On Transfinite Knuth-Bendix Orders

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Abstract. In this paper we discuss the recently introduced *transfinite* Knuth-Bendix orders. We prove that any such order with finite subterm coefficients and for a finite signature is equivalent to an order using ordinals below ω^ω, that is, finite sequences of natural numbers of a fixed length. We show that this result does not hold when subterm coefficients are infinite. However, we prove that in this general case ordinals below ω^{ω^ω} suffice. We also prove that both upper bounds are tight. We briefly discuss the significance of our results for the implementation of first-order theorem provers and describe relationships between the transfinite Knuth-Bendix orders and existing implementations of extensions of the Knuth-Bendix orders.

1 Introduction

The Knuth-Bendix order (KBO for short) is the most common order used in first-order theorem provers. It is implemented in all commonly used resolution theorem provers: Vampire [15, 19], E [16], Otter [21], Spass [20], and in the equational theorem prover Waldmeister [4]. Recently, Ludwig and Waldman [11] introduced a modification of KBO, called *transfinite KBO* (*TKBO* for short), which can use arbitrary ordinals below ϵ_0 instead of natural numbers as symbols weights and subterm coefficients (we give all the necessary definitions in Section 3).

The TKBO can be more expressive than the KBO. However, the increase in expressiveness comes at the cost of a more complex implementation since one has to implement ordinals and two operations on them: the natural sum and the natural product. The natural product is especially hard to implement. One can get rid of the natural product by requiring that subterm coefficients are finite.

This paper is organised as follows. Section 2 gives a brief introduction into ordinals. In Section 3 we define the KBO and the TKBO. Our two main results are proved in Sections 4 and 6, as follows. In Section 4 we show that every instance of the TKBO on finite signatures with finite subterm coefficients is equivalent to a TKBO using ordinals below ω^ω, that is, sequences of natural numbers of a fixed length. Moreover, in Section 6 we prove that every instance

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of the TKBO (with unrestricted subterm coefficients) is equivalent to a TKBO using ordinals below ω^{ω^ω}, that is, sequences of sequences of natural numbers. In Section 5 we show that these results cannot be significantly improved. Note that ordinals below ω^ω are relatively easy to implement. Indeed, such kind of orders have been implemented in Vampire long ago. In Section 7 we discuss the use of KBO and TKBO in theorem provers and termination tools.

2 Preliminaries

We assume basic knowledge of set-theory, in particular of ordinals [5]. We write $>$ to denote the standard order on ordinals, and $<$ to denote the inverse of $>$. Recall that any ordinal $\alpha \neq 0$ can be uniquely represented in *Cantor normal form*, that is, written as a finite sum

$$\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \ ,$$

where $\alpha_1 \geqslant \cdots \geqslant \alpha_n$. Here $\geqslant$ denotes the usual total order on ordinals. We allow the sum in the above equation to be empty, that is, $0 = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ for $n = 0$.

For ordinals below ϵ_0, the Cantor normal form gives a basis for their syntactic representation: any such ordinal can be written in this form by recursively writing the exponents of ω in the same form. For every ordinal α, the set of ordinals strictly less than α is denoted by $\mathcal{O}(\alpha)$. We will simply write $\mathcal{O}$ when $\alpha = \epsilon_0$, that is, $\mathcal{O}$ is the set of all ordinals strictly below ϵ_0. Recall, that ϵ_0 is the smallest solution of the equation $\alpha = \omega^\alpha$.

In the sequel we assume that all ordinals are represented using their Cantor normal form.

To motivate the definitions of *natural sum* and *natural product* given below, we recall that the standard ordinal addition $+$ and ordinal multiplication $\cdot$ are not commutative. Moreover, $\cdot$ does not right-distribute over $+$.

For $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\beta = \omega^{\alpha_{n+1}} + \cdots + \omega^{\alpha_{n+m}}$, we define the *natural sum* $\alpha \oplus \beta$ as $\omega^{\alpha_{\pi(1)}} + \cdots + \omega^{\alpha_{\pi(n+m)}}$, where π is any permutation of the indices $\{1, \ldots, n+m\}$ such that $\alpha_{\pi(1)} \geqslant \alpha_{\pi(2)} \geqslant \cdots \geqslant \alpha_{\pi(n+m)}$. Note that this definition includes the case that β is zero; we set $\alpha \oplus 0 = 0 \oplus \alpha = \alpha$. Likewise, we define the *natural product* $\odot$ of ordinals in $\mathcal{O}$, as follows. For $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$, we define

$$\alpha \odot \beta = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \left(\omega^{\alpha_i \oplus \beta_j} \right) \ .$$

REMARK 2.1 The natural sum and product defined above are respectively called the *Hessenberg addition* and the *Hessenberg product* in [11].

We write $\alpha \cdot n$ as an abbreviation of $\underbrace{\alpha + \cdots + \alpha}_{n \text{ times}}$. Further, we identify the natural numbers with the ordinals below ω. For example, we write 3 instead of $\omega^0 + \omega^0 + \omega^0$.

The following lemma is an immediate consequence of the above given definitions.

LEMMA 2.2 Let α, β, and γ be ordinals in $\mathcal{O}$. Then the following properties hold.

1. $\alpha \oplus \beta = \beta \oplus \alpha$.
2. $\alpha \odot \beta = \beta \odot \alpha$.
3. $\alpha \odot (\beta \oplus \gamma) = \alpha \odot \beta \oplus \alpha \odot \gamma$.
4. If $\alpha > \beta$, then $\alpha \oplus \gamma > \beta \oplus \gamma$. If, in addition, $\gamma > 0$, then $\alpha \odot \gamma > \beta \odot \gamma$. ❑

3 Transfinite KBO

In what follows, we assume that $\mathcal{F}$ is a *finite signature*. We denote by $\mathbb{N}$ the set of natural numbers.

DEFINITION 3.1 Let $\mathcal{F}$ be a signature. A *weight function* for $\mathcal{F}$ is a function $\mathsf{w} : \mathcal{F} \to \mathcal{O}$. A *subterm coefficient function* for $\mathcal{F}$ is a partial function $\mathsf{s} : \mathcal{F} \times \mathbb{N} \to \mathcal{O}$ such that for every $f \in \mathcal{F}$ and every $n > 0$, if n is less than or equal to the arity of f, then $\mathsf{s}(f, n)$ is defined and $\mathsf{s}(f, n) > 0$. A *precedence relation* on $\mathcal{F}$ is any strict total order on $\mathcal{F}$. ❑

DEFINITION 3.2 (ORDER BASIS) An *order basis* is a tuple $(\mathsf{w}, \mathsf{s}, \gg, w_0)$ where:

1. w is a weight function for $\mathcal{F}$;
2. s is a subterm coefficient function for $\mathcal{F}$;
3. $\gg$ is a precedence relation on $\mathcal{F}$;
4. $w_0 \in \mathbb{N}$ and $w_0 > 0$;
5. for every constant $c \in \mathcal{F}$, we have $\mathsf{w}(c) \geqslant w_0$;
6. if $f \in \mathcal{F}$ is a unary function symbol and $\mathsf{w}(f) = 0$, then f is the greatest element in $\mathcal{F}$ w.r.t. $\gg$. ❑

We will extend weight functions to variables and assume that $\mathsf{w}(x) = w_0$ for every variable x.

Given an order basis $(\mathsf{w}, \mathsf{s}, \gg, w_0)$, we define the *weight* of terms as follows.

DEFINITION 3.3 (WEIGHT) Let t be a term. The *weight* of t, denoted by $weight(t)$, is defined inductively as follows.

1. If t is a variable, then $weight(t) \overset{\text{def}}{=} w_0$.
2. $weight(f(t_1, \ldots, t_n)) \overset{\text{def}}{=} \mathsf{w}(f) \oplus (\mathsf{s}(f, 1) \odot weight(t_1))$
   $$\oplus \cdots$$
   $$\oplus (\mathsf{s}(f, n) \odot weight(t_n)) .$$ ❑

In the sequel we will often assume that we have a fixed order basis $(\mathsf{w}, \mathsf{s}, \gg, w_0)$.

The notion of the weight of a term is central for this paper. We will therefore introduce some notation and prove essential properties of term weights.

We will use the standard notion of a *position* in a term, and a *subterm* at a given position [2]. Any position is a sequence of positive integers. The empty position is denoted by ϵ.

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DEFINITION 3.4 (COEFFICIENT) Let t be a term and let p be a position in t. The *coefficient* of p in t, denoted by $coeff(p, t)$, is an ordinal defined inductively as follows.

1. $coeff(\epsilon, t) \stackrel{\mathrm{def}}{=} 1$.
2. $coeff(i.p, f(t_1, \ldots, t_n)) \stackrel{\mathrm{def}}{=} \mathsf{s}(f, i) \odot coeff(p, t_i)$. ❏

Let t be a term. We denote by $\mathcal{V}\mathsf{ar}(t)$ the set of all variables of t, by $\mathsf{Pos}(t)$ the set of positions in t, and by $\mathsf{Pos}_\mathcal{V}(t)$ the set of variable positions in t. If p is a position in t, we denote by $top_p(t)$ the symbol (that is, a function symbol or a variable) of t at the position p. Let x be a variable. The set of positions of x in t is denoted by $\mathsf{Pos}_\mathcal{V}(x, t)$. We call the *variable coefficient* of x in t, denoted by $vcoeff(x, t)$, the ordinal $\bigoplus_{p \in \mathsf{Pos}_\mathcal{V}(x,t)} coeff(p, t)$.

Let us give a useful characterisation of weights of terms using $coeff$.

LEMMA 3.5 For every term t we have

$$weight(t) = \bigoplus_{p \in \mathsf{Pos}(t)} coeff(p, t) \odot \mathsf{w}(top_p(t)).$$

PROOF. By straightforward induction on the depth of t. ❏

DEFINITION 3.6 (TKBO) Let $B = (\mathsf{w}, \mathsf{s}, \gg, w_0)$ be an order basis. The *instance of a transfinite Knuth-Bendix order induced by B*, denoted by $\succ_B$, is defined as follows. For all terms s, t, we have $s \succ_B t$ if the following conditions hold:

1. $\mathcal{V}\mathsf{ar}(s) \supseteq \mathcal{V}\mathsf{ar}(t)$;
2. for all $x \in \mathcal{V}\mathsf{ar}(t)$,
$$vcoeff(x, s) \geqslant vcoeff(x, t) ; \tag{$\dagger$}$$

3. either
   (a) $weight(s) > weight(t)$, or
   (b) $weight(s) = weight(t)$, and one of the following alternatives hold:
       i. t is a variable, and $s = f^n(t)$ for some unary function symbol f and $n > 0$;
       ii. $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, and there exists $i \in \{1, \ldots, n\}$ such that $s_j = t_j$ for all $1 \leqslant j < i$ and $s_i \succ_B t_i$;
       iii. $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, and $f \gg g$. ❏

We will sometimes simply write "a TKBO" instead of "an instance of a TKBO".

For every function whose range is a set of ordinals, we say that the function is *finite* if every value of this function is an ordinal below ω, that is, an element of $\mathbb{N}$. The *standard Knuth-Bendix order* (KBO) is a special case of the TKBO when the weight function is finite and the subterm coefficient function always returns 1. The TKBO is thus more expressive than the KBO, as it allows the use of infinite weight functions and arbitrary finite and infinite subterm coefficient functions.

We recall the following fact about the TKBO from [11].

PROPOSITION 3.7 For any order basis B, the induced TKBO $\succ_B$ is a simplification order. That is, $\succ_B$ is monotone, closed under substitutions, well-founded, and extends the subterm relation. ❏

We will now give two lemmas formulating sufficient conditions for equality and inequality of instances of the TKBO.

In the sequel we will assume that $B = (\mathsf{w}, \mathsf{s}, \gg, w_0)$ and $B' = (\mathsf{w}', \mathsf{s}', \gg', w_0')$ are two order bases. Denote by $weight$ and $weight'$ the term weight functions defined respectively by B and B'. Likewise, denote by $coeff$ and $coeff'$ the coefficients, and by $vcoeff$ and $vcoeff'$ the variable coefficients defined respectively by B and B'.

LEMMA 3.8 Suppose that (i) $\gg$ coincides with $\gg'$; (ii) for every two terms s and t and variable $x \in \mathcal{V}\mathsf{ar}(t)$ we have $vcoeff(x, s) \geqslant vcoeff(x, t)$ iff $vcoeff'(x, s) \geqslant vcoeff'(x, t)$; and (iii) for every two terms s and t we have $weight(s) > weight(t)$ iff $weight'(s) > weight'(t)$.

Then $\succ_B$ coincides with $\succ_{B'}$.

PROOF. Immediate by Definition 3.6. ❏

LEMMA 3.9 Suppose $B = (\mathsf{w}, \mathsf{s}, \gg, w_0)$ and $B' = (\mathsf{w}', \mathsf{s}, \gg', w_0')$ are two order bases, and let $weight$ and $weight'$ be the term weight functions defined respectively by B and B'. Suppose there exist two terms s and t such that $weight(s) > weight(t)$ and $weight'(t) > weight'(s)$.

Then $\succ_B$ does not coincide with $\succ_{B'}$.

PROOF. Let s, t be terms satisfying the conditions of the lemma. Take a fresh variable x and denote by u and v the terms obtained by replacing all variables in, respectively, s and t by x. Then $\mathcal{V}\mathsf{ar}(u) = \mathcal{V}\mathsf{ar}(v) = \{x\}$. Furthermore, as the weights of all variables are the same, we have $weight(u) = weight(s) > weight(t) = weight(v)$. Similarly, we conclude $weight'(v) > weight'(u)$.
Consider now the two possible cases. If $vcoeff(x, u) \geqslant vcoeff(x, v)$, we have $u \succ_B v$, but $weight'(v) > weight'(u)$ makes $u \succ_{B'} v$ impossible, so the two orderings do not coincide. Likewise, if $vcoeff(x, v) > vcoeff(x, u)$, we have $v \succ_{B'} u$, but $weight(u) > weight(v)$ makes $v \succ_B u$ impossible, so the two orderings do not coincide, too. ❏

4 TKBOs with Finite Subterm Coefficient Functions

In this section we consider TKBOs with *finite subterm coefficient functions*. Throughout this section we thus assume that $B = (\mathsf{w}, \mathsf{s}, \gg, w_0)$ is an order basis such that s is finite. The aim of this section is to prove that for every such basis, $\succ_B$ is equivalent to a TKBO using ordinals less than ω^ω. To this end, we will define a new basis $B' = (\mathsf{w}', \mathsf{s}, \gg, w_0')$ which agrees with B on the subterm coefficient function and the precedence relation, and show that B' induces the same transfinite Knuth-Bendix order as B.

DEFINITION 4.1 (Γ_B) Let α be an ordinal such that $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$. Then we define $\Gamma(\alpha) \stackrel{\text{def}}{=} \{\alpha_1, \ldots, \alpha_n\}$. The collection of ordinals Γ_B is defined as $\Gamma_B \stackrel{\text{def}}{=} \Gamma(w_0) \cup \bigcup_{f \in \mathcal{F}} \Gamma(\mathsf{w}(f))$. ❏

In other words, Γ_B is the set of exponents used in w_0 and in the weights of symbols in $\mathcal{F}$. Since $\mathcal{F}$ is finite, clearly the set Γ_B is finite and totally ordered by $>$. This property is used in the next definition.

DEFINITION 4.2 (RANK FUNCTION) We define a *rank function* $\phi \colon \Gamma_B \to \mathbb{N}$ as follows:
$$\phi(\alpha) \stackrel{\text{def}}{=} \max\{\phi(\beta) + 1 \mid \alpha > \beta, \beta \in \Gamma_B\},$$
where we assume that $\max \varnothing \stackrel{\text{def}}{=} 0$. In other words, $\phi(\alpha)$ is the number of ordinals in Γ_B strictly smaller than α. Note that ϕ is only defined on elements of Γ_B. ❏

The next lemma is a direct consequence of the previous definitions.

LEMMA 4.3 (ϕ IS MONOTONE) Let $\alpha, \beta \in \Gamma_B$. Then $\alpha \geqslant \beta$ (respectively, $\alpha > \beta$) if and only if $\phi(\alpha) \geqslant \phi(\beta)$ (respectively, $\phi(\alpha) > \phi(\beta)$). ❏

Note that, if $0 \in \Gamma_B$, then $\phi(0) = 0$. Likewise, if $i \in \mathbb{N} \cap \Gamma_B$, then $\phi(i) \leqslant i$.

Let us also make a trivial but useful observation on how to compare two weights given in Cantor normal form.

LEMMA 4.4 Let
$$\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_k},$$
$$\beta = \omega^{\beta_1} + \ldots + \omega^{\beta_m}$$
be two non-zero ordinals in Cantor normal form. Then $\alpha > \beta$ if and only if $(\alpha_1, \ldots, \alpha_k) > (\beta_1, \ldots, \beta_m)$, where the sequences of ordinals are compared lexicographically. ❏

We now define a new weight function w' and an ordinal w_0'.

DEFINITION 4.5 Let $f \in \mathcal{F}$ and $\mathsf{w}(f) = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$. Then we define
$$\mathsf{w}'(f) \stackrel{\text{def}}{=} \omega^{\phi(\alpha_1)} + \cdots + \omega^{\phi(\alpha_n)}.$$
Likewise, if $w_0 = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, then we define $w_0' \stackrel{\text{def}}{=} \omega^{\phi(\alpha_1)} + \cdots + \omega^{\phi(\alpha_n)}$. ❏

Due to Lemma 4.3, note that the above expressions for $\mathsf{w}'(f)$ and w_0' are in Cantor normal form. Using Definition 4.5, we define $B' \stackrel{\text{def}}{=} (\mathsf{w}', \mathsf{s}, \gg, w_0')$.

LEMMA 4.6 B' is an order basis.

PROOF. We first prove that B' is an order basis. Properties (1)-(3) of Definition 3.2 of order basis are obvious. Property (5) is derived by using Lemma 4.4 in conjunction with Lemma 4.3 on the monotonicity of ϕ. To prove property (6), take an arbitrary $f \in \mathcal{F}$. Let $\mathsf{w}(f) = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$. Then $\mathsf{w}'(f) = \omega^{\phi(\alpha_1)} + \cdots + \omega^{\phi(\alpha_n)}$. Evidently, $\mathsf{w}(f) = 0$ holds if and only if $n = 0$, and likewise for $\mathsf{w}'(f) = 0$. Therefore, $\mathsf{w}(f) = 0$ if and only if $\mathsf{w}'(f) = 0$. From this property (6) follows immediately. By replacing f with w_0 is this proof, we obtain a proof of property (4). ❏

LEMMA 4.7 Let t be a term and $weight(t) = \omega^{\alpha_1} + \ldots + \omega^{\alpha_k}$. Then $weight'(t) = \omega^{\phi(\alpha_1)} + \ldots + \omega^{\phi(\alpha_k)}$. As a consequence, $weight'(t) < \omega^\omega$.

PROOF. Straightforward from Lemma 3.5. ❏

LEMMA 4.8 For all terms s, t of the signature $\mathcal{F}$, we have $s \succ_B t$ if and only if $s \succ_{B'} t$.

PROOF. Use Lemma 3.8. Since properties (i) and (ii) of Lemma 3.8 are trivially satisfied, it suffices to prove for any two terms s and t, we have $weight(s) > weight(t)$ iff $weight'(s) > weight'(t)$.

Assume $weight(s) > weight(t)$. Let $weight(s) = \omega^{\alpha_1} + \ldots + \omega^{\alpha_k}$ and $weight(t) = \omega^{\beta_1} + \ldots + \omega^{\beta_m}$. By Lemma 4.4 we have

$$(\alpha_1, \ldots, \alpha_k) > (\beta_1, \ldots, \beta_m). \tag{1}$$

By Lemma 4.7 we have $weight'(s) = \omega^{\phi(\alpha_1)} + \ldots + \omega^{\phi(\alpha_k)}$ and $weight'(t) = \omega^{\phi(\beta_1)} + \ldots + \omega^{\phi(\beta_m)}$. Applying Lemma 4.3 on monotonicity of ϕ to (1) we obtain $(\phi(\alpha_1), \ldots, \phi(\alpha_k)) > (\phi(\beta_1), \ldots, \phi(\beta_m))$, which by Lemma 4.4 gives $weight'(s) > weight'(t)$. ❏

Lemmas 4.8 and 4.6 imply one of our main results.

THEOREM 4.9 Every instance of a TKBO with finite weight coefficients is equivalent to an instance using weights in $\mathcal{O}(\omega^\omega)$. ❏

5 Lower Bounds on Ordinals

In Section 4 we showed that for a basis with finite subterm coefficients the induced TKBO $\succ_B$ is equivalent to a TKBO using ordinals less than ω^ω. In Subsection 5.1 of this section we will show that this result is essentially optimal. Then, in Subsection 5.2 we extend this result to show that an *arbitrary* TKBO is equivalent to a TKBO using ordinals less than ω^{ω^ω}.

To prove these results we will use *ordering constraints* (in the sequel simply *constraints*), that is, expressions $s \sqsupset t$, where s and t are terms. We say that an order $>$ *satisfies* such a constraint if $s > t$. The way we use constraints is the following. Suppose we have a family F of orders and an order $>$. Suppose also that $>$ satisfies a set S of constraints and each order in F violates at least one of the constraints in S. Then we can conclude that $>$ does not belong to F.

5.1 Finite Term Coefficients

Throughout this subsection we assume finite subterm coefficients. In this subsection a, b, c will denote constants; f, g, maybe with indices, unary function symbols; and h a binary function symbol.

We will now define, for every natural number k, a satisfiable set of constraints that can only be satisfied when the weight of one of the symbols is at least ω^k.

EXAMPLE 5.1 Let $k \in \mathbb{N}$, $\mathcal{F} = \{c, h, f_0, \ldots, f_k\}$. Consider the set consisting of all the constraints $f_i(x) \sqsupset h(x, x)$, where $0 \leqslant i \leqslant k$, and the constraints $f_{i+1}(c) \sqsupset f_i^n(c)$, where $0 \leqslant i < k$ and $n \geqslant 0$. ❏

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LEMMA 5.2 There exists a TKBO $>$ with finite subterm coefficients satisfying all constraints of Example 5.1. Moreover, for every TKBO satisfying these constraints, we have $\mathsf{w}(f_i) \geqslant \omega^i$ for all $1 \leqslant i \leqslant k$.

PROOF. To satisfy the constraints, we define the weights and subterm coefficients of h and c to be 1, the subterm coefficients of each f_i to be 2 and the weight of f_i to be ω^i, for all $0 \leqslant i \leqslant k$. We arbitrarily fix the value of w_0. It is not hard to argue that $weight(f_i^n(c)) < \omega^{i+1} < weight(f_{i+1}(c))$, so all the constraints $f_{i+1}(c) \sqsupset f_i^n(c)$ are satisfied. It is also easy to see that the constraints $f_i(x) \sqsupset h(x,x)$ are satisfied too.

For the second part, take any TKBO $\succ$ that satisfies all constraints of Example 5.1. First, we note that $f_i(x) \succ h(x,x)$ and condition (2) of the TKBO imply

$$\mathsf{s}(f_i, 1) = vcoeff(x, f_i(x)) \geqslant vcoeff(x, h(x,x)) = \mathsf{s}(h,1) \oplus \mathsf{s}(h,2) \geqslant 2.$$

Therefore, the subterm coefficient of every f_i is not less than 2. This implies that for every term t we have $weight(f_i^n(t)) \geqslant 2^n \odot weight(t) \oplus 2^{n-1} \odot \mathsf{w}(f_i)$. As $weight(f_{i+1}(c)) \geqslant weight(f_i^n(c))$, we then have

$$\mathsf{w}(f_{i+1}) \oplus \mathsf{s}(f_{i+1}, 1) \odot \mathsf{w}(c) = weight(f_{i+1}(c)) \geqslant$$
$$weight(f_i^n(c)) \geqslant 2^n \odot \mathsf{w}(c) \oplus 2^{n-1} \odot \mathsf{w}(f_i).$$

Thus, we proved that for all natural numbers $n \geqslant 1$ we have

$$\mathsf{w}(f_{i+1}) \oplus \mathsf{s}(f_{i+1}, 1) \odot \mathsf{w}(c) \geqslant 2^n \odot \mathsf{w}(c) \oplus 2^{n-1} \odot \mathsf{w}(f_i). \qquad (2)$$

Consider the case $i = 0$. In this case (2) implies

$$\mathsf{w}(f_1) \oplus \mathsf{s}(f_1, 1) \odot \mathsf{w}(c) \geqslant 2^n \odot \mathsf{w}(c).$$

Since $\mathsf{s}(f_1, 1)$ is finite, we have $\mathsf{w}(f_1) \geqslant (2^n - \mathsf{s}(f_1, 1)) \odot \mathsf{w}(c)$ for all sufficiently large n. This implies $\mathsf{w}(f_1) \geqslant \omega$. Let us now prove $\mathsf{w}(f_{i+1}) \geqslant \omega^{i+1}$ for all $i = 1, \ldots, k-1$ by induction on i. To this end, note that for sufficiently large n we have $2^n \odot \mathsf{w}(c) > \mathsf{s}(f_{i+1}, 1) \odot \mathsf{w}(c)$. Hence, for sufficiently large n, the inequality (2) implies

$$\mathsf{w}(f_{i+1}) \geqslant 2^{n-1} \odot \mathsf{w}(f_i).$$

The induction hypothesis gives $\mathsf{w}(f_i) \geqslant \omega^i$, so

$$\mathsf{w}(f_{i+1}) \geqslant 2^{n-1} \odot \mathsf{w}(f_i) \geqslant 2^{n-1} \odot \omega^i.$$

Since this holds for all sufficiently large n, we finally conclude $\mathsf{w}(f_{i+1}) \geqslant \omega^{i+1}$.
❏

The next theorem is a direct consequence of Lemma 5.2.

THEOREM 5.3 For every natural number $k > 0$ there exists a TKBO $>$ with finite subterm coefficients satisfying the following conditions: (i) all function symbols have weights less than ω^{k+1}, and (ii) $>$ is not equivalent to any TKBO with finite subterm coefficients in which all function symbols have weights less than ω^k.
❏

Let us emphasise that the constraints defined in Example 5.1 are based on *finite* signatures.

5.2 Arbitrary Weight Coefficients

In the remaining part of this section we prove lower bounds for the case when *arbitrary subterm coefficient functions* are used. To this end, we use the condition (†) of Definition 3.6 to force higher ordinals as lower bounds.

EXAMPLE 5.4 Consider the finite signature $\mathcal{F} = \{g, h, f_1\}$. We define the following set S_1 of constraints: $f_1(x) \sqsupset g^n(x)$ for all $n \geqslant 0$, and $g(x) \sqsupset h(x, x)$. ❏

We show that S_1 can only be satisfied when infinite subterm coefficient functions are used. More precisely, we show that the constraints of this example force $\mathsf{s}(f_1, 1) \geqslant \omega$.

LEMMA 5.5 S_1 is satisfiable. For every TKBO satisfying S_1, we have $\mathsf{s}(f_1, 1) \geqslant \omega$.

PROOF. To satisfy S_1, we set $\mathsf{w}(g) = 2$, and $\mathsf{w}(f_1) = \mathsf{w}(h) = w_0 = 1$. We also set $\mathsf{s}(f_1, 1) = \omega$, $\mathsf{s}(g, 1) = 2$, and $\mathsf{s}(h, 1) = \mathsf{s}(h, 2) = 1$.

Let us prove the second part of the lemma. Take any TKBO $>$ satisfying S_1. Property (†) of Definition 3.6 applied to $g(x) > h(x, x)$ implies $\mathsf{s}(g, 1) \geqslant 2$. Then $f_1(x) > g^n(x)$ implies $\mathsf{s}(f_1, 1) \geqslant 2^n$ for all n, hence $\mathsf{s}(f_1, 1)$ is infinite. ❏

Example 5.4 shows that we can force infinite values for the subterm coefficient functions. The next example uses ideas of Example 5.1 to define, for every positive integer k, constraints over a finite signature that require the use of subterm coefficient functions greater than ω^{ω^k}.

EXAMPLE 5.6 Let $\mathcal{F} = \{g, h, f_1, \ldots, f_k, a, b\}$. Consider the set S_2 of constraints obtained from S_1 by adding, for every $1 \leqslant i < k$ and $n \geqslant 0$, the constraints $f_{i+1}(x) \sqsupset f_i^n(x)$, plus a single constraint $a \sqsupset f_k(b)$. ❏

LEMMA 5.7 S_2 is satisfiable. For every TKBO satisfying S_2, we have $\mathsf{s}(f_i, 1) \geqslant \omega^{\omega^{i-1}}$ for all $1 \leqslant i \leqslant k$, and $weight(a) \geqslant \omega^{\omega^{k-1}}$.

PROOF. To prove satisfiability, we arbitrarily fix the the constants a and b, and change the order basis of the proof of Lemma 5.5 by changing the weights of f_i as follows: $\mathsf{w}(f_i) = 1$ and $\mathsf{s}(f_i, 1) = \omega^{\omega^i}$, for all i. To verify that S_2 requires $\mathsf{s}(f_i, 1) \geqslant \omega^{\omega^{i-1}}$ for all i, we proceed inductively as before. Finally, $weight(a) \geqslant \omega^{\omega^{k-1}}$ follows from $weight(b) \geqslant 1$. ❏

The next theorem is a direct consequence of Lemma 5.7.

THEOREM 5.8 For every positive integer k there exists a TKBO satisfying the following conditions: (i) all terms have weights less than $\omega^{\omega^{k+1}}$, and (ii) this TKBO is not equivalent to any TKBO in which terms have weights less than ω^{ω^k}. ❏

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6 TKBOs with Unrestricted Subterm Coefficient Functions

Throughout this section we assume that $B = (\mathsf{w}, \mathsf{s}, \gg, w_0)$ is an order basis. We will show, using a modification of the construction used in Definition 4.5, that an *arbitrary TKBO* is equivalent to a TKBO using only ordinals less than ω^{ω^ω}. To do so, we first define an analogue of Γ_B given in Definition 4.1 as follows.

DEFINITION 6.1 Let α be an ordinal such that $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, where $\alpha_i = \omega^{\beta_{i1}} + \cdots + \omega^{\beta_{im_i}}$ for each $i \in \{1, \ldots, n\}$. We define $\Delta(\alpha) \stackrel{\text{def}}{=} \{\beta_{11}, \ldots, \beta_{1m_1}, \ldots, \beta_{n1}, \ldots, \beta_{nm_n}\}$. Further, the collection of ordinals Δ_B is defined as:
$$\Delta_B \stackrel{\text{def}}{=} \Delta(w_0) \cup \bigcup_{f \in \mathcal{F}} \Delta(\mathsf{w}(f)) \ \cup \bigcup_{f \in \mathcal{F}, i \in \mathbb{N}} \Delta(\mathsf{s}(f, i)) \, . \qquad \Box$$

In other words, Δ_B is the set of exponents of the exponents used in w_0, in the weights of symbols in $\mathcal{F}$, and in the subterm coefficient functions. Clearly, Δ_B is finite and totally ordered by $>$. Without loss of generality, we assume that $0 \in \Delta_B$.

We next refine the definition of the mapping ϕ given in Definition 4.2.

DEFINITION 6.2 We define a rank function $\psi \colon \Delta_B \to \mathbb{N}$ as follows:

$$\psi(\alpha) \stackrel{\text{def}}{=} \max\{\psi(\beta) + 1 \mid \alpha > \beta, \beta \in \Delta_B\} \, ,$$

where we set $\max \varnothing \stackrel{\text{def}}{=} 0$. $\qquad \Box$

Using the function ψ, below we define an ordinal basis $B' = (\mathsf{w}', \mathsf{s}', \gg, w_0')$ using only ordinals in $\mathcal{O}(\omega^{\omega^\omega})$, and then prove that it defines a TKBO equivalent to B. To this end, we will first develop some results about ordinals.

DEFINITION 6.3 Denote by $\mathcal{O}_B$ the set of all ordinals having the form

$$\omega^{\omega^{\beta_{11}} + \cdots + \omega^{\beta_{1m_1}}} + \cdots + \omega^{\omega^{\beta_{n1}} + \cdots + \omega^{\beta_{nm_n}}} + \omega k + m \, . \tag{3}$$

where for all $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m_i$, $\beta_{ij} \in \Delta_B$. Note that the set $\mathcal{O}_B$ is closed under $\oplus$ and $\odot$. We define an *ordinal mapping* γ with the domain $\mathcal{O}_B$ as follows. For every ordinal α of the form (3) we have

$$\gamma(\alpha) \stackrel{\text{def}}{=} \omega^{\omega^{\psi(\beta_{11})} + \cdots + \omega^{\psi(\beta_{1m_1})}} + \cdots + \omega^{\omega^{\psi(\beta_{n1})} + \cdots + \omega^{\psi(\beta_{nm_n})}} + \omega k + m \, . \qquad \Box$$

The following lemma is the key for all proofs of this section.

LEMMA 6.4 The mapping γ defines an isomorphic embedding of the ordered algebra of ordinals $\mathcal{O}_B$ into $\mathcal{O}$, that is, for every pair (α_1, α_2) of ordinals we have
$$\gamma(\alpha_1 \oplus \alpha_2) = \gamma(\alpha_1) \oplus \gamma(\alpha_2),$$
$$\gamma(\alpha_1 \odot \alpha_2) = \gamma(\alpha_1) \odot \gamma(\alpha_2),$$
$$\alpha_1 \geqslant \alpha_2 \text{ iff } \gamma(\alpha_1) \geqslant \gamma(\alpha_2). \qquad \Box$$

The proof of this lemma is straightforward and left to the reader.

Using γ, we can now define the order base B' which will give us the required order.

DEFINITION 6.5 Define the order basis $B' = (\mathsf{w}', \mathsf{s}', \gg, w_0')$ having the same precedence relation $\gg$ as B, as follows. Let $f \in \mathcal{F}$. Then $\mathsf{w}'(f) \stackrel{\text{def}}{=} \gamma(\mathsf{w}(f))$. Further, let i be a positive integer less than or equal to the arity of f. Then $\mathsf{s}'(f, i) \stackrel{\text{def}}{=} \gamma(\mathsf{s}(f, i))$. Finally, we let $w_0' \stackrel{\text{def}}{=} \gamma(w_0)$. ❑

As usual, we will respectively denote by $coeff'$, $vcoeff'$, and $weight'$ the functions $coeff$, $vcoeff$, and $weight$ induced by B'. The following result is then straightforward.

LEMMA 6.6 B' is an order basis. For all terms t, we have $weight'(t) < \omega^{\omega^\omega}$. ❑

It remains to prove that B' defines the same order as B. To this end, we will use Lemma 6.4.

LEMMA 6.7 Let t be a term, x be a variable, and p a position in s. Then we have the following.

1. $coeff'(p, t) = \gamma(coeff(p, t))$;
2. $vcoeff'(v, t) = \gamma(vcoeff(v, t))$;
3. $weight'(t) = \gamma(weight(t))$.

PROOF. (1) is immediate by the definition of $coeff$ and Lemma 6.4. (2) is immediate by the definition of $vcoeff$, (1) and Lemma 6.4. (3) is immediate by the definition of $weight$, (1) and Lemma 6.4. ❑

LEMMA 6.8 For all s, t, we have $s \succ_B t$ if and only if $s \succ_{B'} t$.

PROOF. Immediate by the definition of TKBO and Lemmas 6.4 and 6.7. ❑
Lemma 6.8 implies the main result of this paper given below.

THEOREM 6.9 Every instance of a TKBO is equivalent to an instance using weights in $\mathcal{O}(\omega^{\omega^\omega})$. ❑

7   Notes on Implementation and Applications

Knuth-Bendix orders have two main applications: automatic proofs of termination and first-order theorem proving. In termination tools, one automatically seeks orders that orient a given set of rewrite rules. For that purpose one can use the ordering algorithm of Korovin and Voronkov [7] to decide whether a given set of rules is compatible with a KBO – see also related results by Zankl et al. in [22].

The transfinite KBO has not been used for this purpose so far. The standard KBO ordering problem can be reduced to a problem of finding weights of symbols and precedences satisfying some conditions that turn out to be decidable (and even solvable in polynomial time). For transfinite KBOs the problem is much more complex, since subterm coefficients create non-linear inequalities and there

Order	Solvable only by such orders	30 seconds difference	10 seconds difference
KBO	163	32	110
TKBO	342	59	190

Table 1. Performance of the TKBO in Vampire on Hard Problems.

is no clear way of searching for ordinals instead of numbers. Our results shed some light on the problem and essentially show that it is sufficient to search for "small" ordinals only, but it is not clear also how the search for such small ordinals can be implemented. This can be a subject of future work.

It is interesting that simple variants of the transfinite KBO have been implemented in theorem provers before paper [11] describing them appeared. The theorem prover Otter [21] allowed for arbitrary finite subterm coefficient functions. The resulting instances of the KBO were not transfinite since the weights were always finite. We do not know exactly when such subterm coefficient functions first appeared in Otter, but they were available already in 1994 (see [13], sections 5.4 and 8.1).

Later in 2004 such orders were implemented in Vampire and immediately abandoned. The reason was that the use of weight coefficient 2 resulted in integer arithmetic overflows. We asked Bill McCune [12] whether he observed a similar behavior in Otter. He replied that Otter uses ordinary C "int"s (that were 32 bits at the time), yet he "have never noticed it, and no one has ever complained about it". He also pointed out that Otter's weighting is used only for very simple things, mainly for slight adjustments to the default weight (symbol count).

Since integer overflow results in incorrect ordering comparisons, using weight coefficients greater than 1 requires the use of arbitrary precision integers. We decided not to use such weight coefficients in Vampire for efficiency reasons. Indeed, theorem provers sometimes make millions of KBO comparisons in a short time, and these comparisons may take considerable time [10]. Therefore, modern theorem provers use a linear KBO comparison algorithm of [10]. Using arbitrary-precision integers incurs potential performance degradation both in time and space.

Nonetheless, a simple special case of TKBO was implemented in Vampire [15, 19] in 1996, and already used in the version of 1999, winning CASC-16 [17]. In Vampire, comparison of atoms is done in the following way. Each predicate symbol is assigned, in addition to the precedence and weight, a *level*, which is a non-negative integer. When we compare two atoms $p(s_1, \ldots, s_m)$ and $q(t_1, \ldots, t_n)$, we first compare the levels of p and q. If the level of p is greater, we decide $p(s_1, \ldots, s_m) \succ q(t_1, \ldots, t_n)$. If we compare two atoms having predicates of the same level, we apply the ordinary KBO. One can see that this way of ordering atoms corresponds to using a TKBO where the weight of a predicate symbol p is $\omega \cdot l + w$, where l is the level and w the weight of p. This ordering scheme is also convenient for the following reason: the use of orders in superposition-based provers normally requires that equalities be smaller than non-equality atoms. We achieve this by assigning equality level 0 and using positive levels for all other predicates. This means that Vampire uses a special case of the TKBO with ordinals below ω^2 and subterm coefficient always set to 1. We will now present

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Category	TKBO	KBO
CNF, non-Horn, with equality (NEQ)	35	28
CNF, Horn, with equality (HEQ)	7	9
CNF, non-Horn, without equality (NNE)	16	3
First-order, with equality (FEQ)	145	79
First-order, without equality (FNE)	10	4
Total	213	123

Table 2. Use of Strategies in the CASC Version of Vampire.

some statistics showing that the use of such ordinals is essential in Vampire's performance.

Hard problems. We have a database containing results of running various proving strategies of Vampire on TPTP problems [18]. We selected all problems solvable by Vampire and belonging to categories having predicate symbols different from equality (otherwise, the use of levels makes no difference). There are 8019 such problems. This set of problems contains many very hard problems: 652 problems, for example, have rating greater than 0.91 which, in most cases, means that these problems are solvable by only one theorem prover.

The database contains results of 1,351,630 test runs of Vampire on these problems. Most of the runs use 60 seconds time limits, but there are other runs ranging from 30 seconds to 3 minutes time limits. It is common in theorem proving that solvable problems are solved by at least one strategy in a very short time (a few seconds) but there are many exceptions.

Table 1 shows the number of problems on which KBO is considerably better than TKBO and vice versa. It turned out that there are 342 problems solvable only by TKBO and 163 problems solvable only by KBO. That is, the 163 problems solvable only by the KBO could not be solved using a TKBO where the subterm coefficient function is not set to 1 and the weight function is not finite. We also considered problems solvable both by the KBO and the TKBO but on which the difference between the best KBO and TKBO results is more than 30 seconds (10 seconds). It turned out that on such problems the TKBO also behaves considerably better than the KBO.

Use of strategies in the CASC version of Vampire. Since 1999, Vampire won 23 World Champion titles in various divisions of CASC. The 2010 version won in three divisions. Given a problem, Vampire runs on it a sequence of strategies, depending on the syntactic class of the problem. For each class of problems (i.e. category) we selected a collection of proving strategies that turned out to be the best on this class in our test runs before the competition. Each strategy uses exactly one instance of the KBO or the TKBO.

Table 2 summarises the number of strategies using, respectively, the KBO and the TKBO. It turns out that the number of strategies using the TKBO almost doubles the number of strategies use the KBO. The difference is especially big on problems without equality.

Note that Vampire does not implement the full TKBO with ordinals below ω^2 since function symbols only have finite weights. Nonetheless, Vampire was probably the first first-order prover using a TKBO with infinite ordinals in CASC. The usage of limited forms of the TKBO in Vampire suggests that other forms of the TKBO may turn out to be useful for solving hard problems. The

TKBO with finite subterm coefficients turned out to be also very useful in [1] for proving algebraic problems by combining resolution theorem proving with quantifier elimination over real closed fields.

Our theoretical results show that one does not need very complex ordinals to obtain arbitrarily complex instances of the TKBO. However, we think it is unrealistic to expect TKBOs with arbitrary subterm coefficients to be used in first-order theorem provers since the overhead of implementing ordinals in $\mathcal{O}$ and especially their natural product is too high. Moreover, as we pointed out, even the use of finite subterm coefficient functions requires arbitrary precision integers. However, implementing TKBOs with ordinals below ω^k for small k (sequences of k non-negative integers ordered lexicographically) and subterm coefficients set to 1 seems relatively inexpensive and requires more experiments to be understood. We believe it makes sense to make experiments in this area since in practice in first-order provers KBO behaves much better than LPO. For example, Waldmeister [4] selects LPO only on a small handful of problems (Waldmeister implements both kinds of orderings and is known for its extensive experiments with finding best orderings and strategies).

One potential use of instances of the TKBO below ω^k would be to assign large weights to symbols "close" to the goal. The theorem prover E [16] has a similar strategy (though based on finite ordinals only); likewise, Vampire chooses the level of a predicate symbol based on the "distance" between the symbol and symbols occurring in the goal. However, to the best of our knowledge nobody so far used TKBO instances in which function (not predicate) symbols have infinite weights. Checking whether a potential gain from using TKBOs outweighs performance overhead arising from their use is an interesting subject for future work.

Yet another potential use of the TKBO is automated termination proofs of sets of rewrite rules that are currently outside the scope of termination tools. Such set R is given for example as an early formalisation of the battle of Hydra and Hercules [6]. The system R, introduced in [3], withstands any attempt so far in proving its termination automatically. The reason is that the termination proof necessarily needs interpretations into $\mathcal{O}$ [14]. Due to our results we cannot hope to define generalisations of TKBO that are compatible with R. Even if we would allow for ordinal weights greater or equal to ϵ_0, our result imply that we cannot use this additional power. However, the TKBO may be successfully applied on restrictions.

It is also worth noting that the use of a TKBO with ordinals below ω^3 was essential in the applications of Vampire in interpolation [9] and loop invariant generation [8].

8 Conclusion

We proved two main results related to the use of transfinite Knuth-Bendix orders with finite signatures. First, we proved that any such order with finite subterm coefficients is equivalent to an order using ordinals below ω^ω, that is, finite sequences of natural numbers of a fixed length. Second, we proved that any such order is equivalent to an order using ordinals below ω^{ω^ω}. We also proved that the

ω^ω and ω^{ω^ω} bounds are tight. Our results show that transfinite Knuth-Bendix orders based on arbitrarily complex ordinals below ϵ_0 can be replaced by such orders using simpler ordinals. For example, when searching for an instance of the TKBO ordering a rewrite rule system, it is enough to search only for such instances using ordinals below ω^{ω^ω}.

We also discuss application and implementation issues of extensions of the Knuth-Bendix orders in first-order theorem provers.

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