DIPLOMARBEIT

Environmental quality and education in an economic growth model with finitely lived agents

Ausgeführt am Institut für

Wirtschaftsmathematik
der Technischen Universität Wien

unter der Anleitung von Univ.Prof. Dipl.-Ing. Dr.techn. Alexia Fürnkranz-Prskawetz

durch

Patrick Cassidy

Hauptstraße 9, 7111 Parndorf

15. 12. 2011 ______________________________

Contents

1 Introduction **2**

2 The Blanchard-Yaari model **4**
- 2.1 Individual consumption . . . 5
- 2.2 Aggregate consumption . . . 7

3 Environmental quality in a Blanchard-Yaari model **10**
- 3.1 Individuals and households . . . 11
- 3.2 The firm sector . . . 12
- 3.3 Market equilibrium . . . 15
- 3.4 The steady state equilibrium . . . 15
- 3.5 Equilibrium dynamics . . . 19

4 Human capital driven growth **29**
- 4.1 The Lucas model . . . 29
- 4.2 Introducing education to the Blanchard-Yaari model with environmental quality . . . 32
  - 4.2.1 Individuals and households . . . 33
  - 4.2.2 The firm sector . . . 36
  - 4.2.3 Market equilibrium . . . 38
  - 4.2.4 The dynamic system and the BGP . . . 39
  - 4.2.5 Stability Analysis and Bifurcations . . . 44
- 4.3 Abatement technology and endogenous labour supply . . . 51
  - 4.3.1 Individuals and households . . . 51
  - 4.3.2 The firm sector . . . 53
  - 4.3.3 The general equilibrium . . . 56
  - 4.3.4 Stability analysis . . . 59
- 4.4 Human capital spillover in abatement technology . . . 61

5 Comparison of the models **67**

6 Summary **70**

7 Bibliography **73**

8 Appendix **74**

1 Introduction

The relationship between the three main topics of this thesis - environmental quality, education and economic growth - is of utmost importance in the modern world. With large cities suffocating in smog, hurricanes, earthquakes and other natural catastrophes becoming more and more frequent and climatic change picking up speed steadily, nearly every branch of science has come to occupy itself with environmental aspects such as pollution control or waste reduction. Obviously, environmental quality influences public welfare to a large extent, so it is natural that economics be one of the major disciplines of science to engage in research related to the environment.
Since Lucas (1988), human capital has become an essential component of modern growth theory. It is commonly accepted nowadays that the acquisition of human capital (by schooling or learning-by-doing, both of which we will refer to as education in this thesis) is one of the driving forces, if not THE driving force behind economic growth, but that's not the end of it. As Wolfgang Lutz put it in his speech at the Wittgenstein-Symposium in September 2011[1], "using brain power is a zero-emission way of accelerating technological change and wealth accumulation, thus investment in education is probably the best long-term strategy to counteract climatic change".

The interaction between environmental quality, education and growth constitutes an enormously large field of study, and surely a lot of research in this direction will be done in coming years. The aim of this thesis is to analyze these interactions in a very specific setting, by using "finite horizon" models, which are in principle models of overlapping generations in continuous time. The finite aspect of this kind of models applies only to individual agents, who face finite lifespans. The economy of course persists irrespective of the constant fluctuation of finitely lived agents.
The findings of these models will sometimes differ immensely from the results of the commonplace neoclassical growth models, as agents of different ages have different levels of wealth and consumption, which makes it a lot more difficult to determine aggregate values for an existing population. Also, there is need for life insurances to avoid unintended bequests.

This thesis is structured as follows: chapter 2 introduces the concept of finite horizons, that is, a way of modelling death as a source of uncertainty to heterogenize the agents of a model. The central model of this kind is

[1] Symposium on "Demography, Education, and Democracy – A Global Perspective", in celebration of the opening of the Wittgenstein Centre for Demography and Global Human Capital. September 29^{th}, Austrian Parliament, Vienna.

the basic Blanchard-Yaari model, which will be briefly presented. Chapter 3 analyzes Pautrel (2009), a more complex model of finite horizons where agents also care about the environment. Chapter 4 takes human capital into account, by first taking a look at the Lucas model, which paved the way for human capital in growth models, and then by examining two more models by Xavier Pautrel, the first combining environmental concerns and education, the second differentiating between the technologies used in final goods production and abatement services and thereby examining the role of the abatement technology. Chapter 5 summarizes the main findings, commonalities and differences between the respective models, chapter 6 concludes. Very lengthy calculations are positioned in the Appendix, as they would otherwise interfere with the legibility of the thesis.
All diagrams were created using MATLAB. All calculations were carried out manually, except for one very cumbersome expression in Appendix C, which was computed using Maple.

2 The Blanchard-Yaari model

The perpetual youth model or Blanchard-Yaari model (see Blanchard (1985) and Yaari (1965)) is a model of overlapping generations in continuous time. It is in many ways related to two very popular growth models, namely the Ramsey model of infinitely lived agents in continuous time (see Ramsey (1928)), and the Diamond model of overlapping generations in discrete time (see Diamond (1965)). Although these two models are among the most influential growth models, they have certain weaknesses, especially in the way they handle the death of an agent - in the Ramsey model there is no such thing as death, whereas in the Diamond model agents die with certainty after a given number of years. The Blanchard-Yaari model now tries to combine the many excellent features of the Ramsey model and the Diamond model, and at the same time to introduce a more realistic way of dying by making death stochastic and assigning individuals certain probabilities of death.
The Blanchard-Yaari model was presented for the first time in Blanchard (1985), with essential insights gained from Yaari (1965). Agents face probabilities of death that are exponentially distributed and take the form

$$\boxed{f(t) = \lambda e^{-\lambda t}} \tag{1}$$

with constant death rate or instantaneous probability of death λ.[2] In the basic model, there is no population growth. In order to have the entire population size normalized to 1, a (sufficiently large) cohort is born at every instant of time, whose size is therefore necessarily equal to the death rate λ, as can easily be seen from the fact that the size of the entire population takes the form $\int_{-\infty}^{t} \lambda e^{-\lambda(t-s)} ds = 1$.[3]

Analyzing (1) more closely, the name "Perpetual Youth Model" becomes apparent: due to the exponential distribution's memorylessness, each individual at every point of time faces exactly the same life expectancy $\int_0^\infty t\lambda e^{-\lambda t} dt = \frac{1}{\lambda}$, therefore the same horizons and the same propensities to consume or save, which makes aggregation quite easy. Moreover, the well-known Ramsey model is a special case of the Blanchard-Yaari model (for $\lambda \to 0$, that is, infinite horizons).

[2]The notation presented differs somewhat from the original to stay consistent throughout this thesis. This will be the case for all models from now on.

[3]The size at time t of a cohort born at time s is simply $\lambda e^{-\lambda(t-s)}$ (non-stochastically).

2.1 Individual consumption

Let $c(s,t)$ denote per capita[4] consumption at time t of an agent born in s. Utility is derived from consumption only. In a first approach, we analyze a logarithmic utility function. At time t, an agent born in s maximizes her expected lifetime utility

$$\mathbb{E}\left[\int_t^\infty log\ c(s,\nu)e^{\theta(t-\nu)}\ d\nu|t\right], \tag{2}$$

where $\theta \geq 0$ denotes the rate of time preference. The agent has to maximize her *expected* lifetime utility, since she is uncertain about the time of her death (and utility in case of death is strictly 0). By recalling the calculation rule for conditional expectations: $\mathbb{E}(X|Y) = \frac{\mathbb{E}(X\cap Y)}{\mathbb{E}(Y)}$ with X denoting utility at time $\nu \geq t$ and Y denoting probability of survival at time $\nu \geq t$,[5] (2) can equivalently be written as

$$\boxed{max_{c(s,t),a(s,t)}\ \int_t^\infty log\ c(s,\nu)e^{(\theta+\lambda)(t-\nu)}\ d\nu} \tag{3}$$

Therefore, the effective discount rate is $(\theta+\lambda)$. As long as $\lambda > 0$, individuals discount the future even for $\theta = 0$ due to the positive probability of death.

Although agents are not altruistic towards their descendants, they are forbidden to be in debt at their time of death (otherwise they would simply go indefinitely into debt and all model solutions would be pathologic). As a consequence of the agents' uncertainty regarding their time of death, these assumptions would lead to a higher savings rate than the optimal rate without uncertainty. To avoid this inefficiency, we allow for life insurances. Of course, these life insurances are quite the opposite of conventional life insurances: throughout their lives, individuals receive payments from the insurance companies and in return leave their entire wealth to the insurance company after their death. As there is no aggregate uncertainty regarding death in the Blanchard-Yaari model, life insurances are not exposed to any kind of risk. It is assumed that there is perfect competition on the insurance market, so at every point in time, insurances receive transfers $\lambda\mathcal{A}$ from the agents that have just died and pay λa to the surviving agents, where

[4]From now on, per capita values will be denoted by small letters with two arguments in brackets, the first refering to the individuals time of birth, the second to the current time period.

[5]The probability of survival at time $\nu \geq t$ is $e^{-\lambda(\nu-t)}$, see (1).

$\mathcal{A}$ stands for aggregate and a for per-capita financial wealth (assets). Furthermore, financial wealth bears interest at rate $r(t)$, so that we have the following budget constraint:

$$\boxed{\dot{a}(s,t) = [r(t) + \lambda]a(s,t) + w(t) - c(s,t)} \tag{4}$$

$w(t)$ denoting labour income (for simplicity, the assumption is made that all agents work and receive the same amount of labour income, irrespective of their age). To avoid a Ponzi-scheme, that is, individuals going in debt indefinitely, we impose the following transversality condition:

$$\boxed{lim_{\nu \to \infty}\ e^{-\int_r^\nu [r(\mu)+\lambda]d\mu} a(s,\nu) = 0} \tag{5}$$

Each agent now maximizes (3) subject to (4) and (5). The current-value Hamiltonian takes the form

$$H = log\ c(s,t) + \mu(t)[(r(t) + \lambda)a(s,t) + w(t) - c(s,t)]$$

From the first order conditions

$$H_c = 0 \Leftrightarrow \frac{1}{c(s,t)} = \mu(t) \tag{6}$$

$$H_a = (\theta + \lambda)\mu(t) - \dot{\mu}(t) \Leftrightarrow \mu(t)[r(t) + \lambda] = (\theta + \lambda)\mu(t) - \dot{\mu}(t) \tag{7}$$

$$H_\mu = \dot{a}(s,t) \Leftrightarrow [(r(t) + \lambda)a(s,t) + w(t) - c(s,t)] = \dot{a}(s,t) \tag{8}$$

the Euler equation can easiliy be derived, as due to (7)

$$\dot{\mu}(t) = \mu(t)[r(t) - \theta], \tag{9}$$

and by differentiating (6) with respect to time, we find that

$$\dot{\mu}(t) = \frac{\dot{c}(s,t)}{c(s,t)^2}. \tag{10}$$

Combining (9) and (10) yields

$$\frac{\dot{c}(s,t)}{c(s,t)^2} = \frac{1}{c(s,t)}[r(t) - \theta]$$

and after multiplying $c(s,t)$ to both sides we get

$$\frac{\dot{c}(s,t)}{c(s,t)} = r(t) - \theta. \tag{11}$$

From (4), (5) and (11) we can derive the actual consumption of any individual as per time t (see Appendix A):

$$c(s,t) = (\lambda + \theta)[a(s,t) + \omega(t)], \tag{12}$$

where $\omega(t)$ stands for per-capita human wealth, which is the discounted present value of all future labour incomes:

$$\omega(t) = \int_t^\infty w(\nu) e^{-\int_t^\nu r(\zeta)+\lambda \ d\zeta} \ d\nu. \tag{13}$$

As we can see from (13), individual consumption depends on the entire individual wealth, with the same propensity $(\lambda+\theta)$ for all agents. The derivation of equation (12) is somewhat complicated and is presented in Appendix A for reasons of legibility.

2.2 Aggregate consumption

Aggregate Variables (which will be written in capital letters throughout this thesis) can be easily calculated thanks to the simple demographic structure of the model. Aggregate consumption for example takes the form

$$C(t) = \int_{-\infty}^t c(s,t) \lambda e^{\lambda(s-t)} \ ds, \tag{14}$$

in the same manner as aggregate labour income $Y(t)$, financial assets $\mathcal{A}(t)$ and human wealth $\Omega(t)$. Aggregate consumption can be expressed in a different and more instructive way as a consequence of equation (12):

$$C(t) = (\lambda + \theta)[\Omega(t) + \mathcal{A}(t)]. \tag{15}$$

Aggregate consumption therefore also depends on the entire aggregate wealth, with the same marginal propensity $(\lambda + \theta)$.

Following (13), $\Omega(t)$ can be written as

$$\Omega(t) = \int_{-\infty}^t \left[\int_t^\infty w(\nu) e^{-\int_t^\nu r(\zeta)+\lambda \ d\zeta} \ d\nu \right] \lambda e^{\lambda(s-t)} \ ds. \tag{16}$$

Changing the order of integration yields

$$\Omega(t) = \int_t^\infty \left[\int_{-\infty}^t w(\nu) \lambda e^{\lambda(s-\nu)} \ ds \right] e^{-\int_t^\nu r(\zeta)+\lambda \ d\zeta} \ d\nu. \tag{17}$$

Aggregate human wealth is therefore equal to the aggregate discounted present value of all future labour incomes accruing to all agents alive in t. Due to the assumption that wages are independent of age, (17) can be written as

$$\Omega(t) = \int_t^\infty Y(\nu) e^{-\int_t^\nu r(\zeta)+\lambda \, d\zeta} \, d\nu \tag{18}$$

with $Y(t) = \int_{-\infty}^t w(t)\lambda e^{\lambda(s-t)} ds$, or, equivalently,

$$\dot{\Omega}(t) = [r(t) + \lambda]\Omega(t) - Y(t) \tag{19}$$

and

$$lim_{\nu\to\infty} \, \Omega(\nu) e^{-\int_t^\nu r(\zeta)+\lambda \, d\zeta} = 0. \tag{20}$$

To fully describe the dynamics of the system, we still need an expression for $\dot{\mathcal{A}}(t)$. Taking the time-derivative of $\mathcal{A}(t) = \int_{-\infty}^t a(s,t)\lambda e^{\lambda(s-t)} \, ds$, we obtain, using the Leibnitz-rule

$$\dot{\mathcal{A}}(t) = \underbrace{a(t,t)}_{=0} - \lambda\mathcal{A}(t) + \int_{-\infty}^t \dot{a}(s,t)\lambda e^{\lambda(s-t)} \, ds \tag{21}$$

Equation (21) describes the change in aggregate financial wealth at an arbitrary point of time and can be interpreted in quite an enlightening way: the first term corresponds to the financial wealth of newborns; as there are no bequests, newborns enter this world without any financial assets, therefore $a(t,t) = 0$. The second term corresponds to the financial wealth of the dying agents in t, the last term captures the fluctuations of financial wealth of all the remaining agents.
Substituting (4) into (21) yields

$$\dot{\mathcal{A}}(t) = r(t)\mathcal{A}(t) + Y(t) - C(t). \tag{22}$$

Comparing (4) with (22), we notice that the rate at which financial wealth accumulates differs between the individual and the aggregate variables: per-capita financial wealth accumulates at rate $(r(t) + \lambda)$, whereas aggregate financial wealth only accumulates at rate $r(t)$. The reason for this is evident: the term λa in (4), describing the payments by the insurance company to the agents, is nothing but a transfer from the dying agents to those who survive, and therefore has no effect on aggregate financial wealth.

Equations (15), (19) and (22) fully describe the dynamic system. However, by taking the time-derivative of (15) and eliminating $\dot{\Omega}(t)$ and $\dot{\mathcal{A}}(t)$, we obtain at first[6]

$$\dot{C} = (\lambda + \theta)[(r + \lambda)\Omega - Y + r\mathcal{A} + Y - C]$$

and after substituting $\Omega = \frac{C}{\lambda+\theta} - \mathcal{A}$ from (15)

$$\boxed{\dot{C} = (r - \theta)C - \lambda(\lambda + \theta)\mathcal{A}} \tag{23}$$

$$\boxed{\dot{\mathcal{A}} = r\mathcal{A} + Y - C} \tag{24}$$

Equations (23) and (24) are equivalent to the equations (15), (19) and (22), but easier to handle, so we will use them instead.
Equation (23) yields the well-known Ramsey model for $\lambda = 0$; for $\lambda > 0$, agents have finite horizons and discount the future more heavily. Aggregate consumption grows more slowly than in the case where agents live forever. The reason for this is that agents differ in their levels of wealth, although their marginal propensities to consume are identical. Old agents with greater wealth die, younger agents with less wealth take their place. The growth rate of aggregate consumption is smaller, the greater λ, i.e. the shorter the agents' life expectancies.
It's important to realize that in spite of all this argumentation, the Euler-equation still applies for individual consumption: $\dot{c} = (r - \theta)c$. Therefore, when $\lambda > 0$ and $r = \theta$, individual consumption stays constant, whereas aggregate consumption declines.

After having analyzed the main features of the Blanchard-Yaari model, which will be the basis of nearly all of the following models, we now turn towards a more sophisticated version of a perpetual youth model with environmental concerns.

[6] From now on, time indices will be left out for reasons of legibility when they are not essential.

3 Environmental quality in a Blanchard-Yaari model

This chapter is based on Pautrel (2009). Starting from the Blanchard-Yaari model, Pautrel introduces a number of different features to the model and examines the impact of environmental taxes on consumption and savings. Our focus lies on a different aspect of the model: we will analyze the implications of demographic changes and retirement schemes on the economy, while leaving the environmental tax rate more or less unaffected. Thus we will get a first insight into how finite horizons affect welfare when environmental care is taken into account.

The growth rate of the population, g_N, is now defined as $g_N \equiv \frac{\dot{N}(t)}{N(t)} = b-p$ with b the birth rate and p the death rate. The population size at time t can hence be derived as follows:

$$\int \frac{\dot{N}(t)}{N(t)}\ dt = \int (b-p)\ dt$$
$$ln\ (N(t)) + c_0 = (b-p)t + c_1$$
$$N(t) = ce^{(b-p)t}, c = \pm\ e^{c_1 - c_0}.$$

Without loss of generality let $N(0) = 1$, so that $c = 1$ and

$$N(t) = e^{(b-p)t}. \tag{25}$$

The size of a cohort of newborns is directly proportional to the current population size via $N(s,s) = bN(s) = be^{(b-p)s}$. The size of a cohort born in s at time t $(t \geq s)$ is

$$N(s,t) = e^{-p(t-s)}N(s,s) = be^{-pt+ps+bs-ps} = be^{bs-pt}. \tag{26}$$

Individual labour supply $\hbar(s,t)$ is age-dependent, with $\hbar(s,t) = \phi e^{-\psi(t-s)}$, $\phi > 0, \psi \geq 0$. ψ denotes age-dependent productivity, that is, the rate at which individual labour supply decreases with age. This age-dependent productivity is a measure of the amount to which elderly people are part of the workforce (the higher ψ, the earlier they retire) and must not be confused with the common concept of labour productivity (units of output per units of labour input).

Aggregate labour supply can hence be written as

$$L(t) = \int_{-\infty}^{t} \hbar(s,t)N(s,t)\ ds\ =\ \int_{-\infty}^{t} \phi e^{-\psi(t-s)}be^{bs-pt}\ ds\ =$$

$$\phi b e^{-t(\psi+p)} \frac{1}{\psi+b}[e^{s(\psi+b)}]_{-\infty}^{t} \;=\; \phi b e^{-t(\psi+p)} \frac{1}{\psi+b}[e^{t(\psi+b)}] \;=$$
$$\frac{\phi b}{\psi+b} e^{t(b-p)} \;=\; \mathcal{L}N(t), \tag{27}$$

with $\mathcal{L} \equiv \frac{\phi b}{b+\psi}$ denoting per-capita labour supply. $\mathcal{L}$ fulfills the following properties:

$$\mathcal{L} \geq 0, \qquad \frac{\partial \mathcal{L}}{\partial b} = \frac{\phi \psi}{(\psi+b)^2} \geq 0, \qquad \frac{\partial \mathcal{L}}{\partial \psi} = -\frac{\phi b}{(b+\psi)^2} < 0.$$

Per-capita labour supply thus increases with the birth rate and decreases with ψ. Both effects are reasonable: when age-dependent productivity is high, people reduce their individual labour supply $\hbar(s,t)$ faster (they "retire" earlier) which of course leads to lower per-capita labour supply. On the other hand, individual labour supply is, regardless of the exact value of ψ, always highest early in life. A high birth rate implies that there are many young workers in the economy and consequently per-capita labour supply increases. The implications of these dependencies will be studied in more detail in chapter 3.5.

3.1 Individuals and households

Again, individuals maximize their expected lifetime utility being uncertain about their time of death. However, consumption is no longer their only source of utility, they also benefit from a clean and functioning environment, so the net flow of pollution, $\mathcal{P}(t)$ now also enters the utility function in a negative way. Agents born at time s now maximize as of time t

$$max_{c(s,t),a(s,t)} \; \mathbb{E}\left[\int_t^\infty (log\ c(s,\nu) - \kappa\ log\ \mathcal{P}(\nu))\ e^{-\theta(t-\nu)}\ d\nu | t\right] \tag{28}$$

or, equivalently,

$$\boxed{max_{c(s,t),a(s,t)} \int_t^\infty [log\ c(s,\nu) - \kappa\ log\ \mathcal{P}(\nu)]\, e^{-(\theta+p)(t-\nu)}\ d\nu} \tag{29}$$

with κ being the relative importance of environmental quality.

The transversality condition (5) stays the same, the budget constraint (4) changes slightly as wage income now depends on individual labour supply[7]:

[7]Note that the insurance payments depend only on the death rate p.

$$\boxed{\dot{a}(s,t) = (r(t)+p)a(s,t) + \hbar(s,t)w(t) - c(s,t)} \qquad (30)$$

Maximizing (29) subject to (30) and (5) yields the identical Hamiltonian, first oder conditions and Euler equation as the basic model (see (6) to (11)). Individual consumption again takes the form

$$c(s,t) = (\theta + p)[a(s,t) + \omega(s,t)], \qquad (31)$$

with the only difference that the expected present value of lifetime income now also depends on individual labour supply, as
$\omega(s,t) = \int_t^\infty \hbar(s,\nu)w(\nu)e^{-\int_t^\nu r(\zeta)+p\ d\zeta}\ d\nu$, and is therefore no longer identical for all agents. The derivation of (31) is identical to that of (12) except for the different expression of $\omega(s,t)$ and can be found in Appendix A.

Aggregate consumption can again be written as in (15). However, taking the time derivate of (15) now leads to a somewhat different and more complicated result:

$$\dot{C}(t) = [r(t) - \theta + b - p + \psi]C(t) - (\psi + b)(\theta + p)\mathcal{A}(t). \qquad (32)$$

As the calculations leading to this result are lengthy, they are presented in Appendix A.

3.2 The firm sector

The productive sector is taken to be a perfectly competitive market. Firms produce output $Y(t)$ according to the (Harrod-neutral) Cobb-Douglas production function

$$\boxed{Y(t) = K(t)^\alpha[A(t)L(t)]^{1-\alpha}, \quad \alpha \in (0,1)} \qquad (33)$$

where $K(t)$ denotes the aggregate stock of capital and $A(t)$ the prevailing level of technology.

It is now supposed that the level of pollution $\mathcal{P}(t)$ increases with output and diminishes with abatement measures $F(t)$:

$$\mathcal{P}(t) = \left[\frac{Y(t)}{F(t)}\right]^\gamma, \ \gamma > 0. \qquad (34)$$

As this model does not examine resource scarcity, firms have no incentive to invest in abatement initially. To overcome this problem, an environmental

tax at rate $\vartheta(t)$ on the net pollution of firms is introduced. After the environmental tax is deducted from each firm, it is fully returned to the firms to subsidize their abatement measures.[8]
Firms chose the amount of labour, capital and abatement that maximizes their profit:

$$\pi(t) = Y(t) - r(t)K(t) - w(t)L(t) - \vartheta(t)\mathcal{P}(t) - F(t) + T^p(t),$$

with $T^p(t)$ the governmental transfer payments (of course, $T^p(t) = \vartheta(t)\mathcal{P}(t)$). As the productive sector is perfectly competitive, firms make zero profit ($\pi = 0$). All production factors (labour, capital and abatement) are determined by carrying out the representative firm's profit optimization problem.

By taking the partial derivatives of the profit function and combining them with (34), we get a formula for the interest rate:

$$\frac{\partial \pi(t)}{\partial K(t)} = 0 \Leftrightarrow$$

$$r(t) = \alpha K(t)^{\alpha-1}(A(t)L(t))^{1-\alpha}\left[1 - \vartheta(t)\gamma K(t)^{\alpha(\gamma-1)}(A(t)L(t))^{(1-\alpha)(1-\gamma)}\frac{1}{F(t)^\gamma}\right]$$

as $\frac{K(t)^{\alpha\gamma-1}}{K(t)^{\alpha-1}} = K(t)^{\alpha(\gamma-1)}$ and $\frac{L(t)^{(1-\alpha)\gamma}}{L(t)^{1-\alpha}} = L(t)^{(1-\alpha)(\gamma-1)}$.

With (33), we can see that

$$K(t)^{\alpha(\gamma-1)}(A(t)L(t))^{(\gamma-1)(1-\alpha)}\frac{1}{F(t)^\gamma} = Y(t)^{\gamma-1}\frac{1}{F(t)^\gamma} = \frac{\mathcal{P}(t)}{Y(t)},$$

hence the interest rate can be expressed by

$$r(t) = \left[1 - \gamma\vartheta(t)\frac{\mathcal{P}(t)}{Y(t)}\right]\alpha K(t)^{\alpha-1}(A(t)L(t))^{1-\alpha}. \tag{35}$$

In the same way, we find an expression for the wage rate:

$$\frac{\partial \pi(t)}{\partial L(t)} = 0 \Leftrightarrow w(t) =$$

$$(1-\alpha)K(t)^\alpha A(t)^{1-\alpha}L(t)^{-\alpha}\left[1 - \vartheta(t)\gamma K(t)^{\alpha(\gamma-1)}(A(t)L(t))^{(1-\alpha)(\gamma-1)}\frac{1}{F(t)^\gamma}\right]$$

[8]Final output is used as numeraire, its price being set equal to 1. Moreover, we suppose that the cost of one unit of abatement equals the cost of one unit of final output and is therefore also set to 1.

so that

$$w(t) = \left[1 - \gamma\vartheta(t)\frac{\mathcal{P}(t)}{Y(t)}\right](1-\alpha)K(t)^{\alpha}A(t)^{1-\alpha}L(t)^{-\alpha}. \tag{36}$$

Finally, the optimization procedure yields

$$\frac{\partial\pi(t)}{\partial F(t)} = 0 \Leftrightarrow \vartheta(t)\gamma Y(t)^{\gamma}\frac{1}{F(t)^{\gamma+1}} - 1 = 0 \Leftrightarrow F(t) = \vartheta(t)\gamma\mathcal{P}(t). \tag{37}$$

From the last result, we obtain

$$\mathcal{P}(t) = \left(\frac{Y(t)}{\vartheta(t)\gamma\mathcal{P}(t)}\right)^{\gamma} \Rightarrow \mathcal{P}(t)^{\gamma+1} = \left(\frac{Y(t)}{\vartheta(t)\gamma}\right)^{\gamma} \Rightarrow \mathcal{P}(t) = \left(\gamma\frac{\vartheta(t)}{Y(t)}\right)^{-\frac{\gamma}{\gamma+1}}$$

We assume that the environmental tax rate $\vartheta(t)$ grows at the same rate as output $Y(t)$. This assumption is reasonable, as net pollution has to be constant in the long run to ensure a constant degree of environmental quality. Due to the specification of net pollution (it also grows with output), the tax rate also needs to grow in order to encourage firms to invest more in abatement measures so that net pollution stays constant in spite of growing output.
By defining the environmental tax rate in terms of final output $\tau \equiv \frac{\vartheta(t)}{Y(t)}$ (which has to be constant according to the considerations above) and $\chi(\tau) \equiv (\gamma\tau)^{\frac{1}{1+\gamma}}$, we get

$$\chi(\tau) = \left(\frac{\gamma\vartheta(t)}{Y(t)}\right)^{\frac{1}{1+\gamma}} \Rightarrow \mathcal{P}(t) = \chi(\tau)^{-\gamma},$$

which particularly means that $\mathcal{P}$ is not time-dependent! Contrarily, $F(t)$ is indeed time-dependent:

$$F(t) = \vartheta(t)\gamma\chi(\tau)^{-\gamma} = (\vartheta(t)\gamma)^{\frac{1}{1+\gamma}}Y(t)^{\frac{\gamma}{1+\gamma}} = \chi(\tau)Y(t).$$

With some rearranging, it is possible to express $\chi(\tau)$ via

$$\gamma\vartheta(t)\frac{\mathcal{P}(t)}{Y(t)} = \gamma\vartheta(t)\chi(\tau)^{-\gamma}Y(t)^{-1} = (\gamma\vartheta(t))^{\frac{1}{1+\gamma}}Y(t)^{\frac{\gamma}{1+\gamma}-1} =$$

$$(\gamma\vartheta(t))^{\frac{1}{1+\gamma}}Y(t)^{\frac{-1}{1+\gamma}} = \chi(\tau),$$

so that we immediately get new expressions for the interest rate

$$\boxed{r(t) = \alpha(1-\chi(\tau))K(t)^{\alpha-1}[A(t)L(t)]^{1-\alpha}} \tag{38}$$

and for the wage rate

$$\boxed{w(t) = (1-\alpha)(1-\chi(\tau))K(t)^{\alpha}A(t)^{1-\alpha}L(t)^{-\alpha}} \tag{39}$$

3.3 Market equilibrium

By substituting $F(t) = \chi(\tau)Y(t)$ and using (38) in the equilibrium conditions for the goods market ($Y(t) = C(t) + \dot{K}(t) + F(t)$) and the financial market ($\mathcal{A}(t) = K(t)$) we get two differential equations that fully characterise the dynamics of the model economy:

$$\boxed{\dot{K}(t) = Y(t) - F(t) + C(t) = (1-\chi(\tau))K(t)^{\alpha}(A(t)L(t))^{1-\alpha} - C(t)} \tag{40}$$

and, due to

$$\dot{C}(t) = [r(t) - \theta + b - p + \psi]C(t) - (\psi + b)(\theta + p)\mathcal{A}(t) \Rightarrow$$

$$\boxed{\begin{aligned}\dot{C}(t) = {} & [\alpha(1-\chi(\tau))K(t)^{\alpha-1}[A(t)L(t)]^{1-\alpha} - \theta + b - p + \psi]C(t) \\ & - (\psi + b)(\theta + p)K(t)\end{aligned}} \tag{41}$$

3.4 The steady state equilibrium

After having set up the model, we now turn to the task of looking for steady states and determining their stability. We abstract from technological progress, i.e. $A(t) \equiv A^{\frac{1}{1-\alpha}}$.

It is important to emphasize that the per-capita variables for consumption and capital, $c(t)$ and $k(t)$, differ from the per-worker variables, $\tilde{c}(t)$ and $\tilde{k}(t)$ due to the assumed age-earning profiles (as long as $\psi \neq 0$). In our analysis of steady states we will use per-worker variables only. Later however, we will also examine the equilibrium behaviour of the per-capita values and discover some striking differences.

Obviously $k(t) = \frac{K(t)}{N(t)} = \frac{K(t)\mathcal{L}}{L(t)} = \mathcal{L}\tilde{k}(t)$.[9]
We can now derive the two differential equations (42) und (43), that describe the dynamics of the stock of capital per-worker and consumption per-worker, respectively:
From $\tilde{k}(t) = \frac{K(t)}{L(t)}$ we get, by taking the logarithm and deriving with respect to time

$$ln\ \tilde{k}(t) = ln\ K(t) - ln\ L(t) \Rightarrow \frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)} =$$

$$(1-\chi(\tau))K(t)^{\alpha-1}AL(t)^{1-\alpha} - \frac{C(t)}{K(t)} - (b-p),$$

as $\frac{\dot{L}(t)}{L(t)} = \frac{\dot{N}(t)}{N(t)} = b - p$, and therefore

$$\dot{\tilde{k}}(t) = (1-\chi(\tau))A\tilde{k}(t)^{\alpha} - \tilde{c}(t) - (b-p)\tilde{k}(t).$$

In the same way $\tilde{c}(t) = \frac{C(t)}{L(t)}$:

$$ln\ \tilde{c}(t) = ln\ C(t) - ln\ L(t) \Rightarrow \frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{\dot{C}(t)}{C(t)} - \frac{\dot{L}(t)}{L(t)} =$$

$$[\alpha(1-\chi(\tau))AK(t)^{\alpha-1}L(t)^{1-\alpha} - \theta + b - p + \psi] - \frac{(\psi+b)(\theta+p)K(t)}{C(t)} - (b-p)$$

$$= [\alpha(1-\chi(\tau))A\tilde{k}(t)^{\alpha-1} - \theta + \psi]\tilde{c}(t) - (\psi+b)(\theta+p)\tilde{k}(t).$$

To summarize, the two central model equations read:

$$\boxed{\dot{\tilde{k}}(t) = (1-\chi(\tau))A\tilde{k}(t)^{\alpha} - \tilde{c}(t) - (b-p)\tilde{k}(t)} \tag{42}$$

$$\boxed{\dot{\tilde{c}}(t) = [\alpha(1-\chi(\tau))A\tilde{k}(t)^{\alpha-1} - \theta + \psi]\tilde{c}(t) - (\psi+b)(\theta+p)\tilde{k}(t)} \tag{43}$$

In a steady state, the conditions $\dot{\tilde{k}}(t) = 0$ and $\dot{\tilde{c}}(t) = 0$ have to be fulfilled. As a consequence, we get the equilibrium value for $\tilde{c}^*$ in quite a simple way by rearranging equation (42):

$$\boxed{\tilde{c}^* = A(1-\chi(\tau))\tilde{k}^{*\alpha} - (b-p)\tilde{k}^*} \tag{44}$$

[9] Note that $\mathcal{L}(t) = \frac{L(t)}{N(t)}$, see (27).

With this result and the insights gained from equation (43), it is now possible to derive an expression for $\tilde{k}^*$:

$$0 = [\alpha(1-\chi(\tau))A\tilde{k}^{*\alpha-1}-\theta+\psi][A(1-\chi(\tau))\tilde{k}^{*\alpha}-(b-p)\tilde{k}^*]-(\theta+p)(b+\psi)\tilde{k}^* =$$
$$[\alpha(1-\chi(\tau))A\tilde{k}^{*\alpha-1}-\theta+\psi][A(1-\chi(\tau))\tilde{k}^{*\alpha-1}-(b-p)]-(\theta+p)(b+\psi) =$$
$$\alpha[(1-\chi(\tau))A\tilde{k}^{*\alpha-1}]^2-(\theta-\psi+\alpha(b-p))[(1-\chi(\tau))A\tilde{k}^{*\alpha-1}]+$$
$$(\theta-\psi)(b-p)-(\theta+p)(b+\psi)$$

Dividing the last expression by $(1-\chi(\tau))$ and collecting terms, we get

$$0 = \tilde{k}^{*2(\alpha-1)}\alpha(1-\chi(\tau))A+\tilde{k}^{*\alpha-1}(-\theta+\psi-\alpha(b-p))-[(1-\chi(\tau))A]^{-1}(\psi+p)(b+\theta).$$

Solving the quadratic equation yields

$$\tilde{k}^{*\alpha-1} = \frac{(\theta-\psi+\alpha(b-p)) \pm \sqrt{(\psi-\alpha(b-p)-\theta)^2+4\alpha(\psi+p)(b+\theta)}}{2\alpha(1-\chi(\tau))A}$$

As the share of capital per-worker cannot be negative, and the denominator is necessarily positive ($\chi(\tau) < 1$), it is obvious that the numerator needs to be positive as well. Consequently, the solution with "-" before the square root is obsolete.
For further analysis, we denote

$$\mathcal{D} \equiv (\theta-\psi+\alpha(b-p)) + \sqrt{(\psi-\alpha(b-p)-\theta)^2+4\alpha(\psi+p)(b+\theta)}.$$

It is now easy to see that

$$\boxed{\tilde{k}^* = [2\alpha(1-\chi(\tau))A]^{\frac{1}{1-\alpha}}\mathcal{D}^{\frac{-1}{1-\alpha}}} \tag{45}$$

If we ignore - just for a moment - the special case where $\tilde{k}^*$ and $\tilde{c}^*$ are both zero, we have $\mathcal{D} > 0$. Moreover, $\mathcal{D}$ is unique and $\mathcal{D} \in \mathbb{R}$. Thus the steady state equilibrium is unique - again ignoring the zero-solution, which of course is an equilibrium as well (although MATLAB does not identify the point $(0,0)$ as an equilibrium[10])

[10]For $\tilde{k} = 0$ and $\tilde{c} = 0$ it is obvious that $\dot{\tilde{k}}(t) = 0$. For $\dot{\tilde{c}}(t)$ this is not quite as clear at first, as the term $\tilde{k}^{\alpha-1}\tilde{c} = \frac{\tilde{c}}{\tilde{k}^{1-\alpha}}$ leads to an expression of the form "$\frac{0}{0}$" (which is precisely the reason why MATLAB fails). Using the rules of de l'Hôspital, we find that $lim_{(\tilde{c},\tilde{k})\to(0,0)} \frac{\tilde{c}}{\tilde{k}^{1-\alpha}} = \frac{1}{(1-\alpha)\tilde{k}^{-\alpha}} = \frac{k^{\alpha}}{1-\alpha} = 0$, which proves, that the point $(0,0)$ really is a proper equilibrium (of course, this finding was intuitively clear from the very beginning, as it is impossible to move away from a point where there is neither capital nor consumption in a two-dimensional model).

The Jacobian evaluated in $(\tilde{k}^*, \tilde{c}^*)$ looks as follows (the special case (0,0) is analyzed in chapter 3.5):

$$\mathcal{J} = \begin{bmatrix} \alpha(1-\chi(\tau))A\tilde{k}^{*\alpha-1} - (b-p) & -1 \\ \Big((\alpha-2)\alpha(1-\chi(\tau))A\tilde{k}^{*\alpha-1} + \psi - \theta\Big)\frac{\tilde{c}^*}{\tilde{k}^*} & \alpha(1-\chi(\tau))A\tilde{k}^{*\alpha-1} - \theta + \psi \end{bmatrix}$$

where $\frac{\tilde{c}^*}{\tilde{k}^*} = A(1-\chi(\tau))\tilde{k}^{*\alpha-1} - (b-p)$.

The derivation of the matrix elements is straightforward, except for the (2,1)-element. For this element, we have

$$\left.\frac{\partial\dot{\tilde{c}}(t)}{\partial\tilde{k}}\right|_{(c^*,k^*)} = \alpha(\alpha-1)(1-\chi(\tau))A\tilde{k}^{*\alpha-2}\tilde{c}^* - (\theta+p)(b+\psi) =$$

$$\alpha(\alpha-1)(1-\chi(\tau))A\tilde{k}^{*\alpha-1}\frac{\tilde{c}^*}{\tilde{k}^*} - \Big[\alpha(1-\chi(\tau))A\tilde{k}^{*\alpha-1} - \theta + \psi\Big]\Big[A(1-\chi(\tau))\tilde{k}^{*\alpha-1} - (b-p)\Big]$$

We have already shown that $\frac{\tilde{c}^*}{\tilde{k}^*} = A(1-\chi(\tau))\tilde{k}^{*\alpha-1} - (b-p)$, therefore we can simplify the above expression so that

$$\left.\frac{\partial\dot{\tilde{c}}(t)}{\partial\tilde{k}}\right|_{(c^*,k^*)} = (\alpha-1)\alpha(1-\chi(\tau))A\tilde{k}^{*\alpha-1}\frac{\tilde{c}^*}{\tilde{k}^*} - \Big[\alpha(1-\chi(\tau))A\tilde{k}^{*\alpha-1} - \theta + \psi\Big]\frac{\tilde{c}^*}{\tilde{k}^*} =$$

$$\Big[(\alpha-2)\alpha(1-\chi(\tau))A\tilde{k}^{*\alpha-1} - \theta + \psi\Big]\frac{\tilde{c}^*}{\tilde{k}^*}.$$

To analyze the stability of the steady state equilibrium, we first need the determinant of the Jacobian:

$$det\ \mathcal{J} = \Big[\alpha(1-\chi(\tau))A\tilde{k}^{*\alpha-1} - \theta + \psi\Big]\Big[\alpha A(1-\chi(\tau))\tilde{k}^{*\alpha-1} -$$

$$(b-p) + \frac{\tilde{c}^*}{\tilde{k}^*}(\alpha-2)\Big] = \left[\underbrace{\alpha(1-\chi(\tau))A\tilde{k}^{*\alpha-1} - \theta + \psi}_{>0}\right]\left[\underbrace{2(\alpha-1)\frac{\tilde{c}^*}{\tilde{k}^*}}_{<0}\right] < 0.$$

The trace of the Jacobian reads

$$tr\ \mathcal{J} = 2\alpha(1-\chi(\tau))A\tilde{k}^{*\alpha-1} - (b-p) + \psi - \theta$$

and is non-negative. Combining the last two results, we conclude that the steady state equilibrium lies in the fourth quadrant of the trace-determinant plane. Consequently, the eigenvalues are real and have opposite signs, which means that the equilibrium is saddlepoint stable.

3.5 Equilibrium dynamics

We now focus on the deeper analysis of the steady state equilibrium. In order to do so, Pautrel (2009) chooses parameter values estimated from US-data in the year 2005: Life expectancy of an average US-citizen was around 77 years in 2005. In the context of this model, this yields $\frac{1}{p} = 77 \Rightarrow p = 0.013$. US-population was growing at roughly 1% per year, so that $b - p = 0.01 \Rightarrow b = 0.023$. The value for $\alpha = 0.3$ has been proven by empirical evidence in many studies. The other parameters are calibrated in a way that they fulfill two central requirements: that per-capita GDP be around 45 700 $, and that the individual effective labour supply at the age of 80 be very close to 0.
The resulting parameter values are listed in Table 1; the values of the relevant steady state variables, given the parameter set in Table 1, are listed in Table 2.

A	α	γ	τ	θ	p	b	ψ	ϕ
3.54	0.3	0.3	0.02	0.05	0.013	0.023	0.05	1

Table 1

$\tilde{k}^*$	$\tilde{c}^*$	$\tilde{y}^*$	k^*	c^*	y^*
110.419	13.131	14.518	34.790	4.137	4.574

Table 2

We see from Table 2, that per-capita GDP, y^*, is very close to its target value (except for some scaling). Individual consumption constitutes over 90% of GDP, the rest being allocated to abatement measures and changes in financial assets.

With these values, the phase portrait can be computed (see Figure 1).

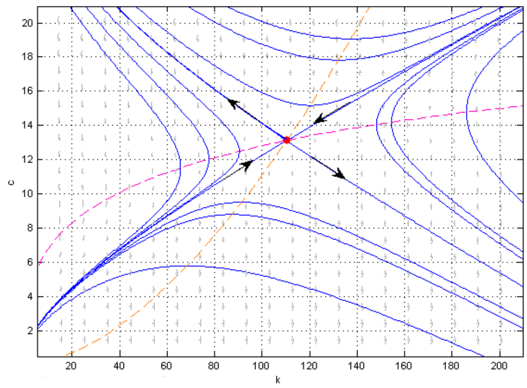

Figure 1: Phase portrait of the steady state equilibrium

The dashed lines in pink and yellow depict the isoclines of the dynamic system, the pink line representing the $\dot{\tilde{k}} = 0$ - locus, the yellow line representing the $\dot{\tilde{c}} = 0$ - locus. The steady state equilibrium lies exactly in the intersection of the two isoclines. The starting point in this diagramm is of utmost importance to the development of the trajectory. By starting above the $\dot{\tilde{k}} = 0$ - locus, per-worker consumption is high and we see from (42) that dissaving takes place, hence the capital stock per-worker declines. Below the $\dot{\tilde{k}} = 0$ - locus, the opposite is true, as consumption is low. Starting to the left of the $\dot{\tilde{c}} = 0$ - locus, the capital stock per-worker is relatively small, thus per-worker consumption increases, see (43) and vice versa to the right of the $\dot{\tilde{c}} = 0$ - locus.

The nearly straight blue lines with arrows depict the characteristic directions of the system, where of course the two lines with arrows pointing in towards the equilibrium are stable manifolds, the other two are instable manifolds. The stable manifolds are the only trajectories that lead into the equilibrium (in forward-time). As, naturally, the stable manifolds have Lebesgue-measure 0, the probability of ending up in the equilibrium by randomly choosing a starting point is equal to 0. If, for example, we start at some point close to (0,0) that is not on the stable manifold, at first consumption and capital will rise according to the arguments above. As trajectories must never intersect, it is important to notice at which side of the stable manifold our starting point was. If it was to the left, the trajectory will eventually intersect the $\dot{\tilde{k}} = 0$ - locus, which means that afterwards, consumption will continue to rise but the capital stock will gradually tend towards 0. If the starting point was to the right of the stable manifold, the trajectory will intersect the $\dot{\tilde{c}} = 0$ - locus, so that consumption will slowly decline while the capital stock continues to grow. For the case that we choose starting points above the steady state, we might even end up with infinite per-worker consumption and/or capital stock. However, the transversality condition ensures that we always start on a stable manifold: with a given stock of per-capita capital, consumption per-capita is adjusted accordingly.[11]

As we have proved in the last chapter, the equilibrium turns out to be a saddlepoint with values $\tilde{k}^* = 110.4$, $\tilde{c}^* = 13.1$ in the steady state. If we examine the system more closely, that is, by trying many different sets of parameter values, it turns out that the equilibrium is always saddlespoint-stable. Only when we use economically non-relevant values that violate the model specifications (such as $\alpha > 2$), different kinds of equilibria occur.

However, as mentioned above, there is another equilibrium point in (0,0).[12] The Jacobian, evaluated in (0,0) takes the form

$$\begin{bmatrix} -(b-p) & -1 \\ (\psi-\theta)(b-p) & -\theta+\psi \end{bmatrix}$$

The eigenvalues can be calculated according to

$$[-(b-p)-\lambda](\psi-\theta-\lambda)+(\psi-\theta)(b-p)=$$

$$-(b-p)(\psi-\theta)-\lambda(\psi-\theta)+\lambda(b-p)+\lambda^2+(b-p)(\psi-\theta)\overset{!}{=}0$$

$$\Leftrightarrow\ \lambda(\lambda+b-p-\psi+\theta)=0\ \Rightarrow\ \lambda_1=0,\ \lambda_2=-(b-p)+(\psi-\theta)$$

With the given parameter values, $\lambda_2 < 0$, which would imply that the linear system consists of a stable straight line of equilibria. However, as the Jacobian, evaluated in (0,0), possesses imaginary eigenvalues ($\lambda_1 = 0$ is strictly imaginary in this case), (0,0) is a non-hyperbolic equilibrium, which throws up a bunch of complications. For a start, the preconditions for the Hartman-Grobman theorem are violated, so that the topological equivalence of the linear and the nonlinear system is not granted in any neighborhood of (0,0), no matter how small. Moreover, we cannot draw any conclusions from the stability of the linearized system to that of the nonlinear system. A possible approach would be to try to find a Ljapunov-function to determine the

[11] This is precisely the reason why a saddlepoint is called a stable equilibrium.

[12] This point is completely ignored by Pautrel (2009). Although an equilibrium where nothing is produced nor consumed is irrelevant from an economic point of view, it should not be neglected when analyzing a dynamic system mathematically.

stability of the equilibrium. Due to the complexity of the system however, this is nearly impossible without an appropriate approach. The best thing we can do is to examine the stability of the other equilibrium point in detail, especially the direction of the trajectories, and to conclude logically on the stability of the point (0,0). With the given parameter values, (0,0) is instable, as can be perceived from figure 1. In fact, by trying numerous different scenarios, the equilibrium appears to be instable, irrespective of the chosen parameter values (of course, only as long as the model specifications are not violated).

Finally, we will study the bifurcations occurring when varying the most important parameters (b, p, ψ). Of course, as we are dealing with a unique steady state, we don't find any further equilibria and therefore no branching points. Still, the following diagrams illuminate the theoretic findings up to now[13].

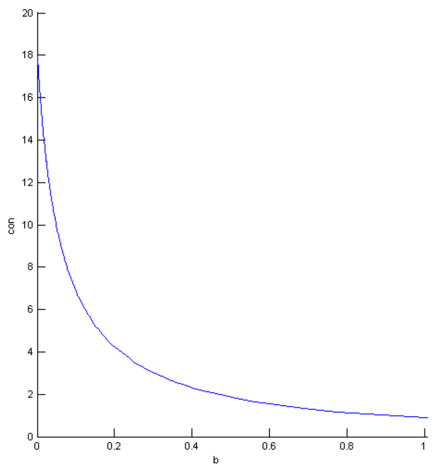

Figure 2: Steady-state per-worker consumption with varying birth rate

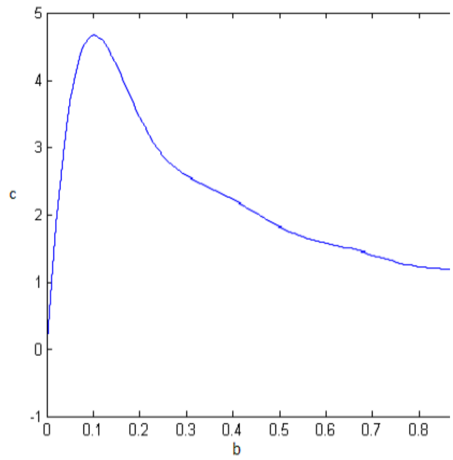

Figure 3: Steady-state per-capita consumption with varying birth rate

Figure 2 shows, that the steady-state per-worker consumption depends negatively on the birth rate, but with diminishing marginal rates. The closer the birth rate approaches 1, the smaller the marginal effect becomes. The higher the birth rate, the greater the labour force and thus GDP, but there are two oppositional forces behind the development of per-worker consumption: on the one hand a positive effect due to higher GDP, but on the other

[13]The respective values for consumption and capital are split into seperate diagrams for matters of clearness.

hand a negative dilution-effect that originates from the greater labour force among which aggregate consumption has to be divided in order to obtain per-worker GDP. Figure 2 shows that the dilution effect dominates, so that a higher birth rate unambiguously leads to smaller per-worker consumption.

It is a very interesting feature of this model, that the sensitivity of per-capita and per-worker values towards the same parameter might differ immensely. The reason for this lies in the assumed age-earning profiles that lead to the exgenous per-capita labour supply $\mathcal{L} = \frac{\phi b}{b+\psi}$ with $c(t) = \mathcal{L}\tilde{c}(t)$ and likewise for $k(t)$. Hence, with respect to productivity and the birth rate, there is a second effect that either amplifies, weakens or even reverses the impact of these parameters on per-capita values as compared to per-worker values. In the case of per-capita consumption and the birth rate, we have

$$c_b^* = \frac{\partial \mathcal{L}}{\partial b}\tilde{c}_b^* + \mathcal{L}\tilde{c}_b^*.$$

As we have just discussed, $\tilde{c}_b^* < 0$ (proven in Appendix B). However, $\frac{\partial \mathcal{L}}{\partial b} = \frac{\phi\psi}{(b+\psi)2} > 0$, therefore the overall effect is ambiguous. An increased birth rate (a "baby-boom") leads to less per-worker consumption, but also to a higher workforce participation rate, as young agents supply more individual labour than old agents (for individual labour supply, we have $\hbar(s,t) = \phi e^{-\psi(t-s)}$, therefore it declines with age). The interaction of these effects is not clear immediately, but can be examined very nicely in Figure 3.

Exactly the same is true for capital. Whereas increased fertility unambiguously decreases per-worker capital, it can have a very positive impact on the per-capita capital stock as per-capita labour supply rises with the birth rate, see figures 4 and 5. Indeed, by examining Figure 3 and Figure 5 in detail, we observe that for low values of b the impact of fertility is strictly positive, both with respect to capital and consumption. The birth rate we assumed initially, based on US-data, was 0.023 and thus far below the consumption- respectively capital-peak at about $b = 0.1$. Thus, for realistic values of b, per-capita consumption and capital depend strictly positive on the birth rate.

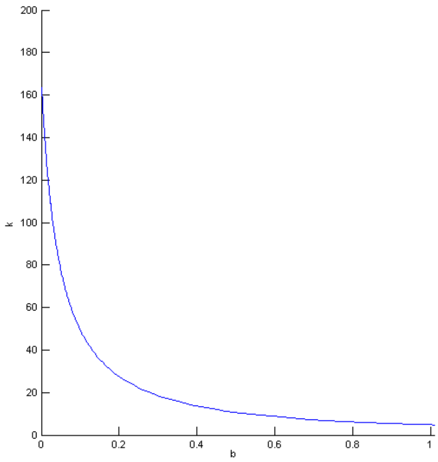

Figure 4: Steady-state per-worker capital with varying birth rate

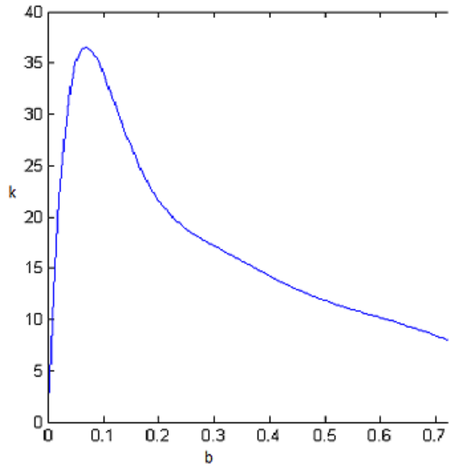

Figure 5: Steady-state per-capita capital with varying birth rate

The equilibrium per-worker capital stock also depends negatively on the death rate, although with smaller sensitivity than on the birth rate. Again, it is the interaction between direct GDP-effects through a demographic change and the dilution effect through a changed workforce that determine the relationship. This time however, the effects go in the opposite direction as compared to the analysis of the birth rate. A higher death rate reduces GDP due to a smaller labour force, but at the same time the smaller labour force has a positive "counter-dilutional" effect on per-worker capital. Contrarily to the previous case, the GDP-effect dominates in this case so that the overall effect is negative again. The result can be seen in Figure 6.

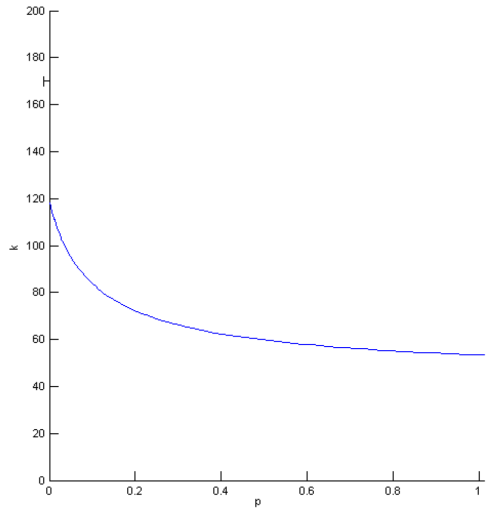

Figure 6: Impact of the death rate on steady-state per-worker capital

As per-capita labour supply $\mathcal{L}$ does not depend on p,[14] the effect a change in mortality has on per-capita capital is qualitatively identical to the effect on per-worker capital, the same being true for consumption. Interestingly, in the steady-state equilibrium, consumption per-worker increases with the death rate, which means that once more the dilution effect (which is of course a counter-dilution effect) dominates. From an economic point of view, this finding is very reasonable: the higher death rate increases the insurance premium paid to living agents by the insurance companies, whereas it decreases their expected lifetime income. As agents face ever shorter lifespans and ever higher uncertainty about their future, they discount the future more heavily (see (29)). All these reasons together explain why agents prefer to consume and gain direct utility from this consumption, than to wait for increased consumption possibilities they might not live to see. Yet, the effect of the death rate is not independent of the chosen parameter values. For very extreme parameter values, the effect of the death rate can be negative (see Appendix B); however, for any realistic parameter values, the relationship between the death rate and steady-state consumption looks qualitatively like in Figure 7.

[14] $k_p^* = \underbrace{\mathcal{L}_p \tilde{k}^*}_{=0} + \mathcal{L}\tilde{k}_p^* = \mathcal{L}\tilde{k}_p^*$.

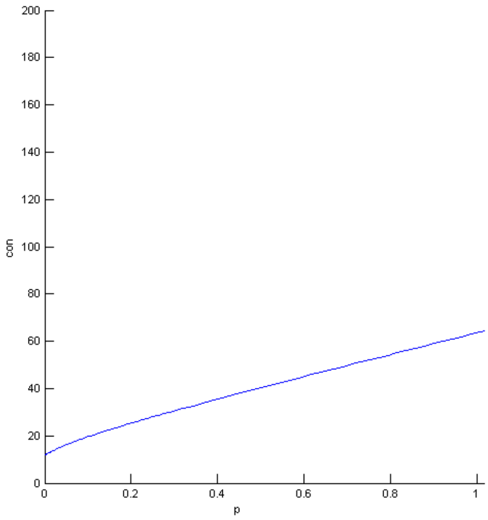

Figure 7: Impact of the death rate on steady-state per-worker consumption

The impact of the age-specific productivity on the steady state equilibrium is positive with respect to both consumption and capital per-worker, but with diminishing marginal rates, as can be seen in Figure 8 and 10. Increased age-specific productivity decreases per-capita labour supply, as $\frac{\partial \mathcal{L}}{\partial \psi} = -\frac{\phi b}{(b+\psi)^2} < 0$, and thereby aggregate labour supply. From (39), we see that wages depend negatively on the work force, which of course means they depend positively on ψ. Higher wages signify more wealth for workers, and thus increased per-worker capital stock and consumption.

The effects of varying age-dependent productivity change drastically when accounting for per-capita values. Figure 9 and Figure 11 show, that both per-capita consumption and the per-capita capital stock in the steady state decline when ψ rises. US-data suggest that ψ be small (we assumed $\psi = 0.05$ in our first analysis of the steady state), therefore the sensitivity of per-capita consumption and capital towards productivity is high. This result is hardly surprising: when there are less people in the workforce, the benefits of higher wages accrues to ever fewer people. Also, GDP as a whole falls. So, while the working population becomes wealthier, the retirees have to cut down on consumption and savings. With increasing ψ, they get more and more numerous and thereby outbalance the workers' positive effects.

Again, these statements do not hold for every possible szenario; very extreme parameter values could change the system variables' dependencies with respect to productivity. However, like in the last section, our analysis is accurate in any realistic setting (see Appendix B for details).

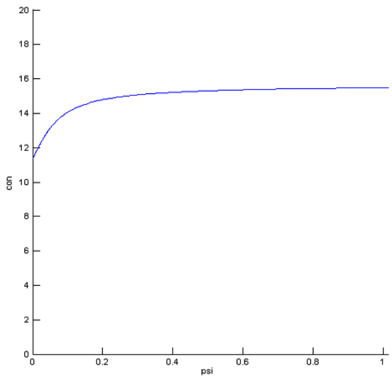

Figure 8: Steady-state per-worker consumption with varying productivity

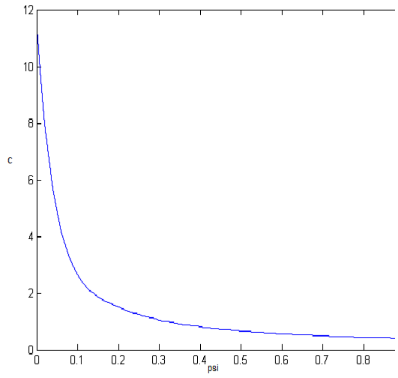

Figure 9: Steady-state per-capita consumption with varying productivity

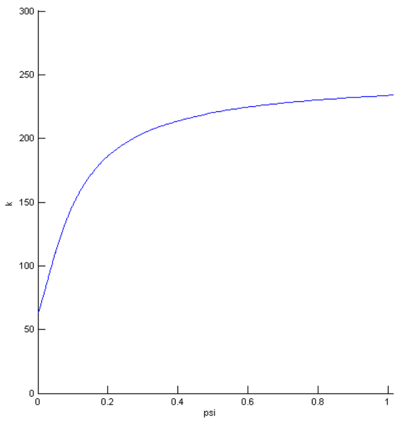

Figure 10: Steady-state per-worker capital with varying productivity

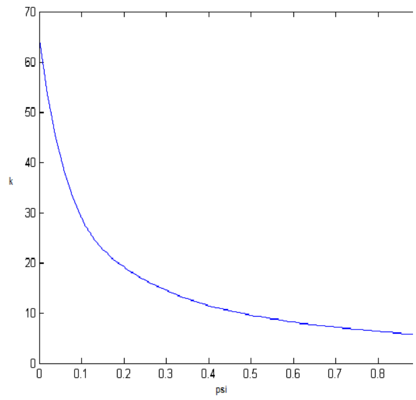

Figure 11: Steady-state per-capita capital with varying productivity

A detailed analysis of the relationship between per-worker and per-capita values is to be found in Appendix B.

In summary, we found that the impact of demographic change on our model economy depends on the precise nature of demographic change. While a "baby-boom", expressed by a higher birth rate, can have positive as well as negative effects on per-capita consumption and capital, the impact of a

lower death rate is unanimously positive on per-capita capital and negative on consumption. The reason lies in the assumed age-earning profiles, which depend on the birth rate, but not on the death rate.
Pautrel's examination of the environmental policy yields similar results: although the impact of the environmental policy on per-capita consumption and capital is always negative, its detrimal impact also depends on the nature of demographic change. Whereas a decrease in the death rate (and thus an increase in life expectancy) unanimously supports the negative impact of the environmental policy, the effect of a lower birth rate is not clear. However, for realistic values of b, an increase in fertility supports the detrimal impact of the environmental policy.

4 Human capital driven growth

The models we have considered up to now, however insightful and well-composed they might be, have one flaw in common: they are not growth models per se. The basic version of the Blanchard-Yaari model presented has no growth components at all (of course, they can be added in countless ways) whereas the Pautrel model possesses two possible sources of growth, namely population growth and technological advancement. But, as death and birth rates as well as technology are exogenous, again no endogenous growth occurs. In order to analyze a Blanchard-Yaari model with environmental concerns in a growth scenario, we will now include a second state variable apart from physical capital - human capital - in the same way as Lucas (1988). Whereas physical capital refers to machines and other durable goods used in production, human capital describes the abilities and the knowledge of a worker that she uses in production. Generally, there are two major ways of accumulating human capital: by schooling activities on the one hand, and by "learning by doing" on the other hand. Lucas (1988) paved the way for human capital accumulation to enter growth models, so we will start by briefly studying his model (and concentrate on the schooling aspect), afterwards we will study two more models by Xavier Pautrel, which combine the Lucas model with the Pautrel (2009) model: the first model (Pautrel (2011b)) introduces finite horizons and environmental concerns to the Lucas model, the second model (Pautrel (2011a)) has a similar approach with a slightly simpler model structure, but differentiates between the technologies used in output production and abatement service production. We start by setting up the basic Lucas model.

4.1 The Lucas model

We begin in the setting of the well-known Ramsey model with CRRA-utility with $\theta \geq 0$ again the rate of time preference and $\sigma > 0$ the coefficient of relative risk aversion (and σ^{-1} the intertemporal elasticity of substitution). Agents maximize

$$\boxed{max_{c(t),u(t),K(t),h(t)} \int_0^\infty e^{-\theta t} \frac{1}{1-\sigma}[c(t)^{1-\sigma} - 1]N(t)\ dt} \tag{46}$$

subject to

$$\boxed{\dot{K}(t) = A(t)K(t)^\beta N(t)^{1-\beta} - c(t)N(t)} \tag{47}$$

and

$$\boxed{lim_{t\to\infty} e^{-\theta t}\iota(t)K(t) = 0} \tag{48}$$

where $\iota(t)$ denotes the co-state variable and describes the "shadow-price" of capital. Technological progress takes place at rate μ, while population grows at rate g_N.
In contrast to the Blanchard-Yaari models considered so far, the Lucas model does not maximize the utility of a single agent but rather of society as a whole. Of course, in the context of the Lucas model, this is easier than in a Blanchard-Yaari setting, as Lucas assumes all individuals to be identical - the neoclassical concept of the "representative individual" - whereas in the models considered so far we would first need to aggregate the utility of the very heterogeneous population, which would raise a lot of (philosophical) questions: how can we compare the utility of agents of different ages, different levels of wealth etc.

Now, we add human capital to the model. In the economy, there are N workers, each with a skill-level somewhere between zero and infinity. Let $h(t)$ be the skill-level (in other words, the human capital) of a worker at time t, we can denote by $N(h)$ the total number of workers with skill-level h, so that

$$N = \int_0^\infty N(h)\ dh.$$

The average skill-level of all the workers in the economy is therefore equal to

$$\boxed{h_a = \frac{\int_0^\infty hN(h)\ dh}{\int_0^\infty N(h)\ dh}}$$

It is important to realize that h_a is a positive external effect of human capital, as every worker benefits from h_a (as we will see soon), but a single worker's human capital accumulation has no measurable impact on h_a.

Agents can accumulate human capital by devoting parts of their non-leisure time to schooling activities, which naturally implies that the amount of time used for production diminishes. Let $u(h)$ be the fraction of non-leisure time a worker with skill-level h uses for production. Instead of the total workforce, we will now consider the effective workforce in production, that is, the amount of skill-weighted manhours devoted to production:

$$N^e = \int_0^\infty u(h)N(h)h\ dh.$$

To simplify the model, all agents are taken to be identical, as in the Ramsey-model. Consequently, $N^e = uhN$ and $u(h) = u$ as all workers have the same skill-level h.

Human capital accumulates according to

$$\dot{h}(t) = h(t)^{\zeta} G(1 - u(t)) \tag{49}$$

with $\zeta \leq 1$ and G monotonically increasing and homogeneous. Human capital accumulates faster, the more time agents invest in schooling. The rate at which it accumulates is $\frac{\dot{h}(t)}{h(t)} = h(t)^{\zeta-1} G(1 - u(t))$. Therefore, except for the case $\zeta = 1$, the accumulation of human capital depends negatively on the prevailing level of knowledge. This is reasonable, as a less developed society can normally increase its state of knowledge much faster than a more developed society by adopting or copying techniques.
In a first basic approach, we set $\zeta = 1$ and $G(x) = \delta x$, so that (49) becomes

$$\boxed{\dot{h}(t) = h(t)\delta[1 - u(t)]} \tag{50}$$

This simplification is to be handled with caution, as now human capital accumulates at the same rate irrespective of the prevailing level of knowledge, which is of course unrealistic. If agents invest all their time in schooling, human capital grows at its maximum rate δ. With human capital, (47) becomes

$$\boxed{\dot{K}(t) = AK(t)^{\beta} N^e(t)^{1-\beta} h_a(t)^{\gamma} - N(t)c(t)} \tag{51}$$

There are now two production factors, namely physical capital and the effective workforce: $Y = F(K, N^e)$. In a strict sense, human capital is not a production factor in itself. However, as a worker's productivity increases with human capital, it does (indirectly) enter the production function.
The reason we introduced the average skill level h_a earlier on becomes apparent now: it enters the capital accumulation function as a positive externality and in that way it is intended to model knowledge spillovers in a society. When Lucas first presented this model, the prevailing model was the standard neoclassical growth model, of which Lucas was very critical. Especially, he criticized the neoclassical model's inability to explain observed diversity across countries. Through this knowledge spillover, Lucas tried to account for this diversity by assuming that workers in technologically advanced countries benefit from the prevailing level of technology (or human capital), whereas workers in less technologically advanced countries suffer from the underdevelopment of their country.

The representative agent's utility maximization problem is to maximize (46) subject to (50), (51) and (48). Due to the existence of knowledge spillovers, the equilibrium outcome, that is, the equilibrium that arises in a perfectly competitive setting, is not identical to the optimal outcome (when a benevolent social planner is in charge of all the proceedings in the economy). The reason lies in the fact that for an individual, h_a represents an externality in her optimization procedure, and it is only afterwards, due to market clearance, that $h(t) = h_a(t)$ adjusts. Contrarily, to the planner, $h(t) = h_a(t)$ from the beginning, as she internalizes the externality and includes it in her optimization procedure. The results of the two approaches differ both with respect to outcome and interpretation.

The main conclusion of the Lucas model is the fact, that, in equilibrium, economies who are poor will remain poor, whereas their rich counterparts will remain rich. The growth rate of all economies is the same in equilibrium, so there is no convergence between economies of different levels of wealth. This finding is completely opposed to the standard neoclassical growth model, which suggests that economies converge with respect to their level of wealth. The reason for this lies in the assumed knowledge spillover, which benefits technologically advanced countries.
The Lucas model proposes that the acquisition of human capital and thereby the development of better technologies is the driving force behind economic growth. This is commonly accepted nowadays. Before this model was first presented, technological advancement was mostly an exogenous parameter, which was most unsatisfactory, as it is the rate of technological progress that effectively determines the economic growth rate and the development of consumption. This basic Lucas model marked the beginning of an era of - often highly complex - models that use human capital accumulation as the key element leading to economic prosperity.

The following model merges the Blanchard-Yaari model and the Lucas model and includes environmental concerns.

4.2 Introducing education to the Blanchard-Yaari model with environmental quality

The model presented in this chapter is an extension and a slight modification of the model analyzed in chapter 3. It is introduced in Pautrel (2011b). Agents now have the opportunity to invest part of their available time in schooling activities à la Lucas (1988) and by doing so to increase their level of

human capital and thereby their future consumption possibilities. Pollution again affects the agents' well-being, with the crucial difference, that the net flow of pollution no longer increases with output, but with the aggregate stock of capital.
The demographic structure is simpler than in Pautrel (2009), in order to not let the model get too complex to handle. Analogously to the original Blanchard-Yaari model, there is no population growth, with death rate and birth rate equal to λ.

4.2.1 Individuals and households

Agents optimize their expected lifetime utility

$$\boxed{max_{c(s,t),u(s,t),z(s,t),a(s,t),h(s,t)} \int_s^\infty \left[log\ c(s,t) - \frac{\zeta}{1+\varphi}\mathcal{P}(t)^{1+\varphi}\right] e^{-(\theta+\lambda)(t-s)}\ dt} \tag{52}$$

Individuals therefore gain utility only from their individual consumption c and perceive pollution $\mathcal{P}$ as a bad. Parameters $\zeta > 0$ and $\varphi > 0$ capture the relative importance of environmental quality in terms of utility. As agents vary with respect to their ages and horizons, all individual variables depend on two time indices, the first refering to the agents' time of birth, the second to the actual time. For example, $c(s,t)$ denotes the consumption at time t of an agent born in s. In the optimization process, agents choose their optimal intertemporal allocation of consumption c, time spent working u, purchased units of educational inputs (i.e. learning materials, education fees, etc.) z,[15] net asset holdings a and human capital h. Thus, we now have three control variables (c,u,z) and two state variables (a,h). The effective discount rate, $(\theta+\lambda)$ is, like in every other finite-horizon model, higher than it would be in a standard neoclassical model, as agents have to take the possibility of their death into account (see Chapter 2).

Agents can accumulate human capital according to

$$\boxed{\dot{h}(s,t) = B[(1-u(s,t))h(s,t)]^{1-\delta}z(s,t)^{\delta}, \qquad \delta \in [0,1)} \tag{53}$$

where parameter B denotes the efficiency or quality of education and $u(s,t) \in (0,1)$ stands for the part of non-leisure time used for work (and consequently $(1-u(s,t))$ for the part of time allocated to schooling activities). Note that for $\delta = 0$, that is, educational inputs are not taken into

[15]For simplicity, one unit of $z(s,t)$ is supposed to cost one unit of output.

account, human capital accumulates in exactly the way proposed by Lucas (1988), which we have analyzed in the last section (see (50)).

The budget constraint is very similar to those considered so far, except that the acquisition of educational inputs needs to be considered as an additional cost and that the wages earned, w, now depend on effective units of labour, i.e. the skill-weighted time devoted to work, $u(s,t)h(s,t)$. A less educated agent can thus compensate her lack of skill by spending a greater amount of her non-leisure time working (although she might be better off spending her time studying to increase her future earnings). Altogether, we have

$$\boxed{\dot{a}(s,t) = [r(t)+\lambda]a(s,t) + u(s,t)h(s,t)w(t) - c(s,t) - z(s,t)} \tag{54}$$

where $[r(t)+\lambda]$ stands for the insurance premium paid to each living agent by the insurance companies at every point of time.
Last but not least, the transversality condition (5) applies.

As usual, we set up the Hamiltonian and derive the necessary first order conditions. We now have three control variables (c, u, z) and two state variables (a, h), which means that this process is somewhat more lengthy than in the models considered so far:

$$H = log\ c(s,t) - \frac{\zeta}{1+\varphi}\mathcal{P}(t)^{1+\varphi} + \mu_1(t)([r(t)+\lambda]a(s,t) + u(s,t)h(s,t)w(t)$$
$$-c(s,t) - z(s,t)) + \mu_2(t)(B[(1-u(s,t))h(s,t)]^{1-\delta}z(s,t)^{\delta})$$

$$H_c = 0 \Leftrightarrow \frac{1}{c(s,t)} = \mu_1(t) \tag{55}$$
$$H_u = 0 \Leftrightarrow \mu_1(t) = \frac{(1-\delta)\mu_2(t)Bh(s,t)^{-\delta}(1-u(s,t))^{-\delta}z(s,t)^{\delta}}{w(t)} \tag{56}$$
$$H_z = 0 \Leftrightarrow \mu_1(t) = \delta\mu_2 z(s,t)^{\delta-1}B[(1-u(s,t))h(s,t)]^{1-\delta} \tag{57}$$
$$H_a = (\theta+\lambda)\mu_1(t) - \dot{\mu}_1(t) \Leftrightarrow (r(t)+\lambda)\mu_1(t) = (\theta+\lambda)\mu_1(t) - \dot{\mu}_1(t) \tag{58}$$
$$H_h = (\theta+\lambda)\mu_2(t) - \dot{\mu}_2(t) \Leftrightarrow \mu_1(t)u(s,t)w(t)$$
$$+(1-\delta)\mu_2(t)Bh(s,t)^{-\delta}(1-u(s,t))^{1-\delta}z(s,t)^{\delta} = (\theta+\lambda)\mu_2(t) - \dot{\mu}_2(t) \tag{59}$$
$$H_{\mu_1} = \dot{a}(s,t) \Leftrightarrow [r(t)+\lambda]a(s,t) + u(s,t)h(s,t)w(t) - c(s,t) - z(s,t) = \dot{a}(s,t) \tag{60}$$
$$H_{\mu_2} = \dot{h}(s,t) \Leftrightarrow B[(1-u(s,t))h(s,t)]^{1-\delta}z(s,t)^{\delta} = \dot{h}(s,t) \tag{61}$$

The derivation of the Euler-equation is once more straightforward. By taking the time-derivative of (55) and substituting (58), we obtain

$$\frac{\dot{c}(s,t)}{c(s,t)} = r(t) - \theta. \tag{62}$$

It can be shown in exactly the same way as before that individual consumption evolves according to

$$c(s,t) = (\theta + \lambda)[a(s,t) + \omega(s,t)], \tag{63}$$

where the present value of lifetime earnings here takes the form $\omega(s,t) \equiv \int_t^\infty [u(s,\nu)h(s,\nu)w(\nu)]e^{-\int_t^\nu [r(\zeta)+\lambda]d\zeta}\, d\nu$ (see the derivation of (12) Appendix A).

A very important feature of this model is the fact that at any point of time, all individuals devote the same amount of their non-leisure time to schooling, which is a consequence of the following considerations:
Rearranging equation (56), we have $\left(\frac{z(s,t)}{(1-u(s,t))h(s,t)}\right)^\delta = \frac{\mu_1(t)w(t)}{\mu_2(t)B(1-\delta)}$. As the right hand side of this equation is independent of s, the same is true for the left hand side. Consequently, we can denote $\tilde{z}(t) \equiv \frac{z(s,t)}{(1-u(s,t))h(s,t)}$. Moreover, by rearranging equation (59) and replacing $\tilde{z}(t)$, we have

$$u(s,t) = \frac{(\theta+\lambda)\mu_2(t) - \dot{\mu}_2(t) - (1-\delta)\mu_2(t)B\tilde{z}(t)^\delta}{\mu_1(t)w(t) - (1-\delta)\mu_2(t)B\tilde{z}(t)^\delta},$$

which shows that $u(s,t) \equiv u(t)$ is independent from s. Utility maximization therefore implies that all agents, irrespective of their age, devote the same amount of their available time to schooling activities, which is of course again an implication of the "perpetual youth"-feature of the Blanchard-Yaari model. The fact that $u(t)$ is identical for all agents is of great importance as we will see many times in the analysis of this model.

Aggregate variables are calculated analogously to the original Blanchard-Yaari model. In particular, for aggregate human capital, we have

$$H(t) = \int_{-\infty}^{t} h(s,t)\lambda e^{-\lambda(t-s)}\, ds. \tag{64}$$

Although there are no bequests in terms of financial wealth, it is assumed that a constant part η (if not all) of the dying agents' human capital is inherited by the newborn generation: $h(t,t) = \eta H(t),\ \eta \in (0,1]$. The remaining part, $(1-\eta)$, is lost to the economy. Of course, the smaller this "knowledge inheritance" η, the slower aggregate human capital accumulates, if it accumulates at all, as with decreasing η death destroys an ever larger share of aggregate human capital at every point of time. We will see this soon when deriving the dynamics of H.

4.2.2 The firm sector

Firms produce final output according to

$$\boxed{Y(t) = K(t)^{\alpha} \left[\int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)}\, ds \right]^{1-\alpha}, \quad \alpha \in (0,1)} \tag{65}$$

This production function differs somewhat from (33), as the amount of output produced now depends on the time dedicated to productive activities and the prevailing level of human capital instead of the level of technology. In addition, the labour force is constant and equal to unity here and therefore does not need to be considered. As we have seen, $u(t)$ is identical for all agents, hence we can simplify $\int_{-\infty}^{t} u(t)h(s,t)\lambda e^{-\lambda(t-s)ds} = u(t)H(t)$ and express the economy's production technology in a much more compact way:

$$\boxed{Y(t) = K(t)^{\alpha}[u(t)H(t)]^{1-\alpha}} \tag{66}$$

Like in chapter 3, pollution is a matter over which the individual agent has no control, although it affects her well-being. The net flow of pollution is defined as

$$\mathcal{P}(t) = \left[\frac{K(t)}{F(t)} \right]^{\gamma}, \ \gamma > 0,$$

with $F(t)$ once more denoting abatement measures which are undertaken to counteract environmental degradation. Pollution increases with physical capital and decreases with abatement, therefore the government imposes an environmental tax at rate $\vartheta(t)$ upon the net flow of pollution, which is fully returned to the firms to fund their abatement activities (without this tax, firms would have no incentive to invest in abatement). We expect the environmental tax rate to increase similarly to aggregate physical capital, which is the source of pollution, in order to provide firms with an incentive to increase their abatement activities and thereby to preserve environmental quality in the long run. Therefore, we now look at the environmental tax rate normalized by physical capital, $\tau \equiv \frac{\vartheta(t)}{K(t)}$, whereas in chapter 3 we were interested in the environmental tax rate normalized by final output.

Firms maximize their profits, which now take the form

$$\pi(t) = Y(t)+T^p(t)-r(t)K(t)-w(t)\left[\int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)}\, ds\right]-\vartheta\mathcal{P}-F(t).$$

A firm's profit hence consists of the final output it produces (the price of output is set equal to 1, all other prices are to be seen relative to final output) plus the transfer payments it receives from the government minus the costs for physical capital and labour used in production, the environmental tax payments and the firm's abatement costs. Naturally, the cost of physical capital is the interest rate and the cost of effective labour is the wage rate. From this profit function, we see the incentive for firms to invest in abatement: if they let the environment degrade rampantly, their tax payments will increase just as rampantly (even more so if the capital stock and thus the tax rate also increases). Note that although the environmental tax is fully returned to the firms, i.e. $T^p = \vartheta\mathcal{P}$, this happens after the profit maximization has taken place.

Carrying out the optimization procedure leads to the following factor rewards:

$$\frac{\partial\pi(t)}{\partial K(t)} = \alpha K(t)^{\alpha-1}H_p(t)^{1-\alpha} - r(t) - \vartheta(t)\gamma K(t)^{\gamma-1}F^{-\gamma} = 0$$

$$\Leftrightarrow \boxed{r(t) = \alpha\frac{Y(t)}{K(t)} - \vartheta\gamma\frac{\mathcal{P}}{K(t)}} \tag{67}$$

$$\frac{\partial\pi(t)}{\partial H_p(t)} = (1-\alpha)K(t)^{\alpha}H_p(t)^{-\alpha} - w(t) = 0$$

$$\Leftrightarrow \boxed{w(t) = (1-\alpha)K(t)^{\alpha}H_p(t)^{-\alpha}} \tag{68}$$

$$\frac{\partial\pi(t)}{\partial F(t)} = \gamma\vartheta(t)K(t)^{\gamma}F(t)^{-\gamma-1} - 1 = 0 \Leftrightarrow \gamma\vartheta(t)\frac{\mathcal{P}(t)}{F(t)} = 1$$

$$\Leftrightarrow \boxed{F(t) = \gamma\vartheta(t)\mathcal{P}(t)} \tag{69}$$

With (69), we can express the net flow of pollution in a different way:

$$\mathcal{P}(t) = \left(\frac{K(t)}{\vartheta(t)\gamma\mathcal{P}(t)}\right)^{\gamma} \Rightarrow \mathcal{P}(t)^{\gamma+1} = \left(\frac{K(t)}{\vartheta(t)\gamma}\right)^{\gamma} \Rightarrow \mathcal{P}(t) = \left(\gamma\frac{\vartheta(t)}{K(t)}\right)^{-\frac{\gamma}{\gamma+1}}$$

Analogously to chapter 3, we define $\chi(\tau) \equiv (\gamma\tau)^{\frac{1}{1+\gamma}}$ so that

$$\chi(\tau) = \left(\frac{\gamma\vartheta(t)}{K(t)}\right)^{\frac{1}{1+\gamma}} \Rightarrow \mathcal{P}(t) = \chi(\tau)^{-\gamma},$$

In particular, this implies once more that $\mathcal{P}$ does not depend on time! By substituting $\mathcal{P} = \chi(\tau)^{-\gamma}$, equation (69) yields

$$F(t) = \vartheta(t)\gamma\chi(\tau)^{-\gamma} = (\vartheta(t)\gamma)^{\frac{1}{1+\gamma}} K(t)^{\frac{\gamma}{1+\gamma}} = \chi(\tau)K(t).$$

If we now combine equations (67) and (69), we find that[16]

$$r(t) = \alpha\left(u(t)\frac{H(t)}{K(t)}\right)^{1-\alpha} - \chi(\tau) \tag{70}$$

4.2.3 Market equilibrium

When the final goods market is cleared, production must be equal to the sum of the economy's expenditures, i.e. aggregate consumption, abatement, purchase of educational inputs and changes in the aggregate stock of physical capital, or, more formally

$$\boxed{Y(t) = C(t) + F(t) + Z(t) + \dot{K}(t)}$$

This representation of the market clearing condition is perfectly valid, but we will use a different version based on the following considerations:
By comparing equations (56) and (57), we get

$$\frac{(1-\delta)\mu_2(t)B\tilde{z}(t)^{\delta}}{w(t)} = \delta\mu_2(t)\tilde{z}(t)^{\delta-1}B$$

and consequently a different expression for $\tilde{z}(t)$:

$$\tilde{z}(t) = \frac{\delta}{1-\delta}w(t).$$

If we substitute the expression for the wage rate from profit maximization at firm level, we obtain

$$\begin{aligned} w(t) = (1-\alpha)K(t)^{\alpha}\left[\int_{-\infty}^{t} u(t)h(s,t)\lambda e^{-\lambda(t-s)}\, ds\right]^{-\alpha} = \\ (1-\alpha)\left(\frac{K(t)}{u(t)H(t)}\right)^{\alpha} = (1-\alpha)\frac{Y(t)}{u(t)H(t)} \end{aligned} \tag{71}$$

[16] Note that $F(t) = \chi(\tau)K(t)$.

$$\Rightarrow \tilde{z}(t) = \Delta \frac{Y(t)}{u(t)H(t)},$$

due to the fact that $K(t)^\alpha = \frac{Y(t)}{(u(t)H(t))^{1-\alpha}}$, with $\Delta \equiv \frac{\delta(1-\alpha)}{1-\delta}$. With this, we can find a relation between $Y(t)$ and $Z(t)$:

$$Z(t) = \int_{-\infty}^{t} z(s,t)\lambda e^{-\lambda(t-s)}\, ds = \int_{-\infty}^{t} \tilde{z}(t)(1-u(t))h(s,t)\lambda e^{-\lambda(t-s)}\, ds =$$

$$\tilde{z}(t)(1-u(t))H(t) = \Delta(1-u(t))\left(\frac{Y(t)}{u(t)}\right).$$

Hence, the market clearing condition $Y(t) = C(t) + \dot{K}(t) + F(t) + Z(t)$ can be written equivalently as

$$\left(1 + \Delta\left(1 - \frac{1}{u(t)}\right)\right) Y(t) = \dot{K}(t) + C(t) + \chi(\tau)K(t), \tag{72}$$

4.2.4 The dynamic system and the BGP

The next step is to set up the dynamic system. This will be achieved by finding expressions for $\dot{x}(t)$, $\dot{b}(t)$ and $\dot{u}(t)$, where $x(t) \equiv \frac{C(t)}{K(t)}$ and $b(t) \equiv \frac{H(t)}{K(t)}$. The reason we are carrying out these transformations lies in the fact that we are dealing with a proper growth model, which means that $C(t), H(t)$ and $K(t)$ are constantly growing, so that it is impossible to find a steady state where $\dot{C} = \dot{H} = \dot{K} = 0$ other than the point $(0,0,0)$. Hence, we seek different forms of equilibria, namely *balanced growth paths* (*BGP*). These balanced growth paths are steady states of the transformed variables, which implies that along a BGP, C, K and H grow at the same rate, as, due to $\dot{x} = \dot{b} = \dot{u} = 0$ we have

$$\dot{x} = \frac{\dot{C}}{K} - \frac{\dot{K}}{K}\frac{C}{K} = 0 \Leftrightarrow \frac{\dot{K}}{K} = \frac{\dot{C}}{C},$$

$$\dot{b} = \frac{\dot{H}}{K} - \frac{\dot{K}}{K}\frac{H}{K} = 0 \Leftrightarrow \frac{\dot{K}}{K} = \frac{\dot{H}}{H}.$$

From the market clearing condition we already have an expression for $\dot{K}(t)$, so we will now turn to deriving expressions for $\dot{H}(t)$ and $\dot{C}(t)$, which is quite easy in both cases.

Taking the time-derivative of (64) yields the rate of accumulation of aggregate human capital:

$$\dot{H}(t) = h(t,t)\lambda + \int_{-\infty}^{t} B[(1-u(t))h(s,t)]^{1-\delta} z(s,t)^{\delta} \lambda e^{-\lambda(t-s)} -$$

$$\lambda^2 h(s,t) e^{-\lambda(t-s)}\, ds \overset{h(t,t)=\eta H(t)}{=} \eta\lambda H(t) + B(1-u(t))\tilde{z}(t)^{\delta} - \lambda H(t)$$

$$\Rightarrow \boxed{\dot{H}(t) = H(t)[B(1-u(t))\tilde{z}(t)^{\delta} - (1-\eta)\lambda]} \tag{73}$$

which can be explained in a very intuitive way: the change in aggregate human capital, $\dot{H}(t)$, is obviously equal to the human capital of the newborn generation, $h(t,t) = \eta\lambda H(t)$, minus the human capital of the dying agents, $\lambda H(t)$, plus the changes in the levels of human capital of those surviving. As we have assumed that $\eta \in (0,1]$, the net change in aggregate human capital due to mortality and reproduction[17], $\lambda H(t)[\eta - 1]$, is less or equal to 0. This generational turnover effect increases with λ, that is, with shorter horizons, as generational turnovers become more frequent.

The rate of accumulation of aggregate consumption can be obtained in a very similar way, with the help of the Euler-equation:

$$\dot{C}(t) = c(t,t)\lambda + \int_{-\infty}^{t} \dot{c}(s,t)\lambda e^{-\lambda(t-s)} - \lambda^2 c(s,t) e^{-\lambda(t-s)}\, ds =$$

$$\lambda c(t,t) + \int_{-\infty}^{t} (r(t)-\theta) c(s,t)\lambda e^{-\lambda(t-s)} ds - \lambda C(t) = \lambda[c(t,t) - C(t)] + (r(t)-\theta)C(t)$$

$$\Rightarrow \boxed{\frac{\dot{C}(t)}{C(t)} = \frac{\dot{c}(s,t)}{c(s,t)} - \frac{1}{C(t)}[\lambda C(t) - \lambda c(t,t)]} \tag{74}$$

The "generational turnover effect" of consumption therefore is equal to $\lambda[c(t,t) - C(t)]$, which is of course negative, as newborns have no financial wealth. Moreover, with shorter horizons, the generational turnover effect regarding consumption increases and thus consumption growth declines more sharply.

By aggregating (63) and differentiating with respect to time, we get a different expression for the growth rate of consumption:[18]

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \theta - (1-\eta)\lambda - \eta\lambda(\theta + \lambda)x(t)^{-1} \tag{75}$$

[17] which Heijdra (2002) calls the "generational turnover effect" of human capital.

[18] The calculations leading to this result are nearly identical to those presented in Appendix A for equation (32) and are straightforward to adapt.

The last result needed to describe the dynamics of the system is an expression for $\dot{u}(t)$. From (71), we obtain $u(t) = \frac{K(t)}{H(t)} \left(\frac{(1-\alpha)}{w(t)}\right)^{\frac{1}{\alpha}}$. Taking the logarithm

$$ln\ u(t) = ln\ K(t) - ln\ H(t) + \frac{1}{\alpha}\ ln\ (1-\alpha) - \frac{1}{\alpha}\ ln\ w(t)$$

and applying the derivative with respect to time yields

$$\frac{\dot{u}(t)}{u(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{H}(t)}{H(t)} - \frac{1}{\alpha}\frac{\dot{w}(t)}{w(t)}.$$

For the growth rate of wages, we see immediately from the necessary conditions, that

$$w(t) \stackrel{(56)}{=} \frac{\mu_2(t)B(1-\delta)\tilde{z}(t)^\delta}{\mu_1(t)}.$$

The same procedure as before yields

$$ln\ w(t) = ln\ \mu_2(t) + ln\ B(1-\delta) + \delta\ ln\ \tilde{z}(t) - ln\ \mu_1(t)$$

and, consequently,

$$\frac{\dot{w}(t)}{w(t)} = \frac{\dot{\mu_2}(t)}{\mu_2(t)} + \delta\frac{\dot{\tilde{z}}(t)}{\tilde{z}(t)} - \frac{\dot{\mu_1}(t)}{\mu_1(t)} \qquad (76)$$

From (59), we know that

$$\dot{\mu_2}(t) = (\theta+\lambda)\mu_2(t) - \mu_1 w(t)u(t) - \mu_2(t)B(1-\delta)(1-u(t))\tilde{z}(t)^\delta$$

$$\stackrel{(57)}{=} (\theta+\lambda)\mu_2(t) - \mu_2\delta B\tilde{z}(t)^{\delta-1}w(t)u(t) - \mu_2(t)B(1-\delta)(1-u(t))\tilde{z}(t)^\delta$$

$$\Rightarrow \frac{\dot{\mu_2}(t)}{\mu_2(t)} = \theta+\lambda-B\tilde{z}(t)^\delta\left[\frac{\delta w(t)u(t)}{\tilde{z}(t)} + (1-\delta)(1-u(t))\right] = \theta+\lambda-B\tilde{z}(t)^\delta(1-\delta),$$

where the last equality follows from $w(t) = \frac{1-\delta}{\delta}\tilde{z}(t)$. Along with this, it is easy to see from (58), that

$$\frac{\dot{\mu_1}(t)}{\mu_1(t)} = \theta - r(t).$$

Thus, we can write (76) equivalently as

$$\frac{\dot{w}(t)}{w(t)} = r(t) + \lambda - B\tilde{z}(t)^\delta(1-\delta) + \delta\frac{\dot{\tilde{z}}(t)}{\tilde{z}(t)}. \qquad (77)$$

Finally, it is possible to replace the term $\frac{\dot{\tilde{z}}(t)}{\tilde{z}(t)}$ in (77), as $\dot{\tilde{z}}(t) = \frac{\delta}{1-\delta}\dot{w}(t)$, so that we get

$$\frac{\dot{w}(t)}{w(t)} = \delta\frac{\dot{w}(t)}{w(t)} - B(1-\delta)\left(\frac{\delta}{1-\delta}w(t)\right)^{\delta} + r(t) + \lambda,$$

and therefore

$$\frac{\dot{w}(t)}{w(t)} = \frac{r(t) + \lambda - B(1-\delta)^{1-\delta}\delta^{\delta}w(t)^{\delta}}{1-\delta} \overset{(71)}{=}$$

$$\frac{r(t) + \lambda - B(1-\delta)^{1-\delta}\delta^{\delta}(1-\alpha)^{\delta}(u(t)b(t))^{-\alpha\delta}}{1-\delta}.$$

Summarizing, we have

$$\frac{\dot{u}(t)}{u(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{H}(t)}{H(t)} - \frac{1}{\alpha}\left(\frac{r(t) + \lambda - B(1-\delta)^{1-\delta}\delta^{\delta}(1-\alpha)^{\delta}(u(t)b(t))^{-\alpha\delta}}{(1-\delta)}\right) \tag{78}$$

Now, the dynamic system can finally be derived:

$$\dot{x}(t) = \frac{\dot{C}(t)}{K(t)} - \frac{\dot{K}(t)}{K(t)}\frac{C(t)}{K(t)}$$

$$\overset{(75)}{=} x(t)[r(t) - \theta - (1-\eta)\lambda] - \eta\lambda(\theta+\lambda) - \frac{\dot{K}(t)}{K(t)}x(t).$$

With (72) and $Y(t) = K(t)^{\alpha}[u(t)H(t)]^{1-\alpha}$, this becomes

$$\dot{x}(t) = x(t)\left[r(t) - \theta - (1-\eta)\lambda - \eta\lambda(\theta+\lambda)x(t)^{-1} - \left(1 + \Delta(1 - \frac{1}{u(t)})\right)(u(t)b(t))^{1-\alpha} + x(t) + \chi(\tau)\right].$$

Remembering that $r(t) = \alpha(u(t)b(t))^{1-\alpha} - \chi(\tau)$, $\dot{x}(t)$ can be expressed equivalently as

$$\boxed{\dot{x}(t) = x(t)\left[-\theta - (1-\eta)\lambda - \eta\lambda(\theta+\lambda)x(t)^{-1} - \left(1 + \Delta(1 - \frac{1}{u(t)}) - \alpha\right)(u(t)b(t))^{1-\alpha} + x(t)\right].} \tag{79}$$

Similarly, we obtain

$$\dot{b}(t) = \frac{\dot{H}(t)}{K(t)} - \frac{\dot{K}(t)}{K(t)}\frac{H(t)}{K(t)}$$

$$\overset{(73)}{=} \left[B(1-u(t))\tilde{z}(t)^{\delta} - (1-\eta)\lambda\right] b(t) - \frac{\dot{K}(t)}{K(t)}b(t).$$

Again, inserting the expression for $\frac{\dot{K}(t)}{K(t)}$ and $\tilde{z}(t) = \Delta\frac{Y(t)}{u(t)H(t)} = \Delta(b(t)u(t))^{-\alpha}$ yields

$$\boxed{\begin{aligned}\dot{b}(t) = \big[&B(1-u(t))\Delta^{\delta}(u(t)b(t))^{-\alpha\delta} - (1-\eta)\lambda \\ &-(1+\Delta(1-\frac{1}{u(t)}))(u(t)b(t))^{1-\alpha} + x(t) + \chi(\tau)\Big] b(t)\end{aligned}} \tag{80}$$

Finally, by rearranging (78) and replacing $r(t)$, we have

$$\begin{aligned}\frac{\dot{u}}{u} &= \frac{\dot{K}}{K} - \frac{\dot{H}}{H} - \frac{1}{\alpha}\left(\frac{r(t)+\lambda - B(1-\delta)^{1-\delta}\delta^{\delta}(1-\alpha)^{\delta}(u(t)b(t))^{-\alpha\delta}}{(1-\delta)}\right) \\ &= \left(1+\Delta(1-\frac{1}{u(t)})\right)(u(t)b(t))^{1-\alpha} - x(t) - \chi(\tau) - B(1-u(t))\tilde{z}^{\delta} \\ &+ (1-\eta)\lambda - \frac{1}{\alpha(1-\delta)}[\alpha(u(t)b(t))^{1-\alpha} - \chi(\tau) + \lambda \\ &- B(1-\delta)^{1-\delta}\delta^{\delta}(1-\alpha)^{\delta}(u(t)b(t))^{-\alpha\delta}]\end{aligned} \tag{81}$$

Observing that $1+\Delta(1-\frac{1}{u(t)}) - \frac{1}{1-\delta} = \Delta\left((1-\frac{1}{u}) - \frac{1}{1-\alpha}\right)$, we can simplify this expression and thus obtain

$$\boxed{\begin{aligned}\dot{u}(t) = \Bigg[&B\Delta^{\delta}(b(t)u(t))^{-\alpha\delta}(\alpha^{-1}-1+u(t)) + \chi(\tau)\left(\frac{1}{\alpha(1-\delta)}-1\right) - x(t) \\ &-\lambda\left(\frac{1}{\alpha(1-\delta)}+\eta-1\right) + (u(t)b(t))^{1-\alpha}\Delta\left((1-\frac{1}{u(t)}) - \frac{1}{1-\alpha}\right)\Bigg] u(t)\end{aligned}}$$

As discussed above, the concept of a steady state equilibrium no longer suffices. Instead we are dealing with a "balanced growth path" (BGP), that is, a trajectory of the dynamic system along which all main model variables - C, K and H - grow at the same rate g^*, which of course needs to be positive. x, b and u stabilize at their BGP-values x^*, b^* and u^*.
Aggregate consumption, human and physical capital however grow at the same rate g^*, which is equal to $\frac{\dot{H}}{H}$ and consequently

$$g^* = B(1-u^*)\tilde{z}^{*\delta} - (1-\eta)\lambda = B(1-u^*)\Delta^\delta(b^*u^*)^{-\alpha\delta} - (1-\eta)\lambda. \quad (82)$$

These analytical findings will be illustrated in the next section. In particular, we will study the impact of varying schooling quality, B, and discount rate, θ, on the BGP-variables and the level of welfare.

4.2.5 Stability Analysis and Bifurcations

To analyze the stability of the BGP-equilibrium, we first need to determine the Jacobian of the dynamic system and its eigenvalues. The Jacobian evaluated in (x^*, b^*, u^*) takes the form[19]

$$\mathcal{J} = \begin{pmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} & \mathcal{J}_{13} \\ \mathcal{J}_{21} & \mathcal{J}_{22} & \mathcal{J}_{23} \\ \mathcal{J}_{31} & \mathcal{J}_{32} & \mathcal{J}_{33} \end{pmatrix}$$

with

$$\begin{aligned}
\mathcal{J}_{11} =& \ \eta\lambda(\theta+\lambda)x^{*-1} + x^* > 0 \\
\mathcal{J}_{12} =& \ -\frac{x^*}{b^*}\frac{(1-\alpha)^2(u^*-\delta)}{(1-\delta)u^*}(u^*b^*)^{1-\alpha} < 0 \\
\mathcal{J}_{13} =& \ -\frac{x^*}{u^*}(1-\alpha)(u^*b^*)^{1-\alpha}\left[\frac{(1-\alpha)u+\alpha\delta}{(1-\delta)u} + \right] < 0 \\
\mathcal{J}_{21} =& \ b^* > 0 \\
\mathcal{J}_{22} =& \ -\alpha\delta(1-u^*)B\Delta^\delta(u^*b^*)^{-\alpha\delta} - (1-\alpha)[1-\Delta(\frac{1}{u^*}-1)](u^*b^*)^{1-\alpha} < 0 \\
\mathcal{J}_{23} =& \ -[1+\alpha\delta(\frac{1}{u^*}-1)]B\Delta^\delta(u^*b^*)^{-\alpha\delta}b^* - (1+\alpha\delta)\left(\frac{1}{u}-1\right)\left[\frac{1-\alpha}{1-\delta}\right](u^*b^*)^{1-\alpha}\frac{b^*}{u^*} < 0 \\
\mathcal{J}_{31} =& \ -u^* < 0 \\
\mathcal{J}_{32} =& \ -\frac{u^*}{b^*}\left(\alpha\delta(\alpha^{-1}-1+u^*)B\Delta^\delta(u^*b^*)^{-\alpha\delta} + (1-\alpha)\Delta\left[\frac{1}{u^*}+\frac{\alpha}{1-\alpha}\right](u^*b^*)^{1-\alpha}\right) < 0 \\
\mathcal{J}_{33} =& \ B\Delta^\delta(u^*b^*)^{-\alpha\delta}[u^*-\delta+\alpha\delta(1-u^*)] + \alpha\Delta\left(\frac{1}{u^*}-1\right)(u^*b^*)^{1-\alpha} > 0
\end{aligned}$$

The determinant of the Jacobian is unambiguously negative, whereas its trace is positive (see Appendix C). Like in chapter 3, the equilibrium lies in

[19] A detailed derivation of the Jacobian's elements is to be found in Appendix C.

the fourth quadrant of the trace-determinant plane and is thus saddlepoint stable.[20]

This equilibrium will now be analyzed using specific data. Again, the chosen parameter values are based on estimates for the United States around the year 2005[21]. They are listed in Table 3:

α	η	θ	τ	B	γ	λ
0.3	0.85	0.025	0.01	0.075	0.3	0.0128

Table 3

The corresponding steady-state values are listed in Table 4:

g^*	$\mathcal{P}$	u^*	x^*	b^*
3.32%	3.82	0.5313	0.2009	0.2532

Table 4

As can be seen, the growth rate amounts to 3.32%, which is maybe a bit optimistic, but at least in non-crisis times quite reasonable. With u^* close to 0.5, people spend half their non-leisure time for work and the other half for education (whether or not this is realistic strongly depends on the exact definition of education). The ratio of human capital to physical capital is only 0.25, which means that there is about four times as much physical capital as human capital used in production. In order to improve environmental quality, efforts should be made to increase this ratio, as human capital does not harm the environment whereas physical capital does.
Next, the impact of different parameters on the BGP will be analyzed in more detail. One of the most interesting parameters of this model is surely the quality of schooling, B. We would expect a more effective education system to increase the stock of human capital as education becomes more rewarding and human capital accumulates easier. At the same time, a higher level of human capital should not increase the stock of physical capital and could even act as a substitute for physical capital in production and thus lower the aggregate stock of physical capital, so that the ratio $b^* = \frac{H^*}{K^*}$ rises. In that way, pollution could be reduced without any negative sideeffects on

[20]There is, analogously to chapter 3, an instable equilibrium in $(x, b, u) = (0, 0, 0)$. As this equilibrium is economicaly not relevant, we will ignore it it the following analysis.

[21]We set $\delta = 0$ to concentrate on other apects, which means that we are in a Lucas-setting where educational inputs are not taken into account.

GDP. This result demonstrates once more that the importance of high quality schooling cannot be overemphaszied! Figure 12 illustrates the relationship between the human/physical capital ratio and the effectiveness of schooling. In fact, B has increasing marginal effects on b^*, which demonstrates how influential the quality of schooling can be.[22]

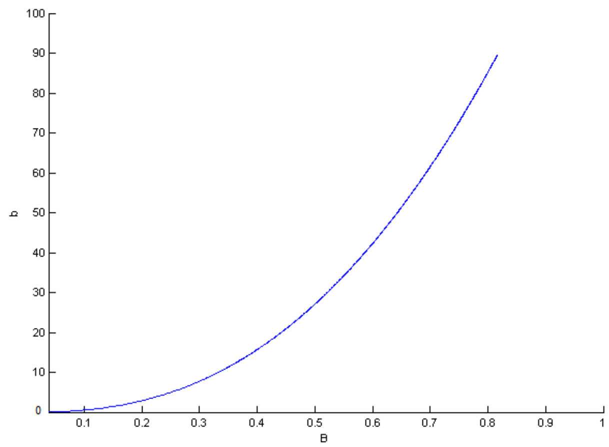

Figure 12: Bifurcation diagram of b^* with varying B

A better educational system cannot have negative effects on welfare, as GDP will surely not decline. Thus, the agents' consumption possibilities rise whereas the stock of physical capital does not rise or even declines. Both effects lead to an increasing consumption/capital ratio, which is depicted in Figure 13.

[22]Note that the abscissa in Figures 12 - 14 starts at 0.04 rather than 0, as for smaller values of B, the share of time devoted to work, u^*, is greater than 1 and thus inadmissible.

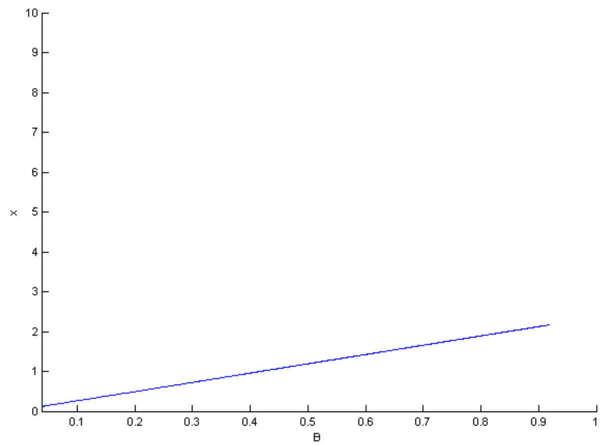

Figure 13: Bifurcation diagram of x^* with varying B

The implications of a rise in B on the time used for production, u^*, are fairly straightforward. As education becomes more and more rewarding, the share of time devoted to schooling, $(1-u^*)$ should rise. With agents acquiring ever more knowledge, they can afford to invest less time in work without having to accept salary losses, as it is effective labour, u^*H^*, that counts. However, a rising B has diminishing marginal effects on u^*, which means that the impact of quality of education on the agents' time allocation is limited after a certain degree. Also, agents cannot rely on their knowledge alone, they must provide some work in order not to have negative effects on GDP. The relationship is depicted in Figure (14).

Another very interesting parameter is the time preference rate θ. In dynamic macroeconomic models, the specific value of θ is very often decisive for the outcome of the model and one of the most influential parameters altogether. In chapter 3, we have neglected the time preference rate in the analysis of the equilibrium in order to concentrate fully on the demographic aspects of the model, but now we will discuss its properties.
The most striking feature of the time preference rate is the enormous impact it has on the equilibrium share of working time. Whereas in the original BGP-equilibrium, an agent devoted about half her non-leisure time to production, this fraction rises significantly with increasing θ, with agents spending all their non-leisure time working with a θ of only around 0.08. When agents care less about the future and prefer to consume now than to have higher consumption possibilities later, they use their time in a way that is most lucrative for them instantly, which is immediate work. They do not care that education would lead to much higher revenues later, as time spent for

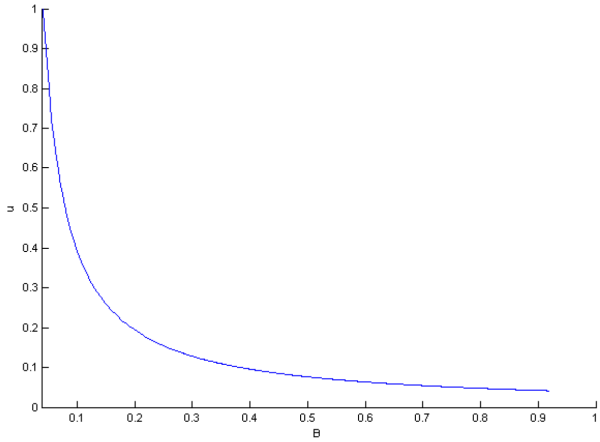

Figure 14: Bifurcation diagram of u^* with varying B

education now is time they cannot use to earn wages. If θ were 0, which means that agents are more indifferent between consumption now and later and only discount future consumption with the death rate, u^* would be around 0.2, in other words, 80% of an agent's non-leisure time would be used for schooling (see Figure 15).[23]

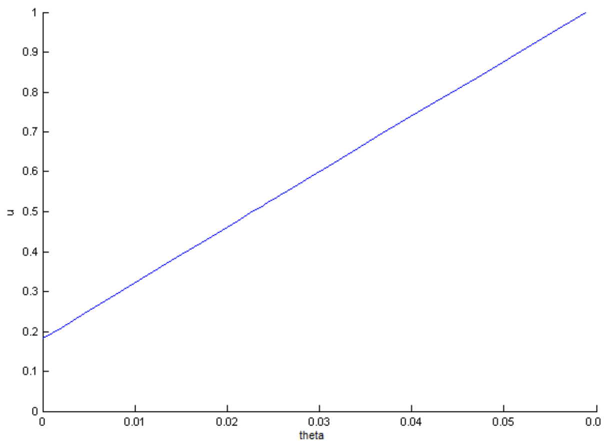

Figure 15: Bifurcation diagram of u^* with varying θ

With the same arguments as above, Figure 16 can be explained: due to

[23]As u^* reacts strongly to θ, the admissible range of Figures 15 - 17 is very limited, beginning at $\theta = 0$ and ending at $\theta = 0.06$.

the heavier discounting of the future, agents invest less time in schooling, thus the aggregate stock of human capital grows ever slower or even diminishes. At the same time, savings drop, as agents prefer to consume instantly than to save for later consumption. However, the human/physical capital ratio also drops, as agents stop learning at a faster pace than they cut down their savings.

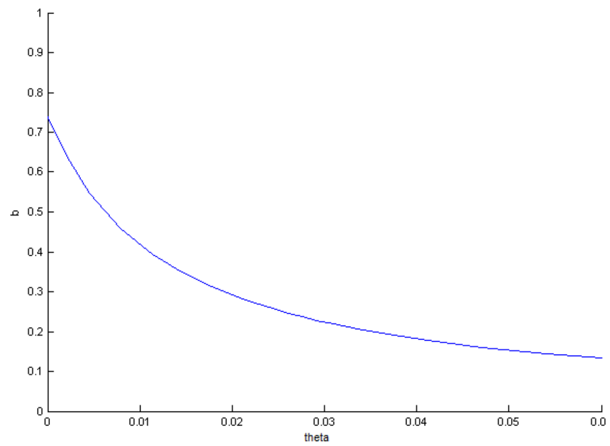

Figure 16: Bifurcation diagram of b^* with varying θ

The effect of an increased time preference rate on the consumption/physical capital ratio is straightforward: agents consume more and fund this higher consumption by saving less. The result - an increasing x - is illustrated in Figure 17.

The model we have just analyzed is an insightful way of dealing with the question how to save the environment while not harming the economy. We have found at least two answers: by increasing the quality of the education system, or by decreasing the time preference rate. Of course, these are no more than mere theoretical considerations, as any - benevolent - government would gladly increase the quality of schooling if this were so easy. In reality, a better schooling system can only be attained at some cost (whereas in the model we have silently assumed that B can be increased without extra costs). Although it is unquestioned that a good education system is profitable in the long run, governments tend not to be overly interested in long run effects. As John Maynard Keynes famously put it: "In the long run, we're all dead". Likewise, it is not possible to change the prevailing discount rate overnight. Nevertheless, the model has provided us with many insights on the relation between economic growth, education, the environment in a

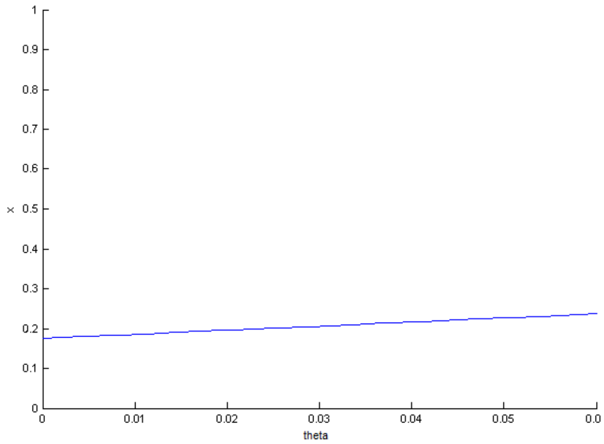

Figure 17: Bifurcation diagram of x^* with varying θ

finite horizon context.

In his analysis of the environmental policy's impact on the model economy, Pautrel finds another interesting result: in a model of finite horizons, the environmental policy can in fact have a positive impact on economic growth, so that a "win-win" situation occurs - raising the environmental tax rate would be beneficial to the state of the environment as well as to the economy's growth rate. This results from the generational turnover effect, which is absent in infinite horizon models. Further, he shows that a higher death rate, that is, a lower life-expectancy and thus an even more frequent generational turnover, increases the environmental policy's positive effect even more.

We will now turn to another model which places more emphasis on the way output and abatement services are generated.

4.3 Abatement technology and endogenous labour supply

In order to examine the role of technology in the abatement sector, Pautrel (2011a) modifies - and simplifies - the model structure of Pautrel (2011b), but adds a new component by distinguishing between the production technology in the output sector and in the abatement sector. The main differences to the Pautrel (2011b) model are the following:

Whereas in Pautrel (2011b) agents had a fixed amount of leisure time and could only choose between devoting time to production or education, Pautrel (2011a) endogenizes labour supply by letting agents decide freely on the time they spend for production, $u(s,t)$, leisure, $l(s,t)$, and schooling activities, $1 - u(s,t) - l(s,t)$. Thus, leisure time enters the utility function as a seperate variable.

4.3.1 Individuals and households

Agents maximize their expected lifetime utility

$$\boxed{max_{c(s,t),u(s,t),l(s,t),h(s,t),a(s,t)} \int_s^\infty [log\ c(s,t) + \xi_l\ log\ l(s,t) - \kappa\ log\ \mathcal{S}(t)] e^{-(\theta+\lambda)(t-s)}\ dt} \tag{83}$$

where κ is once more the weight in the utility function attached to the state of the environment and ξ_l the weight attached to time spent for leisure. Due to the endogenizing of leisure, agents do not only gain utility from consumption, as in the previous models, but also from their leisure time.

Human capital evolves in the following way:

$$\boxed{\dot{h}(s,t) = B[1 - u(s,t) - l(s,t)]h(s,t)} \tag{84}$$

Comparing this with the human capital dynamics of section 4.2, we notice two essential differences: First, we have to bear in mind that leisure time is now endogenous. $u(s,t)$ is hence no longer the share of non-leisure time used for production, but rather the share of time used for production. The share of time devoted to education is therefore, as mentioned above, $1-u(s,t)-l(s,t)$. The second difference is the absence of the factor $z(s,t)$, so educational inputs are no longer taken into account.
Newborns inherit the average aggregate human capital stock from the dying agents: $h(s,s) = H(s)$. This assumption of course implies that, contrarily to

the previous section, no knowledge is lost due to the generational turnover ($\eta = 1$).

Except for the absence of educational inputs, the budget constraint is identical to that of section 4.2:

$$\boxed{\dot{a}(s,t) = [r(t) + \lambda]a(s,t) + u(s,t)h(s,t)w(t) - c(s,t)}, \tag{85}$$

and again the transversality condition (5) to avoid a Ponzi-scheme applies. Agents determine their optimal intertemporal allocation of the three control variables (c, u, l) and two state variables (h, a) by maximizing (83) subject to (84), (85) and (5). Setting up the Hamiltonian

$$H = log\, c(s,t) + \xi_l\, log\, l(s,t) - \kappa\, log\, \mathcal{S}(t) + \mu_1(t)(B(1-u(s,t)-l(s,t))h(s,t)) +$$
$$\mu_2(t)((r(t) + \lambda)a(s,t) + u(s,t)h(s,t)w(t) - c(s,t))$$

we can derive the necessary first oder conditions

$$H_c = 0 \Leftrightarrow \frac{1}{c(s,t)} = \mu_2(t) \tag{86}$$
$$H_u = 0 \Leftrightarrow \mu_2(t)w(t) = \mu_1(t)B \tag{87}$$
$$H_l = 0 \Leftrightarrow \frac{\xi_l}{l(s,t)} = \mu_1(t)Bh(s,t) \tag{88}$$
$$H_h = (\theta + \lambda)\mu_1(t) - \dot{\mu_1}(t) \Leftrightarrow$$
$$\mu_1(t)B(1 - u(s,t) - l(s,t)) + \mu_2(t)u(s,t)w(t) = (\theta + \lambda)\mu_1(t) - \dot{\mu_1}(t) \tag{89}$$
$$H_a = (\theta + \lambda)\mu_2(t) - \dot{\mu_2}(t) \Leftrightarrow \mu_2(t)[r(t) + \lambda] = (\theta + \lambda)\mu_2(t) - \dot{\mu_2}(t) \tag{90}$$

By taking the time-derivative of (86) and substituting (90) we get the Euler-equation

$$\dot{c}(s,t) = [r(t) - \theta]c(s,t).$$

The Euler-equation, together with the budget restriction and the transversality condition enables us to explicitly express the consumption of any agent as per time t:[24]

$$c(s,t) = (\theta + \lambda)[a(s,t) + \omega(s,t)]$$

with $\omega(s,t) = \int_t^\infty [u(s,\nu)h(s,\nu)w(\nu)]e^{-\int_t^\nu [r(\zeta)+\lambda]d\zeta} d\nu$ the present value of lifetime earnings.

[24]The calculations to this result are nearly identical to those of equation (12) and can be found in Appendix A.

From the F.O.C., we can also derive the leisure choice of any agent as per time t: due to (88), we can express an individual's leisure choice as $l(s,t) = \frac{\xi_l}{\mu_1(t)Bh(s,t)}$, and by substituting (87) and (86) we obtain

$$l(s,t) = \xi_l \frac{c(s,t)}{w(t)h(s,t)}. \tag{91}$$

In section 4.2, the share of non-leisure time dedicated to work turned out to be identical for all agents, i.e. $u(s,t) = u(t)\ \forall s$. In this model we find even more: not only is the time dedicated to work identical for all agents, but also their share of leisure and hence their entire time allocation. This fact is remarkable, as it shows, that even with endogenized labour supply, or in other words, an endogenized decision upon leisure time, all individuals devote the same amount of time to schooling and production, despite the fact that they have different levels of wealth and knowledge. The justification for this statement comes from the F.O.C.: Substituting (87) into (89) yields $\mu_2(t)w(t)(1 - l(s,t)) = (\theta + \lambda)\mu_1(t) - \dot{\mu_1}(t)$. By taking the time-derivative of (87) and (86) we furthermore get $\dot{\mu_1}(t) = \frac{\dot{\mu_2}(t)w(t) + \dot{w}(t)\mu_2(t)}{B}$ and $\dot{\mu_2}(t) = (\theta - r(t))\mu_2(t)$. Combining these findings and rearranging, we obtain

$$B[1 - l(s,t)] = r(t) + \lambda - \frac{\dot{w}(t)}{w(t)}, \tag{92}$$

which implies that $l(s,t)$ is independent of s. Consequently, equation (89) shows that $u(s,t)$ is also independent of s.

4.3.2 The firm sector

The interesting changes occur in the production sectors. Whereas in chapter 3 and section 4.2 abatement was produced with output, Pautrel (2011a) distinguishes between the technology in the final goods sector and that in the abatement sector.
Consequently, final goods G are produced according to

$$\boxed{G = (\phi K)^{\alpha}(\psi H_p)^{1-\alpha}, \quad \phi, \psi, \alpha \in (0,1),} \tag{93}$$

with $H_p \equiv \int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)}ds$ the share of aggregate human capital that is used for production.

The production of abatement services F underlies the following technology:

$$\boxed{F = [(1-\phi)K]^{\epsilon}[(1-\psi)H_p]^{1-\epsilon}, \quad \epsilon \in (0,1).} \tag{94}$$

Except for the case $\alpha = \epsilon$, the production technologies differ. However, as both sectors are perfectly competitive, the production factors can be obtained by taking the respective partial derivatives of the firms' profit function. That way, it is possible to determine ϕ and ψ, that is, the fraction of the stock of physical respectively human capital that is devoted to final goods production, by assuming that the production factors in both sectors coincide (which they necessarily do in equilibrium).
National income is measured in terms of final output in this model, so

$$Y = G + P_F F. \tag{95}$$

Here, P_F denotes the relative price of abatement services in terms of final output.

Pollution is now no longer defined as a flow variable: the stock of pollution, denoted by $\mathcal{S}$ in order to make this difference apparent, accumulates according to

$$\dot{\mathcal{S}} = f\left(\frac{Y}{F}\right) - \zeta\mathcal{S} \tag{96}$$

where f is a nonnegative, strictly monotonic increasing, convex function. The stock of pollution therefore increases with production and decreases with abatement as in earlier models, but, due to the stock-definition, a third force plays a role in this process: the environment slowly recovers from existing pollution due to natural forces such as decay or air filtering by trees. This "natural rate of decay" is covered by a parameter $\zeta > 0$. Abatement takes place at firm level, but firms have no incentive to invest in abatement, as pollution only enters the individual agent's utility function. Hence, the government introduces an environmental tax, like in chapter 3. Yet, as this model deals with the stock rather than the flow of pollution, the way the environmental tax (which will be denoted by $t_Y \in (0, 1)$ to make the difference apparent) is being imposed needs to be altered. Because pollution increases with production, t_Y is being imposed upon National income (instead of the net flow of pollution as in earlier models). This approach drastically changes the way abatement is conducted, and it places much more emphasis on the role of the government than Pautrel (2009) and Pautrel (2011b): firms do not abate at all, as they still have no incentive to do so, because the tax is levied on National income rather than on pollution. The tax revenue, $t_Y Y$, is hence no longer transferred to the firms, but used entirely for abatement measures by the government. Consequently, the government basically has a demand monopoly on abatement services.

The environmental tax amounts to $t_Y Y$ and fully funds the abatement sector, which produces abatement services worth $P_F F$, so naturally

$$t_Y Y = P_F F. \tag{97}$$

Equations (95) and (97) combined yield $Y = \frac{G}{1-t_Y}$ and $P_F F = \frac{t_Y G}{1-t_Y}$. These relations will be important for the comparison of the final goods sector and the abatement sector.

Firms in both sectors maximize their profits, which add up to $\pi_G = (1-t_Y)G - r\phi K - w\psi H_p$ in the final goods sector and to $\pi_F = (1-t_Y)P_F F - r(1-\phi)K - w(1-\psi)H_p$ in the abatement sector. Carrying out the profit optimization procedure leads to

$$r = \alpha(1-t_Y)\frac{G}{\phi K} \tag{98}$$

$$w = (1-\alpha)(1-t_Y)\frac{G}{\psi H_p} \tag{99}$$

in the final goods sector. In the abatement sector, profit optimization yields $r = \epsilon(1-t_Y)\frac{P_F F}{(1-\phi)K}$ and $w = (1-\epsilon)(1-t_Y)\frac{P_F F}{(1-\psi)H_p}$. In order to compare these factor rewards with those of the final goods sector, we substitute $P_F F = \frac{t_Y G}{1-t_Y}$ and obtain

$$r = \epsilon t_Y \frac{G}{(1-\phi)K} \tag{100}$$

$$w = (1-\epsilon)t_Y \frac{G}{(1-\psi)H_p} \tag{101}$$

in the abatement sector.
For a market equilibrium, wage rates and interest rates need to be identical in both sectors. Therefore we can express ϕ and ψ by

$$\boxed{\phi = \frac{\alpha(1-t_Y)}{\alpha + t_Y(\epsilon-\alpha)}} \tag{102}$$

$$\boxed{\psi = \frac{(1-\alpha)(1-t_Y)}{(1-\alpha) + t_Y(\alpha-\epsilon)}} \tag{103}$$

Thus the fraction of physical capital used in final goods production depends positively on α, the intensity of physical capital in final goods production, and negatively on ϵ, the intensity of physical capital in the production of abatement services (as can easily be seen by taking the respective partial

derivatives). This is hardly surprising, as factors of production should naturally be used where there are most efficient (relatively seen).
In the special case where $\alpha = \epsilon$ (both sectors use the same technology), we find that $\phi = \psi = 1 - t_Y$. When $\alpha > \epsilon$, that is, the final goods sector produces with a technology that is relatively more intensive in physical capital than the abatement sector, we get $\phi > 1 - t_Y$ and $\psi < 1 - t_Y$, and vice versa when $\alpha < \epsilon$.

4.3.3 The general equilibrium

We now investigate in the dynamics of the economy. Taking the time-derivative of $H(t) = \int_{-\infty}^{t} h(s,t)\lambda e^{-\lambda(t-s)} ds$ leads to the rate of accumulation of aggregate human capital:

$$\dot{H}(t) = h(t,t)\lambda + \int_{-\infty}^{t} B[(1-u(t)-l(t))h(s,t)]\lambda e^{-\lambda(t-s)} - \lambda^2 h(s,t) e^{-\lambda(t-s)}\, ds$$

$$\overset{h(t,t)=H(t)}{=} \lambda H(t) + B(1-u(t)-l(t))H(t) - \lambda H(t) = H(t)[B(1-u(t)-l(t))]$$

$$\Rightarrow \boxed{\frac{\dot{H}(t)}{H(t)} = [B(1-u(t)-l(t))]} \tag{104}$$

With the equilibrium condition on the goods market

$$\boxed{(1-t_Y)Y(t) = C(t) + \dot{K}(t)} \tag{105}$$

or, equivalently, $G(t) = C(t) + \dot{K}(t)$, and the dynamics of aggregate consumption[25]

$$\boxed{\frac{\dot{C}(t)}{C(t)} = r(t) - \theta - \lambda(\theta+\lambda)\frac{K(t)}{C(t)}}, \tag{106}$$

it is possible to summarize the dynamics of the system by three differential equations and one static relation (once more, we define $x \equiv \frac{C}{K}$ and $b \equiv \frac{H}{K}$ and for better legibility $\Psi \equiv \frac{\phi}{\psi} = \frac{\alpha}{1-\alpha}\frac{(1-\alpha)-(\epsilon-\alpha)t_Y}{\alpha+(\epsilon-\alpha)t_Y}$). As $\frac{\dot{x}}{x} = \frac{\dot{C}K - \dot{K}C}{K^2} = \frac{\dot{C}}{K} - \frac{\dot{K}}{K}x$ and $\frac{\dot{b}}{b} = \frac{\dot{H}}{K} - \frac{\dot{K}}{K}b$, we obtain by inserting equations (104)-(106) and rearranging:

$$\boxed{\frac{\dot{x}}{x} = [\alpha(1-t_Y) - \phi]\Psi^{\alpha-1}(bu)^{1-\alpha} - \theta - \lambda(\lambda+\theta)x^{-1} + x} \tag{107}$$

[25]Which can be attained analogously to (32), see Appendix A.

$$\boxed{\frac{\dot{b}}{b} = B[1-u-l] - \phi\Psi^{\alpha-1}(bu)^{1-\alpha} + x} \tag{108}$$

The dynamics of $u(t)$ can be derived by substituting $G = (\phi K)^\alpha(\psi u H)^{1-\alpha}$ in (99) and taking the logarithm, which firstly yields

$$log\ w = log\ (1-\alpha)(1-t_Y) + \alpha\ (log\ \phi + log\ K - log\ \psi - log\ u - log\ H)$$

and, taking the time-derivative and rearranging,

$$\frac{\dot{u}}{u} = \frac{\dot{K}}{K} - \frac{\dot{H}}{H} - \frac{1}{\alpha}\frac{\dot{w}}{w}.$$

We already have an expression for $\frac{\dot{w}}{w}$, see (92), so we can substitute every term above (the interest rate can be expressed as $r = \alpha(1-t_Y)\Psi^{\alpha-1}(ub)^{1-\alpha}$) and express the dynamics of u via

$$\boxed{\frac{\dot{u}}{u} = \alpha^{-1}[B(1-l) - \lambda - \alpha(1-t_Y)\Psi^{\alpha-1}(bu)^{1-\alpha}] - \frac{\dot{b}}{b}} \tag{109}$$

The last relation needed to fully describe the dynamics of the system is

$$\boxed{l = \frac{\xi_l x u}{(1-t_Y)(1-\alpha)\Psi^\alpha(bu)^{1-\alpha}}} \tag{110}$$

Equation (110) comes from the fact, that according to (91), profit maximization yields $l(t) = \xi_l \frac{c(s,t)}{w(t)h(s,t)}$. With (93) and (99), this can equivalently be expressed as $l(t) = \frac{\xi_l u(t)}{(1-t_Y)(1-\alpha)u(t)^{1-\alpha}}\Psi^{-\alpha}\frac{b(t)^\alpha c(s,t)}{h(s,t)}$. As the left hand side of this equation is independent of s, the same must be true for the right hand side, so that $\frac{c(s,t)}{h(s,t)} = \frac{c(t)}{h(t)}$, which is further equal to $\frac{C(t)}{H(t)}$ as the population size is normalized to 1. With this, the expression for l in (110) is obvious.
To obtain expressions for the variables' equilibrium values, a few considerations need to be made. First, from (109) and the fact that $\dot{u} = \dot{b} = 0$ along the BGP, we get

$$b^*u^* = \left(\frac{B(1-l^*) - \lambda}{\Psi^{\alpha-1}\alpha(1-t_Y)}\right)^{\frac{1}{1-\alpha}}$$

As necessarily $b^*u^* > 0$, the above equation implies that we have to restrict $B(1-l^*) > \lambda$, and as $l^* \in [0,1)$ also $B > \lambda$. With (110), it is now easy to see that

$$l^* = \frac{\xi_l x^* u^* \alpha}{(1-\alpha)\Psi(B(1-l^*)-\lambda)}.$$

Solving the last equation for l^*, we find that

$$l^* = \frac{B-\lambda \pm \sqrt{(B-\lambda)^2 - 4\frac{\xi_l \alpha B x^* u^*}{(1-\alpha)\Psi}}}{2B}.$$

l^* needs to be a nonnegative real number, so that it is required that $\frac{\xi_l \alpha B x^* u^*}{(1-\alpha)\Psi} < \frac{(B-\lambda)^2}{4}$. For the case $\xi_l = 0$, we should of course have $l^* = 0$, as leisure is only wasted time when no utility can be gained from it. Therefore, the expression with "-" in front of the square root is the only sensible solution of the quadratic equation.[26] Knowing that $\frac{\dot{x}}{x} - \frac{\dot{b}}{b} = 0$ along the BGP, we get

$$0 = \alpha(1-t_Y)\Psi^{\alpha-1}(b^* u^*)^{1-\alpha} - \theta - \lambda(\lambda+\theta)x^{-1*} - B(1-u^*-l^*)$$

and, by substituting $(b^* u^*)$

$$\boxed{x^* = \frac{\lambda(\lambda+\theta)}{Bu^* - \lambda - \theta}}$$

As x^* needs to be nonnegative, we further have to restrict $u^* > \frac{\lambda+\theta}{B}$. Another - more cumbersome - expression for x^* comes directly from (108):

$$\frac{\dot{b}}{b} = 0 \Rightarrow x^* = -B(1-u^*-l^*) + \phi\Psi^{\alpha-1}\Psi^{1-\alpha}\frac{B(1-l^*)-\lambda}{\alpha(1-t_Y)} \overset{with\ (102)}{=}$$

$$-B + Bu^* + Bl^* + \frac{B - Bl^* - \lambda}{\alpha + (\epsilon-\alpha)t_Y}$$

With (102) and (103), we find that $\frac{1-\alpha}{\alpha}\Psi = \frac{1-\alpha-(\epsilon-\alpha)t_Y}{\alpha+(\epsilon-\alpha)t_Y} = \frac{1}{\alpha+(\epsilon-\alpha)t_Y} - 1$ and therefore

$$x^* = Bu^* - \lambda\left(1 + \frac{1-\alpha}{\alpha}\Psi\right) + \frac{1-\alpha}{\alpha}\Psi B(1-l^*)$$

As it is not possible to express u^* explicitly, we define a function $\Gamma(u)$, which implicitly yields u^* as solution of $\Gamma(u) = 0$. This is achieved by substracting one expression for x^* from the other and so $\Gamma(u)$ reads

$$\Gamma(u) = Bu - \lambda\left(1 + \frac{1-\alpha}{\alpha}\Psi\right) + \frac{1-\alpha}{\alpha}\Psi B(1-l^*) - \frac{\lambda(\lambda+\theta)}{Bu-\lambda-\theta}.$$

[26] Although Pautrel (2011a) comes to the same conclusion, his way of arguing is wrong.

Closer analysis of this implicit function shows, that under one more precondition, $\Gamma(u)$ has exactly got one root: by examining the feasible boundaries of $u^* \in (\frac{(\lambda+\theta)}{B}, 1)$, we find $lim_{u\rightarrow(\lambda+\theta)/B} = -\infty$ and $lim_{u\rightarrow 1} > 0$ if $B - \lambda - \theta > \frac{\lambda(\lambda+\theta)}{\theta}$, which is hence a sufficient condition for the existence of a unique u^* and thus a unique equilibrium. This can be seen by rearranging

$$\Gamma(1) = B - \lambda - \frac{\lambda(\lambda+\theta)}{B-\lambda-\theta} + \frac{1-\alpha}{\alpha}\Psi(\underbrace{B(1-l^*) - \lambda}_{>0})$$

and applying the sufficient condition.

Equipped with all this knowledge, it is now easy to determine the BGP-growth rate g^*: along the BGP, all relevant variables $(C, H$ and $K)$ must grow at the same rate g^*, therefore it suffices to calculate the growth rate of $\frac{\dot{H}}{H}$, which is equal to

$$g^* = B(1 - u^* - l^*), \tag{111}$$

see (104).

4.3.4 Stability analysis

Although Pautrel (2011a) does not analyse the BGP's stability, we will carry out the necessary calculations in order to fully understand the dynamic system. As before, to determine the stability of the BGP-equilibrium, we need to examine the Jacobian of the dynamic system, which reads

$$\mathcal{J} = \begin{pmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} & \mathcal{J}_{13} \\ \mathcal{J}_{21} & \mathcal{J}_{22} & \mathcal{J}_{23} \\ \mathcal{J}_{31} & \mathcal{J}_{32} & \mathcal{J}_{33} \end{pmatrix}$$

with

$$
\begin{aligned}
\mathcal{J}_{11} =&\ \lambda(\theta+\lambda)x^{*-1} + x^* > 0 \\
\mathcal{J}_{12} =&\ (1-\alpha)[\alpha(1-t_Y)-\phi]\Psi^{\alpha-1}(u^*b^*)^{1-\alpha}\frac{x^*}{b^*} \\
\mathcal{J}_{13} =&\ (1-\alpha)[\alpha(1-t_Y)-\phi]\Psi^{\alpha-1}(u^*b^*)^{1-\alpha}\frac{x^*}{u^*} \\
\mathcal{J}_{21} =&\ b^* > 0 \\
\mathcal{J}_{22} =&\ -(1-\alpha)\phi\Psi^{\alpha-1}(u^*b^*)^{1-\alpha} < 0 \\
\mathcal{J}_{23} =&\ -B-(1-\alpha)\phi\Psi^{\alpha-1}(u^*b^*)^{1-\alpha}\frac{b^*}{u^*} < 0 \\
\mathcal{J}_{31} =&\ -u^* < 0 \\
\mathcal{J}_{32} =&\ -(1-\alpha)(u^*b^*)^{1-\alpha}\Psi^{\alpha-1}\frac{u^*}{b^*}[1-t_Y-\phi] \\
\mathcal{J}_{33} =&\ B-(1-\alpha)(u^*b^*)^{1-\alpha}\Psi^{\alpha-1}[1-t_Y-\phi]
\end{aligned}
$$

The derivation of these elements is much simpler than in section 4.2 and follows immediately from the dynamic system. We will now derive the Jacobian's determinant and trace and thereby determine the stability of the BGP-equilibrium.
The determinant can be computed using the cofactor-method:

$$
det\ \mathcal{J} = \underbrace{\mathcal{J}_{11}(\mathcal{J}_{22}\mathcal{J}_{33}-\mathcal{J}_{23}\mathcal{J}_{32})}_{=:\ \mathbb{I}} - \underbrace{\mathcal{J}_{21}(\mathcal{J}_{12}\mathcal{J}_{33}-\mathcal{J}_{32}\mathcal{J}_{13})}_{=:\ \mathbb{II}} + \underbrace{\mathcal{J}_{31}(\mathcal{J}_{12}\mathcal{J}_{23}-\mathcal{J}_{22}\mathcal{J}_{13})}_{=:\ \mathbb{III}}.
$$

$$
\begin{aligned}
\mathbb{I} &= \mathcal{J}_{11}\left((1-\alpha)\Psi^{\alpha-1}(bu)^{1-\alpha}\left[-\phi Bu - \frac{u}{b}(1-t_Y-\phi)\right]\right) \\
&= \lambda(\theta+\lambda)x^{*-1} + x^*\left(-(1-\alpha)\Psi^{\alpha-1}(bu)^{1-\alpha}Bu(1-t_Y)\right) < 0 \\
\mathbb{II} &= \mathcal{J}_{21}\left((1-\alpha)[\alpha(1-t_Y)-\phi]\Psi^{\alpha-1}(bu)^{1-\alpha}\right)\frac{x}{b}Bu \\
&= (1-\alpha)[\alpha(1-t_Y)-\phi]\Psi^{\alpha-1}(bu)^{1-\alpha}Bux \\
\mathbb{III} &= \mathcal{J}_{31}\left((1-\alpha)[\alpha(1-t_Y)-\phi]\Psi^{\alpha-1}(bu)^{1-\alpha}\right)(-Bx) \\
&= (1-\alpha)[\alpha(1-t_Y)-\phi]\Psi^{\alpha-1}(bu)^{1-\alpha}Bux
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow det\ \mathcal{J} = \mathbb{I} - (1-\alpha)[\alpha(1-t_Y)-\phi]\Psi^{\alpha-1}(bu)^{1-\alpha}Bux \\
+(1-\alpha)[\alpha(1-t_Y)-\phi]\Psi^{\alpha-1}(bu)^{1-\alpha}Bux = \mathbb{I} < 0.
\end{aligned}
$$

For the trace, we have

$$
tr\ \mathcal{J} = \mathcal{J}_{11}+\mathcal{J}_{22}+\mathcal{J}_{33} = \lambda(\lambda+\theta)x^{-1}+x+Bu-(1-\alpha)\Psi^{\alpha-1}(bu)^{1-\alpha}(1-t_Y)
$$

which, in equilibrium, is equal to (substituting (bu))

$$tr\ \mathcal{J} = \lambda(\lambda+\theta)x^{-1} + x + Bu - (1-\alpha)\frac{B(1-l)-\lambda}{\alpha} > 0.$$

Therefore, the equilibrium lies once more in the fourth quadrant of the trace-determinant plane and is saddlepoint stable.

We have now seen, that in a model of finite horizons with environmental care and human capital accumulation, all agents devote the same share of their time to production and to education, even if we endogenize leisure and thus labour supply. When we distinguish between the technologies used in final output production and abatement service production, factors of production are used more intensively in the sector where they are relatively more efficient. By imposing a few sensible restrictions on the model parameters, we obtain a unique saddlepoint stable equilibrium. The "win-win" situation of the last section can no longer occur, as final output is now the source of pollution.
Before we turn to a closer analysis of this equilibrium, we will alter the model structure slightly in the next section.

4.4 Human capital spillover in abatement technology

In order to further investigate the role of technology in the abatement sector, we modify the production technology used by Pautrel (94) in a Lucas-like manner: we allow for the average share of human capital used in production, $\overline{H_p} = \frac{\int_{-\infty}^{t} h(s,t)u(t)\lambda e^{-\lambda(t-s)}ds}{\int_{-\infty}^{t} \lambda e^{-\lambda(t-s)}ds}$ to enter the production technology in the abatement sector as a positive externality, according to

$$F = [(1-\phi)K]^{\epsilon}[(1-\psi)H_p]^{1-\epsilon}\overline{H_p}^{\gamma}, \qquad \epsilon, \gamma \in (0,1). \tag{112}$$

There is, however, an important difference between the Lucas-approach and ours: in Lucas (1988), the externality occurred at the individual level, specifically at the agents' human capital dynamics, whereas in our model, the externality only occurrs at firm level. We will consider the various implications of this modification assuming that there is a benevolent social planner in charge of the economy. To the social planner, $\overline{H_p}$ does not represent an externality, as she can decide not only on the allocation of production factors, but also on their disposal. Consequently, the modified production technology in the abatement sector (112) looks slightly different to her:

$$\boxed{F = [(1-\phi)K]^{\epsilon}(1-\psi)^{1-\epsilon}H_p^{1-\epsilon+\gamma}, \qquad \epsilon, \gamma \in (0,1)} \tag{113}$$

Firms in the abatement sector maximize their profits $\pi = (1-t_Y)P_F F - r(1-\phi)K - w(1-\psi)H_p$. Although the social planner internalizes the externality, the interest rate in this setting (and thus ϕ) is precisely the same as before, which can be seen from

$$\frac{\partial \pi}{\partial K} = (1-t_Y)P_F(1-\phi)^{\epsilon}\epsilon K^{\epsilon-1}(1-\psi)^{1-\epsilon}H_p^{1-\epsilon+\gamma} - r(1-\phi) = 0$$

$$\Leftrightarrow r = \epsilon(1-t_Y)\frac{P_F F}{(1-\phi)K} = \frac{\epsilon T_Y G}{K(1-\phi)}.$$

However, the wage rate takes a different form:

$$\frac{\partial \pi}{\partial H_p} = (1-t_Y)P_F[(1-\phi)K]^{\epsilon}(1-\epsilon+\gamma)(1-\psi)^{1-\epsilon}H_p^{-\epsilon+\gamma} - w(1-\psi) = 0$$

$$\Leftrightarrow w = \frac{(1-\epsilon+\gamma)(1-t_Y)P_F F}{H_p(1-\psi)} = \frac{(1-\epsilon+\gamma)t_Y G}{H_p(1-\psi)}.$$

Comparing this with the - unchanged - wage rate in the final goods sector (99), and knowing that the two wage rates need to match in equilibrium, we can express ψ by

$$\frac{(1-\alpha)(1-t_Y)}{\psi} = \frac{(1-\epsilon+\gamma)t_Y}{1-\psi}$$

$$\Rightarrow \boxed{\psi = \frac{(1-\alpha)(1-t_Y)}{(1-\alpha)-(\epsilon-\alpha-\gamma)t_Y}}$$

Comparing this expression for ψ with that in the baseline model, we see that there is an additional positive term in the denominator ($t_Y\gamma$), so ψ in this setting is smaller. This implies, that due to the more human capital intensive production technology in the abatement sector, a larger proportion of effective human capital is allocated there.

Now, how does the BGP-growth rate react to the change in technology? To answer this question, we have to analyze the sensitivity of l^* and u^* to a change in Ψ. In order to do so, we will start with a few considerations. As the spillovers in abatement technology only affect the parameter ψ (and thus of course Ψ), we can use all the insights gained in section 4.3 keeping the alteration of these parameters in mind. Recalling that

$$l^* = \frac{B-\lambda \pm \sqrt{(B-\lambda)^2 - 4\frac{\xi_l \alpha B x^* u^*}{(1-\alpha)\Psi}}}{2B}$$

and

$$\Gamma(u) = Bu - \lambda\left(1 + \frac{1-\alpha}{\alpha}\Psi\right) + \frac{1-\alpha}{\alpha}\Psi B(1-l^*) - \frac{\lambda(\lambda+\theta)}{Bu-\lambda-\theta},$$

we see that l^* depends (amongst others) on u^* and on Ψ. Substituting l^* into $\Gamma(u)$, we observe that $\Gamma(u)$ also depends on u^* and on Ψ. To analyze the effects of a change in Ψ on u^*, we need the implicit function theorem, as we cannot express u^* explicitly. The implicit function theorem states, that (in our situation) $\frac{\partial u^*}{\partial \Psi} = -\frac{\frac{\partial \Gamma}{\partial \Psi}}{\frac{\partial \Gamma}{\partial u^*}}$. By determining the partial derivatives

$$\frac{\partial l^*}{\partial u} = \frac{-\xi_l \alpha}{(1-\alpha)\Psi \frac{\lambda(\lambda+\theta)^2}{(Bu-\lambda-\theta)^2}\sqrt{(B-\lambda)^2 - 4\frac{\xi_l\alpha}{(1-\alpha)\Psi}Bx^*u^*}} < 0$$

$$\frac{\partial l^*}{\partial \Psi} = \frac{-\xi_l \alpha x^* u^*}{(1-\alpha)\Psi^2\sqrt{(B-\lambda)^2 - 4\frac{\xi_l\alpha}{(1-\alpha)\Psi}Bx^*u^*}} < 0$$

$$\frac{\partial \Gamma}{\partial u} = B - \frac{1-\alpha}{\alpha}\Psi B\frac{\partial l^*}{\partial u} + B\frac{\lambda(\lambda+\theta)}{(Bu-\lambda-\theta)^2} =$$

$$B + \frac{\xi_l B\frac{\lambda(\lambda+\theta)^2}{(Bu-\lambda-\theta)^2}}{\sqrt{(B-\lambda)^2 - 4\frac{\xi_l\alpha}{(1-\alpha)\Psi}Bx^*u^*}} + B\frac{\lambda(\lambda+\theta)}{(Bu-\lambda-\theta)^2} > 0$$

$$\frac{\partial \Gamma}{\partial \Psi} = \frac{1-\alpha}{\alpha}\left(B(1-l^*) - \lambda - B\frac{\partial l^*}{\partial \Psi}\Psi\right) =$$

$$\frac{1-\alpha}{\alpha}\left(\underbrace{B(1-l^*)-\lambda}_{>0} + \underbrace{B\frac{\xi_l\alpha x^*u^*}{(1-\alpha)\Psi\sqrt{(B-\lambda)^2 - 4\frac{\xi_l\alpha}{(1-\alpha)\Psi}Bx^*u^*}}}_{>0}\right) > 0$$

it becomes apparent, that $\frac{\partial l^*}{\partial \Psi} < 0$ and that $\frac{\partial u^*}{\partial \Psi} < 0$ by applying the implicit function theorem.
Comparing the new $\Psi_{new} = \frac{\alpha}{1-\alpha}\frac{(1-\alpha)-(\epsilon-\alpha-\gamma)t_Y}{\alpha+(\epsilon-\alpha)t_Y}$ with the old $\Psi_{old} = \frac{\alpha}{1-\alpha}\frac{(1-\alpha)-(\epsilon-\alpha)t_Y}{\alpha+(\epsilon-\alpha)t_Y}$, we find that there is an additional positive term γt_Y in the nominator, so that $\Psi_{new} > \Psi_{old}$. With the knowledge we have just gained about the respective partial derivatives, this implies that $u^*_{new} < u^*_{old}$ and $l^*_{new} < l^*_{old}$ so that, finally,

$$g^*_{new} > g^*_{old}.$$

A more human capital intensive production function is therefore beneficial to the BGP-growth rate. Yet, how beneficial is it really? To answer this question, we will first take a look at the Bifurcation diagrams of the system variables, again with realistically calibrated data:

α	θ	t_Y	B	λ	ϵ	ξ_l
0.3	0.025	0.01	0.085	0.0128	0.3	0.15

Table 5

x^*	b^*	u^*	l^*	g^*
0.158	0.181	0.481	0.088	3.67%

Table 6

We start with $\alpha = \epsilon$, that is, both output and abatement sector use the same technology, to have a clear view on the changes a "knowledge spillover" in abatement technology brings with it. The BGP-growth rate with $\gamma = 0$ is 3.67%. From the considerations above, we know that knowledge spillovers will augment the growth rate, but so far we do not know how much. In fact, Figures 18 - 20 depict a very small impact:

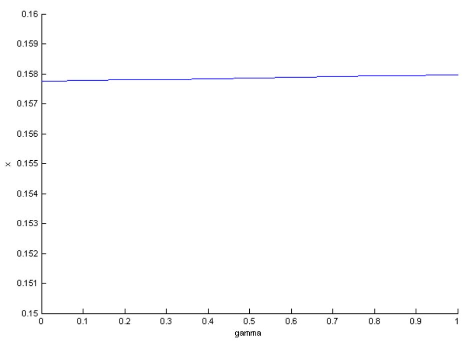

Figure 18: Impact of γ on x^*

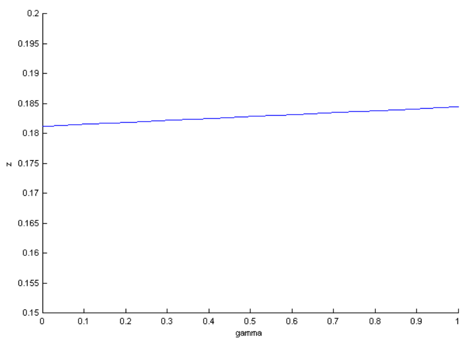

Figure 19: Impact of γ on b^*

For $\gamma = 0$, we are in the situation of section 4.3. Knowledge spillovers can be interpreted similarly to a more effective schooling system, see section 4.2: human capital accumulation becomes more rewarding, so H^* increases, whereas the physical capital stock remains at most unchanged, if not driven back by more human capital. Hence, the human/physical capital ratio b^* rises. In the same way, welfare and thus consumption possibilities increase,

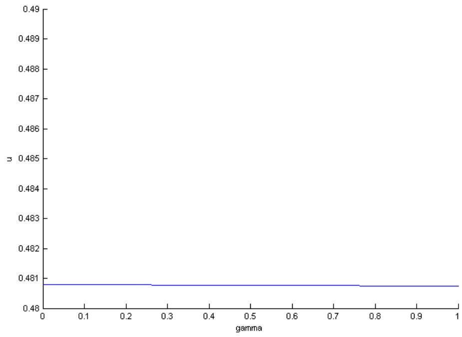

Figure 20: Impact of γ on u^*

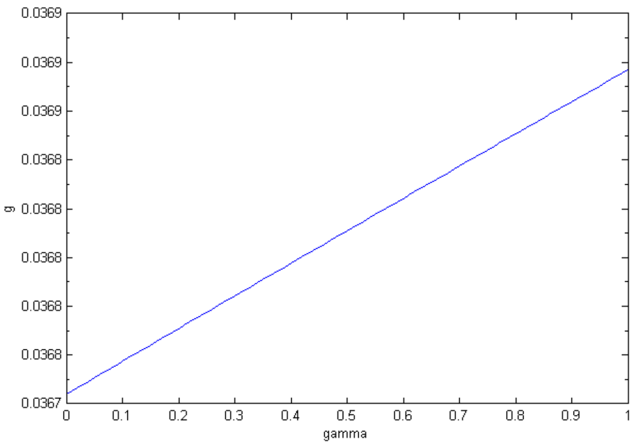

Figure 21: Impact of γ on the BGP-growth rate g^*

which is beneficial to the consumption/capital ratio x^*. And because education is more rewarding and the level of knowledge increases, agents can cut down on their working hours without any income losses, as it is effective labour u^*H^* that determines wages, so u^* decreases. Yet, all these effects happen at a tiny scale compared to an increase of the quality of schooling or even the time preference rate (see again section 4.2). Consequently, the effects of knowledge spillovers in abatement technology on the growth rate are small:

An increase in γ from 0 to 0.5, which would already be enormous, only augments the growth rate by 0.01 points of a percent. There are clearly more efficient ways to boost the economy, but stimulating the growth rate was never our objective when we introduced knowledge spillovers in abate-

ment technology. Lucas himself included knowledge spillovers into his model for one specific reason: to account for the huge differences between countries with respect to technology and the level of wealth. The knowledge spillovers intensify the advantage technologically advanced countries have compared to countries with less advanced technology, but they provoke level effects rather than growth effects. In other words, with level effects, rich countries stay rich, their level of wealth enhanced by the knowledge spillover, while poor countries remain poor, but in equilibrium their growth rates match.
Indeed, we have adapted the model in a way that it embodies this modification proposed by Lucas. Hence, our approach should not be viewed upon as an amelioration of the Pautrel model, but rather as an extension of it.

5 Comparison of the models

Pollution
The way pollution is defined is crucial to the findings of the models. Pautrel (2009) and Pautrel (2011b) study the net flow of pollution, while Pautrel (2011a) models pollution as a stock variable. In Pautrel (2009), pollution increases with output and decreases with abatement activities: $\mathcal{P}(t) = \left[\frac{Y(t)}{F(t)}\right]^{\gamma}$. This implies, that the only way to reduce or stabilize pollution when the economy is growing is to invest in more abatement. Contrarily, Pautrel (2011b) defines the net flow of pollution as $\mathcal{P}(t) = \left[\frac{K(t)}{F(t)}\right]^{\gamma}$, so that physical ("brown") capital rather than output is the source of pollution. This way, it is possible for the economy to grow without harming the environment via the accumulation of human ("green") capital. Investment in human capital therefore bears a double dividend, as it increases both future consumption possibilities and future environmental quality. This definition of pollution is clearly beneficial to the aggregate level of human capital.
The approach in Pautrel (2011a) is of a different form: instead of the flow, the stock of pollution is considered, which increases with production and decreases with abatement and due to natural decay, according to

$$\dot{\mathcal{S}} = f\left(\frac{Y}{F}\right) - \zeta\mathcal{S}.$$

Production and technological progress
The economy in Pautrel (2009) produces output according to

$$Y(t) = K(t)^{\alpha}\left[A(t)L(t)\right]^{1-\alpha}.$$

There is no technological progress (of course it could be introduced in manifold ways), for reasons of simplicity the level of technology is set to $A(t) \equiv A^{\frac{1}{1-\alpha}}$. In Pautrel (2011b), output is generated via

$$Y(t) = K(t)^{\alpha}\left[\int_{-\infty}^{t} u(t)h(s,t)\lambda e^{-\lambda(t-s)}ds\right]^{1-\alpha}$$

and technological progress takes place in the form of human capital accumulation:

$$\dot{h}(s,t) = B[(1-u(t))h(s,t)]^{1-\delta}z(s,t)^{\delta}.$$

$u(t)$ represents the tradeoff between human capital accumulation and production: a unit of time can only be used for either production, which increases present wages, present consumption possibilities and thereby present utility, or human capital accumulation, which increases future consumption possibilities and utility. Due to population dynamics, there is a generational turnover effect in aggregate human capital, as only a fraction η of the human capital of the dying agents is inherited by newborn agents. Aggregate human capital follows

$$\dot{H}(t) = B(1 - u(t))\tilde{z}(t)^{\delta}H(t) - (1 - \eta)\lambda H(t),$$

where the second term represents this turnover effect.
Pautrel (2011a) introduces a second production sector for abatement services, and distinguishes between the production technologies in the two sectors: final goods are produced according to

$$G(t) = (\phi K(t))^{\alpha}(\psi H_p(t))^{1-\alpha},$$

abatement services according to

$$F(t) = [(1 - \phi)K(t)]^{\epsilon}[(1 - \psi)H_p(t)]^{1-\epsilon}.$$

Technological progress again takes place via human capital accumulation:

$$\dot{h}(s,t) = B[1 - u(s,t) - l(s,t)]h(s,t),$$

the generational turnover effect in aggregate human capital no longer exists as newborns inherit the entire human capital from the dying agents (in terms of Pautrel (2011b), $\eta = 1$).

Utility
The utility functions are nearly identical in all the models except for some scaling parameters. Agents gain utility from consumption throughout their lives, present consumption being valued more highly than future consumption due to the probability to die and the rate of time preference. In Pautrel (2011a), individuals also gain utility from their leisure time, while in both of the other models, labour supply is exogenous and thus not part of the utility function. Pollution enters the utility function as a bad (agents like a clean environment), whether pollution increases with output or with physical capital, and whether it is modelled as a stock or a flow variable. If a non-pollutant factor of production exists (as represented by human capital, which is always assumed to be non-polluting), as in Pautrel (2011b) and Pautrel (2011a), the presence of environmental concerns in the utility function enhances the acquisition of the non-pollutant factor as discussed above.

Population dynamics

In Pautrel (2009) agents face exogenous birth (b) and death (p) rates, which make it possible to analyze the effects of demographical change via the growth rate (g_N) of the economy: $g_N = b - p$. Pautrel (2011b) and Pautrel (2011a) abstract from this possibility by simplifying the demographic structure, so that birth and death rates are identical and denoted by λ. Of course, with these assumptions, the total size of the population stays the same forever. The only element that can be adjusted is the life expectancy of individuals ($\frac{1}{\lambda}$).

Results

The models' outcomes depend to a great extent on the definition of pollution. If pollution grows with output, as in Pautrel (2009) and Pautrel (2011a), there is no way the state of the economy and the state of the environment can be ameliorated at the same time; all decisions upon environmental taxes are hence tradeoffs between environmental care and economic stimulation. Only in Pautrel (2001b), where human capital exists as a non-pollutant factor, can both be achieved at the same time. The reason for this lies in the generational turnover effect. By raising the environmental tax rate, both the economy and the environment can benefit under reasonable circumstances.
The impact of demographic change depends on the precise demographic change, at least in Pautrel (2009); in the other models, birth and death rate are identical, so there is little room for demographic change. While a higher birth rate can have positive as well as negative effects on per-capita consumption and capital, the impact of a lower death rate is unanimously positive on per capita capital and negative on consumption in Pautrel (2009). This is because of the assumed age-earning profiles, which depend on the birth rate, but not on the death rate.

6 Summary

In the course of this thesis, we have analyzed and discussed three growth models with environmental and/or educational aspects, and briefly presented the basic models they originate from. What is common to all the models is the assumption that agents have finite horizons, which means they will certainly die at some point but they do not know when. Thus, agents discount the future more heavily. The concept of death with all its implications (annuities paid by life insurances, higher discount rate,...) is what draws a distinction between finite horizon models and the standard neoclassical growth models, where representative individuals live forever. In order to be able aggregate the heterogeneous model population, it is further assumed that all agents face the same instantaneous probability to die. This is probably one of the finite horizon model's greatest shortcomings, as it implies that a newborn and an old man have the same life expectancy.

The first model we considered was Pautrel (2009), a Blanchard-Yaari model with environmental concerns and exogenous technological change. Demographic change is possible, and agents supply labour depending on their age. Life insurances avoid unintended bequests and increase the agents' consumption possibilities throughout their lives. Pollution grows with output and is a public bad. Abatement takes place at firm level, where the government imposes an environmental tax upon firms, which is fully returned to them to fund their abatement activities.
We then looked for steady states and their stability. There are two steady states, although the one at $(k = 0, c = 0)$ is instable and thus economically non-relevant as no poverty trap occurs and hence the economy always converges towards the other equilibrium. We analyzed the other steady state, which is stable, and its sensitivity to various model parameters. We saw, that both steady state consumption and the steady state capital stock depend negatively on the birth rate when accounting for per-worker variables, but positively when accounting for per-capita variables (for realistic values of b). The reason for this lies in the assumed age-specific labour supply, which is also responsible for the diverging sensitivity of per-worker and per-capita variables to productivity. Effectively, in this model economy, by increasing fertility and reducing early retirement (that is, reducing ψ), the steady state could be significantly improved. The state of the environment, at the same time, remains unchanged in the steady state if we assume that the environmental tax rate grows with output to encourage firms to abate more when output rises. A tighter environmental policy would be beneficial to the environment, but at the same time harm the economy's growth rate.

We then turned to Pautrel (2011b), which combines the Blanchard-Yaari model and the Lucas model. Agents can now accumulate human capital, which is a production factor, and thereby increase the level of technology. Demographic change other than varying life-expectancies is no longer possible, as the model is already very complex. Pollution now grows with physical capital, therefore a clean factor of production exists, namely human capital, which bears a double dividend: it is the driving force behind technological advancement, and it doesn't harm the environment. Abatement again takes place at firm level, the government taxing the net flow of pollution, but returning the taxes fully to the firms.
This model is, in contrast to Pautrel (2009), a proper growth model, so we had to look for balanced growth paths rather than steady states. There is one - stable - BGP, which which we analyzed with respect to the quality of schooling and the time preference rate. All model variables react very strongly to changes in these parameters, especially to changes in the time preference rate, which usually plays a decisive role in growth models.
Contrarily to the last model, it is possible in this model (under some reasonable circumstances) to increase the economy's growth rate while at the same time intensifying environmental protection by raising the environmental tax rate. The reason for this lies in the existence of a non-polluting factor of production - human capital. By increasing the aggregate stock of human capital, agents can reduce the stock of - polluting - physical capital without suffering income losses, thus a "win-win" situation occurs.

The last model we analyzed was Pautrel (2011a), which asumes different technologies in the final output sector and the abatement sector. Agents again gained knowledge by accumulating human capital, which they could use as an input factor in production. Leisure time is endogenized, so agents decide freely upon the amount of time they spend working, improving their skills or relaxing. The polluting factor is final output, abatement is no longer carried out by the firms, but rather by the government: firms are taxed proportionally to output. The tax revenue is used by the government to fully subsidize the abatement sector.
Although this model is in many ways similar to Pautrel (2011b), there is one big difference: whereas in Pautrel (2011b) a "win-win" situation could occur, where a tighter environmental policy would at the same time boost the economy's growth rate and ameliorate the state of the environment, this is no longer possible in this model, as no non-pollutant factor of production exists. All decisions upon the level of the environmental tax are therefore tradeoffs between economic growth and environmental care.

We extended the model by modifying the production technology in a Lucas (1988)-way, so that the average level of human capital enters the production technology in the abatement sector as a positive externality - a kind of a knowledge spillover in abatement technology. A benevolent social planner is presumed to be in charge of the entire economy (or at least the abatement sector). We again looked for balanced growth paths - there is exactly one, stable, equilibrium - and compared the modified version to the original model. Thereby, we saw that the externality has a positive effect on the economy's growth rate, although the effect is tiny (0.01 of a percent for huge spillovers).

Generally, we have seen that in finite horizon models, the impact of demographic change on economic variables depends to a great extent on the precise nature of the demographic change. The presence of environmental concerns in the utility function mostly leads to tradeoffs between environmental care and economic growth, except when a non-pollutant factor of production exists. The quality of education turned out to increase welfare in every setting, although we did not account for the costs of higher quality education.

7 Bibliography

Acemoglu, Daron (2009). Introduction to Modern Economic Growth. Princeton University Press.

Blanchard, Olivier J. (1985). Debt, Deficits and Finite Horizons. Journal of Political Economy, vol. 93, no.2.

Diamond, Peter A. (1965). National Debt in a Neoclassical Growth Model. American Economic Review, Vol. 55 (No. 5):1126-1150.

Heijdra, Ben J. and Van der Ploeg, Frederick (2002). Foundations of Modern Macroeconomics. Oxford University Press.

Lucas, Robert E. (1988). On the Mechanics of Economic Development. Journal of Monetary Economics, 22(1):3-42.

Pautrel, Xavier (2009). Macroeconomic Implications of Demography for the Environment: A Life-Cycle Perspective. Fondazione Eni Enrico Mattei Working Papers.

Pautrel, Xavier (2011a). Abatement Technology and the Environment-Growth Nexus with Education. Fondazione Eni Enrico Mattei Working Papers.

Pautrel, Xavier (2011b). Environmental Policy, Education and Growth: A Reappraisal when Lifetime is Finite. Macroeconomic Dynamics.

Ramsey, Frank P. (1928). A Mathematical Theory of Saving. The Economic Journal, Vol. 38 (No. 152):543-559.

Rauscher, Michael (2009). Green R&D versus End-of-Pipe Emission Abatement: A Model of Directed Technical Change. Thünen-Series of Applied Economic Theory, Working Paper No. 106.

Yaari, Menahem E. (1965). Uncertain Lifetime, Life Insurance, and the Theory of the Consumer. Review of Economic Studies 32: 137-50.

8 Appendix

Appendix A - Fundamental Equations

Derivation of equation (12)

For simplicity let $R(t,\nu) = e^{-\int_t^\nu r(\mu)+\lambda \ d\mu}$. We start with the by now well known budget constraint

$$\dot{a}(s,t) = [r(t) + \lambda]a(s,t) + w(t) - c(s,t). \tag{114}$$

Multiplying $R(t,\nu)$ on both sides yields

$$[\dot{a}(s,t) - (r(t) + \lambda)a(s,t)]R(t,\nu) = [w(t) - c(s,t)]R(t,\nu).$$

Using the Leibnitz-rule,

$$\frac{d}{dt}a(s,t)R(t,\nu) = \dot{a}(s,t)R(t,\nu) +$$

$$\left[\underbrace{-\int_t^\nu \frac{\partial}{\partial t}(r(\mu) + \lambda) \ d\mu}_{=0} + \underbrace{(r(\nu) + \lambda)\frac{\partial \nu}{\partial t}}_{=0} - (r(t) + \lambda) \underbrace{\frac{\partial t}{\partial t}}_{=1}\right] a(s,t)R(t,\nu)$$

therefore the budget constraint further yields

$$\frac{d}{dt}[a(s,t)R(t,\nu)] = [w(t) - c(s,t)] \tag{115}$$

By integrating (115) we get, using the transversality condition (5):

$$\int_t^\infty da(s,\nu)R(t,\nu) \ d\nu = \int_t^\infty (w(\nu) - c(s,\nu))R(t,\nu) \ d\nu$$

$$\Rightarrow \underbrace{lim_{\nu\to\infty} e^{-\int_t^\nu (r(\mu)+\lambda) \ d\mu} a(s,\nu)}_{=0} - \underbrace{e^{-\int_t^t (r(\mu)+\lambda) \ d\mu}}_{=1} a(s,t) =$$

$$\omega(t) - \int_t^\infty c(s,\nu)R(t,\nu) \ d\nu,$$

therefore

$$\int_t^\infty c(s,\nu)R(t,\nu) \ d\nu = a(s,t) + \omega(t) \tag{116}$$

From the Euler equation

$$\frac{\dot{c}(s,\nu)}{c(s,\nu)} = r(\nu) - \theta$$

we get in a straightforward way

$$\int_t^\infty \frac{\dot{c}(s,\nu)}{c(s,\nu)}\ d\nu = \int_t^\infty r(\nu) - \theta\ d\nu$$

$$\Rightarrow ln\ c(s,\nu) = \int_t^\infty r(\nu) - \theta\ d\nu \Rightarrow c(s,\nu) = e^{\int_t^\infty r(\nu)-\theta\ d\nu} c_0.$$

As $c(s,t) = c_0$, we can express $c(s,\nu)$ $(\nu \geq t)$ as

$$c(s,\nu) = e^{\int_t^\infty r(\nu)-\theta\ d\nu} c(s,t) \tag{117}$$

Substituting (117) into (116) yields

$$\omega(t) + a(s,t) = \int_t^\infty c(s,t) e^{-\int_t^\nu \lambda+\theta\ d\zeta}\ d\nu = c(s,t) e^{t(\lambda+\theta)} \int_t^\infty e^{-\nu(\theta+\lambda)}\ d\nu =$$

$$c(s,t) e^{t(\theta+\lambda)} \frac{-1}{\theta+\lambda} \left[e^{-\nu(\theta+\lambda)}\right]_t^\infty = c(s,t) \frac{1}{\theta+\lambda}$$

and with this

$$c(s,t) = (\theta+\lambda)[\omega(t) + a(s,t)] \qquad q.e.d. \tag{118}$$

Derivation of equation (32)

We know that

$$C(t) = \int_{-\infty}^t c(s,t) b e^{bs-pt}\ ds = (\theta+p)\ [\Omega(t) + \mathcal{A}(t)] \tag{119}$$

Taking the time-derivative of (119), we get, using the Leibnitz-rule and (12):

$$\dot{C}(t) = c(t,t) b e^{t(b-p)} + \int_{-\infty}^t \dot{c(s,t)} b e^{bs-pt}\ ds - \int_{-\infty}^t c(s,t) p b e^{bs-pt}\ ds =$$

$$c(t,t) b e^{t(b-p)} + (r(t) - \theta) C(t) - pC(t) = (r(t) - \theta - p) C(t) + c(t,t) b e^{t(b-p)}.$$

Due to (12), we can express $c(t,t)$ as

$$c(t,t) = (\theta+p)[\omega(t,t) + a(t,t)],$$

with $a(t,t) = 0$ (newborns have no financial wealth).
The actual difficulty is to determine $\omega(t,t)$. As we know, the expected present value of lifetime labour income is

$$\omega(s,t) = \int_t^\infty \hbar(s,\nu) w(\nu) e^{-\int_t^\nu r(\zeta)+p\ d\zeta}\ d\nu$$

with $\hbar(s,t) = \phi e^{-\psi(t-s)}$ and therefore $\hbar(t,t) = \phi$. Knowing this, we can express $\omega(s,t)$ as

$$\omega(s,t) = \phi e^{\psi s}\hat{w}(t), \tag{120}$$

with $\hat{w}(t) \equiv \int_t^\infty e^{-\psi\nu} w(\nu) e^{-\int_t^\nu r(\zeta)+p\, d\zeta}\, d\nu$. From (120), we now have a handy expression for $\omega(t,t)$:

$$\Omega(t) = \int_{-\infty}^{t} \omega(s,t) b e^{bs-pt}\, ds = \phi\hat{w}(t) b \int_{-\infty}^{t} e^{\psi s + bs - pt} =$$

$$\frac{\phi\hat{w}(t) b e^{-pt+\psi t+bt}}{\psi + b}.$$

As $\omega(t,t) = \phi e^{\psi t}\hat{w}(t)$ due to (120), by substituting into the expression above we get

$$\omega(t,t) = \frac{\Omega(t)(\psi + b)}{b e^{t(b-p)}} \tag{121}$$

With (119) and (121) it is no longer difficult to determine the final expression for $\dot{C}(t)$:

$$\dot{C}(t) = (r(t) - \theta - p)C(t) + (\theta + p)(\psi + b)\Omega(t) =$$

$$(r(t) - \theta + b - p + \psi)C(t) - (\theta + p)(\psi + b)\mathcal{A}(t) \qquad q.e.d.$$

Appendix B - Comparative statics in the Pautrel (2009) - model

We now turn towards the analysis of the per-capita values $k(t)$ and $c(t)$. Due to the relation $\mathcal{L}\tilde{k} = k$ we can use steady state per-worker values (which we already know, see chapter 3.4) to find expressions for $k(t)$ and $c(t)$.

Higher fertility rates

From (45), we know that $\tilde{k}^* = [2\alpha(1-\chi(\tau))A]^{\frac{1}{1-\alpha}}\mathcal{D}^{\frac{-1}{1-\alpha}}$ and thus

$$\frac{\partial\tilde{k}^*}{\partial b} = -\frac{1}{1-\alpha}\left[2\alpha(1-\chi(\tau))A\right]^{\frac{1}{1-\alpha}}\mathcal{D}^{\frac{-1}{1-\alpha}-1}\frac{\partial\mathcal{D}}{\partial b} = -\frac{\frac{1}{1-\alpha}\tilde{k}^*\mathcal{D}_b}{\mathcal{D}} = \frac{1}{1-\alpha}\left(\frac{\tilde{k}^*}{\mathcal{D}}\right)\mathcal{D}_b.$$

As $\mathcal{D} > 0$, $\alpha < 1$ and

$$\mathcal{D}_b = \alpha + \frac{1}{2}\left[(\psi - \alpha(b-p) - \theta)^2 + 4(\psi + p)\alpha(b+\theta)\right]^{-\frac{1}{2}}\left[-2\alpha(\psi - \alpha(b-p) - \theta) + \right.$$

$$4\alpha(\psi + p)] = \alpha \left[\frac{\psi + \theta + \alpha(b-p) + 2p}{\sqrt{(\psi - \alpha(b-p) - \theta)^2 + 4(\psi + p)\alpha(b+\theta)}} + 1 \right] > 0$$

it must be true that $\tilde{k}_b^* < 0$. Consequently, the capital stock per worker decreases with a rise in the birth rate b.

The impact of the birth rate on the per-capita capital stock is less clear, as we have discussed earlier, due to the fact that the per-capita labour supply also depends on the birth rate, but positively:

$$k_b^* = \mathcal{L}_b \tilde{k}^* + \mathcal{L}\tilde{k}_b^* = \frac{\phi\psi}{(b+\psi)^2} \frac{k^*}{\mathcal{L}} - \frac{1}{1-\alpha} \left(\frac{\mathcal{L}\tilde{k}^*}{\mathcal{D}} \right) \mathcal{D}_b =$$

$$k^* \left\{ \underbrace{\frac{\psi}{b(b+\psi)}}_{>0} - \underbrace{\frac{\mathcal{D}_b}{(1-\alpha)\mathcal{D}}}_{>0} \right\} \lesseqgtr 0.$$

Hence, if we assume age-earning profiles ($\psi \neq 0$), an increase in fertility can indeed have a positive impact on the per-capita capital stock.

Precisely the same is true for consumption. While a higher birth rate has a negative effect on per-worker consumption, as we can see from

$$\frac{\partial \tilde{c}^*}{\partial b} = A(1 - \chi(\tau))\alpha \tilde{k}^{*(\alpha-1)} \tilde{k}_b^* - (b-p)\tilde{k}_b^* - \tilde{k}^* =$$

$$\tilde{k}_b^* \left(\frac{\tilde{c}^*}{\tilde{k}^*} \right) - \tilde{k}^* = -\frac{\tilde{c}^*}{(1-\alpha)\mathcal{D}} \mathcal{D}_b - \tilde{k}^* < 0,$$

the impact of increased fertility on per-capita consumption is ambiguous:

$$c_b^* = \mathcal{L}_b \tilde{c}^* + \mathcal{L}\tilde{c}_b^* = \frac{\phi\psi}{(b+\psi)^2} \frac{c^*}{\mathcal{L}} - \frac{\phi b}{b+\psi} \left[\frac{\tilde{c}^*}{(1-\alpha)\mathcal{D}} \mathcal{D}_b + \tilde{k}^* \right] =$$

$$c^* \left\{ \underbrace{\frac{\psi}{b(b+\psi)}}_{>0} - \underbrace{\frac{\mathcal{D}_b}{(1-\alpha)\mathcal{D}}}_{>0} \right\} - k^* \lesseqgtr 0.$$

Shorter horizons

In a very similar way, we can determine the impact of the death rate on the per-capita capital stock:

$$\tilde{k}_p^* = -\frac{1}{1-\alpha}\left(\frac{\tilde{k}^*}{\mathcal{D}}\mathcal{D}_p\right).$$

We further find that

$$\mathcal{D}_p = -\alpha + \frac{1}{2}\left[(\psi-\alpha(b-p)-\theta)^2 + 4(\psi+p)\alpha(b+\theta)\right]^{-\frac{1}{2}}[2\alpha(\psi-\alpha(b-p)-\theta)+$$

$$4\alpha(b+\theta)] = \alpha\left[\frac{\psi+\theta-\alpha(b-p)+2b}{\sqrt{(\psi-\alpha(b-p)-\theta)^2+4(\psi+p)\alpha(b+\theta)}} - 1\right]$$

$\mathcal{D}_p > 0$, which is in fact not quite as obvious as it was for $\mathcal{D}_b$. However, we can show that

$$\mathcal{D}_p > 0 \Leftrightarrow \psi+\theta-\alpha(b-p)+2b > \sqrt{(\psi-\alpha(b-p)-\theta)^2+4(\psi+p)\alpha(b+\theta)} \Leftrightarrow$$

and finally, with some basic calculus,

$$\Leftrightarrow \psi(b+\theta) + \alpha p(b+\theta) + \theta b(1-\alpha) + b^2(1-\alpha) > 0,$$

which is always the case, as due to $\alpha < 1$ the last line contains only non-negative expressions. Thus $\tilde{k}_p^* < 0$, a shorter life expectancy decreases the per-worker capital stock.
Furthermore,

$$k_p^* = \mathcal{L}_p\tilde{k}^* + \mathcal{L}\tilde{k}_p^* = 0\,\frac{k^*}{\mathcal{L}} - \frac{1}{1-\alpha}\left(\frac{\mathcal{L}\tilde{k}^*}{\mathcal{D}}\right)\mathcal{D}_p = -(1-\alpha)^{-1}\frac{k^*}{\mathcal{D}}\mathcal{D}_p < 0.$$

In contrast to k_b^*, the direction of the death rate's impact on steady state per-capita capital is unambiguous, as $\mathcal{L}$ is independent of p. Therefore it also suffices to examine steady-state per worker consumption:

$$\frac{\partial\tilde{c}^*}{\partial p} = A(1-\chi(\tau))\alpha\tilde{k}^{*(\alpha-1)}\tilde{k}_p^* - (b-p)\tilde{k}_p^* + \tilde{k}^* =$$

$$\tilde{k}_p^*\left(\frac{\tilde{c}^*}{\tilde{k}^*}\right) + \tilde{k}^* = -\frac{\tilde{c}^*}{(1-\alpha)\mathcal{D}}\,\mathcal{D}_p + \tilde{k}^* \lessgtr 0.$$

The impact of the death rate on steady-state consumption is thus not clear. In the "generic" case, steady-state consumption depends positively on the death rate, like in Figure 7. It needs a combination of extremly high productivity and discount rate ($\psi, \theta \approx 1$) and low capital intensity in production ($\alpha \approx 0$) to attain a negative relationship. Although this is not impossible in our model structure, we have $c_p^* > 0$ for all realistic szenarios.

Early retirement

The impact of age-specific productivity, which models the agents' "retirement-scheme", on per-worker capital is given by

$$\tilde{k}_\psi^* = -\frac{1}{1-\alpha}\left(\frac{\tilde{k}^*}{\mathcal{D}}\mathcal{D}_\psi\right)$$

with

$$\mathcal{D}_\psi = \frac{\psi - \alpha(b-p) - \theta + 2\alpha(b+\theta)}{\sqrt{(\psi - \alpha(b-p) - \theta)^2 + 4\alpha(\psi+p)(b+\theta)}} - 1 \lesseqgtr 0.$$

Like before, the direction of productivity's impact is ambiguous, although again it needs an unrealistic combination of parameter values ($\alpha, \theta, \psi \approx 1$) to attain a negative relationship. The generic case is therefore $\tilde{k}_\psi^* > 0$.
For per-worker consumption, we have

$$\frac{\partial \tilde{c}^*}{\partial p} = A(1-\chi(\tau))\alpha \tilde{k}^{*(\alpha-1)}\tilde{k}_\psi^* - (b-p)\tilde{k}_\psi^* = \tilde{k}_\psi^*\left(\frac{\tilde{c}^*}{\tilde{k}^*}\right) = -\frac{\tilde{c}^*}{(1-\alpha)\mathcal{D}}\,\mathcal{D}_\psi \lesseqgtr 0,$$

where again the generic case is $\tilde{c}_\psi^* > 0$.

Once more, the situation changes when accounting for per-capita values. Due to $\mathcal{L}_\psi = \frac{-\phi b}{(b+\psi)^2} < 0$, we obtain (concentrating on the realistic case)

$$k_\psi^* = \underbrace{\mathcal{L}_\psi}_{<0}\,\tilde{k}^* + \mathcal{L}\,\underbrace{\tilde{k}_\psi^*}_{>0} \lesseqgtr 0,$$

$$c_\psi^* = \underbrace{\mathcal{L}_\psi}_{<0}\,\tilde{c}^* + \mathcal{L}\,\underbrace{\tilde{c}_\psi^*}_{>0} \lesseqgtr 0.$$

The impact of productivity on the per-capita values is hence not clear without futher knowledge about the parameters. In our steady state, both parameters respond unambiguously negatively to changes in ψ, which is economically reasonable, but not necessarily always the case from a strict mathematic point of view.

Demographic changes with constant population size

Finally, we examine the case where birth and death rate change to the same extent ($db = dp$). This way, the population size remains constant ($dg_N = 0$). Like before, we obtain

$$\tilde{k}^*_{bp} = -\frac{1}{1-\alpha}\left(\frac{\tilde{k}^*}{\mathcal{D}}\right)\mathcal{D}_{bp} < 0.$$

$\mathcal{D}_{bp} = \mathcal{D}_b + \mathcal{D}_p = 2\alpha\frac{\psi+b+p+\theta}{\sqrt{(\psi-\alpha(b-p)-\theta)^2+4(\psi+p)\alpha(b+\theta)}} > 0$, which means that higher birth and death rates that leave the size of the population unchanged, have a negative impact on the capital stock per-worker.

The impact of this population-"shift" on the capital stock per capita however is not clear, as

$$k^*_{bp} = \underbrace{\frac{\partial\mathcal{L}}{\partial b}}_{>0}\tilde{k}^* + \underbrace{\frac{\partial\mathcal{L}}{\partial p}}_{=0}\tilde{k}^* + \mathcal{L}\underbrace{\tilde{k}^*_{bp}}_{<0} = k^*\left\{\frac{\psi}{b(b+\psi)} - \frac{\mathcal{D}_{bp}}{(1-\alpha)\mathcal{D}}\right\} \lesseqgtr 0$$

Appendix C - Stability analysis in the Pautrel (2011b)-model

We first determine the elements of the Jacobian $\mathcal{J}$ for the BGP-equilibrium of the dynamic system

$$\dot{x}(t) = x(t)\left[-\theta - (1-\eta)\lambda - \eta\lambda(\theta+\lambda)x(t)^{-1} - \left(1 + \Delta(1-\frac{1}{u(t)}) - \alpha\right)(u(t)b(t))^{1-\alpha} + x(t)\right].$$

$$\dot{b}(t) = \left[B(1-u(t))\Delta^{\delta}(u(t)b(t))^{-\alpha\delta} - (1-\eta)\lambda - (1+\Delta(1-\frac{1}{u(t)}))(u(t)b(t))^{1-\alpha} + x(t) + \chi(\tau)\right]b(t)$$

$$\dot{u}(t) = \Bigg[B\Delta^\delta (b(t)u(t))^{-\alpha\delta}(\alpha^{-1}(1-\alpha)^{-\delta} - 1 + u(t)) + \chi(\tau)\left(\frac{1}{\alpha(1-\delta)} - 1\right)$$
$$-\lambda\left(\frac{1}{\alpha(1-\delta)} + \eta - 1\right) - x(t) + (u(t)b(t))^{1-\alpha}\Delta\left((1 - \frac{1}{u(t)}) - \frac{1}{1-\alpha}\right)\Bigg] u(t)$$

Some of these elements are easy to derive:[27]

$$\mathcal{J}_{11} = \frac{\partial \dot{x}}{\partial x} = (\eta\lambda(\theta+\lambda)x^{-2} + 1)x + \underbrace{\frac{\dot{x}}{x}}_{=0} = \eta\lambda(\theta+\lambda)x^{-1} + x > 0$$

$$\mathcal{J}_{21} = \frac{\partial \dot{b}}{\partial x} = b > 0$$
$$\mathcal{J}_{22} = \frac{\partial \dot{b}}{\partial b} = \left(-\alpha\delta B(1-u)\Delta^\delta b^{-\alpha\delta-1}u^{-\alpha\delta} - (1-\alpha)\left[1 + \Delta(1-\frac{1}{u})\right]u^{1-\alpha}b^{-\alpha}\right)b + \underbrace{\frac{\dot{b}}{b}}_{=0}$$
$$= -\alpha\delta B(1-u)\Delta^\delta (bu)^{-\alpha\delta} - (1-\alpha)\left[1 + \Delta(1-\frac{1}{u})\right](bu)^{1-\alpha} < 0$$
$$\mathcal{J}_{31} = \frac{\partial \dot{u}}{\partial x} = -u < 0$$
$$\mathcal{J}_{32} = \frac{\partial \dot{u}}{\partial b} = \left(-\alpha\delta B\Delta^\delta b^{-\alpha\delta-1}u^{-\alpha\delta}[\alpha^{-1} - 1 + u] + (1-\alpha)\Delta\left[(1-\frac{1}{u}) - \frac{1}{1-\alpha}\right]b^{-\alpha}u^{1-\alpha}\right)u$$
$$= -\frac{u}{b}\left[\alpha\delta B\Delta^\delta (ub)^{-\alpha\delta}[\alpha^{-1} - 1 + u] + (1-\alpha)\Delta\left[\frac{1}{u} + \frac{\alpha}{1-\alpha}\right](ub)^{1-\alpha}\right] < 0$$

while others are more cumbersome:

$$\mathcal{J}_{12} = \frac{\partial \dot{x}}{\partial b} = -(1-\alpha)\left(1 - \alpha + \frac{\delta - \alpha\delta}{1-\delta}(1-\frac{1}{u})\right)u^{1-\alpha}b^{-\alpha}x$$
$$= -(1-\alpha)^2\left(-1 + \frac{\delta}{1-\delta} - \frac{\delta}{u - u\delta}\right)(bu)^{1-\alpha}\frac{x}{b}$$
$$= -(1-\alpha)^2\left(\frac{u-\delta}{u-u\delta}\right)(bu)^{1-\alpha}\frac{x}{b} < 0$$

[27]The * marking equilibrium values is omitted for convenience.

$$\mathcal{J}_{13} = \frac{\partial \dot{x}}{\partial u} = \left((1-\alpha) \left[\alpha - \left(1 + \Delta(1 - \frac{1}{u}) \right) \right] b^{1-\alpha} u^{-\alpha} - \frac{1}{u^2} \Delta (bu)^{1-\alpha} \right) x$$
$$= -(1-\alpha)(ub)^{1-\alpha} \frac{x}{u} \left[1 - \alpha + \frac{\delta(1-\alpha)(u-1)}{(1-\delta)u} + \frac{\delta}{u(1-\delta)} \right]$$
$$= -(1-\alpha)(ub)^{1-\alpha} \frac{x}{u} \left[\frac{1-\alpha}{1-\delta} + \frac{\alpha\delta}{(1-\delta)u} \right] < 0$$

$$\mathcal{J}_{23} = \frac{\partial \dot{b}}{\partial u} = \left(-B\Delta^{\delta}(ub)^{-\alpha\delta} - \alpha\delta B \Delta^{\delta}(1-u)u^{-\alpha\delta-1}b^{-\alpha\delta} - \frac{1}{u^2}\Delta(ub)^{1-\alpha} \right.$$
$$\left. -(1-\alpha)u^{-\alpha}b^{1-\alpha} \left(1 + \Delta(1 - \frac{1}{u}) \right) \right) b$$
$$= -b[1 + \alpha\delta(\frac{1}{u} - 1)]B\Delta^{\delta}(ub)^{-\alpha\delta} - (ub)^{1-\alpha}\frac{b}{u} \left[\frac{1}{u}\Delta + (1-\alpha)\left(1 + \Delta(1 - \frac{1}{u}) \right) \right]$$
$$= -b[1 + \alpha\delta(\frac{1}{u} - 1)]B\Delta^{\delta}(ub)^{-\alpha\delta} - (ub)^{1-\alpha}\frac{b}{u} \left[\frac{1-\alpha}{1-\delta} \right] \left[1 + \alpha\delta(\frac{1}{u} - 1) \right] < 0$$

$$\mathcal{J}_{33} = \frac{\partial \dot{u}}{\partial u} = \left[-\alpha\delta \left(B\Delta^{\delta} b^{-\alpha\delta} u^{-\alpha\delta-1}(\alpha^{-1} - 1 + u) \right) + B\Delta^{\delta}(ub)^{-\alpha\delta} \right.$$
$$\left. + (1-\alpha)u^{-\alpha}b^{1-\alpha}\Delta \left(1 - \frac{1}{u} - \frac{1}{1-\alpha} \right) + \frac{1}{u^2}(ub)^{1-\alpha}\Delta \right] u + \underbrace{\frac{\dot{u}}{u}}_{=0}$$
$$= B\Delta^{\delta}(ub)^{-\alpha\delta}[u - \delta + \alpha\delta - \alpha\delta u] + (ub)^{1-\alpha}\Delta \left[(1-\alpha)(-\frac{1}{u} - \frac{\alpha}{1-\alpha}) + \frac{1}{u} \right]$$
$$= B\Delta^{\delta}(ub)^{-\alpha\delta}[u - \delta + \alpha\delta(1-u)] + (ub)^{1-\alpha}\alpha\Delta \left[\frac{1}{u} - 1 \right] > 0$$

The calculation of the determinant is extremly cumbersome and therefore carried out in Maple:

$$det\ \mathcal{J} = -\left[(1-\alpha+\Delta)(ub)^{1-\alpha} + \delta B \Delta^{\delta}(ub)^{-\alpha\delta} \right] \left[\frac{\Delta}{ux}\eta\lambda(\lambda+\theta)(ub)^{1-\alpha} \right.$$
$$\left. + uB\Delta^{\delta}(ub)^{-\alpha\delta}(x + \eta\lambda(\theta+\lambda)x^{-1}) \right] < 0$$

The Jacobian's trace is positive:

$$tr\ \mathcal{J} = \eta\lambda(\theta+\lambda)x^{-1} + x + B\Delta^{\delta}(ub)^{-\alpha\delta}(u-\delta) - (ub)^{1-\alpha} \left[(1-\alpha)(1 - \frac{\Delta}{u} + \Delta) - \alpha\frac{\Delta}{u} + \alpha\Delta \right]$$

$$= \eta\lambda(\theta + \lambda)x^{-1} + x + B\Delta^{\delta}(ub)^{-\alpha\delta}(u - \delta) - (ub)^{1-\alpha}\frac{(1-\alpha)(u-\delta)}{(1-\delta)u} > 0$$

Hence, the equilibrium is a stable saddle.