Operations Research Proceedings

Diethard Klatte
Hans-Jakob Lüthi
Karl Schmedders *Editors*

Operations Research Proceedings 2011

Selected Papers of the International Conference on Operations Research (OR 2011), August 30 - September 2, 2011, Zurich, Switzerland

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Springer

Necessary Optimality Conditions for Improper Infinite-Horizon Control Problems

Sergey M. Aseev and Vladimir M. Veliov

Abstract The paper revisits the issue of necessary optimality conditions for infinite-horizon optimal control problems. It is proved that the normal form maximum principle holds with an explicitly specified adjoint variable if an appropriate relation between the discount rate, the growth rate of the solution and the growth rate of the objective function is satisfied. The main novelty is that the result applies to general non-stationary systems and the optimal objective value needs not be finite (in which case the concept of overtaking optimality is employed).

1 Introduction

Infinite-horizon optimal control problems arise in many fields of economics, in particular in models of economic growth. Typically, the utility functional to be maximized is defined as an improper integral of the discounted instantaneous utility on the time interval $[0,\infty)$. The last circumstance gives rise to specific mathematical features of the problems and different pathologies (see [4, 7, 9, 11, 13]).

The contribution of the present paper is twofold. First we extend the version of the Pontryagin maximum principle for infinite-horizon optimal control problems with dominating discount established in [3, 4] to a more general class of non-autonomous problems and relax the assumptions. Second, we adopt the classical needle variations technique [12] to the case of infinite-horizon problems. Thus, the approach in the present paper essentially differs from the ones used in [3, 4, 6]. The needle variations technique is a standard tool in the optimal control theory. The advantage of this technique is that as a rule it produces (if applicable) the most general versions of the Pontryagin maximum principle. Nevertheless, application of needle

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Sergey M. Aseev
International Institute for Applied Systems Analysis, Schlossplatz 1, A-2361 Laxenburg, Austria, and Steklov Mathematical Institute, Gubkina 8, 119991 Moscow, Russia, e-mail: aseev@mi.ras.ru

Vladimir M. Veliov
Institute of Mathematical Methods in Economics, Vienna University of Technology, Argentinierstr. 8/E105-4, A-1040 Vienna, Austria, e-mail: veliov@tuwien.ac.at

D. Klatte et al. (eds.), *Operations Research Proceedings 2011*, Operations Research Proceedings,
DOI 10.1007/978-3-642-29210-1_4,

variations technique is not so straightforward in the case of infinite-horizon problems.

Another important feature of our main result is that it is applicable also for problems where the objective value may be infinite. In this case the notion of overtaking optimality is adapted (see [7]). In contrast to the known results, the maximum principle that we obtain has a normal form, that is, the multiplier of the objective function in the associated Hamiltonian can be taken equal to one.

2 Statement of the problem and assumptions

Let G be a nonempty open convex subset of R^n and U be an arbitrary nonempty set in R^m. Let $f : [0,\infty) \times G \times U \mapsto R^n$ and $g : [0,\infty) \times G \times U \mapsto R^1$.

Consider the following optimal control problem (P):

$$J(x(\cdot),u(\cdot)) = \int_0^\infty e^{-\rho t} g(t,x(t),u(t))\,dt \to \max, \tag{1}$$

$$\dot{x}(t) = f(t,x(t),u(t)), \qquad x(0) = x_0, \tag{2}$$

$$u(t) \in U. \tag{3}$$

Here $x_0 \in G$ is a given initial state of the system and $\rho \in R^1$ is a "discount" rate (which could be even negative).

Assumption (A1): *The functions* $f : [0,\infty) \times G \times U \mapsto R^n$ *and* $g : [0,\infty) \times G \times U \mapsto R^1$ *together with their partial derivatives* $f_x(\cdot,\cdot,\cdot)$ *and* $g_x(\cdot,\cdot,\cdot)$ *are continuous in* (x,u) *on* $G \times U$ *for any fixed* $t \in [0,\infty)$, *and measurable and locally bounded in* t, *uniformly in* (x,u) *in any bounded set.* [1]

In what follows we assume that the class of *admissible controls* in problem (P) consists of all measurable locally bounded functions $u : [0,\infty) \mapsto U$. Then for any initial state $x_0 \in G$ and any admissible control $u(\cdot)$ plugged in the right-hand side of the control system (2) we obtain the following Cauchy problem:

$$\dot{x}(t) = f(t,x(t),u(t)), \qquad x(0) = x_0. \tag{4}$$

Due to assumption $(A1)$ this problem has a unique solution $x(\cdot)$ (in the sense of Carathéodory) which is defined on some time interval $[0,\tau]$ with $\tau > 0$ and takes values in G (see e.g. [9]). This solution is uniquely extendible to a maximal interval of existence in G and is called *admissible trajectory* corresponding to the admissible control $u(\cdot)$.

If $u(\cdot)$ is an admissible control and the corresponding admissible trajectory $x(\cdot)$ exists on $[0,T]$ in G, then the integral

[1] The local boundedness of these functions of t, x and u (take $\phi(\cdot,\cdot,\cdot)$ as a representative) means that for every $T > 0$ and for every bounded set $Z \subset G \times U$ there exists M such that $\|\phi(t,x,u)\| \le M$ for every $t \in [0,T]$ and $(x,u) \in Z$.

$$J_T(x(\cdot),u(\cdot)) := \int_0^T e^{-\rho t} g(t,x(t),u(t))\,dt$$

is finite. This follows from (A1), the definition of admissible control and the existence of $x(\cdot)$ on $[0,T]$.

We will use the following modification of the notion of weakly overtaking optimal control (see [7]).

Definition 1: *An admissible control $u_*(\cdot)$ for which the corresponding trajectory $x_*(\cdot)$ exists on $[0,\infty)$ is locally weakly overtaking optimal (LWOO) if there exists $\delta > 0$ such that for any admissible control $u(\cdot)$ satisfying*

$$\text{meas}\,\{t \geq 0 : u(t) \neq u_*(t)\} \leq \delta$$

and for every $\varepsilon > 0$ and $T > 0$ one can find $T' \geq T$ such that the corresponding admissible trajectory $x(\cdot)$ is either non-extendible to $[0,T']$ in G or

$$J_{T'}(x_*(\cdot),u_*(\cdot)) \geq J_{T'}(x(\cdot),u(\cdot)) - \varepsilon.$$

Notice that the expression $d(u(\cdot),u_*(\cdot)) = \text{meas}\,\{t \in [0,T] : u(t) \neq u_*(t)\}$ generates a metric in the space of the measurable functions on $[0,T]$, $T > 0$, which is suitable to use in the framework of the needle variations technique (see [2]).

In the sequel we denote by $u_*(\cdot)$ an LWOO control and by $x_*(\cdot)$ – the corresponding trajectory.

Assumption (A2): *There exist numbers $\mu \geq 0$, $r \geq 0$, $\kappa \geq 0$, $\beta > 0$ and $c_1 \geq 0$ such that for every $t \geq 0$*

(i) $\|x_*(t)\| \leq c_1 e^{\mu t}$;

(ii) for every admissible control $u(\cdot)$ for which $d(u(\cdot),u_*(\cdot)) \leq \beta$ the corresponding trajectory $x(\cdot)$ exists on $[0,\infty)$ in G and it holds that

$$\|g_x(t,y,u_*(t))\| \leq \kappa(1 + \|y\|^r) \quad \text{for every } y \in \text{co}\,\{x(t),x_*(t)\}.$$

Assumption (A3): *There are numbers $\lambda \in R^1$, $\gamma > 0$ and $c_2 \geq 0$ such that for every $\zeta \in G$ with $\|\zeta - x_0\| < \gamma$ equation (4) with $u(\cdot) = u_*(\cdot)$ and initial condition $x(0) = \zeta$ (instead of $x(0) = x_0$) has a solution $x(\zeta;\cdot)$ on $[0,\infty)$ in G and*

$$\|x(\zeta;t) - x_*(t)\| \leq c_2 \|\zeta - x_0\|\, e^{\lambda t}.$$

The last two assumptions can be viewed as definitions of the constants μ, r and λ, which appear in the key assumption below, called *dominating discount* condition.

Assumption (A4):

$$\rho > \lambda + r\max\{\lambda,\mu\}.$$

For an arbitrary $\tau \geq 0$ consider the following linear differential equation (the linearization of (4) along $(x_*(\cdot),u_*(\cdot))$:

$$\dot{y}(t) = f_x(t, x_*(t), u_*(t))y(t), \quad t \geq 0 \tag{5}$$

with initial condition

$$y(\tau) = y_0. \tag{6}$$

In the next section we present necessary optimality conditions in the form of Pontryagin's maximum principle.

3 Main result

Due to assumption $(A1)$ the partial derivative $f_x(\cdot, x_*(\cdot), u_*(\cdot))$ is measurable and locally bounded. Hence, there is a unique (Carathéodory) solution $y_*(\cdot)$ of the Cauchy problem (8), (6) which is defined on the whole time interval $[0,\infty)$. Moreover,

$$y_*(t) = K_*(t,\tau)y_*(\tau), \quad t \geq 0, \tag{7}$$

where $K_*(\cdot,\cdot)$ is the Cauchy matrix of differential system (8) (see [10]). Recall that

$$K_*(t,\tau) = Y_*(t)Y_*^{-1}(\tau), \quad t,\tau \geq 0,$$

where $Y_*(\cdot)$ is the fundamental matrix solution of (8) normalized at $t = 0$. This means that the columns $y_i(\cdot)$, $i = 1,\dots,n$, of the $n \times n$ matrix function $Y_*(\cdot)$ are (linearly independent) solutions of (8) on $[0,\infty)$ that satisfy the initial conditions

$$y_i^j(0) = \delta_{i,j}, \quad i,j = 1,\dots,n,$$

where

$$\delta_{i,i} = 1, \quad i = 1,\dots,n, \quad \text{and} \quad \delta_{i,j} = 0, \quad i \neq j, \quad i,j = 1,\dots,n.$$

Analogously, consider the fundamental matrix solution $Z_*(\cdot)$ (normalized at $t = 0$) of the linear adjoint equation

$$\dot{z}(t) = -\left[f_x(t, x_*(t), u_*(t))\right]^* z(t). \tag{8}$$

Then $Z_*^{-1}(t) = [Y_*(t)]^*$, $t \geq 0$.

Define the normal-form Hamilton-Pontryagin function $\mathscr{H} : [0,\infty) \times G \times U \times R^n \mapsto R^1$ for problem (P) in the usual way:

$$\mathscr{H}(t,x,u,\psi) = e^{\rho t} g(t,x,u) + \langle f(t,x,u), \psi \rangle, \quad t \in [0,\infty),\ x \in G,\ u \in U,\ \psi \in R^n.$$

The following theorem presents the main result of the paper – a version of the Pontryagin maximum principle for non-autonomous infinite-horizon problems with dominating discount.

Theorem 1. *Assume that* $(A1)$–$(A4)$ *hold. Let* $u_*(\cdot)$ *be an admissible LWOO control and let* $x_*(\cdot)$ *be the corresponding trajectory. Then*

(i) *For any* $t \geq 0$ *the integral*

$$I_*(t) = \int_t^{\infty} e^{-\rho s}[Z_*(s)]^{-1} g_x(s, x_*(s), u_*(s))\, ds \tag{9}$$

converges absolutely.

(ii) *The vector function* $\psi : [0, \infty) \mapsto R^n$ *defined by*

$$\psi(t) = Z_*(t) I_*(t), \quad t \geq 0 \tag{10}$$

is (locally) absolutely continuous and satisfies the conditions of the normal form maximum principle, i.e. $\psi(\cdot)$ *is a solution of the adjoint system*

$$\dot{\psi}(t) = -\mathscr{H}_x(t, x_*(t), u_*(t), \psi(t)) \tag{11}$$

and the maximum condition holds:

$$\mathscr{H}(t, x_*(t), u_*(t), \psi(t)) \stackrel{a.e.}{=} \sup_{u \in U} \mathscr{H}(t, x_*(t), u, \psi(t)). \tag{12}$$

The proof of the above theorem employs approximations with finite-horizon problems in which the solution of the adjoint equation can be defined explicitly. The key point is to use the dominating discount condition (A4) to show locally uniform convergence of the finite-horizon adjoint functions to an infinite-horizon adjoint function for which the normal form maximum principle holds. The proof is too long to be presented in this note, but it is available at http://orcos.tuwien.ac.at/research/research_reports as Research Report 2011-06.

4 Comments

1. The dominating discount condition (A4) is easy to check for the so-called one-sided Lipschitz systems. Namely, assume that in addition to (A1), the following condition is satisfied for some real number λ:

$$\langle f(t,x,u) - f(t,y,u), x - y \rangle \leq \lambda \|x - y\|^2 \qquad \forall\, t \geq 0,\ x, y \in G,\ u \in U. \tag{13}$$

If we assume (for simplification) that the derivative $\|g_x(t,x,u)\|$ is bounded when $t \geq 0$, $x \in G$ and $u \in U$, then the dominating discount condition (A4) reduces to

$$\rho > \lambda.$$

Notice that λ can be negative in (13), in which case the last inequality may even be satisfied for a negative number ρ. A model with $\lambda < \rho < 0$ arising in capital growth theory, to which our result is applicable will presented in a full-size publication.

2. Another way of verifying the dominating discount condition (A4) is presented in [3,4]. It involves calculation (or estimation from above) of the maximal element of the spectrum of the homogeneous part of the linearized dynamics, provided that the latter is regular (see e.g. [6] or [3,4] for the above terms).

In both cases discussed in points 1 and 2 in this section it is possible to prove that, assuming boundedness of g_x as in Point 1, the normal form of the maximum principle is fulfilled with the unique bounded solution $\psi(t)$ of the adjoint equation. The precise formulations and proofs will be given in a full size paper.

5 Acknowledgements

This research was financed by the Austrian Science Foundation (FWF) under grant No I476-N13. The first author was partly supported by the Russian Foundation for Basic Research (grant No 09-01-00624-a).

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