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Convergent Geometric Integrator for the LLG Equation with Magnetostriction

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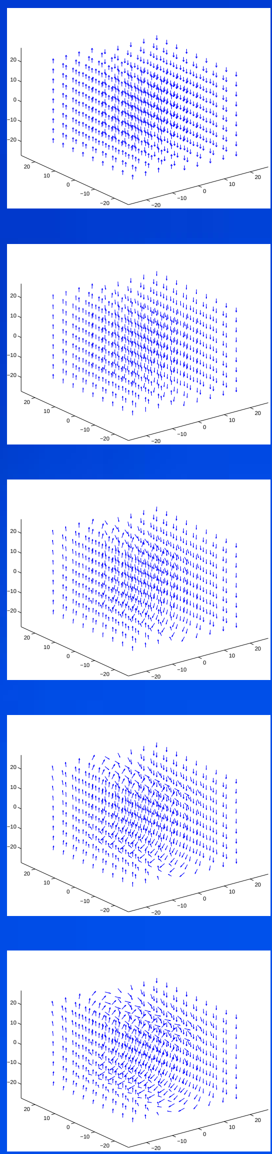
What is all about?

The theoretical understanding and practical prediction of micromagnetic phenomena is of utmost importance for the improvement of existing and development of future magnetic based devices like e.g. storage devices, sensors, or magnetic RAM. However, certain aspects do not need the practical development of prototypes, but can also be well understood by means of numerical simulations. This relies on the mathematical modelling of micromagnetics. In physics, it is well-accepted that the dynamics of micromagnetics is described best by the nonlinear Landau-Lifshitz-Gilbert equation (LLG), where time evolution is driven by the so-called effective field $\mathbf{h}_{\text{eff}}$.

In [1], a numerical integrator is proposed for a simplified effective field, where only the so-called exchange energy is reflected. In our generalization of [1], we further include the cristalline anisotropy energy, the magnetostatic energy, the exterior Zeeman energy, as well as the magnetostrictive energy. The latter couples LLG with the conservation of momentum equation (CM) and includes an additional nonlinearity. This coupling was first analyzed in [2], where a different algorithm was proposed. In our work, we combine the approaches of [1] and [2]. Besides the nonlinearities of LLG and CM, numerical difficulties arise from a non-convex side constraint $|\mathbf{m}| = 1$ in space-time for the magnetization and from a certain non-local, but linear integral operator $\mathcal{P}$ involved for the computation of the demagnetization field.

The developed numerical integrator is linear implicit and treats the known nonlinearities in an effective manner. The key features of our integrator read as follows:

- First, the implicit part only deals with the higher-order term stemming from the exchange energy, whereas the remaining lower-order terms are treated explicitly. In particular, this includes the numerical computation of the demagnetization field which is the most time and memory consuming part of the simulation.
- Second, the integrator decouples LLG and CM. Overall and besides the demagnetization field, this results in the fact that only two linear systems per timestep have to be solved.
- Finally and from utmost importance for reliable simulations, we prove that our integrator is unconditionally convergent as time step-size k and mesh-size h tend to zero.



LLG Equation

Let Ω denote a magnetic body and $\mathbf{m} : (0,\tau)\times\Omega \to \mathbb{R}^3$ with $|\mathbf{m}| = 1$ be the magnetization. With $\alpha > 0$ the damping parameter, the non-dimensional formulation of LLG reads

$$\mathbf{m}_\tau = \frac{-1}{1+\alpha^2}\mathbf{m}\times\mathbf{h}_{\text{eff}} - \frac{\alpha}{1+\alpha^2}\mathbf{m}\times(\mathbf{m}\times\mathbf{h}_{\text{eff}})$$
$$\mathbf{m}(0) = \mathbf{m}_0, \quad \partial_n\mathbf{m} = 0.$$

The total magnetic field $\mathbf{h}_{\text{eff}}$ is given by

$$\mathbf{h}_{\text{eff}}(\mathbf{m}) = \Delta\mathbf{m} + D\Phi(\mathbf{m}) + \mathcal{P}\mathbf{m} - \mathbf{f} + \mathbf{h}^{\boldsymbol{\sigma}}(\mathbf{m}).$$

Here, $\mathcal{P}\mathbf{m}$ refers to the demagnetization field which is induced by the magnetostatic Maxwell's equations, Φ is the anisotropy density, $\mathbf{f}$ is the applied field, and $\Delta\mathbf{m}$ is the exchange. The term $\mathbf{h}^{\boldsymbol{\sigma}}(\mathbf{m})$ denotes the contribution of the magnetostrictive energy which stems from the conservation of momentum equation

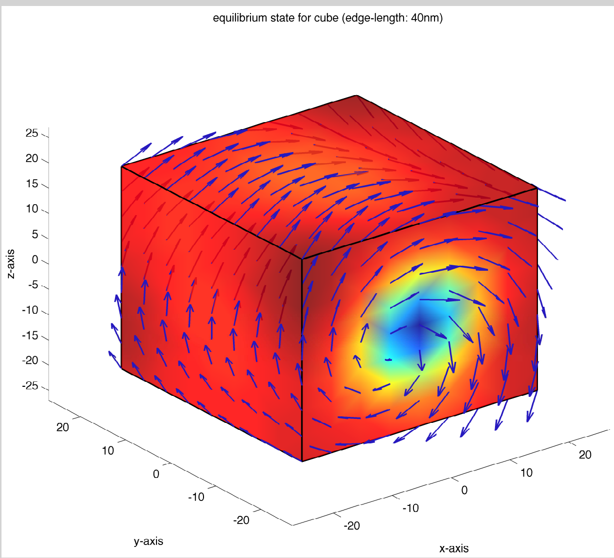
$$\rho\mathbf{u}_{tt} - \nabla\cdot\boldsymbol{\sigma} = 0,$$

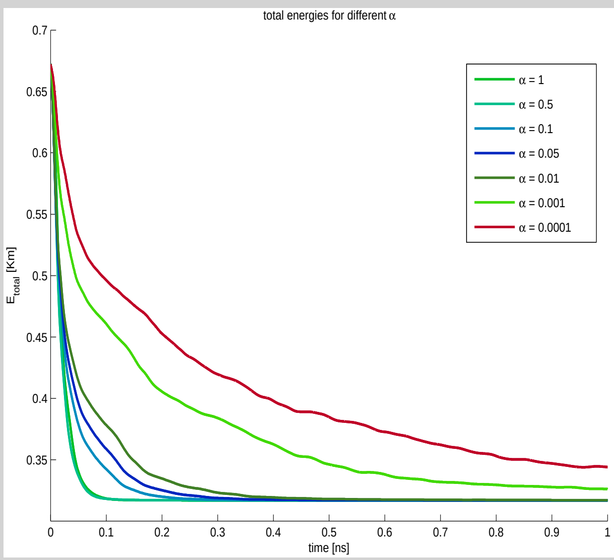
where $\boldsymbol{\sigma}$ denotes the stress tensor. The vector field $\mathbf{u}$ denotes the magnetic displacement and ρ some material parameters.

Algorithm

- **Input**: initial $\mathbf{m}_h^0 \in \mathcal{M}_h, \Pi_h\mathbf{u}(0), \Pi_h\partial_t\mathbf{u}(0)$, damping parameter α, parameter $\theta = 1$
  1. Find $\mathbf{v}_h^j \in \mathcal{K}_{\mathbf{m}_h^j}$ such that for all $\boldsymbol{\psi}_h \in \mathcal{K}_{\mathbf{m}_h^j}$
  $$\alpha(\mathbf{v}_h^j,\boldsymbol{\psi}_h) + \big((\mathbf{m}_h^j\times\mathbf{v}_h^j),\boldsymbol{\psi}_h\big) = -\big(\nabla(\mathbf{m}_h^j+\theta k\mathbf{v}_h^j),\nabla\boldsymbol{\psi}_h\big) + \big(\mathbf{h}_{\text{explicit}}(\mathbf{m}_h^j,\mathbf{u}_h^j,\mathbf{f}),\boldsymbol{\psi}_h\big)$$
  2. Define $\mathbf{m}_h^{j+1} \in \mathcal{M}_h$ nodewise by $\mathbf{m}_h^{j+1}(\mathbf{z}) = \frac{\mathbf{m}_h^j(\mathbf{z})+k\mathbf{v}_h^j(\mathbf{z})}{|\mathbf{m}_h^j(\mathbf{z})+k\mathbf{v}_h^j(\mathbf{z})|}$
  3. Find $\mathbf{u}_h^{j+1} \in \mathbf{V}_h$ such that for all $\boldsymbol{\varphi}_h \in \mathbf{V}_h$
  $$\rho(\delta^2\mathbf{u}_h^{j+1},\boldsymbol{\varphi}_h) + \big(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1}),\boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)\big) = \big(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}),\boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)\big)$$
- **Output**: discrete solutions $\mathbf{v}_h^j \in K_{\mathbf{m}_h^j}, \mathbf{m}_h^j \in M_h$ and $\mathbf{u}_h^j \in \mathbf{V}_h$

Numerical Experiment





Numerical Experiment: We consider a micromagnetic cube with edge-length of $40nm$ consisting of a uniaxial material which is characterized by an exchange constant $A = 1\cdot 10^{-11}$ in $[J/m]$ and an anisotropy constant $K = 3.9788\cdot 10^4$ in $[J/m^3]$. For the initial state, an inhomogeneous magnetization parallel to the easy axis $\mathbf{e} = (0,0,1)$ was chosen *(top)*. In this example, the magnetostrictive effect is neglected and no external field is applied. The five pictures from top to bottom illustrate the dynamic behaviour of the magnetization.

Left: Micromagnetic body Ω with alignment of magnetization in equilibrium state (vortex state). The color visualizes the direction of the magnetization relative to the $\mathbf{y}$-axis.

Right: Variation of the Gibbs Free energy from the beginning of the simulation until equilibrium is reached under consideration of various values for the damping parameter α.

Innovations

Numerical Scheme:
- Including total magnetic field
- Time-splitting for more effective computation
- Only two linear systems to be solved per timestep

Analytical Result:
- Unconditional convergence result for introduced numerical scheme
- Effective treatment of magnetostatic energy
- No regularity assumptions on $\mathbf{m}$

References:

[1] F. ALOUGES, *A new finite element scheme for Landau-Lifchitz equations*, Discrete Contin. Dyn. Syst. Ser. S, 2008

[2] L. BANAS and M. SLODICKA, *Error estimates for Landau-Lifshitz-Gilbert equation with magnetostriction*, Appl. Numer. Math., 2006

[3] P. GOLDENITS et. al, *Effective Simulation of the Dynamics of Ferromagnetism*, work in progress, 2012