Stable self–similar blow up for the semilinear wave equation

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Wave equation with focusing power nonlinearity:

$$\begin{cases} \partial_t^2 \psi - \Delta\psi = |\psi|^{p-1}\psi \\ \psi[0] = (f, g) \end{cases}$$

$\psi : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$. $1 < p \leq 3 \Rightarrow$ energy subcritical.

Known results

- **Local well-posedness** in $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.
- **Fundamental self-similar solution** (ODE blow up)

$$\psi^T(t, x) = (T - t)^{-\frac{2}{p-1}} \kappa_0^{\frac{1}{p-1}}, \quad \kappa_0 = \frac{2(p+1)}{(p-1)^2}.$$

- **Blow up rate**, *F. Merle, H. Zaag (2003, 2005),* $1 < p \leq 3$.
  Any blow up solution blows up at the self similar rate $(T - t)^{-\frac{2}{p-1}}$.
- **Family of self-similar solutions**, *P. Bizon et al. (2010), radial,* $p = 3, 7$.

$$\psi_n^T(t, r) = (T - t)^{-\frac{2}{p-1}} u_n\left(\frac{r}{T-t}\right).$$

- **Numerical experiments**, *P. Bizon, T. Chmaj, Z. Tabor (2004), radial,* $p = 3, 5, 7 \Rightarrow$ Suggestion: ψ^T describes generic blow up behaviour.

Stability of the fundamental self-similar solution

Spherical symmetry, study problem in backward lightcone $\mathcal{C}_T$ of blow up point $(T, 0)$. Consider small perturbations $\psi = \psi^T + \varphi$

$$\varphi_{tt} - \varphi_{rr} - \frac{2}{r}\varphi_r - p(\psi^T)^{p-1}\varphi - N_T(\varphi) = 0 \quad \text{in} \quad \mathcal{C}_T$$
$$\varphi[0] = (f, g) - \psi^T[0]$$

Local energy norm
Energy of free equation : $\int_0^\infty r^2[\varphi_t(t,r)^2 + \varphi_r(t,r)^2]dr \Rightarrow$ not well suited to define *local* norm.

$$E(\varphi) = \int_0^\infty [r\varphi_r(t,r) + \varphi(t,r)]^2 + r^2\varphi_t(t,r)^2 dr$$

Local energy space $(\mathcal{E}(R), \|\cdot\|_{\mathcal{E}(R)})$

$$\|(f,g)\|^2_{\mathcal{E}(R)} := \int_0^R |rf'(r) + f(r)|^2 dr + \int_0^R r^2|g(r)|^2 dr$$

$$\|(\psi^T(t,\cdot), \psi^T_t(t,\cdot))\|_{\mathcal{E}(T-t)} = C_p(T-t)^{-\frac{5-p}{2(p-1)}}.$$

Main result (Donninger, S., 2012)

Fix $\varepsilon > 0$. Let (f, g) be radial initial data sufficiently close to ψ^1 in the local energy topology. Then there exists a T close to 1 such that

$$\begin{cases} \partial_t^2 \psi - \Delta \psi = |\psi|^{p-1}\psi \\ \psi[0] = (f, g) \end{cases}$$

has a unique radial solution $\psi : \mathcal{C}_T \to \mathbb{R}$ which satisfies

$$(T-t)^{\frac{5-p}{2(p-1)}} \|(\psi(t,\cdot), \psi_t(t,\cdot)) - (\psi^T(t,\cdot), \psi_t^T(t,\cdot))\|_{\mathcal{E}(T-t)} \lesssim (T-t)^{|\omega_p|-\varepsilon}$$

for all $t \in [0, T)$ where $\omega_p \in [-1, -\frac{1}{2}]$.

$\Rightarrow$ **The blow up described by ψ^T is stable.**

Sketch of proof

First order formulation

$$\varphi_1 = (T-t)^{\frac{2}{p-1}} r\varphi_t, \quad \varphi_2 = (T-t)^{\frac{2}{p-1}} \partial_r(r\varphi).$$

Similarity coordinates $(t, r) \to (\tau, \rho)$

$$\rho := \frac{r}{T-t}, \quad \tau := -\log(T-t)$$

Operator formulation in local energy space $\mathcal{H} := L^2(0,1) \times L^2(0,1)$

$$\begin{cases} \frac{d}{d\tau}\Phi(\tau) = L\Phi(\tau) + \mathbf{N}(\Phi(\tau)) \text{ for } \tau > -\log T \\ \Phi(-\log T) = \mathbf{u} \end{cases}$$

$\Phi : (-\log T, \infty) \to \mathcal{H}$.

Linear perturbation theory

Semigroup theory. L generates a C_0-semigroup $S : [0,\infty) \to \mathcal{B}(\mathcal{H})$. $\Rightarrow$ well-posedness of linearized equation and growth estimate

$$\|S(\tau)\| \leq e^{\tilde{\omega}_p \tau} \quad \forall \tau \geq 0, \quad \text{where} \quad \tilde{\omega}_p > 0.$$

Spectrum. $\sigma(L) \subset \{\lambda \in \mathbb{C} : Re\lambda \leq \omega_p\} \cup \{1\}$ for $\omega_p \in [-1, -\frac{1}{2}]$.

Unstable eigenvalue. $\lambda = 1$, eigenvector $\mathbf{g}$ (symmetry mode) $\Rightarrow$ instability caused by time translation symmetry.

Spectral projection.

$$P := \frac{1}{2\pi i} \int_\Gamma R_L(\lambda) d\lambda$$

rg$P = \langle \mathbf{g} \rangle$, $\mathcal{N} :=$ ker$P$ is stable subspace, $\sigma(L_\mathcal{N}) = \sigma(L) \backslash \{1\}$. P commutes with $S(\tau)$ and $\mathcal{N}$ is invariant under $S(\tau)$.

Linear time evolution.

$$\|S(\tau)(1-P)\mathbf{f}\| \lesssim_\varepsilon e^{(-|\omega_p|+\varepsilon)\tau} \|(1-P)\mathbf{f}\|$$

for all $\tau \geq 0$ and $\mathbf{f} \in \mathcal{H}$ and

$$S(\tau)P\mathbf{f} = e^\tau P\mathbf{f}.$$

Shifted solution. $\Psi : [0,\infty) \to \mathcal{H}$, $\Psi(\tau) := \Phi(\tau - \log T)$

Rewrite initial data. $\mathbf{u} \approx (f,g) - \psi^T[0] \to$

$$\mathbf{U}(\mathbf{v},T) \approx \mathbf{v} + \psi^1[0] - \psi^T[0], \quad \mathbf{v} \approx (f,g) - \psi^1[0].$$

Symmetry mode. $T \mapsto \mathbf{U}(\mathbf{0},T) \approx \psi^1[0] - \psi^T[0]$

$$D_T\mathbf{U}(\mathbf{0},1) = c_p\mathbf{g}.$$

$\Rightarrow$ $\mathbf{g}$ is tangent vector at $T = 1$.

Nonlinear perturbation theory

Duhamel formula.

$$\Psi(\tau) = S(\tau)\mathbf{U}(\mathbf{v},T) + \int_0^\tau S(\tau-\tau')\mathbf{N}(\Psi(\tau'))d\tau' \quad \text{for} \quad \tau \geq 0$$

$$\mathcal{X} := \left\{\Psi \in C([0,\infty),\mathcal{H}) : \sup_{\tau>0} e^{(|\omega_p|-\varepsilon)\tau}\|\Psi(\tau)\| < \infty\right\}.$$

Estimates for the nonlinearity. $\Rightarrow$ **Restriction** $1 < p \leq 3$**!**. For $\mathbf{u},\mathbf{v}$ small

$$\|\mathbf{N}(\mathbf{u}) - \mathbf{N}(\mathbf{v})\| \lesssim (\|\mathbf{u}\| + \|\mathbf{v}\|)\|\mathbf{u}-\mathbf{v}\|.$$

Subtract element of unstable subspace rg$P$.

$$\Psi(\tau) = S(\tau)\mathbf{U}(\mathbf{v},T) + \int_0^\tau S(\tau-\tau')\mathbf{N}(\Psi(\tau'))d\tau' - e^\tau \mathbf{F}(\mathbf{v},T)$$

Correction: $\mathbf{F}(\mathbf{v},T) := P\left(\mathbf{U}(\mathbf{v},T) + \int_0^\infty e^{-\tau'}\mathbf{N}(\boldsymbol{\Psi}(\mathbf{U}(\mathbf{v},T))(\tau'))d\tau'\right)$.
Banach fixed point theorem $\Rightarrow$ existence of a unique solution for small initial data ($\mathbf{v}$ small and T close to 1) with linear decay $e^{-(|\omega_p|-\varepsilon)\tau}$.

Erase correction by adjusting T.

$\mathbf{U}(\mathbf{0},1) = \mathbf{0} \Rightarrow \mathbf{F}(\mathbf{0},1) = \mathbf{0} \Rightarrow$ extend this to neighbourhood.

For every small $\mathbf{v}$ there exists a T close to 1 such that $\mathbf{F}(\mathbf{v},T) = \mathbf{0} \Rightarrow$ the original equation has a unique solution that decays like $e^{-(|\omega_p|-\varepsilon)\tau}$.

Transform back to original coordinates and variables.

Concluding remarks

- Proof is basically along the lines of similar problem for wave maps equation (Donninger,S., Aichelburg 2011, Donninger 2012).
- $\Rightarrow$ suitable for energy supercritical problems.
- Extension of above result for (NLW) to $p < 5$ straightforward.
- Work in progress: $p \geq 5$.

Thank you for your attention!