitive that the set $\mathcal{P}$ of test probabilities cannot be chosen completely independent of the market situation if we want to avoid situations like in Example 2.2 (see also Remark 2.3). For example assume that there exists a traded D with

$$\pi(D) = 0 \quad \text{and} \quad \Phi_{\mathcal{P},r}(D) = \alpha > 0. \tag{12}$$

Then, for any X, there exists a λ large enough such that

$$\Phi_{\mathcal{P},r}(X + \lambda \cdot D) \geq \Phi_{\mathcal{P},r}(X) + \lambda \cdot \alpha > 0,$$

i.e. “any X can be made acceptable in a self-financed way”. It has been shown in Artzner et al. (1999), Section 4.3 and Delbaen (2000), Chapter 7, that (12) is equivalent to $\mathcal{P} \cap \mathcal{M} = \emptyset$. Therefore, we make the following assumption:

Assumption 3.1 (No Supervisory Arbitrage)

$$\mathcal{P} \cap \mathcal{M} \neq \emptyset. \tag{13}$$

Though we want (13), it would not be wise to have the other extreme $\mathcal{P} \subset \mathcal{M}$, since then $\Phi_{\mathcal{P},r}$ would not distinguish government bonds from dot.com shares in equally priced portfolios A_1, A_1' respectively. A meaningful set of test probabilities $\mathcal{P}$ would therefore satisfy both relations $\mathcal{P} \cap \mathcal{M} \neq \emptyset$ and $\mathcal{P} \setminus \mathcal{M} \neq \emptyset$.

3.2. Two definitions of provision

3.2.1. First definition of the provision of an obligation

Let us apply the Definition 2.2 to the risk-adjusted assessment $\Phi_{\mathcal{P},r}$:

Definition 3.3 (Provision L_0)

$$L_0\,(Z_1) = \inf\left\{\pi(A_1) \mid A_1 \textit{ tradeable with } \Phi_{\mathcal{P},r}(A_1 - Z_1) \geq 0\right\}. \tag{14}$$

$L_0(Z_1)$ is market consistent by Proposition 2.1.

3.2.2. Solvability Condition on (A_0, Z_1)

Definition 3.4 (Solvability Condition)
*(A_0, Z_1) is **solvable** if and only if the available capital $A_0 - L_0(Z_1)$ is positive.*

We shall see by Equation (17) below that an acceptable business plan (A_1, Z_1) not only realizes the solvability condition for (A_0, Z_1), but even satisfies the stronger condition of the positivity of the solvency capital requirement.

3.2.3. Second Definition of Provision

We know by Proposition 2.1 that the provision L_0 is market consistent. Here we give an explicit representation of L_0 in terms of the trading-risk neutral test probabilities. Indeed, one can show that a coherent risk-assessment $\Phi_{\mathcal{P}',r}$ is market consistent if and only if $\mathcal{P}' \subset \mathcal{M}$. The last two phrases give a hint to the fact that L_0 should be in relation to $\Phi_{\mathcal{P}\cap\mathcal{M},r}$. In fact we have the following equality.

Proposition 3.2

$$L_0(Z_1) = -\Phi_{\mathcal{P}\cap\mathcal{M},r}(-Z_1) = \sup_{\mathbb{Q}\in\mathcal{P}\cap\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[Z_1/r]. \tag{15}$$

The proof of (15), given in Artzner and Eisele (2010), is based on the fact that $\Phi_{\mathcal{P}\cap\mathcal{M},r}$ is the convolution of $\Phi_{\mathcal{P},r}$ and $\Phi_{\mathcal{M},r}$.

Remark 3.1 *The positivity of the available capital $A_0 - L_0(Z_1)$ can be written as the condition $\Phi_{\mathcal{P}\cap\mathcal{M},r}(A_0 \cdot r - Z_1) \geq 0$.*

3.3. ALM-risk and solvency capital requirement

Solvency II has already shown a duality of approaches to solvency capital requirement: accounting equality or asset-liability considerations. We have the same opportunity here, with more details for the ALM approach, which we therefore describe first. Asset-Liability-Management (ALM) is guided by the adequacy of coverage between the actual A_1 and Z_1. They are both "centered" around $A_0 \cdot r$ and $L_0(Z_1) \cdot r$ respectively. Recalling $C_1 = A_1 - Z_1$, we therefore introduce the following definition.

Definition 3.5 *The ALM-risk is given by*

$$A_1 - A_0 \cdot r - (Z_1 - L_0(Z_1) \cdot r) = C_1 - \Phi_{\mathcal{P}\cap\mathcal{M},r}(C_1) \cdot r. \tag{16}$$

Its risk-adjusted assessment

$$\begin{aligned} M_0(A_1 - Z_1) = M_0(C_1) &:= -\Phi_{\mathcal{P},r}\left(C_1 - \Phi_{\mathcal{P}\cap\mathcal{M},r}(C_1) \cdot r\right) \\ &= \Phi_{\mathcal{P}\cap\mathcal{M},r}(C_1) - \Phi_{\mathcal{P},r}(C_1) \geq 0 \end{aligned} \tag{17}$$

is the ***solvency capital requirement (SCR)*** *(see Remark 2.1).*

By its very definition, the solvency capital requirement is positive, which implies for an acceptable business plan (A_1, Z_1) the positivity of the available capital.

Remark 3.2 *Since M_0 only deals with "centered" risks, $M_0(A_1 - Z_1) = M_0(A_1 + a \cdot r - (Z_1 + b \cdot r))$ for a and b constant, that is SCR only deals with "unforeseen losses" (CEIOPS, CP20).*

3.4. Supervisory accounting equality

The market consistent definition of provision $L_0(Z_1) = -\Phi_{\mathcal{P}\cap\mathcal{M},r}(-Z_1)$ and the definition of the solvency capital requirement $M_0(C_1) = A_0 + \Phi_{\mathcal{P}\cap\mathcal{M},r}(-Z_1) - \Phi_{\mathcal{P},r}(C_1)$ add up to:

Proposition 3.3 *The* ***supervisory accounting equality*** *is*

$$A_0 = L_0(Z_1) + M_0(C_1) + F_0(C_1). \tag{18}$$

As a consequence of (18) and (15), we find the following equivalences:

Corollary 3.4 *The business plan (A_1, Z_1) is acceptable if and only if*

$$A_0 \geq L_0(Z_1) + M_0(C_1), \tag{19}$$

or equivalently, if the available capital $A_0 - L_0(Z_1)$ is greater than the solvency capital requirement $M_0(C_1)$.

3.5. Optimal replicating portfolios

Since $\Phi_{\mathcal{P}\cap\mathcal{M},r}$ is the convolution of $\Phi_{\mathcal{P},r}$ and $\Phi_{\mathcal{M},r}$, one can show that

$$\Phi_{\mathcal{P}\cap\mathcal{M},r}(X) = \sup_{D\in N_{\mathcal{M}}} \Phi_{\mathcal{P},r}(X + D).$$

Applying (15) gives

$$L_0(Z_1) = \inf_{D\in N_{\mathcal{M}}} - \Phi_{\mathcal{P},r}(D - Z_1). \tag{20}$$

In general, even for a finite probability space, the infimum in (20) is not attained. For a condition which guarantees the existence of an optimal replicating portfolio, we refer to Artzner and Eisele (2010). Nevertheless, it is important to specify portfolios whose trading-risk exposure D^* minimizes (20):

Definition 3.6 *An asset portfolio $A_1^* = r \cdot A_0 + D_1^*$ whose trading-risk exposure $D_1^* = A_1^* - r \cdot A_0$ for $-Z_1$ satisfies*

$$L_0(Z_1) = -\Phi_{\mathcal{P},r}(D^* - Z_1), \tag{21}$$

is called an ***optimal replicating portfolio*** *for the obligation Z_1.*

We have two characterizations of an optimal replicating portfolio.

Proposition 3.5 *A_1^* is an optimal replicating portfolio for Z_1 if and only if one of the following equivalent conditions is satisfied:*

1. *the free capital is equal to the available capital:*

$$\Phi_{\mathcal{P},r}(A_1^* - Z_1) = A_0 - L_0(Z_1), \tag{22}$$

2. *the solvency capital requirement is zero:*

$$M_0(A_1^* - Z_1) = 0. \tag{23}$$

It is remarkable that, in the context of the "cost of capital principle", the Swiss Solvency Test (SST) refers to optimal replicating portfolios as portfolios which "immunize the liability cash-flows against all changes in the underlying market risk factors".

3.6. Three different regions of solvency

The question whether the initial asset value A_0 is sufficient to cover provision and solvency capital requirement or not determines, for any solvency system, three regions described below in terms of the available capital. Moreover, the use of coherent risk measures allows to introduce at some point the risk management choice of "more capital brought by shareholders" or "rebalancing the asset portfolio".

1. The region of negative available capital:

$$A_0 - L_0(Z_1) < 0. \tag{24}$$

   Since the positivity of the available capital is a "conditio sine qua non", the only possibility to avoid closure of the company by the supervisor is to gather (from the shareholders) a new amount of cash of at least $c \geq L_0(Z_1) - A_0$ and thus to meet this condition.

2. Region of positive available capital, but not satisfying the solvency condition:

$$0 \leq A_0 - L_0(Z_1) < M_0(A_1 - Z_1). \tag{25}$$

   There are now two possible solutions:

   (a) Under appropriate assumptions on the set $\mathcal{P}$ of test probabilities, the solvency capital requirement $M_0(A_1 - Z_1)$ can be diminished to zero by rebalancing the assets to an equally priced optimal replicating portfolio A_1^*. Thus, one satisfies the solvency condition:

$$A_0 - L_0(Z_1) \geq \underbrace{M_0(A_1^* - Z_1)}_{=0}.$$

   (b) If the company does not want to change its trading-risk exposure $D_1 = A_1 - A_0 \cdot r$, it has to get an additional amount of cash $c \geq -\Phi_{\mathcal{P},r}(D_1 - Z_1) - A_0$ from the capital market, and to invest c in the numéraire r. The risk exposure D_1 is then left unchanged and

$$\Phi_{\mathcal{P},r}(A_1 + c \cdot r - Z_1) = A_0 + c + \Phi_{\mathcal{P},r}(D_1 - Z_1) \geq 0.$$

   The company thereby avoids a new assessment of the final future net position and, possibly, a new requirement!

3. If $A_0 - L_0(Z_1) \geq M_0(A_1 - Z_1)$, the company meets the solvency condition.

Remark 3.3 *Roughly speaking, the three regions above correspond in Solvency II to*

1. $A_0 <$ *technical provision + minimum capital requirement (MCR) ,*

2. *technical provision + MCR* $\leq A_0 <$ *technical provision + solvency capital requirement (SCR),*

3. *technical provision + SCR* $\leq A_0$.

4. CONCLUSION

Starting from an analysis of Solvency II, we realized that free capital is the key concept in solvency considerations. We think that a supervisory assessment of obligation has to be deduced from the free capital functional, the supervisory assessment of future cash flows. Such a derivation will immediately provide market consistency of provisions.
Value-at-risk methods present difficulties along these lines, which coherent risk measures do not.

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CONTRIBUTED TALKS

PRICING OF A CATASTROPHE BOND RELATED TO SEASONAL CLAIMS

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Abstract

This paper proposes a method to price catastrophe bonds paying multiple coupons, when the number of claims is under the influence of a stochastic seasonal effect. The claim arrival process is modeled by a Poisson Process whose intensity is the sum of an Ornstein Uhlenbeck process and a periodic function. The size of claims is assumed to be a positive random variable, independent of the intensity process. The expected number of claims is deduced from the probability generating function, while the calculation of the fair coupon relies on the Fourier Transform.

1. INTRODUCTION

During the last two decades, we have attended to the emergence of a new category of assets, primarily developed to hedge the costs of insuring natural catastrophes. In this context, catastrophe insurance derivatives have been introduced at the Chicago Board of Trade in the early nineties. The value of those securities is directly related to indexes that account for the total insurance losses due to natural catastrophes in US, by regions. Reinsurers have also started to propose a wide range of insurance bonds, based upon the mechanism of securitization. Those products offer two advantages. Firstly, they transfer a part of insurance risks from the reinsurers to other potential investors, allowing to increase the reinsurers' volume of transactions. Secondly, insurance derivatives are efficient diversification tools for institutional investors, which are in their core business not exposed to catastrophe risks. Indeed, contrary to credit derivatives, the securities linked to insurance events are not at all correlated with financial markets.

However, the valuation of catastrophe derivatives is obviously more complex compared to the pricing of purely financial securities. The first problematic element is the incompleteness of the insurance linked securities. It does not make sense to appraise those contracts based on non arbitrage arguments, given that the underlying risks are not tradeable, by nature. This point has been underlined by many authors, see e.g. Murmann (2001) and Charpentier (2008). To summarize, the incompleteness entails that there exist more than one risk neutral measure, and that the price is not unique. A second issue related to pricing is the complexity of the aggregate losses process, that could hardly be modeled by standard financial tools. The first attempts of pricing were done by Cummins and Geman (1994) and Geman and Yor (1997). They model the underlying catastrophe indexes by a geometric Brownian motion with jumps. Aase (1999, 2001) and Christensen and Schmidli (2000) have proposed to model the aggregated losses as a compound Poisson process with stochastic size of claims. In a recent work, Biagini et al. (2008) have studied the valuation of catastrophe derivatives on a loss index but with a reestimation of the total aggregated claims. In those papers, the intensity of the Poisson process determining the number of claims, is either constant or a deterministic function of time. Jang (2000) and Dassios and Jang (2003) have improved the modeling of the aggregated losses by assuming that the claims arrival process is driven by a Poisson process whose intensity is a stochastic shot noise process. The pricing of the insurance derivatives is done under the Esscher measure.

The contribution of this paper is to propose a method to price an insurance bond which pays multiple coupons and whose nominal depends upon claims under the influence of a stochastic seasonal effect. The interest for modeling the seasonality is particularly obvious for claims such as hurricanes, storms, flooding or even car accidents, which are more frequent during certain periods of the year. For long term insurance securities, it is then crucial to integrate this trend in the pricing. To achieve this goal, the claims arrival process is modeled by a doubly stochastic process, whose intensity is the sum of one deterministic seasonal function and of one mean reverting stochastic process. This mean reversibility can also be useful to model the influence of long term climate changes on claims frequency.

In the following section, we present details on the claims arrival process and we provide a recursion to compute the expected number of claims. In section 3, the aggregated claims process and the insurance bond are defined, and the general formula to value the coupon rate of such bonds is presented. In section 4, we explain how the Fast Fourier Transform can help us to price bonds. Finally, this paper includes a numerical application that underlines the feasibility of our approach.

2. THE CLAIMS ARRIVAL PROCESS

In this paper it is assumed that the number of claims observed till time t is a Poisson process, denoted by N_t, with stochastic intensity. This class of processes is called doubly stochastic. It has already been widely used to model the process of credit events, and it seems well adapted to model the arrivals of claims. The process N_t is defined on a filtration $\mathcal{F}_t$, in a probability space Ω coupled to a probability measure, denoted by Q. This measure is assumed to be the risk neutral measure, used for pricing purposes (the relation between the modeling under P, the real measure, and Q is developed in Appendix B). The intensity of N_t is a non observable stochastic process, denoted by λ_t. It is defined on a filtration $\mathcal{H}_t$ such that, conditionally to $\mathcal{H}_t \vee \mathcal{F}_0$, the process N_t is a Poisson process for which the probability of observing k jumps is given by the formula:

$$P(N_t = k \,|\, \mathcal{H}_t \vee \mathcal{F}_0) = \frac{\left(\int_0^t \lambda_u du\right)^k}{k!} e^{-\int_0^t \lambda_u du}. \tag{1}$$

For more details on doubly stochastic processes, the interested reader is referred to Bremaud (1981) and Bielecki and Rutkowski (2004), chapter 6. The frequency of many claims exhibits both stochasticity and seasonality. This is obviously the case for claims related to natural calamities such as storms or floodings, but it is also the case for car accidents for which the frequency climbs during the winter period. In order to capture this double characteristic, stochasticity and seasonality, in the pricing of insurance bonds, the intensity of our Poisson Process is modeled as the sum of a cyclical deterministic function $\lambda(t)$ and a stochastic process λ_t^{OU}:

$$\lambda_t = \lambda(t) + \lambda_t^{OU}. \tag{2}$$

The deterministic cyclical function is defined by three real constant parameters $\delta, \beta, \gamma \in \mathbb{R}$:

$$\lambda(t) = \delta + \beta \cos\left((t+\gamma)2\pi\right), \tag{3}$$

while the stochastic component of the intensity, λ_t^{OU}, is an Ornstein Uhlenbeck process, whose speed, level of mean reversion and volatility are real constants, respectively denoted as a, b, $\sigma \in \mathbb{R}$. The dynamics of λ_t^{OU} is ruled by the following stochastic differential equation:

$$d\lambda_t^{OU} = a(b - \lambda_t^{OU})dt + \sigma dW_t, \tag{4}$$

where W_t is a Brownian motion defined on the filtration $\mathcal{H}_t$. Figure 1 presents an example of a trajectory followed by the intensity process. The dotted line is the function $\lambda(t)$. The continuous line is a sample path of the intensity λ_t. Note that by choosing a low speed of mean revertion, a, the mean reverting feature of λ_t may be used to model the influence on long term climate changes on the frequency of claims. The distribution of λ_t is detailed in the next proposition.

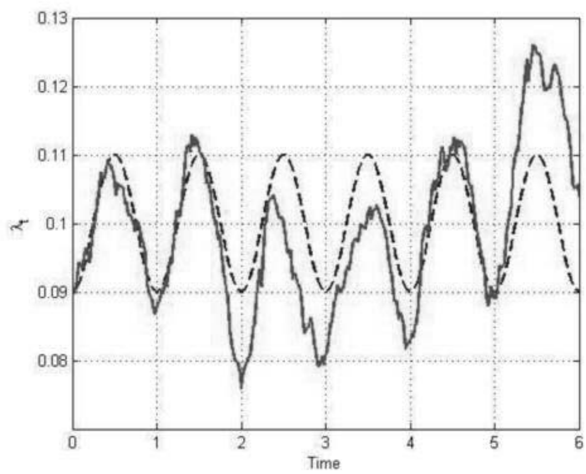

Figure 1: Example of a path for λ_t.

Proposition 2.1 *In our model, the process λ_t as defined in equation (2) is a Gaussian random variable conditionally on $\mathcal{H}_s$, $s \leq t$, whose average $\mu^\lambda(s,t)$ and variance $\left(\sigma^\lambda(s,t)\right)^2$ are given by the following expressions:*

$$\begin{aligned} \mu^\lambda(s,t) &= \lambda(t) + e^{-a(t-s)}\lambda_s^{OU} + b(1 - e^{-a(t-s)}) & (5) \\ \left(\sigma^\lambda(s,t)\right)^2 &= \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right). & (6) \end{aligned}$$

Proof. We just sketch the proof because this result is rather standard and we refer the reader to Musiela and Rutkowski (1997), chapter 12 - p.289, for details. The first step consists of differentiating the process $Z_t = e^{at}(b - \lambda_t^{OU})$ to show that

$$Z_t = Z_s - \int_s^t e^{au}\sigma dW_u; \tag{7}$$

as $\lambda_t^{OU} = b - e^{-at}Z_t$, one infers from equation (7), that

$$\lambda_t^{OU} = e^{-a(t-s)}\lambda_s^{OU} + b(1 - e^{-a(t-s)}) + \int_s^t e^{-a(t-u)}\sigma dW_u. \tag{8}$$

The results of the proposition directly follow from this last relation. ■

The intensity process λ_t defined in equations (2) and (4), is a Gaussian random variable, and the probability of observing a negative value for this process differs from zero. However, if the annual average level δ of $\lambda(t)$ and level of mean reversion b are sufficiently high compared to the volatility σ, this probability should be nearly zero, and λ_t may be used as intensity for N_t. We now present two propositions that will allow us to determine the probability generating function of N_t.

Proposition 2.2 *The integral of λ_u from t_1 to t_2, is a Gaussian random variable conditionally on $\mathcal{H}_s$, $s \le t_1 \le t_2$, whose average $\mu^{\int \lambda du}(s, t_1, t_2)$ and variance $\left(\sigma^{\int \lambda du}(s, t_1, t_2)\right)^2$ are given by the following expressions:*

$$\begin{aligned}
\mu^{\int \lambda du}(s, t_1, t_2) &= \delta(t_2 - t_1) + \frac{1}{2\pi}\beta \sin\left(2\pi(t_2 + \gamma)\right) - \frac{1}{2\pi}\beta \sin\left(2\pi(t_1 + \gamma)\right) \\
&\quad + \lambda_s^{OU} e^{-a(t_1-s)}B(t_1, t_2) + b\left((t_2 - t_1) - e^{-a(t_1-s)}B(t_1, t_2)\right) \qquad (9) \\
\left(\sigma^{\int \lambda du}(s, t_1, t_2)\right)^2 &= \frac{\sigma^2}{2a}B(t_1, t_2)^2\left(1 - e^{-2a(t_1-s)}\right) \\
&\quad + \frac{\sigma^2}{a^2}\left((t_2 - t_1) - B(t_1, t_2) - \frac{1}{2}a\, B(t_1, t_2)^2\right), \qquad (10)
\end{aligned}$$

where the function $B(t_1, t_2)$ is defined as follows:

$$B(t_1, t_2) = \frac{1}{a}\left(1 - e^{-a(t_2-t_1)}\right).$$

Proof. The integral of λ_u is the sum of the integrals of $\lambda(u)$ and of λ_u^{OU}. The integral of $\lambda(u)$ from t_1 to t_2 is worth:

$$\int_{t_1}^{t_2} \lambda(u)du = \delta(t_2 - t_1) + \frac{1}{2\pi}\beta \sin\left(2\pi(t_2 + \gamma)\right) - \frac{1}{2\pi}\beta \sin\left(2\pi(t_1 + \gamma)\right), \tag{11}$$

whereas the integral of the process λ_u^{OU} is obtained by integrating equation (8) from t_1 to t_2:

$$\begin{aligned}
\int_{t_1}^{t_2} \lambda_u^{OU} du &= \lambda_s^{OU} e^{-a(t_1-s)}B(t_1, t_2) + b\left((t_2 - t_1) - e^{-a(t_1-s)}B(t_1, t_2)\right) \\
&\quad + \int_s^{t_1} \frac{\sigma}{a}\left(e^{-at_1} - e^{-at_2}\right)e^{au}dW_u + \int_{t_1}^{t_2} \sigma B(u, t_2)dW_u. \qquad (12)
\end{aligned}$$

The results of the proposition directly follow from this last relation. ■

According to equation (1), the process $N_{t_2} - N_{t_1}$ is a Poisson process conditionally on $\mathcal{H}_{t_2} \vee \mathcal{F}_{t_1} \supset \mathcal{F}_s$. This property allows us to deduce that the probability of observing k jumps in a certain interval of time is equal to the following expectation, where I is an indicator variable:

$$\begin{aligned} P(N_{t_2} - N_{t_1} = k \,|\, \mathcal{F}_s) &= \mathbb{E}\left[I_{N_{t_2}-N_{t_1}=k} | \,\mathcal{F}_s\right] \\ &= \mathbb{E}\Big[\mathbb{E}\left[I_{N_{t_2}-N_{t_1}=k} | \,\mathcal{H}_{t_2} \vee \mathcal{F}_{t_1}\right] \Big| \,\mathcal{F}_s\Big] \\ &= \mathbb{E}\Big[P\left(N_{t_2} - N_{t_1} = k \,|\, \mathcal{H}_{t_2} \vee \mathcal{F}_{t_1}\right) \Big| \,\mathcal{F}_s\Big] \\ &= \mathbb{E}\left[\frac{\left(\int_{t_1}^{t_2} \lambda_u du\right)^k}{k!} e^{-\int_{t_1}^{t_2} \lambda_u du} \middle| \,\mathcal{F}_s\right]. \end{aligned} \tag{13}$$

Except for $k = 0$, no analytical expression exists for this last expectation. However, it will be shown in the remainder of this section that the probabilities of observing $k > 0$ jumps, can be computed by means of an iterative procedure based upon the probability generating function (pgf) of N_t as defined in the next proposition.

Proposition 2.3 *The pgf of N_t defined by*

$$\mathrm{pgf}(x, s, t_1, t_2) = \mathbb{E}\Big[x^{N_{t_2}-N_{t_1}} \Big| \mathcal{F}_s\Big] \tag{14}$$

with $s \leq t_1 \leq t_2$ is given by

$$\begin{aligned} \mathrm{pgf}(x, s, t_1, t_2) &= \mathbb{E}\left[e^{\int_{t_1}^{t_2} \lambda_u du\,(x-1)} \Big| \mathcal{F}_s\right] \\ &= \exp\left((x-1)\mu^{\int \lambda du}(s, t_1, t_2) + \frac{1}{2}(x-1)^2 \left(\sigma^{\int \lambda du}(s, t_1, t_2)\right)^2\right). \end{aligned} \tag{15}$$

Proof. Given that $\mathcal{H}_{t_2} \vee \mathcal{F}_{t_1} \supset \mathcal{F}_s$, one can rewrite the probability generating function as follows:

$$\mathrm{pgf}(x, s, t_1, t_2) = \mathbb{E}\left[x^{N_{t_2}-N_{t_1}} | \mathcal{F}_s\right] = \mathbb{E}\left[\mathbb{E}\left[x^{N_{t_2}-N_{t_1}} | \mathcal{H}_{t_2} \vee \mathcal{F}_{t_1}\right] \middle| \mathcal{F}_s\right].$$

Conditionally on $\mathcal{H}_{t_2} \vee \mathcal{F}_{t_1}$, the jump process is Poisson with known pgf, resulting in

$$\mathrm{pgf}(x, s, t_1, t_2) = \mathbb{E}\Big[\mathbb{E}\left[x^{N_{t_2}-N_{t_1}} | \mathcal{H}_{t_2} \vee \mathcal{F}_{t_1}\right] \Big| \mathcal{F}_s\Big] = \mathbb{E}\left[e^{\int_{t_1}^{t_2} \lambda_u du\,(x-1)} \Big| \mathcal{F}_s\right]. \tag{16}$$

The integral of the intensity is Gaussian according to proposition 2.2, and therefore the expectation in equation (16) is the expectation of a lognormal variable, as given in equation (15). ■

The pgf is a powerful tool that gives us the possibility to infer by a recursive method the probabilities of jumps of $N_{t_2} - N_{t_1}$, conditionally on the filtration $\mathcal{F}_s$ at time $s \leq t_1 \leq t_2$.

Proposition 2.4 *The probability not to observe any jumps in the interval of time $[t_1, t_2]$, conditionally on $\mathcal{F}_s$, $s \leq t_1 \leq t_2$, is equal to*

$$\begin{aligned} P(N_{t_2} - N_{t_1} = 0 \,|\, \mathcal{F}_s) &= \left. \operatorname{pgf}(x, s, t_1, t_2) \right|_{\{x=0\}} \\ &= \exp\left(-\mu^{\int \lambda du}(s, t_1, t_2) + \frac{1}{2}\left(\sigma^{\int \lambda du}(s, t_1, t_2)\right)^2 \right). \end{aligned} \tag{17}$$

The probability that the process N_t exhibits exactly one jump in the interval of time $[t_1, t_2]$ is equal to:

$$\begin{aligned} P(N_{t_2} - N_{t_1} = 1 \,|\, \mathcal{F}_s) &= \left. \frac{\partial}{\partial x} \operatorname{pgf}(x, s, t_1, t_2) \right|_{\{x=0\}} \\ &= P(N_{t_2} - N_{t_1} = 0 \,|\, \mathcal{F}_s) \left(\mu^{\int \lambda du}(s, t_1, t_2) - \left(\sigma^{\int \lambda du}(s, t_1, t_2)\right)^2 \right). \end{aligned} \tag{18}$$

The probability of observing more than one jump can be computed iteratively as follows:

$$P(N_{t_2} - N_{t_1} = k \,|\, \mathcal{F}_s) = \frac{1}{k!} \left. \frac{\partial^k}{\partial x^k} \operatorname{pgf}(x, s, t_1, t_2) \right|_{\{x=0\}}, \tag{19}$$

where

$$\begin{aligned} & \left. \frac{\partial^k}{\partial x^k} \operatorname{pgf}(x, s, t_1, t_2) \right|_{\{x=0\}} \\ &= \left(\mu^{\int \lambda du}(s, t_1, t_2) - \left(\sigma^{\int \lambda du}(s, t_1, t_2)\right)^2 \right) \left. \frac{\partial^{k-1}}{\partial x^{k-1}} \operatorname{pgf}(x, s, t_1, t_2) \right|_{\{x=0\}} \\ &\quad + (k-1) \left(\sigma^{\int \lambda du}(s, t_1, t_2)\right)^2 \left. \frac{\partial^{k-2}}{\partial x^{k-2}} \operatorname{pgf}(x, s, t_1, t_2) \right|_{\{x=0\}}. \end{aligned} \tag{20}$$

Proof. The proof of this proposition directly results from the exponential form of the probability generating function. ∎

Proposition 2.4 provides us with an important tool in order to calibrate our model parameters $(\delta, \beta, \gamma, a, b, \sigma)$ to real data. The calibration can be done by maximizing the likelihood of observed numbers of claims observed during several seasons.

3. THE SIZE OF CLAIMS AND THE PRICING OF BONDS

The risk faced by an insurance bondholder is inherent to his exposure to accumulated insured property losses. This process of accumulated losses, which is denoted by X_t in the sequel of this work, depends both on the frequency of the claims N_t and on the magnitude of the claims. The size of the j^{th} claim, occurring at time t_j, is modeled by a positive random variable, Y_j, defined on the filtration $\mathcal{F}_{t_j}$, but assumed to be independent from previous claims and from the frequency. There are no other constraints on the choice of Y_j. As for the claims arrival process N_t, we directly work with the distribution of Y_t under the risk neutral measure Q (again we refer the interested reader to appendix B for details about the relation between the modeling under P , the real measure, and Q). The process of aggregated losses, is defined by the following expression:

$$X_t = \sum_{t_j<t} Y_j = \sum_{i=1}^{N_t} Y_j.$$

We now describe the characteristics of an insurance bond and introduce the method to price such kind of assets. The insurance bond periodically pays a coupon equal to a constant percentage of the nominal, reduced by the amount of aggregated losses, exceeding a certain trigger level. At maturity, what is left of the nominal value is repaid. In order to compensate for this eventual loss of the nominal value, the coupon rate always exceeds the risk free rate. In case a few claims occur, the bondholder is then rewarded at a higher rate than the one obtained by investing in risk free assets with the same maturities. On the contrary, in case of catastrophic losses, the nominal of the bond can fall to zero and the payment of coupons can be interrupted. In order to understand how the spread of this bond is priced, we need to introduce some additional mathematical notations.

Let us use the notation BN for the initial nominal value of the bond. The level above which the excess of aggregated losses is deduced from the nominal, is denoted by K_1. If the total insured losses reach the amount of $K_2 = K_1 + BN$ before maturity, the bond stops to deliver coupons and the nominal is depleted. The bond, issued at time t_0, pays n coupons, at regular intervals of time, Δt, ranging from t_1 to t_n. The coupon rate is the sum of the constant risk free rate with maturity t_n, and of a spread: they are respectively denoted as r and sp. The coupons paid at times t_i, $i = 1...n$, are denoted as $\mathrm{cp}(t_i)$ and defined as follows:

$$\mathrm{cp}(t_i) = (r + sp)\,\Delta t \underbrace{\left[(K_2 - K_1)\, I_{X_{t_i}\in[0,K_1]} + (K_2 - X_{t_i})\, I_{X_{t_i}\in(K_1,K_2]}\right]}_{BN_{t_i}}. \tag{21}$$

The term between brackets is the (stochastic) nominal of the bond at time t_i and is written as BN_{t_i} in the sequel of our developments. Note that BN_{t_0} is equal to BN. Based upon the principle of absence of arbitrage, the spread of the insurance bond is chosen such that the expectations of future discounted spreads and of future discounted cutbacks of nominal are equal under the risk neutral pricing Q. The expectations of future discounted spreads and reductions of nominal are respectively called the “spreads leg” and the “claims leg” (this terminology is in fact inspired from the one for credit derivatives). These legs are defined by the following expressions:

$$\mathrm{SpreadLeg}(t_0) = sp\,\Delta t \sum_{t_i=t_1}^{t_n} e^{-r\,(t_i-t_0)}\mathbb{E}\left[BN_{t_i} \,|\, \mathcal{F}_{t_0}\right], \tag{22}$$

$$\mathrm{ClaimsLeg}(t_0) = \sum_{t_i=t_1}^{t_n} e^{-r\,(t_i-t_0)}\mathbb{E}\left[BN_{t_{i-1}} - BN_{t_i} \,|\, \mathcal{F}_{t_0}\right]. \tag{23}$$

By equating equation (22) with equation (23), we infer the following fair spread rate that should be added to the risk free rate, at the issuance of the insurance bond:

$$sp = \frac{\sum_{t_i=t_1}^{t_n} e^{-r\,(t_i-t_0)}\mathbb{E}\left[BN_{t_{i-1}} - BN_{t_i}\,|\,\mathcal{F}_{t_0}\right]}{\Delta t\,\sum_{t_i=t_1}^{t_n} e^{-r\,(t_i-t_0)}\mathbb{E}\left[BN_{t_i}\,|\,\mathcal{F}_{t_0}\right]}. \tag{24}$$

Despite the apparent simplicity of this last expression, the expected future nominals are not calculable by means of a closed form expression, and we have to rely on numerical methods to appraise them. Among the numerical tools available, we have chosen to use the Fourier transform.

4. PRICING BY FOURIER TRANSFORM

As explained in the previous paragraph, the pricing of an insurance bond requires the valuation of the expected future value of nominal. According to equation (21), this expectation may be split into two components,

$$\mathbb{E}\left[BN_{t_i}\,|\,\mathcal{F}_{t_0}\right] = \mathbb{E}\left[BN\, I_{X_{t_i}\in[0,K_1]}\,|\,\mathcal{F}_{t_0}\right] + \mathbb{E}\left[(K_2 - X_{t_i})\, I_{X_{t_i}\in(K_1,K_2]}\,|\,\mathcal{F}_{t_0}\right]. \tag{25}$$

As done by Carr and Madan (1999), each component can be reformulated in terms of their Fourier transforms. The two next propositions deal with this point.

Proposition 4.1 *If α_1 is a strictly positive real constant, chosen to ensure the stability of subsequent numerical computations, the following holds:*

$$\begin{aligned}&\mathbb{E}\left[BN\, I_{X_{t_i}\in[0,K_1]}\,|\,\mathcal{F}_{t_0}\right]\\ &\quad = \frac{BN}{\pi}\int_0^{+\infty} \varphi^1(u) e^{-(\alpha_1+iu)X_{t_0}} \sum_{k=0}^{+\infty} P(N_{t_i}-N_{t_0}=k)\Big(\mathbb{E}\left[e^{-(\alpha_1+iu)Y}\right]\Big)^k du,\end{aligned} \tag{26}$$

where $\varphi^1(u)$ is the Fourier transform of the function $e^{\alpha_1 x} I_{x\in[0,K_1]}$, $x\in\mathbb{R}^+$,

$$\varphi^1(u) = \frac{1}{\alpha_1+iu}\left(e^{(\alpha_1+iu)K_1} - 1\right). \tag{27}$$

The probabilities $P(N_{t_i}-N_{t_0}=k)$ can be retrieved from proposition 2.4 whereas the expectation $\mathbb{E}\left[e^{-(\alpha_1+iu)Y}\right]$ is the Laplace transform of the claim size Y, evaluated at α_1+iu.

Proof. Let us denote by $q_{X_{t_i}|\mathcal{F}_{t_0}}(x)$ the density of the aggregated losses process X_{t_i}, conditionally on the filtration $\mathcal{F}_{t_0}$. If we choose a positive constant α_1, the expectation in equation (34) can be written as follows:

$$\begin{aligned}\mathbb{E}\left[\,I_{X_{t_i}\in[0,K_1]}\,|\,\mathcal{F}_{t_0}\right] &= \mathbb{E}\left[e^{-\alpha_1 X_{t_i}}\, e^{\alpha_1 X_{t_i}}\, I_{X_{t_i}\in[0,K_1]}\,|\,\mathcal{F}_{t_0}\right].\\ &= \int_0^{+\infty} e^{-\alpha_1 x}\, e^{\alpha_1 x}\, I_{x\in[0,K_1]}\, q_{X_{t_i}|\mathcal{F}_{t_0}}(x)\, dx.\end{aligned} \tag{28}$$

Now, define $\varphi^1(u)$ as the Fourier transform of the function $e^{\alpha_1 x} I_{x\in[0,K_1]}$, $x \in \mathbb{R}^+$:

$$\varphi^1(u) = \int_{-\infty}^{+\infty} e^{\alpha_1 x} I_{x\in[0,K_1]} e^{iux} dx;$$

it can be easily checked that this last integral is equal to the expression in equation (27). If α_1 is strictly positive, the function $\varphi^1(u)$ is well defined for $u = 0$. The function $e^{\alpha_1 x} I_{x\in[0,K_1]}$ can be retrieved by inverting the Fourier transform $\varphi^1(u)$:

$$\begin{aligned} e^{\alpha_1 x} I_{x\in[0,K_1]} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi^1(u) e^{-iux} du \\ &= \frac{1}{\pi} \int_{0}^{+\infty} \varphi^1(u) e^{-iux} du, \end{aligned} \tag{29}$$

where the second equality results from the symmetry of the integrand, which is itself due to the fact that the function $e^{\alpha_1 x} I_{x\in[0,K_1]}$ is real (no imaginary component). The combination of equation (28) and equation (29) allows us to infer that:

$$\begin{aligned} \mathbb{E}\left[I_{X_{t_i}\in[0,K_1]} \,|\, \mathcal{F}_{t_0} \right] &= \frac{1}{\pi} \int_{0}^{+\infty} \int_{0}^{+\infty} \varphi^1(u) e^{-(\alpha_1+iu)x} \, q_{X_{t_i}|\mathcal{F}_{t_0}}(x) \, dx \, du \\ &= \frac{1}{\pi} \int_{0}^{+\infty} \varphi^1(u) \, \mathbb{E}\left[e^{-(\alpha_1+iu)X_{t_i}} \,|\, \mathcal{F}_{t_0} \right] du. \end{aligned} \tag{30}$$

The integrand of this last equation contains the Laplace transform of the aggregated losses process, evaluated at $\alpha_1 + iu$. This can be worked out as follows:

$$\begin{aligned} \mathbb{E}\left[e^{-(\alpha_1+iu)X_{t_i}} \,|\, \mathcal{F}_{t_0} \right] &= e^{-(\alpha_1+iu)X_{t_0}} \mathbb{E}\left[e^{-(\alpha_1+iu)\sum_{i=N_{t_0}}^{N_{t_i}} Y_i} \,\middle|\, \mathcal{F}_{t_0} \right] \\ &= e^{-(\alpha_1+iu)X_{t_0}} \sum_{k=0}^{+\infty} P(N_{t_i} - N_{t_0} = k) \Big(\mathbb{E}\left[e^{-(\alpha_1+iu)Y} \right] \Big)^k. \end{aligned} \tag{31}$$

■

Proposition 4.2 *If α_2 is a strictly positive real constant, chosen to ensure the stability of subsequent numerical computations, the following holds:*

$$\begin{aligned} &\mathbb{E}\left[(K_2 - X_{t_i}) \, I_{X_{t_i}\in(K_1,K_2]} \,|\, \mathcal{F}_{t_0} \right] \\ &= \frac{1}{\pi} \int_{0}^{+\infty} \varphi^2(u) e^{-(\alpha_2+iu)X_{t_0}} \sum_{k=0}^{+\infty} P(N_{t_i} - N_{t_0} = k) \Big(\mathbb{E}\left[e^{-(\alpha_2+iu)Y} \right] \Big)^k du, \end{aligned} \tag{32}$$

where $\varphi^2(u)$ is the Fourier transform of the function $e^{\alpha_2 x} (K_2 - x) I_{x\in[K_1,K_2]}$, $x \in \mathbb{R}^+$,

$$\begin{aligned} \varphi^2(u) &= \frac{K_1}{\alpha_2 + iu} e^{(\alpha_2+iu)K_1} - \frac{K_2}{\alpha_2 + iu} e^{(\alpha_2+iu)K_1} \\ &\quad + \frac{1}{(\alpha_2 + iu)^2} \left(e^{(\alpha_2+iu)K_2} - e^{(\alpha_2+iu)K_1} \right). \end{aligned} \tag{33}$$

The probabilities $P(N_{t_i} - N_{t_0} = k)$ can be retrieved from proposition 2.4 whereas the expectation $\mathbb{E}\left[e^{-(\alpha_1+iu)Y} \right]$ is the Laplace transform of the claim size Y, valued at $\alpha_1 + iu$.

Proof. The proof is analogous to the proof of proposition 4.1. It may be checked quickly that $\varphi^2(u)$, the Fourier transform of the function $e^{\alpha_2 x}\left(K_2 - x\right) I_{x\in[K_1,K_2]}$, $x \in \mathbb{R}^+$, or

$$\varphi^2(u) = \int_{-\infty}^{+\infty} e^{\alpha_2 x}\left(K_2 - x\right) I_{x\in[K_1,K_2]} e^{iux} dx,$$

is equal to the expression of equation (33). Moreover it is true that for α_2 strictly positive, the function $\varphi^2(u)$ is well defined for $u = 0$. ∎

The calculation of the integrals (26) and (32) can be done numerically by the Fast Fourier Transform algorithm. The FFT algorithm computes in only $O(n \log n)$ operations, for any input array $\{\mathrm{IN}(j)\ :\ j = 0, \ldots, N_S - 1\}$, the following output array:

$$\mathrm{OUT}(m) = \sum_{j=0}^{N_S-1} e^{-\frac{2\pi i}{N_S} mj} \cdot \mathrm{IN}(j), \qquad m = 0, \ldots, N_S - 1.$$

The first step to use this numerical method, consists in discretizing the integrals (26) and (32). We use the notation Δu for the step of discretization and N_S for the number of steps. The mesh of discretization is defined as follows:

$$\{u_j\} = \{j\,\Delta u\ \in \mathbb{R}^+ \,|\, 0 \le j \le N_S - 1\}\ .$$

Next, we define a discretization mesh for the values of X_{t_0}, spaced by Δx, and counting the same number N_S of elements as $\{u_j\}$ (this is a necessary condition in order to use the FFT algorithm):

$$\{x_m\} = \{m\,\Delta x\ \in \mathbb{R}^+ \,|\, 0 \le m \le N_S - 1\}\ .$$

On the condition that steps of discretization Δu and Δx satisfy the equality $\Delta u \Delta x = \frac{2\pi}{N_S}$, the discrete versions of equalities (26) and (32), for all x_m, with $m = 0 ... N_S - 1$, can be reformulated into suitable forms for the FFT algorithm as follows:

$$\underbrace{\frac{\pi}{BN} e^{\alpha_1 m\Delta x} \mathbb{E}\left[\, I_{X_{t_i}\in[0,K_1]} \,|\, \mathcal{F}_{t_0},\, X_{t_0} = m\Delta x\right]}_{\mathrm{OUT}_1(m)}$$
$$\approx \sum_{j=0}^{N_S-1} e^{-i\frac{2\pi}{N_S}\,j\,m}\, \underbrace{\varphi^1(u_j) \sum_{k=0}^{N_J} P(N_{t_i} - N_{t_0} = k)\Big(\mathbb{E}\left[e^{-(\alpha_1 + iu_j)Y}\right]\Big)^k \Delta u}_{\mathrm{IN}_1(j)} \tag{34}$$

$$\underbrace{\pi e^{\alpha_2 m\Delta x} \mathbb{E}\left[\left(K_2 - X_{t_i}\right) I_{X_{t_i}\in(K_1,K_2]} \,|\, \mathcal{F}_{t_0}, X_{t_0} = m\Delta x\right]}_{\mathrm{OUT}_2(m)}$$
$$\approx \sum_{j=0}^{N_S-1} e^{-i\frac{2\pi}{N_S}\,j\,m}\, \underbrace{\varphi^2(u_j) \sum_{k=0}^{N_J} P(N_{t_i} - N_{t_0} = k)\Big(\mathbb{E}\left[e^{-(\alpha_2 + iu_j)Y}\right]\Big)^k \Delta u}_{\mathrm{IN}_2(j)}\ , \tag{35}$$

where N_J is an upper bound chosen such that the probability of observing N_J claims in the interval of time $[t_0, t_i]$ is negligible. The left hand terms of equations (34) and (35) are the output vectors computed by a standard FFT algorithm. By combining them, one can calculate the expected values of future nominal, for a wide range of initial values for X_{t_0} (see equation (25)):

$$\mathbb{E}\left[BN_{t_i}\,|\,\mathcal{F}_{t_0}, X_{t_0} = m\Delta x\right] = \frac{BN}{\pi} e^{-\alpha_1 m\Delta x}\,\mathrm{OUT}_1(m) + \frac{1}{\pi} e^{-\alpha_1 m\Delta x}\,\mathrm{OUT}_2(m), \qquad (36)$$

$\forall m = 1, ..., N_S - 1$. Note that the calculation of the spread by equation (24) at the issuance of the bond, only requires to determine the expected future nominal when $X_{t_0} = 0$. However, the knowledge of expected future nominal when $X_{t_0} > 0$, may be useful to reappraise an insurance bond, issued before t_0, and also when some claims already occurred.

5. NUMERICAL APPLICATIONS

This section illustrates by means of a numerical example the feasibility of the pricing method developed in the first part of this paper. In particular, we compute the fair spreads that insurance bonds of maturities ranging from 1 to 3 years should pay above the risk free rate, here set at 3%. The nominal, NB, is of 70 million. This nominal is decreased if the aggregated losses rise above 10 million. The coupons are paid annually. Other bonds characteristics are summarized in table 1.

r	3%	t_n	1,2,3
K_1	10	t_1	1
K_2	80	Δt	1

Table 1: Parameters of bonds.

Table 2 presents the parameters chosen for the claims arrival process. From Proposition 2.4, we can deduce the probability density function of N_t after 1, 2 and 3 years. The densities are plotted in figure 2. From these densities, we can calculate the averages and standard deviations of the number of claims, after 1, 2 and 3 years; these presented in Table 3. Per year, one foresees on average 11 claims, and the one-year volatility is situated around 3 claims.

α	0.4	δ	10
b	0.1	β	0.01
σ	0.01	γ	0.5

Table 2: Parameters of the claims arrival process.

	$t = 1$	$t = 2$	$t = 3$
$E(N_t)$	11.02	21.06	31.13
$\sigma(N_t)$	3.17	4.48	5.49

Table 3: Means and deviations of N_t.

We have chosen to model the size of claims by a Gamma random variable whose parameters are $\theta = 2$ and $k = 0.5$. The average size of claims is of 1.0 million and the standard deviation is of 2.0 million. The Gamma distribution is the most common distribution to model claims (see appendix A for details), but nothing prevents us to work with another distribution, on the condition that its Laplace transform has a closed form expression. The parameters used in the FFT algorithm are provided in Table 4. Note that the definitions of α_1 and α_2 have been chosen empirically on the basis of several tests. It seems that this choice leads to a good precision of calculations.

N_S	2^{14}	dx	0.5
α_1	$\frac{1}{K_1}$	α_2	$\frac{1}{K_2-K_1}$
du	$\frac{2\pi}{N\,dx}$		

Table 4: FFT parameters.

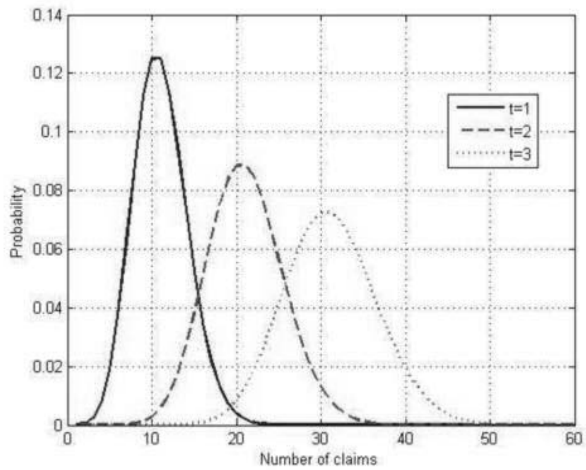

Figure 2: Distribution of claims by maturity.

Figure 3 presents the spreads of bonds by maturities (1, 2 or 3 years), for a set of initial values of aggregated losses, $X_{t=0}$. As expected, the higher the initial value of the total claims, the higher is the spread. This is a direct consequence of the fact that the aggregated losses at time $t = 0$ directly reduce the nominal. The spread is positively correlated with the maturity of the bond: the spreads of 1, 2 and 3 years bonds respectively quote 2,46%, 7,56% and 10,90%, when $X_0 = 0$.

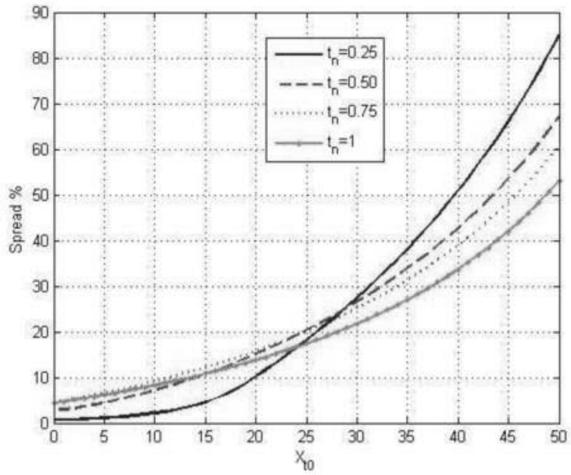

Figure 3: Spreads by maturities and by X_{t_0}.

Figure 4 presents the expected remaining nominal after 1, 2 and 3 years. This expected nominal decreases with time, given that the expected number of occurred claims rises with the time horizon. The expected nominal is also inversely proportional to the total initial amount of claims X_0. When X_0 tends to 70, the nominal falls to zero (and the spread tends to infinity). Note that the expected nominal after one year is slightly higher than zero for values of X_0 around 70, while it should be equal to zero. This can be explained by the imprecision introduced by the discretization of integrals (26) and (32).

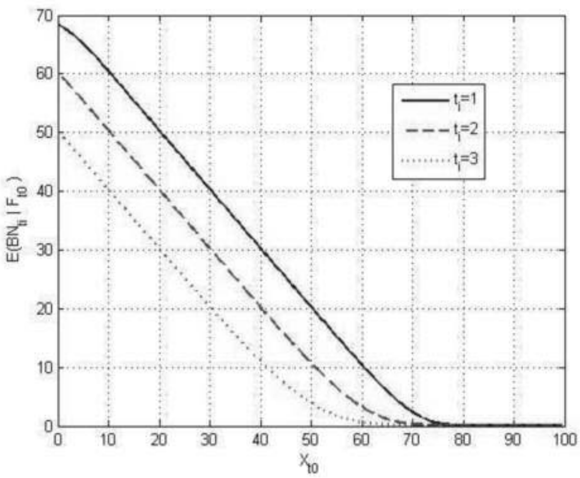

Figure 4: Expected remaining nominal by maturities and by X_{t_0}.

6. CONCLUSIONS

This paper proposes a method to price catastrophe bonds paying multiple coupons, when the number of claims is under the influence of a stochastic seasonal effect. Modeling the seasonality is particularly important for insurance securities, whose valuation is related to claims with an intensity which is rising during certain periods of the year, such as storms or hurricanes. Another important feature of this work is the presence of a mean reverting process, embedded in the intensity of the claims arrival process. This stochastic process may be calibrated so as to reflect the influence of long term climate changes on the frequency of claims. Despite the apparent complexity of the claims arrival process, we have established a simple recursion for the computation of the probability distribution of the number of claims.

The insurance bond periodically pays a coupon equal to a constant percentage of the nominal, reduced by the amount of aggregated losses, exceeding a certain trigger level. At maturity, what is left of the nominal is repaid. In order to compensate for this eventual loss of nominal, the coupon rate always exceeds the risk free rate. The calculation of the spread above the risk free rate requires the appraisal of future expected remaining nominals. As no closed form expression exists for the expected future nominals, we showed how to compute them by the Fast Fourier Transform Algorithm. This approach is also shown to be an efficient method for the reappraisal of insurance bonds when some claims have occurred.

Catastrophe bonds offer an interesting alternative for investors who wish to diversify their exposure to risks, and this paper provides an efficient computational method to price those assets. Yet, many issues and uncertainties about the underlying assumptions remain unsolved. In particular, the shortcoming that consists to assume the independence between frequency and size of claims probably should be dropped. This point should be investigated in future research.

APPENDIX A

A common distribution used to model the size of claims Y, is the Gamma distribution. The density of a Gamma random variable is determined by two real positive parameters k and θ. The probability density function is given by the next expression:

$$f(y) = y^{k-1}\frac{e^{-\frac{y}{\theta}}}{\Gamma(k)\,\theta^k}I_{y\geq 0}.$$

The mean and variance are respectively equal to $k\theta$ and $k\theta^2$. The Laplace transform of a Gamma distributed random variable is given by:

$$\mathbb{E}\left[e^{-uY}\right] = \frac{1}{(1+\theta u)^k}.$$

APPENDIX B

This appendix describes the dynamics of the model presented in this paper under the real measure, and the links between parameters defining the aggregated losses process under real and risk neutral measures. Let us use the notation P for the physical measure. Under P, the intensity of the claims arrival process is given by :

$$\lambda_t^P = \lambda^P(t) + \lambda_t^{OU,P},$$

where $\lambda^P(t)$ is defined by three real constant parameters δ^P, β^P, $\gamma^P \in \mathbb{R}$:

$$\lambda(t) = \delta^P + \beta^P \cos\left((t+\gamma^P)2\pi\right),$$

and where $\lambda_t^{OU,P}$ is an Ornstein Uhlenbeck process, whose speed, level of mean reversion and volatility are real constants, respectively denoted by a^P, b^P, $\sigma^P \in \mathbb{R}$. The dynamics of $\lambda_t^{OU,P}$ is ruled by the following stochastic differential equation:

$$d\lambda_t^{OU} = a^P(b^P - \lambda_t^{OU})dt + \sigma^P dW_t^P. \tag{37}$$

Under P, the size of claims are positive random variables with density $f_Y^P(y)$. The changes of measure from P to a risk neutral measure Q leading to the model developed in section 2 and 3, are defined by the following Radon-Nikodym derivative (for details see e.g. Shreve (2004), chapter 11):

$$\begin{aligned}\frac{dQ}{dP} &= \exp\left(\sum_{k=1}^{N_t} \ln(\kappa g(Y_k)) + \int_0^t \lambda_u(1-\kappa)du\right) \\ &\quad \cdot \exp\left(-\frac{1}{2}\int_0^t \xi^2 du - \int_0^t \xi dW_u^P\right),\end{aligned}$$

where κ is a nonnegative constant and ξ is here a real constant. The function $g(.)$ is measurable $g : \mathbb{R}^+ \to \mathbb{R}$ and satisfies the relation

$$\int_0^{+\infty} g(y) f^P(y) dy = 1.$$

Under Q, $dW_u = dW_u^P + \xi du$ is a Brownian motion, the intensity of the claims arrival process is multiplied by κ and the probability density function of claims is multiplied by $g(y)$. The following relations between twe parameters of our model under Q and P then hold:

$$\begin{aligned}\delta &= \kappa\delta^P & a &= a^P \\ \beta &= \kappa\beta^P & b &= \kappa\left(b^P - \tfrac{\sigma^P\xi}{a^P}\right) \\ \gamma &= \gamma^P & \sigma &= \kappa\sigma^P.\end{aligned}$$

The probability density function of Y_t under Q is equal to:

$$f(y) = g(y) f^P(y).$$

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CMS SPREAD OPTIONS IN THE MULTI-FACTOR HJM FRAMEWORK

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Abstract

Constant maturity swaps (CMS) and CMS spread options are analyzed in the multi-factor Heath-Jarrow-Morton (HJM) framework. For Gaussian models, which include some Libor Market Models (LMM) and the G2++ model, explicit approximated pricing formulae are provided. Two approximating approaches are proposed: an exact solution to an approximated equation and an approximated solution to the exact equation. The first approach borrows from previous literature on other models; the second is new. The price approximation errors are smaller than in the previous literature and negligible in practice. The approach is being used here to price standard CMS and CMS spreads and can be used for some European exotic products.

1. INTRODUCTION

Constant Maturity Swap (CMS) spread options are relatively popular. Their pay-off depends on the difference between two swap rates. Models where only the global level of the curve is modeled would not do a good job in pricing those instruments. Quite naturally one uses two-factor (or more generally multi-factor) models to price those instruments.

The pricing of rate spread instruments in the HJM framework was considered in Fu (1996) and Miyazaki and Yoshida (1998). The former considers the spread between zero-coupon rates with annual compounding, payment on exercise date and strike 0. The options are analyzed in a Gaussian two-factor HJM model. The latter considers a very specific Gaussian two-factor HJM model and spread between continuously compounded zero coupon rates. Due to convention and payment dates, those results can not be used to price actual CMS spread options.

More recently Wu and Chen (2009) analyzed the spread options in the log-normal LMM. The options they consider have strike 0 and payment on exercise date. In practice the payment takes place at the end of an accrual period and not on the fixing date. Their results would need to be adapted to be used in practice. Moreover the majority of spread options are dealt at non-zero strikes. In Antonov and Arneguy (2009), CMS products are analyzed in a LMM with stochastic volatility. The instruments analyzed have non-zero strike and a payment lag. Due to the model complexity their results are not fully explicit and require a characteristic function and one or two-dimensional Laplace transforms.

The CMS spread instruments depend, as their name indicates, on swap rates. One approach is to obtain approximated dynamics for the swap rates in the model. If the approximated dynamics are simple enough, the spread options can be valued as an exchange option. This is the technique used in Wu and Chen (2009) in the log-normal LMM model where the swap rates are approximately log-normal. In Antonov and Arneguy (2009), the swap dynamics are obtained through a technique called *Markov projection*.

A standard approach to price CMS options is to use replication arguments. In that case the price is based on the prices of vanilla cash-settled swaptions (see for example Hagan (2003) or Mercurio and Pallavicini (2005)). Based on the CMS prices and the implied CMS rate, one often prices the CMS spreads as the spread on two log-normal assets with correlation (see for example Berrahoui (2004)). Our proposal could replace this approach. A two-factor model could be calibrated to CMS and CMS spread prices and could be used to price more exotic products. Our approach gives more freedom on the pay-off description than the standard bi-log-normal approach.

Here we first develop results in the G2++ model using the approximated rate dynamics approach. The swap rate dynamics are approximately normal. The exact option price in the approximated dynamics is the first price approximation proposed. The price formula is explicit and similar to the Bachelier formula. The options we consider have a free strike, use the market convention and have a free payment date. The first result is interesting in itself. In particular the smile of the G2++ model is closer to the market smile than a flat log-normal LMM (absence of) smile.

Then we move one step further. We would like to obtain similar results in a general Gaussian separable multi-factor HJM framework. This framework contains in particular the LMM version called *Bond Market Model* and the G2++ model. In that model very efficiently approximated European swaption prices are available, as described in Henrard (2008b). One could extend the approximated swap dynamics methodology; the drawback is that one can only price instruments dependent on (one or two) swap rates; the pay-offs are limited to pay-offs of the form $f(S^1, S^2)1_{(g(S^1,S^2)>0)}$. We work in a term-structure model, modeling the full yield curve but our approach would restrict us to use only two of those rates. This, for example, excludes a swaption with CMS spread trigger, i.e with pay-off $\sum c_i P(\theta, t_i) 1_{(S^1-S^2>0)}$, or similar products.

For that reason we propose a second approach where we use an approximated solution to the exact equation. The exact solution of the discount bond in the HJM framework is computed. With those exact solutions, the pay-offs are too complex to lead to explicit solutions. We use pay-off approximations (to the first and second order) to obtain explicit prices. The price formula contains terms similar to the Bachelier formula for the first order and some additional terms for the second order. The approximating explicit solution to exotic options contrasts with the Monte-Carlo techniques often used in that context.

2. MODELS

In general, a term structure model describes the behavior of $P(t,u)$, the price in t of the zero-coupon bond paying 1 in u ($0 \leq t \leq u \leq T$). When the discount curve $P(t,.)$ is differentiable (in a weak sense), there exists $f(t,u)$ such that

$$P(t,u) = \exp\left(-\int_t^u f(t,s)ds\right). \tag{1}$$

Let $N_t = \exp(\int_0^t r_s ds)$ be the cash-account numeraire with $(r_s)_{0\leq s\leq T}$ the short rate $r_t = f(t,t)$. Except if otherwise stated, the model equations are in the numeraire measure associated to N_t.

2.1. G2++

The G2++ model is a two-factor interest rate model which can be viewed as a two-factor extended Vasicek model. The model is usually introduced as a short rate model with

$$r_t = x_t^1 + x_t^2 + \phi(t)$$

where the stochastic processes are given by

$$dx_t^i = -a_i x_t^i dt + \eta_i(t) dW_t^i.$$

The Brownian processes W_t^1 and W_t^2 are correlated with correlation ρ. The initial values of the processes are $x_0^i = 0$. The a_i are two positive constants and the functions η_i are deterministic. The deterministic function $\phi(t)$ is given by the initial interest rate curve. The description of the model and its analytical formulae can be found in Brigo and Mercurio (2006).

2.2. Gaussian HJM (multi-factor)

The idea of Heath et al. (1992) was to model f with a stochastic differential equation

$$df(t,u) = \sigma(t,u)\cdot(\rho\nu(t,u))\,dt + \sigma(t,u)\cdot dW_t$$

where

$$\nu(t,u) = \int_t^u \sigma(t,s)ds$$

for some suitable σ. The random processes $W_t = (W_t^1,\ldots,W_t^n)$ are Brownian motions with correlation matrix ρ. The special form of the drift is required to ensure the arbitrage-free property. Note that with respect to standard writing we use a correlated Brownian motion to simplify the writing in the developments. This is the origin of the correlation matrix appearing in the drift term. The volatility and the Brownian motion are n-dimensional while the rates are 1-dimensional. The model technical details can be found, among other places, in the original paper or in the chapter *Dynamical term structure model* of Hunt and Kennedy (2004).

We study this model under the separability hypothesis, see e.g. Henrard (2008a)

$$\sigma_i(s,u) = g_i(s)h_i(u).$$

Separability conditions have been widely used in interest rate modeling. The results cover the Markov processes in Carverhill (1994), explicit swaption formulas in Gaussian HJM models in Henrard (2003) and efficient approximations for LMM in Pelsser et al. (2004) and Bennett and Kennedy (2005).

In line with the separability hypothesis, we denote

$$\gamma_{i,j} = \int_0^\theta g_i(s)g_j(s)ds \quad \text{and} \quad H_i(t) = \int_0^t h_i(s)ds.$$

So we obtain that the price of a zero-coupon bond can be written (Henrard 2007, Lemma A.1), after a change of numeraire to $P(.,\theta)$ as

$$P(\theta,u)=\frac{P(0,u)}{P(0,\theta)}\exp\left(-\sum_{j=1}^n \alpha_j(u)X_j - \frac{1}{2}\tau^2(u)\right) \tag{2}$$

with

$$\alpha_j^2(u) = \int_0^\theta (\nu_j(s,u)-\nu_j(s,\theta))^2 ds = (H_j(u)-H_j(\theta))^2\gamma_{j,j}$$

and

$$\int_0^\theta (\nu(s,u)-\nu(s,\theta))\cdot dW_s^\theta = \sum_{j=1}^n (H_j(u)-H_j(\theta))\int_0^\theta g_j(s)dW_s^{\theta,j} = \sum_{j=1}^n \alpha_j(u)X_j,$$

with Brownian motion W_s^θ, and where $\tau^2(u) = \alpha^T(u)\bar\rho\alpha(u)$ is the total variance with the correlation $\bar\rho$ is given by

$$\bar\rho_{j,k} = \rho_{j,k}\frac{\gamma_{j,k}}{\sqrt{\gamma_{j,j}}\sqrt{\gamma_{k,k}}},$$

and the random variable $X = (X_1,\ldots,X_n) \sim N(0,\bar\rho)$. Notice that X is the same for all payment dates u.

The G2++ model is a special case of the HJM framework with

$$\sigma_i(s,u) = \eta_i(s)\exp(-a_i(u-s)).$$

Those functions satisfy the separability condition $\sigma_i(s,u) = g_i(s)h_i(u)$ with $g_i(s) = \eta_i(s)\exp(a_i s)$ and $h_i(u) = \exp(-a_i u)$. In the numerical tests, a piecewise constant G2++ is considered. The equivalence of G2++ to the special HJM case described above is analyzed in (Brigo and Mercurio 2006, Section 5.2).

3. CMS PRODUCTS DESCRIPTION

The CMS underlying is a swap with a given tenor and the market conventions. For a tenor k, the payment dates are denoted $t_{k,i}$ $(1 \le i \le n_k)$ with n_k the number of payments. The settlement date, used for the first floating period, is denoted t_0. For a fixing date, the settlement date is the same for all swaps and there is no need to index the settlement date by the tenor. The associated accrual fractions (i.e. the coverage of the interval $[t_{k,i-1}, t_{k,i}]$) is denoted $\delta_{k,i}$. We will denote the swap (forward) rate for tenor k at the date θ by $S_{k,\theta}$. The swap rate is given by

$$S_{k,\theta} = \frac{P(\theta, t_0) - P(\theta, t_{k,n_k})}{\sum_{i=1}^{n_k} \delta_{k,i} P(\theta, t_{k,i})}.$$

3.1. CMS options

A CMS option is characterized by an expiry or fixing date θ, a CMS underlying, a strike K and a payment date $t_p \geq \theta$. The pay-off is $(S_{k,\theta} - K)^+$ paid in t_p. The generic value in the $P(.,\theta)$ numeraire is

$$P(0,\theta)\,\mathrm{E}^{\theta}\left[P(\theta, t_p)(S_{k,\theta} - K)^+\right].$$

3.2. CMS spread options

A CMS spread option is characterized by an expiry date θ, two CMS underlyings with weights (β_1, β_2), a strike K and a payment date $t_p \geq \theta$. The CMS underlyings are swaps with given tenors and the market convention. The tenors are denoted by k_1 and k_2. The pay-off is $(\beta_1 S_{k_1,\theta} - \beta_2 S_{k_2,\theta} - K)^+$ paid in t_p.

4. APPROXIMATED EQUATION

4.1. CMS options

In the numeraire $P(., t_p)$, the option on a CMS fixed at time θ is

$$P(0, t_p)\,\mathrm{E}^{t_p}\left[(S_{k,\theta} - K)^+\right].$$

Defining the Libor forward rate $L_s^{k,i}$ between $t_{k,i}$ and $t_{k,i+1}$, observed at time s, the swap rate is

$$S_{k,\theta} = \sum_{i=1}^{n_k} w_\theta^{k,i} L_\theta^{k,i-1} \quad \text{with} \quad w_\theta^{k,i} = \frac{\delta_{k,i} P(\theta, t_{k,i})}{\sum_{j=1}^{n_k} \delta_{k,j} P(\theta, t_{k,j})}.$$

As the coefficients $w_\theta^{k,i}$ have small variabilities compared to $L_\theta^{k,i}$'s variabilities (see Lognormal Swap Rate model in Brigo and Mercurio (2006)), the swap rate can be approximated by

$$S_{k,\theta} \simeq \sum_{i=1}^{n_k} w_0^{k,i} L_\theta^{k,i-1} \tag{3}$$

where the $w_0^{k,i}$ are computed with the initial zero coupon curve.

To compute the distribution of $S_{k,\theta}$, we compute the discrete forward rate process in a G2++ model. The link between the discrete forward rate $L_s^{k,i-1}$ and the instantaneous forward rate $f(t,u)$ is

$$L_s^{k,i-1} = \frac{1}{\delta_{k,i}}\left[e^{\int_{t_{k,i-1}}^{t_{k,i}} f(s,u)du} - 1\right].$$

From the analytical zero-coupon bond price in the G2++ model, see e.g. Brigo and Mercurio (2006), the process of $L_s^{k,i-1}$ can be deduced.

Omitting the variables (t,T) and introducing deterministic functions P_i, Q_{i1}, Q_{i2}, with ρ the correlation between the two Brownian motions $W_t^{t_p,1}, W_t^{t_p,2}$ which are Brownian motions defined in Section 2.1, under the t_p-measure

$$\begin{aligned} dL_t^{k,i-1} &= \left(\frac{1+\delta_{k,i}L_t^{k,i-1}}{\delta_{k,i}}\right) \\ &\quad \times \left[\left(P_{k,i} + \frac{1}{2}Q_{k,i1}^2 + \frac{1}{2}Q_{k,i2}^2 + \rho Q_{k,i1}Q_{k,i2}\right)dt + Q_{k,i1}dW_t^{t_p,1} + Q_{k,i2}dW_t^{t_p,2}\right]. \end{aligned}$$

The equation evolution can be approximated by freezing the first term to some acceptable value. As the moments of the distribution will be computed under the numeraire $P(.,t_p)$, the idea is to replace the stochastic variable $L_t^{k,i-1}$ in the first term by its expectation under the numeraire $P(.,t_p)$. On a first order approximation, the drift is neglected

$$E^{t_p}\left[L_t^{k,i-1}\right] \simeq L_0^{k,i-1}.$$

Equation (3) leads then to a Gaussian Swap Rate process

$$\begin{aligned} dS_{k,\theta} &\simeq \sum_{i=1}^{n_k}\left(\frac{1+\delta_{k,i}L_0^{k,i-1}}{\delta_{k,i}}\right) w_0^{k,i}\left[\left(P_{k,i} + \frac{1}{2}Q_{k,i1}^2 + \frac{1}{2}Q_{k,i2}^2 + \rho Q_{k,i1}Q_{k,i2}\right)dt \right. \\ &\quad \left. + Q_{k,i1}dW_t^{t_p,1} + Q_{k,i2}dW_t^{t_p,2}\right] \\ &\simeq \mu^{k,\theta}(t)dt + \sigma_1^{k,\theta}(t)dW_t^{t_p,1} + \sigma_2^{k,\theta}(t)dW_t^{t_p,2}, \end{aligned}$$

where the functions $\mu^{k,\theta}(t)$ and $\sigma_j^{k,\theta}(t)$ are properly defined. In order to obtain a Black normal formula, we need the first two moments of the $S_{k,\theta}$ distribution, which can be computed analytically, namely

$$\begin{aligned} M = \mathrm{E}^{t_p}\left[S_{k,\theta}\right] &= S_{k,0} + \int_0^\theta \mu^{k,\theta}(u)du, \\ V^2 = \mathrm{Var}^{t_p}\left[S_{k,\theta}\right] &= \int_0^\theta \left(\sigma_1^{k,\theta}(u)\right)^2 du + \int_0^\theta \left(\sigma_2^{k,\theta}(u)\right)^2 du + 2\rho\int_0^\theta \sigma_1^{k,\theta}(u)\sigma_2^{k,\theta}(u)du. \end{aligned}$$

Knowing the mean and standard deviation of the Gaussian distribution leads to simple analytical prices for CMS options and CMS spread options.

Theorem 4.1 *The price of the CMS option with an approximated Gaussian swap rate is given by*

$$P(0,t_p)\left[(M-K)\Phi\left(\frac{M-K}{V}\right)+\frac{V}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{M-K}{V}\right)^2}\right] \tag{4}$$

with K the option strike, M the mean and V^2 the variance of the swap rate. $\Phi(\cdot)$ denotes as usually the cumulative density function of the standard normal law.

4.2. CMS spread options

The same technique can be used to compute the distribution of a spread of CMS rates. With the same approximation, the spread $S_\theta = S_{k_1,\theta} - S_{k_2,\theta}$ is also a Gaussian process which can be computed analytically.

Theorem 4.2 *The price of the CMS spread option with an approximated Gaussian swap rate is given by*

$$P(0,t_p)\left[(M-K)\Phi\left(\frac{M-K}{V}\right)+\frac{V}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{M-K}{V}\right)^2}\right] \tag{5}$$

with K the option strike, $M = \mathrm{E}^{t_p}\left[S_{k_1,\theta}\right] - \mathrm{E}^{t_p}\left[S_{k_2,\theta}\right]$ the mean and V^2 the variance of the spread given by

$$\begin{aligned} V^2 &= Var^{t_p}\left[S_\theta\right] = \int_0^\theta \left(\sigma_1^{k_1,\theta}(u) - \sigma_1^{k_2,\theta}(u)\right)^2 du + \int_0^\theta \left(\sigma_2^{k_1,\theta}(u) - \sigma_2^{k_2,\theta}(u)\right)^2 du \\ &+ 2\rho \int_0^\theta \left(\sigma_1^{k_1,\theta}(u) - \sigma_1^{k_2,\theta}(u)\right)\left(\sigma_2^{k_1,\theta}(u) - \sigma_2^{k_2,\theta}(u)\right) du. \end{aligned}$$

5. APPROXIMATED SOLUTION

5.1. CMS options

In the multifactor HJM model, the CMS rate in θ is, from Equation (2),

$$\begin{aligned} &S_{k,\theta}(X) \\ &= \frac{P(0,t_0)\exp(-\sum\alpha_j(t_0)X_j - \tau^2(t_0)/2) - P(0,t_{k,n_k})\exp(-\sum\alpha_j(t_{k,n_k})X_j - \tau^2(t_{k,n_k})/2)}{\sum_{i=1}^{n_k} P(0,t_{k,i})\exp(-\sum\alpha_j(t_{k,i})X_j - \tau^2(t_{k,i})/2)}. \end{aligned}$$

The CMS rate will be approximated by a first or second order Taylor approximation:

$$S_{k,\theta}(X) - K \simeq A + BX + \frac{1}{2}X^TCX.$$

To clarify some computations, we make a change of variable. Let $Y = (Y_1,\ldots,Y_n)$ be distributed as a standard normal with $b_1Y_1 = BX$ (i.e. $b_1 = B^T\bar{\rho}B$). The change of variable is denoted by $Y = DX$. After the change of variables, one has

$$S_{k,\theta}(X) - K \simeq A + bY + \frac{1}{2}Y^TcY$$

with $b = (b_1, 0, \ldots, 0)$ and $c = (D^{-1})^T C D^{-1}$. The CMS option condition is $S_{k,\theta}(X) - K > 0$ which is approximated to the first order by $Y_1 > -\frac{A}{b_1} = \kappa$. Let $-\alpha(t_p)D^{-1} = e$. Using the first order approximation for the exercise condition and the second order approximation for the pay-off, the price of the CMS option is given by

$$P(0,t_p)\,\mathrm{E}^\theta\left[\left(A + b_1 Y_1 + \frac{1}{2}Y^T c Y\right) 1_{\{Y_1>\kappa\}} \exp\left(eY - \frac{1}{2}\tau_p^2\right)\right],$$

where τ_p denotes in the following $\tau(t_p)$.

The value of the expected value is written in the form of a technical lemma.

Lemma 5.1 *If Y is a standard (multivariate) normal distribution,*

$$\begin{aligned}
&\mathrm{E}\left[(A + b_1 Y_1 + \frac{1}{2}Y^T c Y)1_{\{Y_1>\kappa\}} \exp\left(eY - \frac{1}{2}\tau_p^2\right)\right] \\
&= \exp\left(-\frac{1}{2}\tau_p^2 + \frac{1}{2}e^2\right)\left[\left(b_1 + c_{1,.}e + \frac{1}{2}c_{1,1}(\kappa - e_1)\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(\kappa - e_1)^2\right)\right. \\
&\quad \left. + \left(A + b_1 e_1 - \frac{1}{2}e^T c e + \frac{1}{2}\sum_{j=1}^n c_{j,j}\right)\Phi(e_1 - \kappa)\right].
\end{aligned}$$

Proof. The expected value is expressed by using two nested integrals

$$\frac{1}{(2\pi)^{n/2}}\int_{y_1\geq\kappa}\int_{\mathbb{R}^{n-1}}\left(A + b_1 y_1 + \frac{1}{2}y^T c y\right)\exp\left(ey - \frac{1}{2}\tau_p^2\right)\exp\left(-\frac{1}{2}|y|^2\right)dy_2 dy_1.$$

The first step is to compute the inner integral over $\mathbb{R}^{n-1}$ taking advantage of the symmetries. We first write all the integrands in terms of $y - e$:

$$\left(A + b_1 y_1 + \frac{1}{2}(y-e)^T c(y-e) + (y-e)^T ce - \frac{1}{2}e^T ce\right)\exp\left(-\frac{1}{2}(y-e)^2\right)\exp\left(\frac{1}{2}e^2 - \frac{1}{2}\tau_p^2\right).$$

Using the integrals over the $n-1$ dimensional space, the remaining integrand is

$$\left(\left(A + b_1 e_1 - \frac{1}{2}e^T ce + \frac{1}{2}\sum_{j=2}^n c_{j,j}\right) + (b_1 + c_{1,.}e)(y_1 - e_1) + \frac{1}{2}c_{1,1}(y_1 - e_1)^2\right)\exp\left(-\frac{1}{2}(y_1 - e_1)^2\right).$$

The result is obtained by decomposing the above integral on $y_1 \geq \kappa$ in three parts according to the approximation order. The definition of cumulative normal distribution is used together with the integral of the normal density against order one and two exponent. In particular for order two we use $\frac{1}{\sqrt{2\pi}}\int_\kappa^\infty x^2\exp\left(-\frac{1}{2}x^2\right)dx = \frac{1}{\sqrt{2\pi}}\kappa\exp(-\kappa^2/2) + N(-\kappa)$. ■

Theorem 5.2 *The price of the CMS option is approximated to the second order by*

$$\begin{aligned}
&P(0,t_p)\exp\left(-\frac{1}{2}\tau_p^2 + \frac{1}{2}e^2\right)\left[\left(b_1 + c_{1,.}e + \frac{1}{2}c_{1,1}\tilde{\kappa}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\tilde{\kappa}^2\right)\right. \\
&\quad \left. + \left(A + e_1 b_1 - \frac{1}{2}e^T ce + \frac{1}{2}\sum_{j=1}^n c_{j,j}\right)\Phi(-\tilde{\kappa}))\right]
\end{aligned}$$

where $\tilde{\kappa} = \kappa - e_1$, A, b, c and e are described above.

Note that if the first order approximation is used ($C = 0$) the price formula structure is very similar to the one in Baviera (2006) and the one presented in the previous section. The (small) multiplicative adjustment is different and the coefficients are slightly different depending on the freeze technique used in the approach.

5.2. CMS spread options

In the previous section a second order approximation is used for the CMS rate. As the second order approximation of a difference is the difference of the approximations, the same technique can be used for CMS spreads:

$$\beta_1 S_{k_1,\theta}(X) - \beta_2 S_{k_2,\theta}(X) - K \simeq A + BX + \frac{1}{2} X^T C X.$$

Theorem 5.3 *The price of the CMS spread option is approximated to the second order by*

$$P(0,t_p) \exp\left(-\frac{1}{2}\tau_p^2 + \frac{1}{2}e^2\right) \left[\left(b_1 + c_{1,.}e + \frac{1}{2}c_{1,1}\tilde{\kappa}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\tilde{\kappa}^2\right) + \left(A + e_1 b_1 - \frac{1}{2}e^T c e + \frac{1}{2}\sum_{j=1}^{n} c_{j,j}\right) \Phi(-\tilde{\kappa}))\right]$$

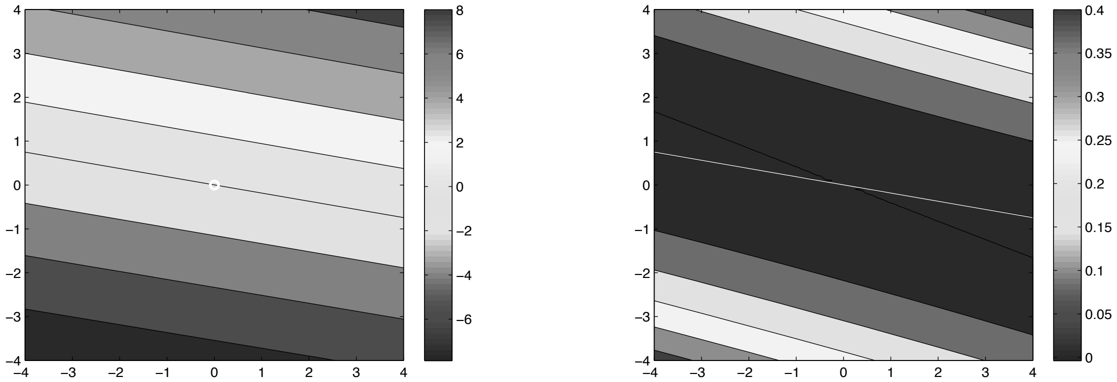
where $\tilde{\kappa} = \kappa - e_1$, A, b, c and e are computed as in the CMS option case.

6. NUMERICAL RESULTS ANALYSIS

The quality of the different approximations is analyzed. The analysis is done for the G2++ model. In Figure 1(a) the level curves for $S_{k,\theta}(X) - A$ are displayed for a four standard deviation square. The level curves are similar to lines and the first order approximation for them may be justified. In the second graph the level curves of $S_{k,\theta}(X) - (A + BX)$ are represented. The first order boundary is represented in white. Second order and higher terms have little impact around the exercise boundary, justifying the use of the first order boundary. In the perpendicular direction, the higher order terms have a non negligible effect, justifying the second order approximation usage.

The first option analyzed is a cap fixing in five years on a ten years CMS with payment one year after fixing. The market data are as of 31 July 2009. The model is calibrated to the at-the-money 5Yx10Y volatility. The parameters are $a_1 = 0.10$, $a_2 = 0.01$, $\eta_1 = 0.27\%$, $\eta_2 = 0.81\%$ and $\rho = -0.30$. For the graphs, the strike is one percent below the money. Table 1 proposes the prices for strikes in a four percent range.

The error with the approximated equation is below 0.50 basis point in price which corresponds roughly to 0.1 ATM Black vega. The first order approximation with the second method has a slightly larger error, which corresponds to 0.2 vega. The error with the second order approximation is below 0.01 basis point and almost invisible; it is certainly more than good enough for any practical purpose.



(a) Order 0 (b) Order 1

Figure 1: Level curves for the swap rate residual after different order approximations.

		Approx. Equation			Approx. Solution		
Strike	**NI price**	**Price**	**Diff.**	**Order 1**	**Diff. 1**	**Order 2**	**Diff. 2**
			One CMS option				
2.69	183.43	183.89	0.47	182.57	-0.85	183.42	-0.008
3.69	114.26	114.87	0.62	113.58	-0.67	114.25	-0.003
4.69	60.97	61.45	0.48	60.34	-0.63	60.97	-0.006
5.69	26.93	27.09	0.16	26.32	-0.61	26.92	-0.005
6.69	9.57	9.51	-0.06	9.11	-0.47	9.57	-0.008
			Five Year CMS option				
2.50	781.96	783.02	1.06	779.87	-2.10	781.96	-0.015
3.50	364.93	367.26	2.33	363.43	-1.51	364.93	-0.013
4.50	144.88	146.71	1.83	143.52	-1.37	144.88	-0.011
5.50	45.28	45.76	0.48	44.16	-1.12	45.27	-0.012
6.50	11.21	11.14	-0.08	10.58	-0.64	11.20	-0.012

Strikes in percent. Prices in basis points.

Table 1: Options on 10Y CMS. Price using full numerical integration, approximated equation method and approximated solution to order 1 and 2. Market data as of 31 July 2009.

The table contains also the figures for a five year swap with annual payments. The same model parameters are used. The error is up to 2.5 basis points for the first approach and first order approximation and at most 0.02 basis points for the second order approximation. Tests have been done with other market data and piecewise constant volatilities and leaded to similar error levels.

The second test run concerns CMS spread options. The CMS used are 10 years and 2 years; the weights are one for each rate. The results are provided in Table 2 for a one period CMS spread and a five year transaction. For the first approach and order one approximation, the error is below 0.10 basis point. Note a compensation between the errors on different payments leading to a smaller total error. Those errors are acceptable. The errors are below the ones presented in Wu and Chen

(2009) where the error is up to 0.40 basis points on a five year option with strike 0 (their approach does not allow other strikes). In Lutz and Kiesel (2008) the error is up to 0.70 basis point on a five years option and 1.00 basis point for ten years. For the second order approximation, we were forced to add an extra decimal to avoid having only zeros; the error is below 0.001 basis point.

		Approx. Equation		**Approx. Solution**			
One CMS Spread option							
Strike	**NI**	**Approx.**	**Diff.**	**Order 1**	**Diff.**	**Order 2**	**Diff.**
-1.00	128.87	128.83	-0.04	128.81	-0.06	128.87	0.0002
-0.50	87.06	87.03	-0.03	87.01	-0.06	87.06	0.0002
0.00	45.26	45.22	-0.04	45.20	-0.06	45.26	0.0002
0.50	6.18	6.01	-0.17	6.15	-0.03	6.18	0.0019
Five Year CMS Spread option							
-1.00	960.285	960.283	-0.002	960.177	-0.107	960.285	0.0003
-0.50	726.838	726.836	-0.002	726.731	-0.108	726.839	0.0003
0.00	493.392	493.390	-0.002	493.285	-0.107	493.392	0.0003
0.50	260.637	260.580	-0.057	260.552	-0.085	260.638	0.0009
1.00	93.820	93.740	-0.081	93.804	-0.016	93.821	0.0009
1.50	29.912	29.913	0.000	29.912	-0.000	29.912	0.0000
Strikes in percent. Prices in basis points							

Table 2: Options on CMS Spread (10Y - 2Y). Price using full numerical integration, approximation equation methodology and approximation solution to order 1 and 2. Market data as of 31 July 2009.

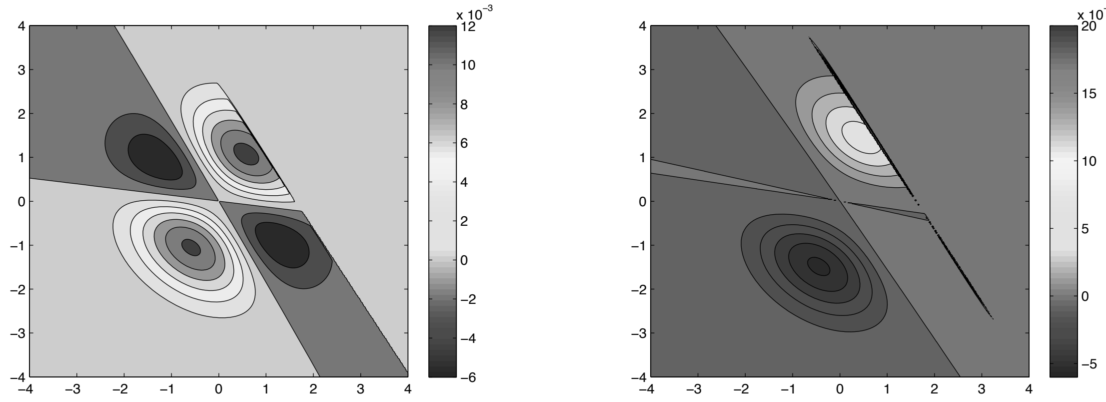
In Figure 2, the approximation error multiplied by the density is provided for CMS spreads to the first and second order. For the second order approximation, note the relatively high error around the exercise boundary. On a very thin band the exercise strategy is not the same between the exact function and the first order approximation. The probability is very small (10^{-4}% in the example). On that small set the error is of the order of magnitude of the value of the first order error. That small set is visible in Figure 2(b) as the thin dark double ellipse on the right.

7. CONCLUSION

We propose two different approaches to CMS pricing in HJM Gaussian models. The first one is based on the exact solution of an approximated equation. The approximation is obtained by an initial freeze of some low variance quantities. A similar approach for other models can be found in the literature. The second approach is based on a solution approximation for the exact equation. The second approach allows higher order approximations.

We showed that our different approximations lead to small approximation errors. The first order approximation errors are of the order of magnitude of 0.1 vega and as precise as the best input data. The second order approximation is one or two order of magnitude more precise and can be as small as 0.001 vega.

Moreover our second approach allows to price very generic European pay-off based instru-



(a) Order 1 (b) Order 2

Figure 2: Approximation error multiplied by the density. CMS spread 10Y-2Y for a 5Y maturity. Strike ATM-10 bps.

ments of CMS-like rates. Exotic options can be priced in a multi-factor HJM with explicit formulae. The formulae for CMS and CMS spreads are also explicit and allow efficient model calibration to those products if necessary. The traditional pricing of CMS spreads using two log-normal assets can be replaced by the pricing in a full term structure model.

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STRONG TAYLOR APPROXIMATION OF STOCHASTIC DIFFERENTIAL EQUATIONS AND APPLICATION TO THE LÉVY LIBOR MODEL

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Abstract

In this article we develop a method for the strong approximation of stochastic differential equations (SDEs) driven by Lévy processes or general semimartingales. The main ingredients of our method is the perturbation of the SDE and the Taylor expansion of the resulting parameterized curve. We apply this method to develop strong approximation schemes for LIBOR market models. In particular, we derive fast and precise algorithms for the valuation of derivatives in LIBOR models which are more tractable than the simulation of the full SDE. A numerical example for the Lévy LIBOR model illustrates our method.

1. INTRODUCTION

The main aim of this paper is to develop a general method for the strong approximation of stochastic differential equations (SDEs) and to apply it to the valuation of options in LIBOR models. The method is based on the perturbation of the initial SDE by a real parameter, and then on the Taylor expansion of the resulting parameterized curve around zero. Thus, we follow the line of thought of Siopacha and Teichmann (2010) and extend their results from continuous to general semimartingales. The motivation for this work comes from LIBOR market models; in particular, we consider the Lévy LIBOR model as the basic paradigm for the development of this method.

The LIBOR market model has become a standard model for the pricing of interest rate derivatives in recent years. The main advantage of the LIBOR model in comparison to other approaches is that the evolution of discretely compounded, market-observable forward rates is modeled directly and not deduced from the evolution of unobservable factors. Moreover, the log-normal LIBOR model is consistent with the market practice of pricing caps according to Black's formula (cf. Black (1976)). However, despite its apparent popularity, the LIBOR market model has certain well-known pitfalls.

On the one hand, the log-normal LIBOR model is driven by a Brownian motion, hence it cannot be calibrated adequately to the observed market data. An interest rate model is typically calibrated to the implied volatility surface from the cap market and the correlation structure of at-the-money swaptions. Several extensions of the LIBOR model have been proposed in the literature using jump-diffusions, Lévy processes or general semimartingales as the driving motion (cf. e.g. Glasserman and Kou (2003), Eberlein and Özkan (2005), Jamshidian (1999)), or incorporating stochastic volatility effects (cf. e.g. Andersen and Brotherton-Ratcliffe (2005)).

On the other hand, the dynamics of LIBOR rates are not tractable under every forward measure due to the random terms that enter the dynamics of LIBOR rates during the construction of the model. In particular, when the driving process has continuous paths the dynamics of LIBOR rates are tractable under their corresponding forward measure, but they are not tractable under any other forward measure. When the driving process is a general semimartingale, then the dynamics of LIBOR rates are not even tractable under their very own forward measure. Consequently:

1. if the driving process is a *continuous* semimartingale caplets can be priced in closed form, but *not* swaptions or other multi-LIBOR derivatives;
2. if the driving process is a *general* semimartingale, then even caplets *cannot* be priced in closed form.

The standard remedy to this problem is the so-called "frozen drift" approximation, where one replaces the random terms in the dynamics of LIBOR rates by their deterministic initial values; it was first proposed by Brace et al. (1997) for the pricing of swaptions and has been used by several authors ever since. Brace et al. (2001), Dun et al. (2001) and Schlögl (2002) argue that freezing the drift is justified, since the deviation from the original equation is small in several measures.

Although the frozen drift approximation is the simplest and most popular solution, it is well-known that it does not yield acceptable results, especially for exotic derivatives and longer horizons. Therefore, several other approximations have been developed in the literature; in one line of research Daniluk and Gątarek (2005) and Kurbanmuradov et al. (2002) are looking for the best lognormal approximation of the forward LIBOR dynamics; cf. also Schoenmakers (2005). Other authors have been using linear interpolations and predictor-corrector Monte Carlo methods to get a more accurate approximation of the drift term (cf. e.g. Pelsser et al. (2005), Hunter et al. (2001) and Glasserman and Zhao (2000)). We refer the reader to Joshi and Stacey (2008) for a detailed overview of that literature, and for some new approximation schemes and numerical experiments.

Although most of this literature focuses on the lognormal LIBOR market model, Glasserman and Merener (2003a,b) have developed approximation schemes for the pricing of caps and swaptions in jump-diffusion LIBOR market models.

In this article we develop a general method for the approximation of the random terms that enter into the drift of LIBOR models. In particular, by perturbing the SDE for the LIBOR rates and applying Taylor's theorem we develop a generic approximation scheme; we concentrate here on the first order Taylor expansion, although higher order expansions can be considered in the same framework. At the same time, the frozen drift approximation can be embedded in this method as the zero-order Taylor expansion, thus offering a theoretical justification for this approximation. The method we develop yields more accurate results than the frozen drift approximation, while being computationally simpler than the simulation of the full SDE for the LIBOR rates. Moreover, our method is universal and can be applied to any LIBOR model driven by a general semimartingale.

However, we focus on the Lévy LIBOR model as a characteristic example of a LIBOR model driven by a general semimartingale.

The article is structured as follows: in section 2 we review time-inhomogeneous Lévy process, and in section 3 we revisit the Lévy LIBOR model. In section 4 we describe the dynamics of log-LIBOR rates under the terminal martingale measure and express them as a Lévy-driven SDE. In section 5 we develop the strong Taylor approximation method and apply it to the Lévy LIBOR model. Finally, section 6 contains a numerical illustration.

2. LÉVY PROCESSES

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a complete stochastic basis, where $\mathcal{F} = \mathcal{F}_{T_*}$ and the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T_*]}$ satisfies the usual conditions; we assume that $T_* \in \mathbb{R}_{\geqslant 0}$ is a finite time horizon. The driving process $H = (H_t)_{0\le t\le T_*}$ is a *process* with *independent increments* and *absolutely continuous* characteristics; this is also called a *time-inhomogeneous Lévy process*. That is, H is an adapted, càdlàg, real-valued stochastic process with independent increments, starting from zero, where the law of H_t, $t \in [0, T_*]$, is described by the characteristic function

$$\mathbb{E}\big[\mathrm{e}^{iuH_t}\big] = \exp\left(\int_0^t \Big[ib_s u - \frac{c_s}{2}u^2 + \int_{\mathbb{R}} (\mathrm{e}^{iux} - 1 - iux)F_s(\mathrm{d}x)\Big]\mathrm{d}s\right); \tag{1}$$

here $b_t \in \mathbb{R}$, $c_t \in \mathbb{R}_{\geqslant 0}$ and F_t is a Lévy measure, i.e. satisfies $F_t(\{0\}) = 0$ and $\int_{\mathbb{R}}(1\wedge|x|^2)F_t(\mathrm{d}x) < \infty$, for all $t \in [0, T_*]$. In addition, the process H satisfies Assumptions ($\mathbb{AC}$) and ($\mathbb{EM}$) given below.

Assumption ($\mathbb{AC}$). The triplets (b_t, c_t, F_t) satisfy

$$\int_0^{T_*} \left(|b_t| + c_t + \int_{\mathbb{R}} (1 \wedge |x|^2)F_t(\mathrm{d}x)\right)\mathrm{d}t < \infty.$$

Assumption ($\mathbb{EM}$). There exist constants $M, \varepsilon > 0$ such that for every $u \in [-(1 + \varepsilon)M, (1 + \varepsilon)M] =: \mathbb{M}$

$$\int_0^{T_*} \int_{\{|x|>1\}} \mathrm{e}^{ux} F_t(\mathrm{d}x)\mathrm{d}t < \infty.$$

Moreover, without loss of generality, we assume that $\int_{\{|x|>1\}} \mathrm{e}^{ux} F_t(\mathrm{d}x) < \infty$ for all $t \in [0, T_*]$ and $u \in \mathbb{M}$.

These assumptions render the process $H = (H_t)_{0\le t\le T_*}$ a *special* semimartingale, therefore it has the canonical decomposition (cf. Jacod and Shiryaev (2003, II.2.38), and Eberlein et al. (2005))

$$H = \int_0^\cdot b_s\mathrm{d}s + \int_0^\cdot \sqrt{c_s}\mathrm{d}W_s + \int_0^\cdot \int_{\mathbb{R}} x(\mu^H - \nu)(\mathrm{d}s, \mathrm{d}x), \tag{2}$$

where μ^H is the random measure of jumps of the process H, ν is the $\mathbb{P}$-compensator of μ^H, and $W = (W_t)_{0\leq t\leq T_*}$ is a $\mathbb{P}$-standard Brownian motion. The *triplet of predictable characteristics* of H with respect to the measure $\mathbb{P}$, $\mathbb{T}(H|\mathbb{P}) = (B, C, \nu)$, is

$$B = \int_0^\cdot b_s \mathrm{d}s, \qquad C = \int_0^\cdot c_s \mathrm{d}s, \qquad \nu([0,\cdot] \times A) = \int_0^\cdot \int_A F_s(\mathrm{d}x)\mathrm{d}s,$$

where $A \in \mathcal{B}(\mathbb{R})$; the triplet (b, c, F) represents the *local characteristics* of H. In addition, the triplet of predictable characteristics (B, C, ν) determines the distribution of H, as the Lévy–Khintchine formula (1) obviously dictates.

We denote by κ_s the *cumulant generating function* associated to the infinitely divisible distribution with Lévy triplet (b_s, c_s, F_s), i.e. for $z \in \mathbb{M}$ and $s \in [0, T_*]$

$$\kappa_s(z) := b_s z + \frac{c_s}{2} z^2 + \int_{\mathbb{R}} (\mathrm{e}^{zx} - 1 - zx) F_s(\mathrm{d}x). \tag{3}$$

Using Assumption $(\mathbb{EM})$ we can extend κ_s to the complex domain $\mathbb{C}$, for $z \in \mathbb{C}$ with $\Re z \in \mathbb{M}$, and the characteristic function of H_t can be written as

$$\mathbb{E}\big[\mathrm{e}^{iuH_t}\big] = \exp\left(\int_0^t \kappa_s(iu)\mathrm{d}s\right). \tag{4}$$

If H is a Lévy process, i.e. time-homogeneous, then (b_s, c_s, F_s) – and thus also κ_s – do not depend on s. In that case, κ equals the cumulant (log-moment) generating function of H_1.

3. THE LÉVY LIBOR MODEL

The Lévy LIBOR model was developed by Eberlein and Özkan (2005), following the seminal articles of Sandmann et al. (1995), Miltersen et al. (1997) and Brace et al. (1997) on LIBOR market models driven by Brownian motion; see also Glasserman and Kou (2003) and Jamshidian (1999) for LIBOR models driven by jump processes and general semimartingales respectively. The Lévy LIBOR model is a *market model* where the forward LIBOR rate is modeled directly, and is driven by a time-inhomogeneous Lévy process.

Let $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = T_*$ denote a discrete tenor structure where $\delta_i = T_{i+1} - T_i$, $i \in \{0, 1, \ldots, N\}$. Consider a complete stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P}_{T_*})$ and a time-inhomogeneous Lévy process $H = (H_t)_{0\leq t\leq T_*}$ satisfying Assumptions $(\mathbb{AC})$ and $(\mathbb{EM})$. The process H has predictable characteristics $(0, C, \nu^{T_*})$ or local characteristics $(0, c, F^{T_*})$, and its canonical decomposition is

$$H = \int_0^\cdot \sqrt{c_s}\mathrm{d}W_s^{T_*} + \int_0^\cdot \int_{\mathbb{R}} x(\mu^H - \nu^{T_*})(\mathrm{d}s, \mathrm{d}x), \tag{5}$$

where W^{T_*} is a $\mathbb{P}_{T_*}$-standard Brownian motion, μ^H is the random measure associated with the jumps of H and ν^{T_*} is the $\mathbb{P}_{T_*}$-compensator of μ^H. We further assume that the following conditions are in force.

(LR1) For any maturity T_i there exists a bounded, continuous, deterministic function $\lambda(\cdot, T_i) : [0, T_i] \to \mathbb{R}$, which represents the volatility of the forward LIBOR rate process $L(\cdot, T_i)$. Moreover,

$$\sum_{i=1}^{N} \left|\lambda(s, T_i)\right| \leq M,$$

for all $s \in [0, T_*]$, where M is the constant from Assumption ($\mathbb{EM}$), and $\lambda(s, T_i) = 0$ for all $s > T_i$.

(LR2) The initial term structure $B(0, T_i)$, $1 \leq i \leq N+1$, is strictly positive and strictly decreasing. Consequently, the initial term structure of forward LIBOR rates is given, for $1 \leq i \leq N$, by

$$L(0, T_i) = \frac{1}{\delta_i}\left(\frac{B(0, T_i)}{B(0, T_i + \delta_i)} - 1\right) > 0.$$

The construction starts by postulating that the dynamics of the forward LIBOR rate with the longest maturity $L(\cdot, T_N)$ is driven by the time-inhomogeneous Lévy process H and evolve as a martingale under the terminal forward measure $\mathbb{P}_{T_*}$. Then, the dynamics of the LIBOR rates for the preceding maturities are constructed by backward induction; they are driven by the same process H and evolve as martingales under their associated forward measures.

Let us denote by $\mathbb{P}_{T_{i+1}}$ the forward measure associated to the settlement date T_{i+1}, $0 \leq i \leq N$. The dynamics of the forward LIBOR rate $L(\cdot, T_i)$, for an arbitrary T_i, is given by

$$L(t, T_i) = L(0, T_i) \exp\left(\int_0^t b^L(s, T_i)\mathrm{d}s + \int_0^t \lambda(s, T_i)\mathrm{d}H_s^{T_{i+1}}\right), \tag{6}$$

where $H^{T_{i+1}}$ is a special *semimartingale* with canonical decomposition

$$H_t^{T_{i+1}} = \int_0^t \sqrt{c_s}\mathrm{d}W_s^{T_{i+1}} + \int_0^t \int_{\mathbb{R}} x(\mu^H - \nu^{T_{i+1}})(\mathrm{d}s, \mathrm{d}x). \tag{7}$$

Here $W^{T_{i+1}}$ is a $\mathbb{P}_{T_{i+1}}$-standard Brownian motion and $\nu^{T_{i+1}}$ is the $\mathbb{P}_{T_{i+1}}$-compensator of μ^H. The dynamics of an arbitrary LIBOR rate again evolves as a martingale under its corresponding forward measure; therefore, we specify the drift term of the forward LIBOR process $L(\cdot, T_i)$ as

$$b^L(s, T_i) = -\frac{1}{2}\lambda^2(s, T_i)c_s - \int_{\mathbb{R}} \left(\mathrm{e}^{\lambda(s, T_i)x} - 1 - \lambda(s, T_i)x\right)F_s^{T_{i+1}}(\mathrm{d}x). \tag{8}$$

The forward measure $\mathbb{P}_{T_{i+1}}$, which is defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T_{i+1}})$, is related to the terminal forward measure $\mathbb{P}_{T_*}$ via

$$\frac{\mathrm{d}\mathbb{P}_{T_{i+1}}}{\mathrm{d}\mathbb{P}_{T_*}} = \prod_{l=i+1}^{N} \frac{1 + \delta_l L(T_{i+1}, T_l)}{1 + \delta_l L(0, T_l)} = \frac{B(0, T_*)}{B(0, T_{i+1})} \prod_{l=i+1}^{N} \left(1 + \delta_l L(T_{i+1}, T_l)\right). \tag{9}$$

The $\mathbb{P}_{T_{i+1}}$-Brownian motion $W^{T_{i+1}}$ is related to the $\mathbb{P}_{T_*}$-Brownian motion via

$$\begin{aligned} W_t^{T_{i+1}} &= W_t^{T_{i+2}} - \int_0^t \alpha(s,T_{i+1})\sqrt{c_s}\mathrm{d}s = \dots \\ &= W_t^{T_*} - \int_0^t \left(\sum_{l=i+1}^{N} \alpha(s,T_l) \right) \sqrt{c_s}\mathrm{d}s, \end{aligned} \tag{10}$$

where

$$\alpha(t,T_l) = \frac{\delta_l L(t-,T_l)}{1+\delta_l L(t-,T_l)} \lambda(t,T_l). \tag{11}$$

The $\mathbb{P}_{T_{i+1}}$-compensator of μ^H, $\nu^{T_{i+1}}$, is related to the $\mathbb{P}_{T_*}$-compensator of μ^H via

$$\begin{aligned} \nu^{T_{i+1}}(\mathrm{d}s,\mathrm{d}x) &= \beta(s,x,T_{i+1})\nu^{T_{i+2}}(\mathrm{d}s,\mathrm{d}x) = \dots \\ &= \left(\prod_{l=i+1}^{N} \beta(s,x,T_l) \right) \nu^{T_*}(\mathrm{d}s,\mathrm{d}x), \end{aligned} \tag{12}$$

where

$$\beta(t,x,T_l,) = \frac{\delta_l L(t-,T_l)}{1+\delta_l L(t-,T_l)} \left(\mathrm{e}^{\lambda(t,T_l)x} - 1 \right) + 1. \tag{13}$$

Remark 3.1 *Notice that the process $H^{T_{i+1}}$, driving the forward LIBOR rate $L(\cdot,T_i)$, and $H = H^{T_*}$ have the same* martingale *part and differ only in the* finite variation *part (drift). An application of Girsanov's theorem for semimartingales yields that the $\mathbb{P}_{T_{i+1}}$-finite variation part of H is*

$$\int_0^\cdot c_s \sum_{l=i+1}^{N} \alpha(s,T_l)\mathrm{d}s + \int_0^\cdot \int_{\mathbb{R}} x \left(\prod_{l=i+1}^{N} \beta(s,x,T_l) - 1 \right) \nu^{T_*}(\mathrm{d}s,\mathrm{d}x).$$

Remark 3.2 *The process $H = H^{T_*}$ driving the most distant LIBOR rate $L(\cdot,T_N)$ is – by assumption – a time-inhomogeneous Lévy process. However, this is* not *the case for any of the processes $H^{T_{i+1}}$ driving the remaining LIBOR rates, because the* random *terms $\frac{\delta_l L(t-,T_l)}{1+\delta_l L(t-,T_l)}$ enter into the compensators $\nu^{T_{i+1}}$ during the construction; see equations* (12) *and* (13).

4. TERMINAL MEASURE DYNAMICS AND LOG-LIBOR RATES

In this section we derive the stochastic differential equation that the dynamics of log-LIBOR rates satisfy under the terminal measure $\mathbb{P}_{T_*}$. This will be the starting point for the approximation method that will be developed in the next section. Of course, we could consider the SDE as the defining point for the model, as is often the case in stochastic volatility LIBOR models, cf. e.g. Andersen and Brotherton-Ratcliffe (2005).

Starting with the dynamics of the LIBOR rate $L(\cdot, T_i)$ under the forward martingale measure $\mathbb{P}_{T_{i+1}}$, and using the connection between the forward and terminal martingale measures (cf. eqs. (10)–(13) and Remark 3.1), we have that the dynamics of the LIBOR rate $L(\cdot, T_i)$ under the terminal measure are given by

$$L(t, T_i) = L(0, T_i) \exp \left(\int_0^t b(s, T_i) \mathrm{d}s + \int_0^t \lambda(s, T_i) \mathrm{d}H_s \right), \tag{14}$$

where $H = (H_t)_{0 \le t \le T_*}$ is the $\mathbb{P}_{T_*}$-time-inhomogeneous Lévy process driving the LIBOR rates, cf. (5). The drift term $b(\cdot, T_i)$ has the form

$$\begin{aligned} b(s, T_i) = &-\frac{1}{2} \lambda^2(s, T_i) c_s - c_s \lambda(s, T_i) \sum_{l=i+1}^{N} \frac{\delta_l L(s-, T_l)}{1 + \delta_l L(s-, T_l)} \lambda(s, T_l) \\ &- \int_{\mathbb{R}} \left(\left(\mathrm{e}^{\lambda(s, T_i) x} - 1 \right) \prod_{l=i+1}^{N} \beta(s, x, T_l) - \lambda(s, T_i) x \right) F_s^{T_*}(\mathrm{d}x), \end{aligned} \tag{15}$$

where $\beta(s, x, T_l)$ is given by (13). Note that the drift term of (14) is random, therefore we are dealing with a general semimartingale, and not with a Lévy process. Of course, $L(\cdot, T_i)$ is not a $\mathbb{P}_{T_*}$-martingale, unless $i = N$ (where we use the conventions $\sum_{l=1}^{0} = 0$ and $\prod_{l=1}^{0} = 1$).

Let us denote by Z the log-LIBOR rates, that is

$$\begin{aligned} Z(t, T_i) &:= \log L(t, T_i) \\ &= Z(0, T_i) + \int_0^t b(s, T_i) \mathrm{d}s + \int_0^t \lambda(s, T_i) \mathrm{d}H_s, \end{aligned} \tag{16}$$

where $Z(0, T_i) = \log L(0, T_i)$ for all $i \in \{1, \ldots, N\}$. We can immediately deduce that $Z(\cdot, T_i)$ is a semimartingale and its triplet of predictable characteristics under $\mathbb{P}_{T_*}$, $\mathbb{T}(Z(\cdot, T_i) | \mathbb{P}_{T_*}) = (B^i, C^i, \nu^i)$, is described by

$$\begin{aligned} B^i &= \int_0^\cdot b(s, T_i) \mathrm{d}s \\ C^i &= \int_0^\cdot \lambda^2(s, T_i) c_s \mathrm{d}s \\ 1_A(x) * \nu^i &= 1_A\big(\lambda(s, T_i) x\big) * \nu^{T_*}, \qquad A \in \mathcal{B}(\mathbb{R} \backslash \{0\}). \end{aligned} \tag{17}$$

The assertion follows from the canonical decomposition of a semimartingale and the triplet of characteristics of the stochastic integral process; see, for example, Proposition 1.3 in Papapantoleon (2007).

Hence, the log-LIBOR rates satisfy the following linear SDE

$$\mathrm{d}Z(t, T_i) = b(t, T_i) \mathrm{d}t + \lambda(t, T_i) \mathrm{d}H_t, \tag{18}$$

with initial condition $Z(0, T_i) = \log L(0, T_i)$.

Remark 4.1 *Note that the martingale part of $Z(\cdot, T_i)$, i.e. the stochastic integral $\int_0^\cdot \lambda(s, T_i) \mathrm{d}H_s$, is a time-inhomogeneous Lévy process. However, the random drift term destroys the Lévy property of $Z(\cdot, T_i)$, as the increments are no longer independent.*

5. STRONG TAYLOR APPROXIMATION AND APPLICATIONS

The aim of this section is to *strongly approximate* the stochastic differential equations for the dynamics of LIBOR rates under the terminal measure. This pathwise approximation is based on the strong Taylor approximation of the random processes $L(\cdot, T_l)$, $i+1 \leq l \leq N$ in the drift $b(\cdot, T_i)$ of the semimartingale driving the LIBOR rates $L(\cdot, T_i)$; cf. equations (14)–(16). The idea behind the strong Taylor approximation is the perturbation of the initial SDE by a real parameter and a classical Taylor expansion around this parameter, with usual conditions for convergence (cf. Definition 5.1).

5.1. Definition

We introduce a parameter $\epsilon \in \mathbb{R}$ and will approximate the terms

$$L(t-, T_l)$$

which cause the drift term to be random, by their *first-order strong Taylor approximation*; cf. Lemma 5.1. Note that the map $x \mapsto \frac{\delta_l x}{1+\delta_l x}$, appearing in the drift, is globally Lipschitz with Lipschitz constant $\delta^* = \max_l \delta_l$.

The following definition of the strong Taylor approximation is taken by Siopacha (2006); see also Siopacha and Teichmann (2010). Consider a smooth curve $\epsilon \mapsto W_\epsilon$, where $\epsilon \in \mathbb{R}$ and $W_\epsilon \in L^2(\Omega; \mathbb{R})$.

Definition 5.1 *A* strong Taylor approximation *of order $n \geq 0$ is a (truncated) power series*

$$\mathbf{T}^n(W_\epsilon) := \sum_{k=0}^{n} \frac{\epsilon^k}{k!} \frac{\partial^k}{\partial \epsilon^k}\Big|_{\epsilon=0} W_\epsilon$$

such that

$$\mathbb{E}\left[|W_\epsilon - \mathbf{T}^n(W_\epsilon)|\right] = o(\epsilon^n),$$

holds true as $\epsilon \to 0$.

Then, for Lipschitz functions $f : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ with Lipschitz constant k we get the following error estimate:

$$\mathbb{E}[|f(W_\epsilon) - f(\mathbf{T}^n(W_\epsilon))|] \leq k\mathbb{E}[|W_\epsilon - \mathbf{T}^n(W_\epsilon)|] = ko(\epsilon^n). \tag{19}$$

Remark 5.1 *It is important to point out that motivated by the idea of the Taylor series we perform an expansion around $\epsilon = 0$ and the estimate* (19) *is valid. However, for the pathwise approximation of LIBOR rates we are interested in the region $\epsilon \approx 1$, and hope that the expansion yields adequate results; for $\epsilon = 0$ we would simply recover the "frozen drift" approximation. Numerical experiments show that this approach indeed yields better results than the "frozen drift" approximation; cf. section 6.*

5.2. Strong Taylor approximation

In this section we develop a strong Taylor approximation scheme for the dynamics of log-LIBOR rates. Let us introduce the auxiliary process $X^\epsilon(\cdot, T_i) = (X^\epsilon(t, T_i))_{0\leq t\leq T_i}$ with initial values $X^\epsilon(0, T_i) = L(0, T_i)$ for all $i \in \{0, \dots, N\}$ and all $\epsilon \in \mathbb{R}$. The dynamics of $X^\epsilon(\cdot, T_i)$ is described by perturbing the SDE of the log-LIBOR rates by the perturbation parameter $\epsilon \in \mathbb{R}$:

$$\mathrm{d}X^\epsilon(t, T_i) = \epsilon\Big(b(t, T_i; X^\epsilon(t))\mathrm{d}t + \lambda(t, T_i)\mathrm{d}H_t\Big), \tag{20}$$

where the drift term $b(\cdot, T_i; X^\epsilon(\cdot))$ is given by (15). The term $X^\epsilon(\cdot)$ in $b(\cdot, T_i; X^\epsilon(\cdot))$ emphasizes that the drift term depends on all subsequent processes $X^\epsilon(\cdot, T_{i+1}), \dots, X^\epsilon(\cdot, T_N)$, which are also perturbed by ϵ. Note that for $\epsilon = 1$ the processes $X^1(\cdot, T_i)$ and $Z(\cdot, T_i)$ are *indistinguishable*.

Remark 5.2 *In the sequel we will use the notation* $\mathbf{T}$ *as shorthand for* $\mathbf{T}^1$.

Lemma 5.1 *The* first-order strong Taylor approximation *of the random variable* $X^\epsilon(t, T_i)$ *is given by:*

$$\mathbf{T}\big(X^\epsilon(t, T_i)\big) = \log L(0, T_i) + \epsilon\frac{\partial}{\partial\epsilon}\big|_{\epsilon=0}X^\epsilon(t, T_i), \tag{21}$$

where the first variation process $\frac{\partial}{\partial\epsilon}|_{\epsilon=0}X^\epsilon(\cdot, T_i) =: Y(\cdot, T_i)$ *of* $X^\epsilon(\cdot, T_i)$ *is a* time-inhomogeneous Lévy process *with local characteristics*

$$\begin{aligned} b_s^{Y_i} &= b(s, T_i; X(0)) \\ c_s^{Y_i} &= \lambda^2(s, T_i)c_s \\ \int 1_A(x)F_s^{Y_i}(\mathrm{d}x) &= \int 1_A\big(\lambda(s, T_i)x\big)F_s^{T_*}(\mathrm{d}x), \quad A \in \mathcal{B}(\mathbb{R}). \end{aligned} \tag{22}$$

Proof. By definition, the first-order strong Taylor approximation is given by the truncated power series

$$\mathbf{T}\big(X^\epsilon(t, T_i)\big) = X^0(t, T_i) + \epsilon\frac{\partial}{\partial\epsilon}\big|_{\epsilon=0}X^\epsilon(t, T_i). \tag{23}$$

Since the curves $\epsilon \mapsto X^\epsilon(t, T_i)$ are smooth, and $X^\epsilon(t, T_i) \in L^2(\Omega)$ by Assumption $(\mathbb{EM})$, we get that strong Taylor approximations of arbitrary order can always be obtained, cf. Kriegl and Michor (1997, Chapter 1). In particular, for the first-order expansion we have that

$$\mathbb{E}\left[\big|X^\epsilon(t, T_i) - \mathbf{T}\big(X^\epsilon(t, T_i)\big)\big|\right] = o(\epsilon).$$

The zero-order term of the Taylor expansion trivially satisfies

$$X^0(t, T_i) = X^0(0, T_i) \text{ for all } t, \text{ since } \mathrm{d}X^0(t, T_i) = 0.$$

Of course, the initial values of the perturbed SDE coincide with the initial values of the unperturbed SDE, hence $X^0(0, T_i) = L(0, T_i) =: X(0, T_i)$.

The first variation process of $X^\epsilon(\cdot, T_i)$ with respect to ϵ is derived by differentiating (20); hence, the dynamics is

$$\begin{aligned} \mathrm{d}\Big(\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}X^\epsilon(t, T_i)\Big) &= b(t, T_i; X^\epsilon(t))|_{\epsilon=0}\mathrm{d}t + \lambda(t, T_i)\mathrm{d}H_t \\ &= b(t, T_i; X(0))\mathrm{d}t + \lambda(t, T_i)\mathrm{d}H_t. \end{aligned} \tag{24}$$

We can immediately notice that in the drift term $b(\cdot, T_i; X(0))$ of the first variation process, the random terms $X^\epsilon(t, T_i)$ are replaced by their deterministic initial values $X(0, T_i) = Z(0, T_i)$.

Let us denote by $Y(\cdot, T_i)$ the first variation process of $X^\epsilon(\cdot, T_i)$. The solution of the linear SDE (24) describing the dynamics of the first variation process yields

$$Y(t, T_i) = \int_0^t b(s, T_i; X(0))\mathrm{d}s + \int_0^t \lambda(s, T_i)\mathrm{d}H_s. \tag{25}$$

Since the drift term is deterministic and H is a time-inhomogeneous Lévy process we can conclude that $Y(\cdot, T_i)$ is itself a time-inhomogeneous Lévy process. The local characteristics of $Y(\cdot, T_i)$ are described by (22). ∎

To summarize, by setting $\epsilon = 1$ in Lemma 5.1, we have developed the following approximation scheme for the logarithm of the random terms $X^1(\cdot, T_i) = Z(\cdot, T_i)$ entering the drift:

$$\mathbf{T}X(t, T_i) = \log L(0, T_i) + \int_0^t b(s, T_i; X(0))\mathrm{d}s + \int_0^t \lambda(s, T_i)\mathrm{d}H_s. \tag{26}$$

Comparing (26) with (16) it becomes evident that we are approximating the semimartingale $Z(\cdot, T_i)$ with the time-inhomogeneous *Lévy process* $\mathbf{T}X(\cdot, T_i)$.

Remark 5.3 *A consequence of this approximation scheme is that we can embed the "frozen drift" approximation into our method. Indeed, the "frozen drift" approximation is the zero-order Taylor approximation, i.e.* $X^1(t, T_i) \approx \log L(0, T_i)$. *The dynamics of LIBOR rates using this approximation will be denoted by* $\widehat{L}^0(\cdot, T_i)$.

5.3. Application to LIBOR models

In this section, we will apply the strong Taylor approximation of the log-LIBOR rates $Z(\cdot, T_i)$ by $\mathbf{T}X(\cdot, T_i)$ in order to derive a *strong*, i.e. pathwise, approximation for the dynamics of log-LIBOR rates. That is, we replace the random terms in the drift $b(\cdot, T_i; Z(\cdot))$ by the Lévy process $\mathbf{T}X(\cdot, T_i)$ instead of the semimartingale $Z(\cdot, T_i)$. Therefore, the dynamics of the *approximate* log-LIBOR rates $\widehat{Z}(\cdot, T_i)$ are given by

$$\widehat{Z}(t, T_i) = Z(0, T_i) + \int_0^t b(s, T_i; \mathbf{T}X(s))\mathrm{d}s + \int_0^t \lambda(s, T_i)\mathrm{d}H_s, \tag{27}$$

where the drift term is provided by

$$\begin{aligned} b(s, T_i; \mathbf{T}X(s)) = &-\frac{1}{2}\lambda^2(s, T_i)c_s - c_s\lambda(s, T_i)\sum_{l=i+1}^{N} \frac{\delta_l \mathrm{e}^{\mathbf{T}X(s-, T_l)}}{1 + \delta_l \mathrm{e}^{\mathbf{T}X(s-, T_l)}}\lambda(s, T_l) \\ &- \int_{\mathbb{R}} \left(\left(\mathrm{e}^{\lambda(s, T_i)x} - 1\right) \prod_{l=i+1}^{N} \widehat{\beta}(s, x, T_l) - \lambda(s, T_i)x \right) F_s^{T_*}(\mathrm{d}x), \end{aligned} \tag{28}$$

with

$$\widehat{\beta}(t, x, T_l,) = \frac{\delta_l \exp\left(\mathbf{T}X(t-, T_l)\right)}{1 + \delta_l \exp\left(\mathbf{T}X(t-, T_l)\right)} \left(\mathrm{e}^{\lambda(t, T_l)x} - 1\right) + 1. \tag{29}$$

T	0.5 Y	1 Y	1.5 Y	2 Y	2.5 Y
$B(0,T)$	0.9833630	0.9647388	0.9435826	0.9228903	0.9006922
T	3 Y	3.5 Y	4 Y	4.5 Y	5 Y
$B(0,T)$	0.8790279	0.8568412	0.8352144	0.8133497	0.7920573

Table 1: Euro zero coupon bond prices on February 19, 2002.

The main advantage of the strong Taylor approximation is that the resulting SDE for $\widehat{Z}(\cdot,T_i)$ can be simulated more easily than the equation for $Z(\cdot,T_i)$. Indeed, looking at (18) and (15) again, we can observe that each LIBOR rate $L(\cdot,T_i)$ depends on all subsequent rates $L(\cdot,T_l)$, $i+1 \leq l \leq N$. Hence, in order to simulate $L(\cdot,T_i)$, we should start by simulating the furthest rate in the tenor and proceed iteratively from the end. On the contrary, the dynamics of $\widehat{Z}(\cdot,T_i)$ depend only on the Lévy processes $\mathbf{T}X(\cdot,T_l)$, $i+1 \leq l \leq N$, which are independent of each other. Hence, we can use *parallel computing* to simulate all approximate LIBOR rates simultaneously. This significantly increases the speed of the Monte Carlo simulations while, as the numerical example reveals, the empirical performance is very satisfactory.

Remark 5.4 *Let us point out that this method can be applied to any LIBOR model driven by a general semimartingale. Indeed, the properties of Lévy processes are not essential in the proof of Lemma 5.1 or in the construction of the LIBOR model. If we start with a LIBOR model driven by a general semimartingale, then the structure of this semimartingale will be "transferred" to the first variation process, and hence also to the dynamics of the strong Taylor approximation.*

6. NUMERICAL ILLUSTRATION

The aim of this section is to demonstrate the accuracy and efficiency of the Taylor approximation scheme for the valuation of options in the Lévy LIBOR model compared to the "frozen drift" approximation. We will consider the pricing of caps and swaptions, although many other interest rate derivatives can be considered in this framework.

We revisit the numerical example in Kluge (2005, pp. 76-83). That is, we consider a tenor structure $T_0 = 0, T_1 = \frac{1}{2}, T_2 = 1 \ldots, T_{10} = 5 = T_*$, constant volatilities

$$\begin{array}{lll} \lambda(\cdot,T_1) = 0.20 & \lambda(\cdot,T_2) = 0.19 & \lambda(\cdot,T_3) = 0.18 \\ \lambda(\cdot,T_4) = 0.17 & \lambda(\cdot,T_5) = 0.16 & \lambda(\cdot,T_6) = 0.15 \\ \lambda(\cdot,T_7) = 0.14 & \lambda(\cdot,T_8) = 0.13 & \lambda(\cdot,T_9) = 0.12 \end{array}$$

and the discount factors (zero coupon bond prices) as quoted on February 19, 2002; cf. Table 1. The tenor length is constant and denoted by $\delta = \frac{1}{2}$.

The driving Lévy process H is a normal inverse Gaussian (NIG) process with parameters $\alpha = \bar{\delta} = 1.5$ and $\mu = \beta = 0$. We denote by μ^H the random measure of jumps of H and by $\nu(\mathrm{d}t,\mathrm{d}x) = F(\mathrm{d}x)\mathrm{d}t$ the $\mathbb{P}_{T_*}$-compensator of μ^H, where F is the Lévy measure of the NIG process. The necessary conditions are satisfied because $M = \alpha$, hence $\sum_{i=1}^{9} |\lambda(\cdot, T_i)| = 1.44 < \alpha$ and $\lambda(\cdot, T_i) < \frac{\alpha}{2}$, for all $i \in \{1, \dots, 9\}$.

The NIG Lévy process is a pure-jump Lévy process and, for $\mu = 0$, has the canonical decomposition

$$H = \int_0^{\cdot} \int_{\mathbb{R}} x(\mu^H - \nu)(\mathrm{d}s, \mathrm{d}x).$$

The cumulant generating function of the NIG distribution is

$$\kappa(u) = \bar{\delta}\alpha - \bar{\delta}\sqrt{\alpha^2 - u^2},$$

for all $u \in \mathbb{C}$ with $|\Re u| \leq \alpha$.

6.1. Caplets

The price of a caplet with strike K maturing at time T_i, using the relationship between the terminal and the forward measures cf. (9), can be expressed as

$$\begin{aligned}
\mathbb{C}_0(K, T_i) &= \delta B(0, T_{i+1})\, \mathbb{E}_{\mathbb{P}_{T_{i+1}}}[(L(T_i, T_i) - K)^+] \\
&= \delta B(0, T_{i+1})\, \mathbb{E}_{\mathbb{P}_{T_*}}\Big[\frac{\mathrm{d}\mathbb{P}_{T_{i+1}}}{\mathrm{d}\mathbb{P}_{T_*}}\big|_{\mathcal{F}_{T_i}} (L(T_i, T_i) - K)^+\Big] \\
&= \delta B(0, T_*)\, \mathbb{E}_{\mathbb{P}_{T_*}}\Big[\prod_{l=i+1}^{N} \big(1 + \delta L(T_i, T_l)\big)(L(T_i, T_i) - K)^+\Big].
\end{aligned} \tag{30}$$

This equation will provide the actual prices of caplets corresponding to simulating the full SDE for the LIBOR rates. In order to calculate the first-order Taylor approximation prices for a caplet we have to replace $L(\cdot, T_\cdot)$ in (30) with $\widehat{L}(\cdot, T_\cdot)$. Similarly, for the frozen drift approximation prices we must use $\widehat{L}^0(\cdot, T_\cdot)$ instead of $L(\cdot, T_\cdot)$.

We will compare the performance of the strong Taylor approximation relative to the frozen drift approximation in terms of their implied volatilities. In Figure 1 we present the difference in implied volatility between the full SDE prices and the frozen drift prices, and between the full SDE prices and the strong Taylor prices. One can immediately observe that the strong Taylor approximation method performs much better than the frozen drift approximation; the difference in implied volatilities is very low across all strikes and maturities. Indeed, the difference in implied volatility between the full SDE and the strong Taylor prices lies always below the 1% threshold, which deems this approximation accurate enough for practical implementations. On the contrary, the difference in implied volatilities for the frozen drift approximation exceeds the 1% level for in-the-money options.

6.2. Swaptions

Next, we will consider the pricing of swaptions. Recall that a payer (resp. receiver) swaption can be viewed as a put (resp. call) option on a coupon bond with exercise price 1; cf. section 16.2.3

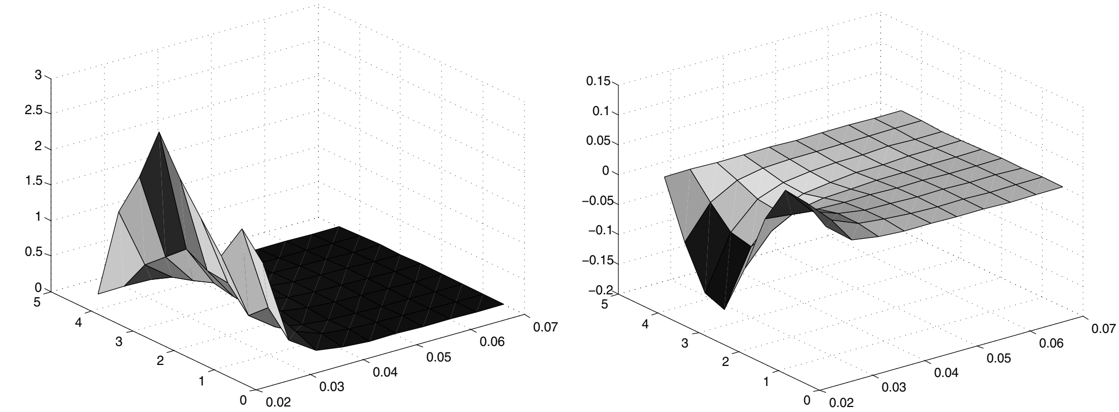

Figure 1: Difference in implied volatility between the full SDE and the frozen drift prices (left), and the full SDE and the strong Taylor prices (right).

and 16.3.2 in Musiela and Rutkowski (1997). Consider a payer swaption with strike rate K, where the underlying swap starts at time T_i and matures at T_m ($i < m \leq N$). The time-T_i value is

$$\begin{aligned}\mathbb{S}_{T_i}(K, T_i, T_m) &= \left(1 - \sum_{k=i+1}^{m} c_k B(T_i, T_k)\right)^+ \\ &= \left(1 - \sum_{k=i+1}^{m} \left(c_k \prod_{l=i}^{k-1} \frac{1}{1 + \delta L(T_i, T_l)}\right)\right)^+, \end{aligned} \tag{31}$$

where

$$c_k = \begin{cases} K, & i+1 \leq k \leq m-1, \\ 1+K, & k = m. \end{cases}$$

Then, the time-0 value of the swaption is obtained by taking the $\mathbb{P}_{T_i}$-expectation of its time-T_i value, that is

$$\begin{aligned}\mathbb{S}_0 &= \mathbb{S}_0(K, T_i, T_m) \\ &= B(0, T_i)\, \mathbb{E}_{\mathbb{P}_{T_i}} \left[\left(1 - \sum_{k=i+1}^{m} \left(c_k \prod_{l=i}^{k-1} \frac{1}{1 + \delta L(T_i, T_l)}\right)\right)^+\right] \\ &= B(0, T_*)\, \mathbb{E}_{\mathbb{P}_{T_*}} \left[\prod_{l=i}^{N} \big(1 + \delta L(T_i, T_l)\big) \left(1 - \sum_{k=i+1}^{m} \left(c_k \prod_{l=i}^{k-1} \frac{1}{1 + \delta L(T_i, T_l)}\right)\right)^+\right], \end{aligned}$$

hence

$$\mathbb{S}_0 = B(0, T_*)\, \mathbb{E}_{\mathbb{P}_{T_*}} \left[\left(-\sum_{k=i}^{m} \left(c_k \prod_{l=k}^{N} (1 + \delta L(T_i, T_l))\right)\right)^+\right], \tag{32}$$

where $c_i := -1$. Once again, this equation will provide the actual prices of swaptions corresponding to simulating the full SDE for the LIBOR rates. In order to calculate the first-order Taylor

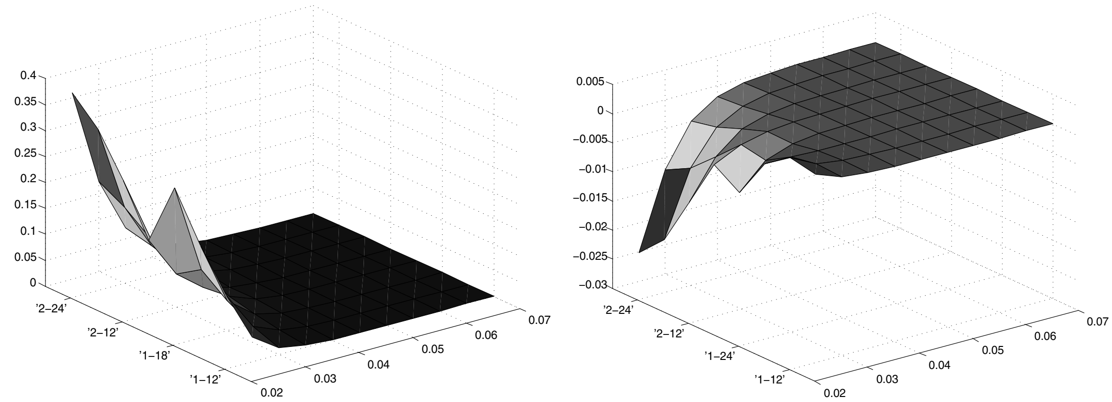

Figure 2: Difference in swaption prices between the full SDE and the frozen drift method (left), and the full SDE and the strong Taylor method (right).

approximation prices we have to replace $L(\cdot, T.)$ with $\widehat{L}(\cdot, T.)$, and for the frozen drift approximation prices we must use $\widehat{L}^0(\cdot, T.)$ instead of $L(\cdot, T.)$.

We will price eight swaptions in our tenor structure; we consider 1 year and 2 years as option maturities, and then use 12, 18, 24 and 30 months as swap maturities for each option. Similarly to the simulations we performed for caplets, we will simulate the prices of swaptions using all three methods and compare their differences; these can be seen in Figure 2. Once again we observe that the strong Taylor method is performing very well across all strikes, option maturities and swap maturities, while the performance of the frozen drift method is poor for in-the-money swaptions and seems to be deteriorating for longer swap maturities. This observation is in accordance with the common knowledge that the frozen drift approximation is performing worse and worse for longer maturities.

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POSTER SESSION

THRESHOLD PROPORTIONAL REINSURANCE: AN ALTERNATIVE STRATEGY TO REDUCE THE INITIAL INVESTMENT IN A NON-LIFE INSURANCE PORTFOLIO

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We introduce a dynamic proportional reinsurance strategy in a non-life portfolio in which the retention level k, instead of being constant, depends on the level of the surplus. With this strategy the insurer can reduce the probability of ruin for some predetermined level of initial capital relative to the option of not reinsuring and the option of applying a proportional reinsurance. Therefore, if the manager of the portfolio chooses this dynamic reinsurance strategy, a reduction in the initial capital investment can be achieved while maintaining the solvency level.

1. INTRODUCTION

In a non-life insurance portfolio, the manager must make decisions that ensure the business's technical solvency, i.e., that ensure a sufficient level of technical reserves to cover the payment of claims. Ruin theory models allow taking the long-term view in analyzing a portfolio's current solvency and making the necessary decisions. In this sense, these models are in many aspects superior to models that focus on the short term. In particular, classical ruin theory represents the level of reserves $R(t)$ at any given time t as

$$R(t) = u + ct - S(t)$$

where u is the initial level of reserves, i.e., the initial capital which allows the portfolio to operate, c is the cash inflow rate from premiums on the portfolio at each instant, and $S(t)$ is the sum of the claims that have occurred up to time t. The aggregate sum of the claims up to time t is estimated as $S(t) = \sum_{i=1}^{N(t)} X_i$, where $N(t)$ is the stochastic process of the number of claims up to time t, and X_i the amount of the ith claim. The classical hypotheses are that the amounts of the individual claims are identical and independently distributed, and that they are independent of the number of claims, so that $S(t)$ is a compound process. If it also assumed, as is usual, that $N(t)$ is a Poisson process of rate λ, then $S(t)$ is a compound Poisson process.

The premium is calculated as the expected loss rate increased by a "loading" coefficient, $\rho > 0$ to satisfy the "net profit" condition, and is given by

$$c = \lambda E[X](1 + \rho).$$

One of the measures used to assess the solvency of the portfolio is the probability of ruin $\psi(u)$, i.e., the probability that the reserves are negative,

$$\psi(u) = P[R(t) < 0 | R(0) = u].$$

With a reinsurance treaty (or contract), the insurer cedes a part of the premiums to the reinsurers, and accordingly also a part of the losses. In a proportional treaty, the insurer keeps a certain fraction of all claims $k \in (0, 1]$, the retention ratio, and the reinsurers takes the remaining $(1 - k)$.

In their treaty with the insurer, the reinsurers also apply a loading coefficient $\rho_R > 0$, so that the net reinsurance premium for the insurer is

$$c' = c - (1 - k)(1 + \rho_R)\lambda E[X].$$

It is normally assumed that $\rho_R > \rho > 0$, because if $\rho_R \leq \rho$, the insurer would simply cede his entire portfolio to the reinsurers, a situation which would be senseless. This net premium thus defines a new real loading coefficient for the insurer,

$$\rho_N = \frac{c'}{k\lambda E[X]} - 1 = \rho_R - \frac{\rho_R - \rho}{k}.$$

The "net profit" condition for the loading coefficient ($\rho_N > 0$) imposes a natural limit on the proportion retained by the insurer, so that

$$\frac{\rho_R - \rho}{\rho_R} < k \leq 1, \ \rho_R > \rho > 0.$$

Studies of the effect of reinsurance strategy on solvency measures have concentrated their attention on the ultimate ruin probability. Several of them analyze the effect of reinsurance on the adjustment coefficient or Lundberg exponent, see Waters (1979), Centeno (2002), Hesselager (1990).

Many authors have considered the problem of determining the optimal level and/or type of reinsurance, where optimal is defined in terms of some stability criterion, mainly the probability of ruin, see e.g. Waters (1983), Goovaerts et al. (1989), Verlaak and Beirlant (2003), Schmidli (2002), Hipp and Vogt (2003), Taksar and Markussen (2003).

The reinsurance strategy considered may be static or dynamic. In the first case, it is assumed that the level and type of reinsurance remain constant throughout the period considered, where this period in many cases is infinite, see Centeno (2005), Dickson and Waters (1996). In the dynamic case, we can find papers assuming that for a fixed type of reinsurance the level of reinsurance can change continuously, see Hojgaard and Taksar (1998), Schmidli (2002), Hipp and Vogt (2003). In these papers, optimal stochastic control tools in continuous time are used. Dickson and Waters

(2006) assume that the insurer can change the type and/or level of reinsurance at the start of each year, so they studied a discrete time stochastic control problem.

In this paper we introduce a dynamic reinsurance strategy and we obtain the ruin probability when the individual claim amount is distributed exponentially. Before that, we remind the expression of ruin probability when the individual claim amount follows a unit-mean exponential distribution,

$$\psi(u) = \frac{1}{1+\rho} e^{-\frac{\rho}{1+\rho}u}, \ \forall u \geq 0. \tag{1}$$

If we include a proportional reinsurance, the ruin probability can be obtained directly from (1),

$$\psi_R(u,k) = \frac{1}{1+\rho_N} e^{-\frac{\rho_N}{k(1+\rho_N)}u}, \ \forall u \geq 0.$$

If the manager's objective is to optimize the portfolio's solvency by minimizing the probability of ruin, the control variable related to reinsurance is the retention ratio.

2. THRESHOLD PROPORTIONAL REINSURANCE

In this paper we introduce a dynamic proportional reinsurance strategy in which the retention ratio, k, instead of being constant, depends on the level of the surplus. Then the threshold proportional strategy we define is as follows: the retention ratio applied is k_1 whenever the reserves are less than a determined threshold b, and k_2 otherwise.

Since, for the insurer, reinsurance is a tool for controlling the solvency of the portfolio, it seems natural that the retention ratio should depend on the current level of surplus. The threshold proportional reinsurance strategy that we propose is a straightforward and transparent way to include this dependence. The fractions k_1 and k_2 define two new net premium loading coefficients for the insurer, ρ_1 and ρ_2. For a graphical illustration, see Figure 1.

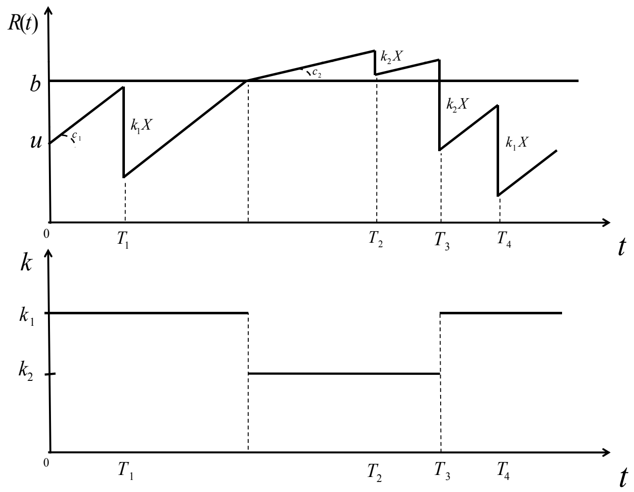

Figure 1: Threshold Proportional Reinsurance

With this new reinsurance strategy, the probability of ruin behaves differently, depending on whether its initial surplus u is below or above the level b,

$$\psi(u) = \begin{cases} \psi_1(u) & 0 \leq u < b \\ \psi_2(u) & u \geq b \ . \end{cases}$$

The explicit expression of the ruin probability can be found in Mármol et al. (2009) when the individual claim amount is distributed as an exponential with unitary mean.

Threshold proportional reinsurance includes standard proportional reinsurance ($k_1 = k_2 = k$) as a particular case, and therefore also the option of not reinsuring ($k_1 = k_2 = k = 1$).

For fixed values of the parameters (λ, ρ and ρ_R), we can find the optimal threshold proportional reinsurance that minimize the probability of ruin in a numerical way. In the optimal solution, k_1 is different from k_2, so with this new threshold reinsurance strategy the insurer can reduce his probability of ruin for some predetermined level of initial capital relative to the other options: no reinsurance or applying proportional reinsurance. This optimality of threshold proportional reinsurance also implies that, if the manager wants to obtain this minimal probability of ruin but with proportional reinsurance or no reinsurance, he will need more initial capital. The relative increase in the initial reserves to achieve this optimal probability of ruin can be considered as the cost of the options of proportional reinsurance and no reinsurance as against threshold proportional reinsurance. For example, when $\lambda = 1$, $\rho = 0.15$ and $\rho_R = 0.25$, for $u = 4$, if the manager chooses the optimal threshold proportional reinsurance he achieves a probability of ruin of 49.8%. If he chooses proportional reinsurance with the optimal fixed retention percentage, to achieve this probability of ruin he will need a 4.075% more initial capital, and if he chooses not to reinsure, his financial requirement to initiate the business would be increased by 6.8%.

3. CONCLUSION

The threshold proportional reinsurance strategy consists of applying a retention ratio that depends on the level of reserves. It allows a better management of the initial capital compared with the options of proportional reinsurance and no reinsurance. With this new strategy, one achieves either a higher level of solvency while maintaining the same initial capital invested in the portfolio, or a reduction in the initial capital investment while maintaining the same level of solvency.

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LOCAL VOLATILITY PRICING MODELS FOR LONG-DATED FX DERIVATIVES

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In recent years the market in long-dated FX options has grown considerably. Currently the most traded and liquid long-dated FX Hybrid products are so-called Power-Reverse Dual-Currency swaps (PRDC) (see for example Piterbarg (2006)) as well as vanilla or exotic long-dated products such as barrier options. While for short-dated options (less than 1 year), assuming constant interest rates does not lead to significant mispricing, for long-dated options the effect of interest rate volatility becomes increasingly pronounced with increasing maturity and can become as important as FX spot volatility. Most dealers are using a three-factor pricing model for long-dated FX products where the FX spot is locally governed by a geometric Brownian motion, and where each of the domestic and foreign interest rates follows a Hull-White one factor Gaussian model (Hull and White (1993)). However using such a model does not allow the volatility smile and skew effect encountered in the FX market to be taken into account, and is therefore not appropriate to price and hedge long-dated FX products.

Different methods exist to incorporate smile and skew effects in the three-factor pricing model. In the literature, one can find different approaches which consist of either using a stochastic volatility for the FX spot or a local volatility. There are many processes that can be used for the stochastic volatility and their choices will generally depend on their tractability. Before being used for pricing all these models should be calibrated over the market. In general the calibration is based on calculating prices of liquid products for different strikes and maturity and the parameters of the model are adjusted until these prices match sufficiently with the market. However, in most cases it is difficult to derive analytical formulae, and consequently the calibration procedure often remains approximative or computationally demanding.

Local volatility models were introduced in 1994 by Dupire (Dupire (1994)) and Derman and Kani (Derman and Kani (1994)) for equity based products. As compared to stochastic volatility models they have the advantage that they are Markovian in only one factor (because the local volatility is a deterministic function of both the FX spot and time) implying that they are more appropriate for hedging strategies. Local volatility models also have the advantage that they are calibrated on the complete implied volatility surface, and hence they usually capture more precisely the surface of the implied volatilities than stochastic volatility models. However, a local volatility model has the drawback that it predicts unrealistic dynamics for the stock volatility since the volatilities observed in the market are really stochastic, capable of rising without a movement in spot FX prices. In Bossens et al. (2006), the authors compare short-dated barrier option market prices with the corresponding prices derived from either a Dupire local volatility or a Heston stochastic volatility model both calibrated on the vanilla smile and skew. It appears from their study that in a simplified world, where exotic option prices are derived either from Dupire local volatility or from a Heston stochastic volatility dynamics, an FX market characterised by a mild skew (USDCHF) exhibits mainly a stochastic volatility behavior, and that FX markets characterised by a dominantly skewed implied volatility (USDJPY) exhibit a stronger local volatility component. This observation also underlines that calibrating a stochastic model to the vanilla market is by no means a guarantee that exotic options will be priced correctly (Schoutens et al. (2004)). This is because the vanilla market carries no information about the smile dynamics. The market dynamics could be better approximated by a hybrid volatility model that contains both stochastic volatility dynamics and local volatility ones. To the best of our knowledge this approach has never been studied in a stochastic interest rates framework, but it is also known that it provides positive results for short dated options when interest rates are assumed to be constant (see for example Lipton (2002), Lipton and McGhee (2002), Madan et al. (2007), Tavella et al. (2006)). Once the local volatility surface is available, the new mixed volatility can be computed by multiplying this local volatility with a ratio of integrals that depend on the joint density of the FX spot and the stochastic volatility. This density can be determined by numerically solving the associated Kolmogorov forward PDE.

The study of the local volatility and its calibration to a three-factor model can then be motivated by hedging arguments but is also considerably useful for the calibration of hybrid volatility models. In Deelstra and Rayee (2010), we derive the Dupire's like formula in a three-factor model where we have three sources of randomness: the FX spot with a local volatility and the domestic as well as the foreign interest rates. We present two ways to derive the expression of the local volatility. The first one is based on the method used by Dupire in a simple one factor Gaussian model (Dupire (1994)) and consists of deriving the call prices with respect to the strike and the maturity. In the second approach we use the results of Atlan (Atlan (2006)) as well as Tanaka's formula, to show that we can also obtain the Dupire's like formula for the three-factor model with local volatility.

In a one-factor Gaussian model, the local volatility surface is generally built by using the Dupire's formula where partial derivatives of call options with respect to strikes and maturities are calculated by finite differences and where the real implied volatility surface is an interpolation of a finite set of market call prices. In a three-factor framework with local volatility, the Dupire's like formula becomes more complicated. This is because it also depends on a particularly complicated expectation for which no closed form expression exists and which is not directly related to European call prices or other liquid products. Of course such expectation can be evaluated using numerical integration methods. Then, to enable realisations of the numerical integrations one needs the forward probability distribution of the spot FX rate and the domestic and foreign interest rates up to maturity. In Deelstra and Rayee (2010), we have derived the forward PDE associated to this forward probability density function. However, in a three factor framework, numerical integration methods and PDE solvers are computationally demanding: numerical resolutions of high dimensional PDEs are sometimes unstable.

An alternative approach is to calibrate the local volatility from stochastic volatility models by establishing links between local and stochastic volatility. Extracting the local volatility surface from a stochastic volatility model rather than using the market implied volatility surface seems to be a preferred approach. Indeed, let us first observe that the market implied volatility surface can in practice only be an interpolation of a finite set of available market data. Consequently a local volatility surface built from an approximative implied volatility surface is often unstable. In contrast stochastic volatility models can be calibrated by using fast algorithms like Fast Fourier Transforms (FFT) (see for example van Haastrecht et al. (2008)) and the local volatility surface extracted from the calibrated stochastic volatility model will be really smooth. In Deelstra and Rayee (2010), we present some mimicking properties that links the three-factor model with a local volatility to the same model with a stochastic volatility rather than a local volatility. These properties will allow us to obtain explicit expressions to construct the local volatility surface.

Finally, we study two different extensions of the local volatility model with stochastic interest rates. First, following the ideas developed by Derman and Kani (1998) and by Dupire (2004), we assume that the dynamic behavior of the local volatility obeys a stochastic process driven by some additional Brownian motion. Second, using Gyöngy's result (Gyöngy (1986)), we derive some links between the three-factor model with local volatility and a hybrid volatility model where the volatility of the spot FX rate mixes a stochastic volatility with a local volatility. Knowing the local volatility function associated to the three-factor model with local volatility, we propose a calibration method for the local volatility in the four-factor hybrid volatility model.

The paper Deelstra and Rayee (2010) is organised as follows. We begin by defining the three-factor model with local volatility. Then, we derive the local volatility expression for this model by using two different techniques. First, we derive the Fokker-Plank equation for the forward probability density function of the FX spot and the domestic and foreign interest rates at maturity. This PDE is used in the derivation of the local volatility function by differentiating European call price expressions with respect to the strike and the maturity. The second approach is based on Tanaka's formula.

Afterwards, we focus on the calibration of this local volatility function. We obtain a link between the local volatility function derived in a three-factor framework and the one that stems from the simple one-factor Gaussian model.

Next, we derive a link between the three factor model with a stochastic volatility for the spot FX rate and the one where the spot FX rate volatility is a local volatility. This link provides a relationship between our local volatility function and future instantaneous spot FX rate volatilities. We also propose two extensions of the three-factor model with local volatility. The first extension is obtained by introducing a stochastic structure on the local volatility surface. We show that in that case local volatilities are risk-adjusted expectations of future instantaneous volatilities with respect to the K-strike and T-maturity forward risk-adjusted measure.

Finally, we consider a hybrid volatility model, where the volatility of the spot FX rate is the product of a local volatility and a stochastic volatility. Thanks to Gyöngy's result we obtain equations that link the local volatility function associated to the three-factor model with the local volatility function of the four-factor hybrid volatility model and propose a calibration procedure for the local volatility function associated to this four-factor hybrid volatility model.

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PRICING AND HEDGING OF INTEREST RATE DERIVATIVES IN A LEVY DRIVEN TERM STRUCTURE MODEL

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We study the pricing of forward payer swaptions and derive hedging strategies for these on the basis of investments in zero-coupon bonds. As framework we consider the Lévy driven Heath-Jarrow-Morton model for the term structure and we determine the delta-hedge and the mean-variance hedge which is a quadratic hedge. The pricing formula and the hedging strategies are derived as closed-form expressions in terms of Fourier transforms.

1. INTRODUCTION

Nowadays models for the term structure of interest rates that are driven by the Brownian motion are widely used in practice. However serious shortcomings of those models, in particular concerning the smile effect, are well known. Therefore Eberlein and Kluge (2006) extended the Heath-Jarrow-Morton model of Heath et al. (1992), to a model in which the forward rate is driven by a time-inhomogeneous Lévy process. Kluge (2005) showed that such a model allows to reproduce the so-called smile surface.

In this article we will investigate pricing and hedging of swaptions, which are options on swaps. Our contribution to the pricing of these products, see e.g. Eberlein and Kluge (2006), consists in a compact representation of the price by using the Jamshidian decomposition, introduced in Jamshidian (1989) for a Vasiček model.

Further, we will focus on hedging strategies with investments in zero-coupon bonds. In particular we will examine delta-hedges, which are mainly used in practice and which make the portfolio risk-neutral for changes in the underlying of the claim.

In literature many other hedging approaches are discussed, such as utility minimization and quadratic strategies. These methods are based on quantifying the risk and minimizing it over a certain set of strategies. We will study the quadratic strategies that minimize the square of the difference

between the claim at maturity and the portfolio. This so-called mean-variance hedging strategy has the appealing advantage of being self-financing.
We will state here the main results. For the proofs we refer to Glau et al. (2010) and Vandaele (2010).

2. LEVY DRIVEN HEATH–JARROW–MORTON MODEL

We assume that the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is given. We give a short introduction of the model under consideration. Denote by $T^* > 0$ the fixed horizon date, this means that trading will only happen during the interval $[0, T^*]$. The value at time t of a zero-coupon bond paying 1 unit at maturity T is given by $B(t,T)$ and, of course, $B(T,T)$ equals 1. The forward interest rate at date $t \leq T$, $f(t,T)$, is the instantaneous risk-free interest rate for borrowing or lending at time T seen from time t. When we have a family of forward interest rates $f(t,T)$, we can easily derive the prices of the zero-coupon bonds: $B(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$. The short-term interest rate r_t is described by $f(t,t)$ and $B_t = \exp(\int_0^t r_u du)$ represents the savings account.
Denote by $\mathbb{F}$ the natural filtration generated by the one-dimensional time-(in)homogeneous Lévy process L with Lévy-Khintchine triplet (b_s, c_s, F_s) and associated cumulant θ_s such that

$$\int_0^{T^*} \left(|b_s| + |c_s| + \int_{\mathbb{R}} (x^2 \wedge 1) F_s(dx) \right) ds < \infty.$$

Then, the dynamics of the forward interest rates and the zero-coupon bonds under the measure $\mathbb{P}$ are given by

$$\begin{aligned} df(t,T) =& \alpha(t,T)dt - \sigma(t,T)dL_t \\ B(t,T) =& B(0,T)\exp\Big(\int_0^t (r_s - A(s,T))ds + \int_0^t \Sigma(s,T)dL_s\Big), \end{aligned} \tag{1}$$

with α, σ $\mathbb{R}$-valued adapted stochastic processes and, for s belonging to $[0, T^*]$,

$$A(s,T) = \int_{s\wedge T}^T \alpha(s,u)du \qquad \text{and} \qquad \Sigma(s,T) = \int_{s\wedge T}^T \sigma(s,u)du. \tag{2}$$

The dynamics of the zero-coupon bonds can be rewritten as:

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left(-\int_0^t (A(s,T) - A(s,t))ds + \int_0^t (\Sigma(s,T) - \Sigma(s,t))dL_s \right). \tag{3}$$

We assume the following integrability condition on the measures F_s to ensure in particular that L is an exponential special semimartingale:

Axiom 1 (EM) *There are constants $M, \epsilon > 0$ such that for every $u \in [-(1+\epsilon)M, (1+\epsilon)M]$:*

$$\int_0^{T^*} \int_{\{|x|>1\}} \exp(ux) F_s(dx) ds < \infty.$$

The discounted bond prices are martingales and, hence, the model is described under the unique martingale measure $\mathbb{P}^*$, also called the spot martingale measure, see Eberlein et al. (2005), if

$$A(s,T) = \theta_s(\Sigma(s,T)) \quad \forall T \in [0,T^*]. \tag{4}$$

Concerning the volatility structure we impose the following additional axioms.

Axiom 2 (DET) *The volatility structure σ is bounded and deterministic. Furthermore we assume*

$$0 \le \Sigma(s,T) \le M' < M \qquad \textit{for all } 0 \le s,\, T \le T^*$$

with M the constant defined in Axiom 1 and Σ given in (2).

Axiom 3 (VOL) *For all $T \in [0,T^*]$ we assume that $\sigma(\cdot\,,T) \not\equiv 0$ and*

$$\sigma(s,T) = \sigma_1(s)\sigma_2(T) \quad 0 \le s \le T,$$

where $\sigma_1 : [0,T^*] \to \mathbb{R}^+$ and $\sigma_2 : [0,T^*] \to \mathbb{R}^+$ are continuously differentiable. Furthermore we assume that $\inf_{s\in[0,T^*]} \sigma_1(s) \ge \underline{\sigma}_1 > 0$.

Important for pricing and hedging interest rate derivative products is the forward measure.

Definition 2.1 *The forward measure is linked with a settlement date T, such that the forward price of any financial asset (in our case any zero-coupon bond) is a (local) martingale. The forward price at time t of an asset S is given by $S_t/B(t,T)$.*

The change of measure from the spot martingale measure $\mathbb{P}^*$, which equals $\mathbb{P}$ in our setting, to the forward martingale measure linked with the settlement date T is according to (1)

$$\left.\frac{d\mathbb{P}_T}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{B(t,T)}{B_t B(0,T)} = \exp\left(-\int_0^t A(s,T)ds + \int_0^t \Sigma(s,T)dL_s\right). \tag{5}$$

From Proposition 10 and Lemma 11 of Eberlein and Kluge (2006) we conclude that L is also a time-inhomogeneous Lévy process under the forward measure $\mathbb{P}_T$, that L is again special under this measure and its characteristics $(b_s^{\mathbb{P}_T}, c_s^{\mathbb{P}_T}, F_s^{\mathbb{P}_T})$ under $\mathbb{P}_T$ can be expressed in terms of those under $\mathbb{P}$.

3. PAYER SWAPTION

A (plain vanilla) interest rate swap is a contract to exchange a fixed interest rate against a floating reference rate, like the Libor. Both rates are based on the same notional amount and for the same period of time. In the case of a payer swap the investor pays the fixed rate and receives the floating rate, where the fixed rate is chosen such that the contract is worth zero at the initial date.
A forward swap is an agreement to enter into a swap at a future date T_0 with a pre-specified fixed rate κ, while a payer swaption gives the owner the right to enter the forward payer swap at T_0. Musiela and Rutkowski (2004) showed that the payer swaption can be seen as a put option with strike price 1 on a coupon-bearing bond. Therefore we can write the payer swaption's payoff as $(1 - \sum_{j=1}^n c_j B(T_0,T_j))^+$, where $T_1 < T_2 < \ldots < T_n$ are the payment dates of the swap with $T_1 > T_0$. We denote the length of the accrual periods $[T_{j-1},T_j]$, $j = 1,\ldots,n$ by $\delta_j := T_j - T_{j-1}$. The coupons c_i equal $c_i = \kappa\delta_i$ for $i = 1,\ldots,n-1$ and $c_n = 1 + \kappa\delta_n$ where κ is the fixed interest rate of the swap.

4. PRICING OF THE PAYER SWAPTION

We present the derivation for the payer swaption in a slightly different way than in Eberlein and Kluge (2006) to make the application of the so called Jamshidian trick more visible. This allows to interpret the payer swaption as a weighted sum of put options with different strikes on bonds with different maturities. This was already noticed by Annaert et al. (2007) in a continuous setting for a general interest rate model where the zero-coupon bond prices are comonotonic.
The fair price of the payer swaption is given by $\mathrm{PS}_t = B_t E[\frac{1}{B_{T_0}}(1-\sum_{j=1}^n c_j B(T_0,T_j))^+|\mathcal{F}_t]$, where the expectation is taken under the risk-neutral measure $\mathbb{P}$. We change to the forward measure $\mathbb{P}_{T_0}$ eliminating the instantaneous interest rate B_{T_0} under the expectation in this way:

$$\mathrm{PS}_t = B(t,T_0) E^{\mathbb{P}_{T_0}}\Big[(1-\sum_{j=1}^n c_j B(T_0,T_j))^+\Big|\mathcal{F}_t\Big] = B(t,T_0) E^{\mathbb{P}_{T_0}}\Big[(1-\sum_{j=1}^n c_j \widetilde{D}_{T_0}^{T_j} e^{\widetilde{\Sigma}_{T_0}^{T_j} X_{T_0}})^+\Big|\mathcal{F}_t\Big]$$

with according to (2), (3), (4) and Axiom 3

$$\widetilde{D}_{T_0}^{T_j} = \frac{B(0,T_j)}{B(0,T_0)} \exp\left(\int_0^{T_0} [\theta_s(\Sigma(s,T_0)) - \theta_s(\Sigma(s,T_j))]\, ds\right)$$
$$\widetilde{\Sigma}_{T_0}^{T_j} = \int_{T_0}^{T_j} \sigma_2(u)du \quad \text{and} \quad X_{T_0} = \int_0^{T_0} \sigma_1(s) dL_s.$$

Theorem 4.1 *Under the Axioms 1, 2, 3 and if $|\sigma_1| < \overline{\sigma}_1$ for a certain $\overline{\sigma}_1 \in \mathbb{R}$, the price at time t of a forward payer swaption is given by a weighted sum of put options on bonds*

$$\begin{aligned} PS_t &= B(t,T_0)\sum_{j=1}^n c_j E^{\mathbb{P}_{T_0}}\big[(b_j - B(T_0,T_j))^+\big|\mathcal{F}_t\big] \\ &= B(t,T_0)\sum_{j=1}^n c_j \frac{e^{-RX_t}}{2\pi}\int_{\mathbb{R}} e^{iuX_t}\varphi_{X_{T_0}-X_t}^{\mathbb{P}_{T_0}}(u+iR)\hat{v}^j(-u-iR)du, \end{aligned} \tag{6}$$

where $\varphi_{X_{T_0}-X_t}^{\mathbb{P}_{T_0}}$ is the characteristic function of $X_{T_0}-X_t$ under the measure $\mathbb{P}_{T_0}$ and where

$$\hat{v}^j(-u-iR) = \frac{b_j e^{(-iu+R)z^* \widetilde{\Sigma}_{T_0}^{T_j}}}{(-iu+R)(-iu+\widetilde{\Sigma}_{T_0}^{T_j}+R)}$$

for R in $]0,\frac{M}{\overline{\sigma}_1}]$ and b_j such that $g(T_0,T_j,z^*) = b_j$, where z^* is the solution to the equation $\sum_{j=1}^n c_j g(T_0,T_j,z^*) = 1$ with g the non-decreasing function defined as $g(s,t,x) = \widetilde{D}_s^t e^{\widetilde{\Sigma}_s^t x}$.

5. DELTA-HEDGING OF THE PAYER SWAPTION

We determine the delta-hedge for a short position in the payer swaption when one zero-coupon bond is used for hedging.

Theorem 5.1 *Under the axioms of Theorem 4.1 and if* $|u|\cdot|\varphi^{\mathbb{P}_{T_0}}_{X_{T_0}-X_t}(u+iR)|$ *is integrable then the optimal amount, denoted by* Δ^j_t*, to invest in the zero-coupon bond with maturity* T_j *to delta-hedge a short position in the forward payer swaption is given by:*

$$\Delta^j_t = -\frac{B(t,T_0)}{B(t,T_j)\widetilde{\Sigma}^{T_j}_t}\sum_{k=1}^{n} c_k\Big(\widetilde{\Sigma}^{T_0}_t H^k(t,X_t) + \frac{\partial}{\partial X_t}H^k(t,X_t)\Big),$$

with

$$\begin{aligned} H^k(t,X_t) =& \frac{e^{-RX_t}}{2\pi}\int_{\mathbb{R}} e^{iuX_t}\varphi^{\mathbb{P}_{T_0}}_{X_{T_0}-X_t}(u+iR)\hat{v}^k(-u-iR)du. \\ \frac{\partial H^k(t,X_t)}{\partial X_t} =& \frac{1}{2\pi}\int_{\mathbb{R}} e^{(-R+iu)X_t}\varphi^{\mathbb{P}_{T_0}}_{X_{T_0}-X_t}(u+iR)\hat{v}^k(-u-iR)(-R+iu)du. \end{aligned}$$

The integrability condition of Theorem 5.1 is satisfied for a wide range of processes, see Glau et al. (2010).

6. MEAN-VARIANCE HEDGING STRATEGY FOR THE PAYER SWAPTION

In the interest rate derivatives market it is unrealistic to hedge with the risk-free interest rate product in contrast to the stock market. Therefore we choose the bond $B(\cdot,T_0)$ as numéraire to develop a hedging strategy for the payer swaption.

For the explicit determination of the strategy in terms of the cumulant process we use ideas of Hubalek et al. (2006) adapted to our setting. They determine the variance-optimal hedging strategy for an exponential Lévy process, which is not necessarily a martingale. We work under the forward measure $\mathbb{P}_{T_0}$ which ensures that the discounted asset $B(\cdot,T_j)/B(\cdot,T_0)$ is a martingale. Hence determining the strategy reduces to finding the Galtchouk-Kunita-Watanabe decomposition of the claim H.

On the other hand Hubalek et al. use time-homogeneous processes, while the driving process in our setting is a time-inhomogeneous Lévy process. Instead of generalizing the argumentation given in Hubalek et al. (2006), we give a new proof in Glau et al. (2010) based on properties of the cumulant process of time-inhomogeneous Lévy processes and the Galtchouk-Kunita-Watanabe decomposition.

Following a mean-variance hedging strategy we determine the optimal number we have to invest in the zero-coupon bond with maturity T_j to hedge the forward swaption:

Theorem 6.1 *Under the assumptions of Theorem 4.1 and if* $3M' \leq M$ *and if* R *is chosen in the interval* $]0,\frac{M}{2\sigma_1}]$ *then the Galtchouk-Kunita-Watanabe decomposition of the forward payer swaption* (6) *exists. The optimal number to invest in the zero-coupon bond with maturity* T_j *is according to the mean-variance hedging strategy given by*

$$\xi^j_t = \int_{\mathbb{R}} \exp\left(\int_t^{T_0}\kappa^{\widetilde{X}^j}_s\Big(\frac{iu-R}{\widetilde{\Sigma}^{T_j}_{T_0}}\Big)ds\right)\widetilde{B}(t-,T_j)^{\frac{iu-R}{\widetilde{\Sigma}^{T_j}_{T_0}}-1}\frac{\kappa^{\widetilde{X}^j}_t\left(\frac{iu-R}{\widetilde{\Sigma}^{T_j}_{T_0}}+1\right)-\kappa^{\widetilde{X}^j}_t\left(\frac{iu-R}{\widetilde{\Sigma}^{T_j}_{T_0}}\right)}{\kappa^{\widetilde{X}^j}_t(2)}\Pi(du),$$

with

$$
\begin{aligned}
\kappa_s^{\widetilde{X}^j}(w) &= \theta_s(w\Sigma(s,T_j) + (1-w)\Sigma(s,T_0)) - w\theta_s(\Sigma(s,T_j)) - (1-w)\theta_s(\Sigma(s,T_0)),\\
\Pi(du) &= \sum_{k=1}^{n} \frac{c_k}{2\pi} (f_{T_0}^j)^{\frac{iu-R}{\widetilde{\Sigma}_{T_0}^{T_j}}} \hat{v}^k(-u-iR)du,\\
f_{T_0}^j &= \frac{B(0,T_0)}{B(0,T_j)} \exp\left(\int_0^{T_0} [\theta_s(\Sigma(s,T_j)) - \theta_s(\Sigma(s,T_0))]ds\right).
\end{aligned}
$$

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OPTIMAL TRADING STRATEGIES AND THE BESSEL PROCESS

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It is shown that delta hedging provides the optimal trading strategy in terms of minimal required initial capital to replicate a given terminal payoff in a continuous-time Markovian context. This holds true in market models where no equivalent local martingale measure exists but only a square-integrable market price of risk. A new probability measure is constructed, which takes the place of an equivalent local martingale measure. In order to ensure the existence of the delta hedge, sufficient conditions are derived for the necessary differentiability of expectations indexed over the initial market configuration. For a precise statement of the assumptions, proofs of the statements, further references and results we refer to Ruf (2010).

1. STOCK PRICE MODEL AND WEALTH PROCESSES

We use the notation $\mathbb{R}_+^n := \{s = \begin{pmatrix} s_1 & \cdots & s_n \end{pmatrix}^\mathsf{T} \in \mathbb{R}^n, s_i > 0, \text{ for all } i = 1, \ldots, n\}$, fix a time horizon T and assume a market where the stock price processes are modelled as positive continuous Markovian semimartingales. That is, we consider a financial market $S(\cdot) = \begin{pmatrix} S_1(\cdot) & \cdots & S_n(\cdot) \end{pmatrix}^\mathsf{T}$ of the form

$$dS_i(t) = S_i(t) \left(\mu_i(t, S(t))dt + \sum_{k=1}^{n} \sigma_{i,k}(t, S(t))dW_k(t) \right) \tag{1}$$

for all $i = 1, \ldots, n$ and $t \in [0, T]$ starting at $S(0) \in \mathbb{R}_+^n$ and a money market $B(\cdot)$. Here $\mu : [0, T] \times \mathbb{R}_+^n \to \mathbb{R}^n$ denotes the mean rate of return and $\sigma : [0, T] \times \mathbb{R}_+^n \to \mathbb{R}^{n \times n}$ the volatility. Both functions are assumed to be measurable. For the sake of convenience we only look at discounted (forward) prices and set the interest rates constant to zero, that is, $B(\cdot) \equiv 1$. The flow of information is modelled by a right-continuous filtration $\mathcal{F}(\cdot)$ such that $W(\cdot) = \begin{pmatrix} W_1(\cdot) & \cdots & W_n(\cdot) \end{pmatrix}^\mathsf{T}$ is an n-dimensional Brownian motion with independent components. We only consider mean rates of return μ and volatilities σ which imply that the stock prices $S_1(\cdot), \ldots, S_n(\cdot)$ exist and are unique

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and strictly positive. We denote by $a(\cdot,\cdot) = \sigma(\cdot,\cdot)\sigma^{\mathsf{T}}(\cdot,\cdot)$ the covariance process of the stocks in the market.

Furthermore, we assume here that $\sigma(t, S(t))$ is invertible for all $t \in [0,T]$ and that the market price of risk

$$\theta(t, S(t)) := \sigma^{-1}(t, S(t))\mu(t, S(t))$$

satisfies the integrability condition $\int_0^T \|\theta(t, S(t))\|^2 dt < \infty$ almost surely.

Based upon the market price of risk, we are now ready to define the stochastic discount factor as

$$Z^\theta(t) := \exp\left(-\int_0^t \theta^{\mathsf{T}}(u, S(u))dW(u) - \frac{1}{2}\int_0^t \|\theta(u, S(u))\|^2 du\right)$$

for all $t \in [0,T]$. In classical no-arbitrage theory, $Z^\theta(\cdot)$ represents the Radon-Nikodym derivative which translates the "real-world" measure into the generic "risk-neutral" measure with the money market as the underlying. Since in this work we explicitly want to allow a "Free Lunch with Vanishing Risk", we shall not assume that the stochastic discount factor $Z^\theta(\cdot)$ is a true martingale. Thus, we can only rely on a local martingale property of $Z^\theta(\cdot)$.

We denote the number of shares held by an investor with initial capital $v > 0$ at time t by $\eta(t) = \begin{pmatrix}\eta_1(t) & \cdots & \eta_n(t)\end{pmatrix}^{\mathsf{T}}$ and the associated wealth process by $V^{v,\eta}(\cdot)$. To wit,

$$dV^{v,\eta}(t) = \sum_{i=1}^n \eta_i(t) dS_i(t)$$

for all $t \in [0,T]$. We call η a *trading strategy* or in short, a *strategy*. To ensure that $V^{v,\eta}(\cdot)$ is well-defined and to exclude doubling strategies we restrict ourselves to progressively measurable trading strategies which satisfy $V^{1,\eta}(t) \geq 0$ for all $t \in [0,T]$.

If Y is a nonnegative $\mathcal{F}(T)$-measurable random variable such that $\mathbb{E}[Y|\mathcal{F}(t)]$ is a function of $S(t)$ for all $t \in [0,T]$, we use the Markovian structure of $S(\cdot)$ to denote conditioning on the event $\{S(t) = s\}$ by $\mathbb{E}^{t,s}[Y]$.

2. HEDGING

In the following, we shall call $(t,s) \in [0,T] \times \mathbb{R}_+^n$ a *point of support for* $S(\cdot)$ if there exists some $\omega \in \Omega$ such that $S(t,\omega) = s$. We define for any measurable function $p : \mathbb{R}_+^n \to [0,\infty)$ a candidate $h^p : [0,T] \times \mathbb{R}_+^n \to [0,\infty)$ for the hedging price of the corresponding European option:

$$h^p(t,s) := \mathbb{E}^{t,s}\left[\frac{Z^\theta(T)}{Z^\theta(t)} p(S(T))\right]. \tag{2}$$

Equation (2) has appeared as the "real-world pricing formula" in the Benchmark approach, compare Platen and Heath (2006), Equation (9.1.30). Applying Itô's rule to Equation (2) yields the following result. Here we write D_i and $D_{i,j}^2$ for the partial derivatives with respect to the variable s.

Theorem 2.1 (Markovian representation for non path-dependent European claims) *Assume that we have a contingent claim of the form $p(S(T)) \geq 0$ and that the function h^p of Equation (2) is sufficiently differentiable, or more precisely, for all points of support (t, s) for $S(\cdot)$ we have $h^p \in C^{1,2}(\mathcal{U}_{t,s})$ for some neighborhood $\mathcal{U}_{t,s}$ of (t, s). Then, with $\eta_i^p(t, s) := D_i h^p(t, s)$, for all $i = 1, \ldots, n$ and $(t, s) \in [0, T] \times \mathbb{R}_+^n$, and with $v^p := h^p(0, S(0))$, we get*

$$V^{v^p,\eta^p}(t) = h^p(t, S(t))$$

for all $t \in [0, T]$. The strategy η^p is optimal in the sense that for any $\tilde{v} > 0$ and for any strategy $\tilde{\eta}$ whose associated wealth process is nonnegative and satisfies $V^{\tilde{v},\tilde{\eta}}(T) \geq p(S(T))$, we have $\tilde{v} \geq v^p$. Furthermore, h^p satisfies the PDE

$$\frac{\partial}{\partial t} h^p(t, s) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j a_{i,j}(t, s) D_{i,j}^2 h^p(t, s) = 0 \tag{3}$$

at all points of support (t, s) for $S(\cdot)$.

Next, we will provide sufficient conditions under which the function h^p is sufficiently smooth. For that we need the following definition.

Definition 2.1 (Locally Lipschitz and bounded) *We call a function $f : [0, T] \times \mathbb{R}_+^n \to \mathbb{R}$ locally Lipschitz and bounded on $\mathbb{R}_+^n$ if for all $s \in \mathbb{R}_+^n$ the function $t \to f(t, s)$ is right-continuous with left limits and for all $M > 0$ there exists some $C(M) < \infty$ such that*

$$\sup_{\substack{\frac{1}{M} \leq \|y\|, \|z\| \leq M \\ y \neq z}} \frac{|f(t, y) - f(t, z)|}{\|y - z\|} + \sup_{\frac{1}{M} \leq \|y\| \leq M} |f(t, y)| \leq C(M),$$

for all $t \in [0, T]$.

Using the theory of stochastic flows and Schauder estimates, we obtain the necessary differentiability of h^p.

Theorem 2.2 *We assume that the functions θ_k and $\sigma_{i,k}$ are for all $i, k = 1, \ldots, n$ locally Lipschitz and bounded. We furthermore assume that for all points of support (t, s) for $S(\cdot)$ there exist $C_1, C_2 > 0$ and some neighborhood $\mathcal{U}$ of (t, s) such that $\sum_{i,j=1}^{n} a_{i,j}(u, y)\xi_i \xi_j > C_1 \|\xi\|^2$ for all $\xi \in \mathbb{R}^n$ and $h^p(u, y) \leq C_2$ for all $(u, y) \in \mathcal{U}$. Then, there exists for all points of support (t, s) for $S(\cdot)$ some neighborhood $\tilde{\mathcal{U}}$ of (t, s) such that the function h^p defined in Equation (2) is in $C^{1,2}(\tilde{\mathcal{U}})$.*

3. CHANGE OF MEASURE

To simplify the computation of h^p, one can perform a change of measure after making some technical assumptions. For that, we rely on the techniques developed by Föllmer (1972), Meyer (1972), and Delbaen and Schachermayer (1995), Section 2.

Theorem 3.1 (Generalized change of measure) *There exists a measure $\mathbb{Q}$ such that for all $\mathcal{F}(T)$-measurable random variables $Y \geq 0$ we have*

$$\mathbb{E}^{\mathbb{P}}\left[Z^{\theta}(T)Y\right] = \mathbb{E}^{\mathbb{Q}}\left[Y\mathbf{1}_{\{1/Z^{\theta}(T)>0\}}\right]$$

where $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation with respect to the new measure $\mathbb{Q}$. That is, $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$. Under this measure $\mathbb{Q}$, the process $\widetilde{W}(\cdot) = \left(\widetilde{W}_1(\cdot) \quad \cdots \quad \widetilde{W}_n(\cdot)\right)^{\mathsf{T}}$ with

$$\widetilde{W}_k(t \wedge \tau^{\theta}) := W_k(t \wedge \tau^{\theta}) + \int_0^{t\wedge\tau^{\theta}} \theta_k(u, S(u))du$$

for all $k = 1, \ldots, n$ and $t \in [0, T]$ is an n-dimensional Brownian motion stopped at time $\tau^{\theta} := \lim_{i\to\infty} \inf\{t \in [0, T] : Z^{\theta}(t) \geq i\}$.

Furthermore, it is now easy to show that we have, up to the stopping time τ^{θ}, the following dynamics for $S(\cdot)$ and $1/Z^{\theta}(\cdot)$ under $\mathbb{Q}$:

$$dS_i(t) = S_i(t)\sum_{k=1}^{n} \sigma_{i,k}(t, S(t))d\widetilde{W}_k(t),$$
$$d\left(\frac{1}{Z^{\theta}(t)}\right) = \frac{1}{Z^{\theta}(t)}\sum_{k=1}^{n} \theta_k(t, S(t))d\widetilde{W}_k(t),$$

for all $i = 1, \ldots, n$ and $t \in [0, T]$. One can also prove a generalization of Bayes' rule for Girsanov-type measure changes to the measure change suggested by Theorem 3.1.

4. THREE-DIMENSIONAL BESSEL PROCESS

We illustrate the techniques presented here with a toy model. Let $n = 1$ and $S(\cdot)$ be a three-dimensional Bessel process. To wit,

$$dS(t) = \frac{1}{S(t)}dt + dW(t)$$

for all $t \in [0, T]$. For any payoff function $p(\cdot) \geq 0$ we obtain from Theorem 3.1 that $h^p(t, s) := \mathbb{E}^{\mathbb{Q},t,s}[p(S(T))\mathbf{1}_{\{S(T)>0\}}]$, where $S(\cdot)$ is now a $\mathbb{Q}$-Brownian motion stopped at zero. For example, if $p(s) \equiv s$, that is, the stock itself, then $h^p(t, s) = \mathbb{E}^{\mathbb{Q},t,s}[S(T)] = s$. To wit, the hedging price of the stock is exactly its price and the optimal strategy is to hold the stock. However, if $p(s) \equiv 1$, then we compute

$$h^p(t, s) = \mathbb{Q}^{t,s}(S(T) > 0) = 2\Phi\left(\frac{s}{\sqrt{T-t}}\right) - 1 < 1.$$

There is a trading strategy η^p, which yields exactly one monetary unit at time T and costs $h^p(0, s)$ at time 0 if the stock price equals s. By Theorem 2.1, there is no other strategy which needs less initial capital and leads to a nonnegative wealth process. Furthermore, we have the representation

$$\eta^p(t,s) = \frac{2}{\sqrt{T-t}}\phi\left(\frac{s}{\sqrt{T-t}}\right),$$

where ϕ denotes the standard normal density function.

5. FURTHER RESULTS AND A VERY INCOMPLETE LIST OF REFERENCES

The hedging results of Theorem 2.1 also hold when the number of Brownian motions is larger than the number of stocks. However, in this case one has to pay attention to the choice of the market price of risk, which is no longer unique. The PDE (3) usually allows for several solutions satisfying the same boundary conditions and being of polynomial growth. The function h^p can be characterized as the minimal nonnegative solution of that PDE.

This work is motivated by the desire to better understand the question of hedging in stochastic portfolio theory and in the Benchmark process. For an overview of the former, we recommend the survey paper by Fernholz and Karatzas (2009). Furthermore, in Fernholz and Karatzas (2010) optimal trading strategies to hold the market portfolio at time horizon T are discussed. For an introduction to the Benchmark process, developed by Eckhard Platen and co-authors, we refer to the monograph by Platen and Heath (2006). In particular, Theorem 2.1 generalizes Platen and Hulley (2008), Proposition 3, where the same statement is shown for a one-dimensional market with a time-transformed squared Bessel process of dimension four modelling the stock price process.

The results presented here also yield optimal trading strategies for models where the stock price has a bubble. A stock is said to have a bubble if its price does not equal its "intrinsic value". We refer to Jarrow et al. (2007) for a precise definition and further references.

Theorem 2.2 generalizes recent Feynman-Kac type theorems by Heath and Schweizer (2000), Janson and Tysk (2006), and Ekström and Tysk (2009) for the stock price models presented here.

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THE TERM STRUCTURE OF INTEREST RATES AND IMPLIED INFLATION

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The yield-curve models developed by macroeconomists and financial economists are very different in form and fit due to particular demands and different motives. While macroeconomists focus on the role of expectations of inflation and future real economic activity in the determination of yields, financial economists avoid any explicit role for such determinants. These different attitudes cause a gap between the yield curve models developed. A joint macro-finance modelling strategy would provide the most comprehensive understanding of the term structure of interest rates (Diebold et al. 2005). There are various recent papers which aim to bridge this gap by formulating and estimating a yield curve model that integrates macroeconomic and financial factors (Dewachter and Lyrio (2006), Diebold et al. (2006), Diebold and Li (2006), Diebold et al. (2007), Lildholdt et al. (2007), Ang et al. (2008)).

The starting point of most of the macro-finance models is the short-term interest rates. Short-term interest rates have different meanings from a macroeconomic perspective and a finance perspective. From a macroeconomic perspective, the short-term interest rate is a policy instrument directly controlled by the central bank to achieve its economic stabilization goals. From a finance perspective, the short rate is a fundamental building block for yields of other maturities, which are just risk-adjusted averages of expected future short rates. Focusing on short-term interest rates to construct a yield-macro model is consistent with the Taylor (1993) policy rules. The Taylor rule is a monetary-policy rule that stipulates how much the central bank should change the nominal interest rate in response to divergences of actual gross domestic product (GDP) from potential GDP and of actual inflation rates from target inflation rates.

This study aims to model the UK term structures of interest rates and the term structure of implied inflation simultaneously using the additional macroeconomic variables in a way that is consistent with macroeconomic theory. To construct such a model which we will call 'yield-macro' model, we use nominal government spot rates extracted from the conventional gilt market and real and implied inflation spot rates extracted from the index-linked gilt market by the Bank of England. We use all available maturities i.e. 50 different maturities for nominal rates (starting from 6 month and ending with 25 years) and 46 maturities for real rates and implied inflation (starting from 2.5 years and ending with 25 years). As for the macroeconomic variables we use annual realized inflation obtained from the UK Retail Price Index and the output gap ($= \frac{\text{actual GDP} - \text{potential GDP}}{\text{potential GDP}}$) provided by the OECD Economic Outlook publications.

1. FITTING THE CAIRNS MODEL ON YIELD CURVES

In order to fill in the gaps in the data we fit the Cairns (1997) model described in Equation (1) below before applying any analysis to the term structures. The curve is a flexible model with four exponential terms and nine parameters in total. However, four of these parameters (the exponential rates) are fixed which reduces the risk of multiple solutions. If the value of c_i where $i = 1, 2, 3, 4$ is small then the relevant value of b_i affects all durations whereas if c_i is large then the relevant value of b_i only affects the shortest durations.

$$R(t, t+s) = b_0(t) + b_1(t)\frac{1-e^{-c_1 s}}{c_1 s} + b_2(t)\frac{1-e^{-c_2 s}}{c_2 s} + b_3(t)\frac{1-e^{-c_3 s}}{c_3 s} + b_4(t)\frac{1-e^{-c_4 s}}{c_4 s} \quad (1)$$

where $R(t, t+s)$ is the spot rate at time t whose maturity is $t+s$. We fit the Cairns model on to the daily nominal spot rates to decide the best set of exponential parameters ($C = (0.1, 0.2, 0.4, 0.8)$) for the nominal yield curve and use this set of parameters for the other yield curves (implied inflation and real spot rates) as well. We use fitted curves to construct the yield-macro model.

2. PRINCIPAL COMPONENT ANALYSIS ON YIELD CURVES

Principal component analysis (PCA) attempts to describe the behaviour of a range of correlated random variables (in this case, the various spot yields for different times to maturity) in terms of a small number of uncorrelated principal components. This approach was first applied to bond yields by Litterman and Scheinkman (1991), who found three common factors called 'level', 'slope' and 'curvature' which influenced the returns on all treasury bonds.

To construct a yield-macro model, we decreased the dimension of the yield curves by applying PCA on quarterly values of nominal, real and implied inflation spot rates for the period 1995 and 2007. Therefore, we extract the three most important components of these yield curves. Figure 1 shows the loadings of these principal components. According to the PCA, the first three factors explain more than 99.9% of the variability for each yield curve.

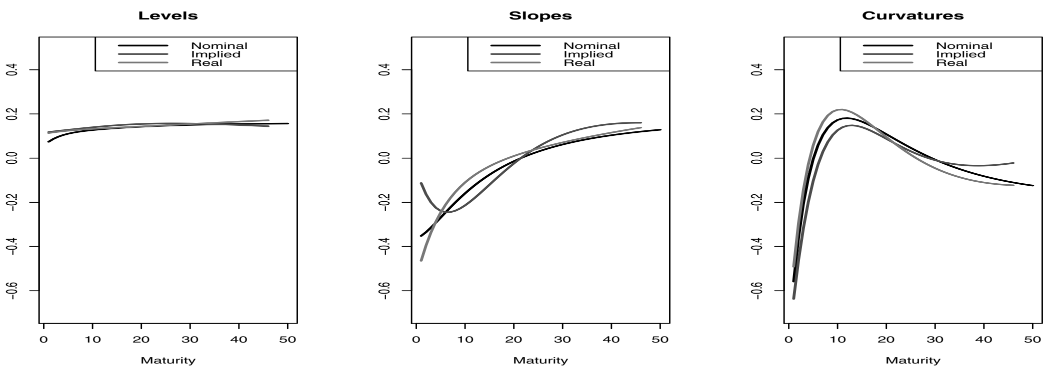

Figure 1: Loadings of the PCs

3. YIELD-MACRO MODEL

Once we obtained the principal components (PCs) of the three yield curves, we examined the correlations between these components and the macroeconomic variables. We have found significant auto- and cross-correlations between the variables.

All variables have exponentially decreasing auto-correlation functions which indicate AR processes. There are high simultaneous and lagged correlations between the levels, slopes and curvatures factors. The level factors of the yield curves are negatively correlated with the output gap which is consistent with the macroeconomic theory that explains the link between the goods market and financial markets. Accordingly, an increase in interest rates lowers the investment and thus reduces the output. On the other hand, the only yield curve factor which has significant correlation with the realized inflation is the nominal curvature factor.

After examining the correlations between the variables we fitted a vector autoregressive model (VAR). We aimed for parsimony through a systematic analysis of a wide range of models and used only the first two lags of the variables to fit the VAR model.

To construct the 'yield-macro' model, we used quarterly nominal spot rates, implied inflation spot rates, real spot rates, annual realized inflation and output gap data over the period 1995 to 2007 [1].

Let $Y(52 \times 11)$ be the matrix of the PCs of the yield curves and the macroeconomic variables where N, I, R, RI and OG are abbreviations for nominal spot rates, implied inflation spot rates, real spot rates, realized inflation rates and output gap, and L, S and C are abbreviations for level, slope and curvature respectively.

The VAR structure of the 'yield-macro' model is as below:

$$Y[t] - \mu_Y = A_1 \ (Y[t-1] - \mu_Y) + A_2 \ (Y[t-2] - \mu_Y) + \epsilon[t]$$

where:
μ_Y is the matrix of long run means of the variables, A_1 and A_2 are the coefficient matrices for the first and second lags of the explanatory variables respectively and $\epsilon[t] \sim N(0, \Sigma)$, i.e. normally distributed residuals with zero mean and Σ variance-covariance matrix.

$$Y^t = \left[\begin{array}{ccccccccccc} Y_{N_L} & Y_{N_S} & Y_{N_C} & Y_{I_L} & Y_{I_S} & Y_{I_C} & Y_{R_L} & Y_{R_S} & Y_{R_C} & Y_{RI} & Y_{OG} \end{array} \right]$$

$$\mu_Y^t = \left[\begin{array}{ccccccccccc} -6.76 & 0 & 0 & -1.47 & 0 & 0 & -6.99 & 0 & 0 & 2.88 & 0 \end{array} \right]$$

[1] Since the output gap data are subject to continuous revision which may take three years to get the latest estimate, the data period in this modelling work is restricted by 2007.

$$A_1 = \begin{bmatrix} 0.92 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.78 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.96 & 0 & 0 & 0 & 0 & 0 & 0 & -0.15 & 0 \\ 0 & 0 & 0 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.56 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.62 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.95 & 0 & 0 & 0 & 0 \\ 0 & 0.27 & 0 & 0 & 0 & 0 & 0 & 0.49 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.86 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.92 & 0 \\ 0 & -0.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.89 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.21 & 0 & -0.41 \\ 0 & 0 & -0.34 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.32 & 0 & 0 & 0 & 1.38 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.09 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.34 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 4.54 & & & & & & & & & & \\ -0.62 & 0.76 & & & & & & & & & \\ -0.50 & 0.07 & 0.14 & & & & & & & & \\ 2.51 & -0.22 & -0.30 & 2.28 & & & & & & & \\ -0.17 & 0.25 & 0.07 & -0.18 & 0.27 & & & & & & \\ 0.09 & -0.11 & 0.01 & 0.07 & -0.05 & 0.08 & & & & & \\ 1.88 & -0.30 & -0.19 & 0.20 & 0.05 & 0.00 & 1.61 & & & & \\ -0.53 & 0.30 & 0.05 & -0.10 & -0.02 & 0.02 & -0.40 & 0.29 & & & \\ -0.20 & 0.06 & 0.03 & -0.02 & -0.02 & -0.02 & -0.17 & 0.05 & 0.05 & & \\ 0.07 & 0.10 & -0.05 & 0.07 & 0.03 & -0.01 & 0.02 & 0.02 & -0.01 & 0.17 & \\ -0.03 & 0.03 & 0.01 & 0.04 & 0.01 & 0.01 & -0.06 & 0.02 & 0.01 & 0.02 & 0.06 \end{bmatrix}$$

As A_1 and A_2 indicate, the level factors and the curvature factor of real interest rates can be modelled as AR(1) processes. The diagonal structure of A_1 and the few values in A_2 can be explained by the existence of the high autocorrelations in the variables.

4. TESTING THE YIELD-MACRO MODEL

To compare the goodness of fit of our models with a random walk (RW) and an autoregressive process of order one (AR(1)) we calculate the following ratios.

$$R^2_{\mathrm{RW}^*} = 1 - \frac{SS_{\mathrm{model}}}{SS_{\mathrm{RW}}}\frac{df_{\mathrm{RW}}}{df_{\mathrm{model}}} \qquad R^2_{\mathrm{AR}(1)^*} = 1 - \frac{SS_{\mathrm{model}}}{SS_{\mathrm{AR}(1)}}\frac{df_{\mathrm{AR}(1)}}{df_{\mathrm{model}}}$$

Models	$R^2_{\mathrm{RW}^*}$	$R^2_{\mathrm{AR}(1)^*}$
Nominal Spot Rates		
Level	0.16	0
Slope	0.31	0.26
Curvature	0.30	0.20
Implied Inflation		
Level	0.13	0
Slope	0.51	0.43
Curvature	0.24	0.10
Real Spot Rates		
Level	0.10	0
Slope	0.26	0.21
Curvature	0.07	0
Realized Inflation	0.25	0.19
Output Gap	0.14	0.08

Table 1: Model Comparison

Zeros in the third column of the above table indicate that the fitted models are already AR(1). According to the Table 1, nominal spot rate models explain a significant amount of variability compared with the RW and AR(1) models. Implied inflation slope model improves the explained variability by about 51% and 43% comparing with the RW and AR(1) respectively. On the other hand, the real slope model shows a significant improvement while real level and curvature do not. It is also seen that the realized inflation model performs better than the RW and AR(1) when it includes the nominal curvature and output gap lagged values as explanatory variables. Finally, the output gap model performs slightly better than the RW and AR(1) with the help of the nominal slope factor as an explanatory variable.

We also test the one quarter ahead forecasts obtained by fitting the VAR model to show that the Fisher relation [2] is still satisfied. This enables us to decrease the number of models and derive one of the yield curves by modelling the other two.

[2]The Fisher relation indicates that the nominal interest rates can be decomposed into two parts: real rates and expected future inflation.

5. CONCLUSIONS

In this work, the three term structures (nominal, implied inflation and real spot rates) and two macroeconomic variables (realized inflation and output gap) have been modelled by considering the bidirectional relations between the two sets of variables. The correlations between the variables show that the yield curve factors highly depend on their lagged values and the other yield curves' corresponding factors as well as the output gap. A parsimonious VAR model has been fitted to the data to forecast the yield curves simultaneously. The VAR model has performed better than the RW and AR(1) processes in terms of the explained variability in the data. One quarter ahead forecasts are well within the 95% confidence limits for all yield curves and maturities. Furthermore, the model is consistent with the Fisher relation.

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De handelingen van deze contactfora vormen een aparte publicatiereeks van de Academie.

Contactforum "Actuarial and Financial Mathematics Conference" (4-5 februari 2010, Prof. M. Vanmaele)

De "Actuarial and Financial Mathematics Conference 2010" was de 8ste editie van het "Actuarial and Financial Mathematics" contactforum dat ondertussen zijn plaats veroverd heeft tussen de internationale conferenties die focussen op de wisselwerking tussen financieel en actuarieel wiskundige technieken. De eerste dag was opgebouwd rond het thema *Market Consistent Valuation* met voordrachten door gerenommeerde internationale sprekers gevolgd door een leerrijke en levendige paneldiscussie. Op dag twee mochten we naast genodigde sprekers ook sprekers uit de praktijk verwelkomen met onderwerpen zoals Solvency II en spread opties. Gedurende beide dagen was er ook een postersessie tijdens de welke jonge onderzoekers de mogelijkheid kregen om hun onderzoeksresultaten voor te stellen aan een ruim publiek bestaande uit academici uit binnen- en buitenland alsook collega's uit de bank- en verzekeringswereld. In deze publicatie vindt u een samenvatting van een van de voordrachten over *Market Consistent Valuation*. Verder zijn er bijdragen over het prijzen van afgeleide producten zoals *catastrophe bonds, spread options* en *swaptions*, over *optimal trading strategies, proportional reinsurance, strong Taylor approximation of stochastic differential equations* en *implied inflation*.