A Conversation About Fuzzy Logic and Vagueness

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Chris Fermüller: At the LoMoReVI conference, last September, in Čejkovice you gave an interesting presentation entitled *Vagueness and fuzzy logic—can logicians learn from philosophers and can philosophers learn from logicians?* ...

Petr Hájek: ... but I pointed out that my title in fact just added '*and can philosophers learn from logicians?*' to a title of an earlier paper of yours!

CF: I am flattered by this reference to my own work. But we should let our intended audience know a bit a more about the background of those two contributions. I still remember that I imagined to be quite bold and provocative by submitting a paper to a workshop on *Soft Computing*—organized by you, by the way, in 2003—that suggested already in the title that logicians should not just presume that the are properly dealing with vagueness when they investigate fuzzy logics, but should pay attention to the extended discourse on so-called 'theories of vagueness' in philosophy to understand the various challenges for correct reasoning in face of vagueness. I was really surprised when my submission was not only accepted, but when you even decided to make me an invited speaker, which entailed a longer presentation. A version of the contribution soon afterwards appeared as [7], again on your invitation.

PH: Don't forget that I also want to ask the reverse question: '*Can philosophers learn from logicians?*' I think that philosophers are often badly informed about what fuzzy logic in the narrow sense of formal development of many-valued calculi, often called just mathematical fuzzy logic, has to offer.

CF: I agree with you, of course, but my original audience consisted people working in fuzzy logic. I saw no point in explaining to them how philosophers could profit from a better knowledge of their field. But the LoMoReVI conference was an excellent opportunity to ask the 'reverse question', since we had experts from quite different areas: logic, mathematics, cognitive science, linguistics, but also philosophy. So what are the main features of fuzzy logic, that you think philosophers should learn about?

PH: First of all one should recall the distinction between fuzzy logic in the broad and in

[1]Christian G. Fermüller was supported by the grant I143-G15 of the Austrian Science Foundation (FWF) Petr Hájek was supported by the grant ICC/08/E018 of the Czech Science Foundation (both these grant are part of ESF Eurocores-LogICCC project FP006). Petr Hájek also acknowledges the support of the Institutional Researc Plan AV0Z10300504.

the narrow sense as presented by several authors, among them Wang who writes in [25]:

> Fuzzy logic in the narrow sense is formal development of various logical systems of many-valued logic. In the broad sense it is an extensive agenda whose primary aim is to utilize the apparatus of fuzzy set theory for developing sound concepts, principles and methods for representing and dealing with knowledge expressed by statements in natural language.

I want to focus on fuzzy logic in the narrow sense, often called just mathematical fuzzy logic.

CF: Your monograph [13], published in 1998, has been and to a large extent still is *the* major source for research in mathematical fuzzy logic. In preparation for this conversation I checked the corresponding `google.scholar` entry, where it is currently listed as cited 2133 times—quite an achievement for a book entitled "Metamathematics of Fuzzy Logic" and that is certainly not a student textbook nor amounts to easy reading. I am glad that I had the chance to witness the evolution of its major concepts in the mid and late 1990's, when various collaborations, in particular also the *COST Action 15* on "Many-valued Logics for Computer Applications", gave ample opportunity to present and discuss what one can call the t-norm based approach to deductive logic. Let us try to summarize the essential ingredients very briefly.

PH: Well, a binary operation $*$ on the real unit interval $[0,1]$ is a t-norm if it is commutative ($x*y = y*x$), associative ($x*(y*z) = (x*y)*z$), non-decreasing in both arguments ($x \leq y$ implies $x*z \leq y*z$ and consequently $z*x \leq z*y$), and where 1 is the unit element ($x*1 = x$). I suggest to consider *any continuous t-norm* as a candidate of a truth function for conjunction.

CF: Sorry for interrupting already at this stage, but I think the intended general audience should take note of a few 'design decisions' that are implicit in choosing this starting point. First of all we have decided to consider not just 0 for 'false' and 1 for 'true' as formal truth values, but also all real numbers in between. In other words we have decided to allow for arbitrarily many intermediate truth values and insist that those values are densely linearly ordered. Moreover we stipulated that the semantics of conjunction as a logical connective can be modeled by some function over those 'degrees of truth'. This means that the semantic status of a conjunctive proposition *A&B*, i.e. its degree of truth depends only on the degrees assigned to *A* and *B*, respectively, but not on any material relation between the propositions *A* and *B*. In other words we stipulate *truth functionality*. This move alone implies that whatever 'degrees of truth' are, they must be something very different from 'degrees of belief' and from probabilities that certainly do not propagate functionally over conjunction.

PH: Sure, but regarding the choice of $[0,1]$ as sets of truth value one should point that in investigations of mathematical fuzzy logic one frequently makes a move that is obvious to mathematicians, namely *generalizing* to algebraic structures that are less particular than $[0,1]$ equipped by the standard arithmetical operations and order. This gives the so-called *general semantics* of the logic in question. Regarding truth functionality, we can agree with the following critical statement of Gaifman [11]:

> There is no denying the graded nature of vague predicates—i.e. that the aptness of applying them can be a matter of degree—and there is no denying the gradual decrease in degree. More than other approaches degree theory does justice to these facts. But from this to the institution of many-valued logic, where connectives are interpreted as functions over truth degree there is a big jump.

However, I want to point out that various detailed suggestions on how to deal with truth functionality have been made. For example, Jeff Paris [22] investigates conditions that justify truth functionality. Also the philosophers N.J.J. Smith in has monograph "Vagueness and Degrees of Truth" [24] on vagueness is positive on truth degrees and on truth functionality under some conditions.

CF: Let us go further down the list of 'design choices' made by mathematical fuzzy logic. So far we have only mentioned possible truth functions of conjunction. But a considerable part of the mathematical beauty of the t-norm based approach that you have developed consists in the fact that one can derive truth function from all other logical connectives from given t-norms by quite straightforward assumptions on their relation to each other.

PH: A central tenet of this approach is to observe that any continuous t-norm has a unique residuum that we may take as corresponding implication. This is derived from the principle that $\|A \,\&\, B\| \leq \|C\|$ if and only if $\|A\| \leq \|B \to C\|$, where $\|X\|$ denotes the truth value assigned to proposition X. Thus the truth function of implication is given by $x \to y = \max_z\{x * z \leq y\}$, where $*$ is the corresponding t-norm. Negation, in turn, can be defined by $\neg A = A \to \bot$, where $\bot$ always receives the minimal truth value 0. Disjunction can be derived in various ways, e.g., by dualizing conjunction. Moreover, the popular choice of min for conjunction arises in two ways. First min is a one of three fundamental examples of a t-norm. The corresponding residual implication is given by $\|A \to B\| = 1$ for $\|A\| \leq \|B\|$ and $\|A \to B\| = \|B\|$ otherwise. The corresponding truth function for disjunction, dual to conjunction ($\vee$), is max, of course. This logic is called Gödel logic in our community, because of one of those famous short notes of Kurt Gödel from the 1930s, where he essentially defines these truth functions. But the logic is so fundamental that has been re-discovered and re-visited many times. Furthermore min as conjunction—or 'lattice conjunction' as our algebraically minded colleagues like to call it—arises in the t-norm based approach is the fact that it is *definable* in *all* those logics. i.e., even if we have chosen a another continuous t-norm as truth function for conjunction (&), min-conjunction ($\wedge$) is implicitly presented, e.g., by taking $A \wedge B$ to abbreviate $A \,\&\, (A \to B)$. In this sense all t-norm based logics—except Gödel logics, of course—have two conjunctions.

Having defined Gödel logics explicitly, we should also mention two other fundamental logics: namely, Łukasiewicz logic, arising from the continuous t-norm $a * b = \max(0, a + b - 1)$, and Product logic [16, 18], arising from standard multiplication over $[0,1]$ as underlying t-norm. Moreover, the three mentioned logics can be combined into a single system, called $\text{Ł}\Pi\frac{1}{2}$ [6].

CF: But probably the most characteristic move that you have made in establishing t-norm based fuzzy logic as a rich research field is to ask: which logic arises if we do not fix any particular t-norm, but let conjunction vary over all continuous t-norms? You understandably call the resulting logic of all continuous t-norms "basic logic" [13, 4]. But since also other logics can and have been called "basic" in different contexts, it is best known as "Hájek's BL" nowadays. In this context one should probably also mention the logic MTL [5], that arises if one generalizes continuity to left-continuity, which is still sufficient to guarantee the existence of unique residua for corresponding t-norms.

We could indeed spend many hours discussing interesting and important results of mathematical fuzzy logic. At least one more basic fact should be mentioned: also first order versions, as well as various kinds of extensions, e.g., including modal operators, for the mentioned logics are well investigated by now.

But we should return to our motivating question: can fuzzy logic contribute to theories of vagueness as investigated by philosophers? "Theories of Vagueness", as you know, is in fact the title of a book [20] by the philosopher Rosanna Keefe, who is very critical about degree based approaches to vagueness. I don't think that pointing out that Keefe has not been aware of contemporary developments in mathematical fuzzy logic when she wrote her book suffices to deflect the worries that she and others have voiced about fuzzy logic in this context.

PH: Keefe characterizes the phenomena of vagueness quite neutrally focusing on so-called borderline cases, fuzzy boundaries, and susceptibility to the sorites paradox. I find it very acceptable, when she writes

> Vague predicates lack well-defined extensions. [They] are naturally described as having fuzzy, or blurred boundaries. Theorists should aim to find the best balance between preserving as many as possible of our judgments or opinions of various different kinds and meeting such requirements on theories as simplicity. There can be disputes about what is in the relevant body of opinions. Determining the counter-intuitive consequences of a theory is always a major part of its assessment.

Regarding the intermediary truth values of fuzzy logic she writes later in [20]:

> . . . perhaps assignment of numbers in degree theory can be seen merely as a useful instrumental device. But what are we to say about the real truth-value status of borderline case predictions? The modeler's approach is a mere hand waving . . . surely the assignment of numbers is central to it? Only order is important?

My comment here is, that we can indeed say that truth degrees are just a "model": the task is not to assign concrete numerical values to given sentences (formulas); rather the task is to study the notion of consequence (deduction) in presence of imprecise predicates. One should not conflate the idea that, in modeling logical consequence and validity, we *interpret* statements over structures where formulas are evaluated in $[0,1]$ with the much stronger claim that we actually single out a particular such interpretation as the "correct" one, by *assigning* concrete values to atomic statements.

CF: I see a rather fundamental methodological issue at play here. Philosophers often seem to suppose that any proposed theory of vagueness is either correct or simply wrong. Moreover, all basic features of a "correct" model arising from such a theory are required to correspond to some feature of the modeled part of "the real world". An exception to this general tendency is Stewart Shapiro, who in his book "Vagueness in Context" [23] has a chapter on the role of model theory, where he leaves room for the possibility that a model includes elements that are not intended to directly refer to any parameter of the modeled scenarios. Truth values from the unit interval are explicitly mentioned as an example. Nevertheless Shapiro finally rejects fuzzy logic based models of vagueness for other reasons.

PH: In [14] I have taken Shapiro's book as a source for investigating some of the formal concepts he introduces in his contextualist approach to vagueness. No philosophical claims are made, but I demonstrate that Shapiro's model, that is based Kleene's three valued logic, can be rather straightforwardly generalized to BL as an underlying many valued logic.

CF: As you indicate yourself, this leaves open the question how to interpret the role of intermediate truth values. After all truth values from $[0,1]$ or from some more general algebraic structure are the central feature of any fuzzy logic based model.

PH: Let me point out an analogy with subjective probability here. By saying "Probably I will come" you assume that there is some concrete value of your subjective probability without feeling obliged to "assign" it to what you say.

By the way, "probably" may be viewed as fuzzy modality, as explained in [13], Section 8.4, and in [12], as well as in many follow-up papers by colleagues. But, whereas the semantics of the logic of "probably" is specified truth functionally, probability itself of course is not truth functional. There is no contradiction here. Two levels of propositions are cleanly separated in the logic: the boolean propositions that refer to (crisp) sets of states measured by probabilities, and the fuzzy propositions that arise from identifying the probability of A with the truth value of "Probably A".

CF: The analogy with subjective probability is indeed illuminating. It also provides an occasion to recall a central fact about logical models of reasoning that is shared with probability theory. Quite clearly, the aim is not to model actually observed behavior of (fallible) human reasoners in face of vague, fuzzy, or uncertain information. As is well known, human agents are usually not very good in drawing correct inferences from such data and often behave inconsistently and rather unpredictably when confronted with such tasks.

While studying systematic biases and common pitfalls in reasoning under uncertainty and vagueness is a relevant topic in psychology with important applications, e.g., in economics and in medicine, the task of logic, like that of probability theory is quite different in this context: there is a strong *prescriptive* component that trumps *descriptive* adequateness. Thus, in proposing deductive fuzzy logic as a model of reasoning with vague expressions—at least of a certain type, namely gradable adjectives like "tall" or "expensive" when used in a fixed context—one does not predict that ordinary language users behave in a manner that involves the assignment of particular values to elementary propositions or the computation of truth values for logically complex sentence using particular truth functions. Rather fuzzy logic (in the narrow sense) suggests that we obtain a formal tool that generalizes classical logic in a manner, that allows one to speak of preservation of degrees of truth in inference in a precise and systematic manner. Such tools are potentially useful in engineering contexts, in particular in information processing. Whether the resulting "models" are also useful in philosophy and linguistics is a different question. Linguists seem to be unhappy about a naive application of fuzzy logics, because empirical investigations suggest that no truth function matches the way in which speakers tend to evaluate logically complex sentences involving gradable or vague adjectives. (See, e.g., Uli Sauerland's and Galit Sasson's contributions to this volume.)

Indeed, I think that, given the linguists' findings, truth functionality is best understood as a feature that it is *prescribed*, rather than "*predicted*" (to use a linguistic keyword). This actually already applies to classical logic. While we arguably *ought* to respect classical logic in drawing inferences, at least in realms like classical mathematics, logicians don't claim that ordinary language use of words like "and", "or", "implies", "for all" directly corresponds to the formal semantics codified in the corresponding truth tables, whether classical or many-valued.

Note that, if I am right about the prescriptive aspect of logic, this does not at all exclude the usefulness of truth functional logics also in the context of descriptive models. However it implies that, in order to arrive at a more realistic formal semantics of vague natural language, fuzzy logic will certainly have to be supplemented by various intensional features and also by mechanism that model the dynamics of quickly shifting contexts, as described, e.g., by Shapiro [23] but also by many linguists investigating vagueness, e.g., [1]. Actually much work done in the context of LoMoReVI is of this nature, namely *combining* deductive fuzzy logics with other types of logical models.

But then again, the role of classical logics in linguistics is analogous: it is routinely extended by intensional features, concepts from type theory and lambda calculus, generalized to so-called dynamic semantics, etc. Experts agree that there is no naive and direct translation from natural language into classical logic if we want to respect the apparent complexity of natural language expressions. Of course, the same applies to fuzzy logic. In any case, albeit the influential criticism of Kamp [19] and others, I'd say that the question of whether fuzzy logic can be usefully employed in linguistics is still open.

PH: Your terms "prescribed" and "predicted" are new for me; I find them interesting but cannot say much about this distinction. I think that the relation of mathematical fuzzy logic to natural language is very similar to that of classical mathematical logic and its relation to natural language: both deal with symbolic sentences (formulas), not with sentences of a natural language.

You say that the question of whether fuzzy logic can be usefully employed in linguistics is still open. My formulation would be "how far" instead of "whether" since I think that to some extent it has been shown already, e.g., by [24], that fuzzy logic can be usefully applied to the analysis of vague natural language.

CF: Indeed, Nick Smith [24] develops a theory of vagueness that puts fuzzy logic in its very center. Although he mainly addresses his colleagues in philosophy, I agree that it is also of direct relevance to linguistics.

However we have also mentioned that Rosanna Keefe in “Theories of Vagueness” [20] prominently criticizes an approach to vagueness that involves functions on degrees of truth as models for logical connectives. You have shortly discussed some of Keefe’s objections in [15]. Since those objections are not only advanced by Keefe, but are rather widespread in philosophical discussions about fuzzy logic, I suggest to briefly look again into some concrete issues.

PH: One of the things Keefe complains about—in the sense of judging it to be counter-intuitive—is that, if A is a perfectly “half-true” proposition, i.e., if $\|A\| = 0.5$ then we have $\|A \to A\| = \|A \to \neg A\| = 1$, assuming the Łukasiewicz truth functions for implication ($\|A \to B\| = \min(1, 1 - \|A\| + \|B\|)$) and negation ($\|\neg A\| = 1 - \|A\|$). But I think that this ceases to be problematic if we view a “half-true” statement as characterized by receiving the same truth value as its negation and remember that, like in classical logic, we declare $\|A \to B\|$ to be 1 whenever $\|A\| \leq \|B\|$.

CF: Still, I understand why Keefe thinks that modeling implication in this way clashes with some intuitions about the informal meaning of “if ... then ...”. I guess that she would point out that it is hard to accept, previous to exposure to fuzzy logic, that “If it is cold then it is not cold” has the same semantic status as “If it is cold then it is cold” in a borderline context with respect to temperature. This, of course, is a consequence of truth functionality and of the rather innocent assumption that the truth value of a perfect “half-truth” is identical to that of its negation.

I think that the reliance on pre-theoretic intuitions is at least as problematic here as it is in the case of the so-called paradoxes of material implication for classical logic. That the formula $A \to B$ is true according to classical logic, whenever A is false or B is true, only emphasizes the well known fact that there is a mismatch between (1) the precise formal meaning of $\to$ as stipulated by the corresponding truth function and (2) the conditions under which an utterance of the form “If ... then ...” successfully conveys information among speakers of English. We have to keep in mind that *material* implication is not supposed to refer to any content-related dependency between its arguments, but only refers to the (degrees of) truth of the corresponding sub-formulas.

Your reply to Keefe’s criticism points out that it is perfectly coherent to *define* the meaning of the connective “$\to$” in the indicated manner, if we are prepared to abstract away from natural language and from pre-formal intuitions. The main motivation for doing so is to arrive at a mathematically robust and elegant realm of logics that we can study in analogy to classical logic, right?

PH: Right. Let me once more emphasize that the metamathematics that arises from this particular generalization of classical logic is deep and beautiful indeed, as not only my book [13], but dozens, if not hundreds of papers in contemporary mathematical fuzzy logics can testify.

CF: I certainly agree. But this still leaves room for the possibility that mathematical fuzzy logic is just a niece piece of pure mathematics without much relevance for how we actually reason or should reason with vague notions and propositions.

PH: Well, then let us consider some sentences from natural language that may illustrate some properties of fuzzy logic.

Compare “I love you” with “I love you and I love you and I love you”. Clearly the latter implies the former; but not necessarily conversely. If we model “and” by a non-idempotent t-norm then indeed a A is not equivalent to $A\&A$, matching the indicated intuition.

Moreover: “Do I like him? Oh, yes and no”. Doesn’t this mean that the truth value of “I like him” is neither 1 nor 0? Why shouldn’t it be one half (0.5) in such a case?

CF: You might remember from earlier conversations that I actually have a different opinion about these examples. Let me briefly spell it out here once more.

As to repetition: I think that this is better analyzed as a pragmatic and not as a semantic phenomenon. To repeat a statement in the indicated manner is a way to *emphasize* the corresponding assertion. I don’t think that conjunctive repetition in natural language entails the idea that the conjunction of identical statements may be less true than that the unrepeated statement. Note that linguists take it for granted that by asserting a declarative sentence S (in usual contexts) a speaker wants to convey that the proposition p_S expressed by S is true in the given context. Emphasis, hesitation, doubt, etc., about p_S may be expressed explicitly or implicitly by different means, but the impact of such qualifications should better not be conflated with the semantic status, i.e., the asserted truth of p_S itself.

As to “Yes and No”: it is indeed not unusual to provide such an answer to the question whether (or to the suggestion that) a statement A holds. But it seems to me that this answer is a short form of expressing something like: “Yes, in some respect (i.e., in some relevant interpretation of the used words) A is indeed true; but in another, likewise relevant respect A is not true.” If I am correct in this analysis, then degrees of truth do not enter the picture here. At least not in any direct manner.

PH: What about hedges like “very”, “relatively”, “somewhat”, “definitely” etc.? Extending standard first order fuzzy logics, one may consider, e.g., “very” as a predicate modifier. Syntactically this amounts to the stipulation that for every sentence $P(a)$, where P is a fuzzy predicate and a is an individual, $very(P)(a)$ is also a well-formed sentence. Semantically, the extension of the predicate $very(P)$ is specified as a fuzzy set that can be obtained from the fuzzy set that represents the extension of the predicate P. This can be done in a simple and uniform manner, for example by squaring the membership function for P ($\mu_{very(P)}(a) = (\mu_P(a))^2$. Obviously there is great flexibility in this approach and one can study the logic of such “truth stressers”, and similarly “truth depressors”, over given fuzzy logics, like BL or MTL, both proof theoretically and from an algebraic point of view (see, e.g., [2, 3]).

CF: These are certainly good examples of research in contemporary fuzzy logic that is inspired by looking at words like "very", "relatively" etc. I have to admit that I am fascinated, but also a bit puzzled by the fact that one can retrieve literally hundreds of papers in fuzzy logic by searching for "linguistic hedges" in `google.scholar`. (Actually more than 27,100 entries are listed in total.) But if one looks at linguistic literature on the semantics of such words one finds quite different models. While gradability of adjectives and properties of corresponding order relations are investigated in this context, a methodological principle seems to be in place—almost universally accepted among linguists—that at the level of truth conditions one should stick with bivalent logic. I think that there are indeed good reasons, mostly left implicit, for sticking with this principle. If I understand linguists correctly, then a very important such reason is that their models should always be checked with respect to concrete linguistic data. But those data usually only allow to categorize linguistic expressions as being accepted or not accepted by competent language users. Indeed, it is hard to imagine how one could use a standard linguistic corpus to extract information about degrees of acceptability in connection with logical connectives.

My remarks are not intended to imply that there can't be a role for fuzzy logic in linguistics. In recent work with Christoph Roschger [9] we explicitly talk about potential bridges between fuzzy logic and linguistic models. But these "bridges" do not directly refer to deductive t-norm based fuzzy logics. We rather looked at ways to systematically extract fuzzy sets from given contextual models, as they are used in so-called dynamic semantics. Of course, one could also generalize the underlying bivalent models to fuzzy ones. But the price, in terms of diminished linguistic significance, is hardly worth paying, unless one can show that mathematical structures arise that are interesting enough to be studied for their own sake.

A direct role for logics like Łukasiewicz, Gödel, Product logic, and more fundamental deductive fuzzy logics, like BL and MTL, in linguistic contexts may arise if we insist on the linguistic fact, that "true" itself is sometimes used as gradable adjective, just like "tall", "clever", "heavy" etc. The various fuzzy logics then correspond to (prescriptive) models of reasoning that take perfectly comprehensible talk about statements being only "somewhat true", "more true" than others, or "definitely true" at face value. Of course, we thereby abstract away from individual utterances and idealize actual language use in a manner that is familiar from classical logic.

PH: Your last remark may bring us back to philosophy. There the *sorites* paradox is considered whenever one discusses the role of logic in reasoning with vague notions. In [17] an analysis of *sorites* is offered using a hedge *At*—"almost true". Consider the axioms: $bold(0)$ and $(\forall n)(bold(n) \to At(bold(n+1)))$, where $(bold(n)$ represents the proposition that a man with n hears on his head is bold. This is augmented by further natural axioms about At. Based on basic logic BL we obtain a simple and clear degree based semantics for *At* and for *bold* that does not lead to contradiction or to counter-intuitive assumptions.

CF: This is indeed a nice example of how fuzzy logic can be used as a *prescriptive* tool of reasoning. The paradox simply disappears, which of course implies that the model is not be understood *descriptively*. If people actually find themselves to be in a sorites like scenario, they will feel the tendency to end up with contradicting assumptions. In other words they do not use fuzzy logic to start with. After all we ("competent speakers") do understand that such a scenario can is "paradoxical". Your models show that one can avoid or circumvent the difficulty by considering "near-truth" in a systematic manner.

Shapiro [23] offers an alternative analysis that moves closer to observable behavior of speakers. He invites us to imagine a community of conversationalists that are successively confronted with members of a sorites series, e.g., a series of 1000 men, starting with Yul Brynner and ending with Steve Pinker, where each man is indistinguishable from his neighbors in the series in respect of boldness. Shapiro's model predicts, that if the conversationalists are forced to judge the boldness of each of those men one by one, they will try to maintain consistency with their earlier (yes/no) judgments. However at some point they will realize that this is not possible if they don't want to call Steve Pinker bold, which is absurd, as anyone that has ever seen a picture of Pinker can testify. Thus they will retract earlier judgments made along their forced march through the sorites series and thereby "jump" between different (partial) truth assignments. Shapiro uses precisification spaces based on Kleene's three valued logic to model the resulting concept of inference formally.

As you have already mentioned, you have shown in [14] how Shapiro's model can be generalized to placing fuzzy instead of of a three valued interpretations at its core. In [8] I indicate that this can be understood as abstracting away from a concretely given sorites situation towards a model that summarizes in a static picture what can be observed about the overall dynamics of many individual instances of forced marches through a sorites series. In that interpretation degrees of truth emerge as measures of likelihood of "jumps", i.e., of revisions of binary judgments. Truth-functionality is preserved, because for complex statement we don't consider the likelihood of, say, judging $A\&B$ to be true, but rather the (properly regulated) degree of truth of the statement "A is likely to be judged true and B is likely to be judged true". (There is some similarity to the earlier mentioned logic of "probably" as a fuzzy modality.)

PH: We should not give the wrong impression that fuzzy logic in its broader sense of dealing with problems and applications arising from a graded notion of membership in a set is mainly used to analyze vague language. The areas of fuzzy controlling, soft computing, and inference using "fuzzy if-then rules" have not only attracted a lot of research, but can point to many interesting applications in engineering, decision making, data mining, etc. (see, e.g., [25]). The simple idea to model an instruction like "If the pressure is rather high. then turn the valve slightly to left" by reference to fuzzy sets rather than to fixed threshold values has proved to be effective and useful.

With hindsight it is hard to understand why Zadeh's proposal to generalize the classical notion of a set ("crisp set") to a fuzzy set by allowing intermediate degrees of membership [21] has been met with so much resistance from traditional mathematics and engineering. Presumably many found it unacceptable to declare that vagueness is not necessarily a defect of language, and that it may be adequate and useful to deal with it mathematically instead of trying to eliminate it. There is a frequently encountered misunderstanding here: fuzzy logic provides precise mathematical means to talk about impreciseness, but it does not advocate imprecise or vague mathematics.

CF: As Didier Dubois in his contribution to this volume reminds us, Zadeh insisted that a proposition is vague if, in addition to being fuzzy, i.e., amenable to representation by fuzzy sets and relations, "it is insufficiently specific for a particular purpose" [26]. I am not sure that this characterization of vagueness is robust enough to assist useful formal models. But in any case, it is clear that fuzziness and vagueness are closely related and might not always be distinguishable in practice. At the very least there is some kind of dependency: fuzzy notions systematically give rise to vague language.

PH: Let us finally return to our two-fold question: can logicians learn from philosophers and can philosophers learn from logicians? I think we both agree that the answer should be "yes".

CF: Certainly. Moreover, thanks also to the activities in LoMoReVI and our sister LogICCC project VAAG, we may include linguists in the circle of mutual learning regarding appropriate theorizing about vagueness.

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