Generalized Dynamic Factor Models - The Single- and the Multifrequency Case

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Outline

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GDFMs - The Model Class

$$y_t^N = \hat{y}_t^N + u_t^N$$

- y_t^N ... observations
- $\hat{y}_t^N$... latent variables, strongly dependent in the cross-sectional dimension
- u_t^N ... (wide sense) idiosyncratic noise, weakly dependent

Assumptions

- $(\hat{y}_t^N)$, (u_t^N) wide sense stationary with absolutely summable covariances
- $\mathbb{E}\left(\hat{y}_t^N \left(u_s^N\right)'\right) = 0$
- $\mathbb{E}\left(\hat{y}_t^N\right) = \mathbb{E}\left(u_s^N\right) = 0$

Corresponding Spectral Densities

$$f_y^N(\lambda) = f_{\hat{y}}^N(\lambda) + f_u^N(\lambda)$$

- Asymptotic analysis for both $T \to \infty$ and $N \to \infty$
- Sequence of models
  - Nested elements of $\hat{y}_t^N$ and u_t^N do not depend on N
  - Identifiability is obtained only asymptotically

Assumptions

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1. Strong dependence of $(\hat{y}_t^N)_{t\in\mathbb{Z}}$:
   - The first q eigenvalues of $f_{\hat{y}}^N(\lambda)$ diverge to infinity for all frequencies λ, as $N \to \infty$ and the others are zero.
2. Weak dependence of $(u_t^N)_{t\in\mathbb{Z}}$:
   - The largest eigenvalue of $f_u^N(\lambda)$ is uniformly bounded for all frequencies λ and all N
3. $f_{\hat{y}}^N(\lambda)$ is a rational spectral density with constant rank $q < N$, and of McMillan degree $2n < N$;
   - q and n do not depend on N.
   - Additional (but justified) restriction

Main Early References for GDFMs

- Forni, Hallin, Lippi, Reichlin, between 2000 and 2005
  - Representation Theory
  - Identification and Estimation
  - One-sided Estimation and Forecasting
  - Consistency and Rates
- Stock and Watson, 2002
  - Forecasting Using Principal Components from a Large Number of Predictors
  - Macroeconomic Forecasting Using Diffusion Indexes

Recent References

- Doz, Giannone, Reichlin, JoE, 2011
  - A two-step estimator for large approximate dynamic factor models based on Kalman filtering
- Forni, Hallin, Lippi, Zaffaroni 2011 - One-Sided Representations of Generalized Dynamic Factor Models

Special References for Structure Theory

- Deistler, Anderson, Filler, Zinner, Chen, European Journal of Control, 2010
  - Generalized Dynamic Factor Models - An Approach via Singular Autoregression
- Anderson, Deistler 2008/2009
  - Properties of Zero-Free Transfer Function Matrices
  - Properties of Zero-Free Spectral Matrices
- Deistler, Filler, Funovits, CIS, 2011
  - AR Systems and AR Processes: The Singular Case
- Anderson, Deistler, Felsenstein, Funovits, Zadrozny, Eichler, Chen, Zamani, CDC 2012 accepted
  - Identifiability of regular and singular multivariate autoregressive models from mixed frequency data

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Approach

In reality starting point is a set of high dimensional data

$$\left\{ y_t^{(i)} | i \in \{1, \ldots, N\},\ t \in \{1, \ldots, T\} \right\}$$

We start from an idealized setting (a partial problem which is important for the problem as a whole).
We start from the population second moments of the latent variables

$$\left(\left(\hat{y}_t^{(i)} \right)_{t \in \mathbb{Z}} \right)_{i \in \mathbb{N}},$$

i.e. from
autocovariance function $\gamma_{\hat{y}}^N(s) = \mathbb{E}\left[\hat{y}_t^N \left(\hat{y}_{t-s}^N \right)^T \right],\ s \in \mathbb{Z},\ N \in \mathbb{N}$ or
spectral density $f_{\hat{y}}^N(\lambda) = \sum_{k=-\infty}^{\infty} \gamma_{\hat{y}}^N(k) e^{-i\lambda k},\ \lambda \in (-\pi, \pi]$
Makes sense because latent variables are obtained through a previous denoising step.

Major Steps in Structure Theory 1

- Factorization of $f_{\hat{y}}^N(\lambda)$, the rational spectral density of the latent variables of rank q:

$$\underbrace{f_{\hat{y}}^N(\lambda)}_{N\times N, rk=q} = \underbrace{w^N(e^{-i\lambda})}_{N\times q} w^N(e^{-i\lambda})^*$$

  where $w^N(z)$ is a stable miniphase factor
- Realization of a "tall" spectral factor by a state space model (F, G, H^N) with state dimension n

$$\begin{aligned} x_t &= Fx_{t-1} + G\varepsilon_t \\ \hat{y}_t^N &= H^N x_t \end{aligned}$$

Note that under our assumptions F, G, x_t and ε_t do not depend on N from a certain N_0 onwards.

- Minimal, stable, and miniphase

Major Steps in Structure Theory 2

- Obtain a minimal static factor (z_t) of cross-sectional dimension r which has the same dynamics as the latent variable $\hat{y}_t^N = M^N z_t$

$$\begin{aligned} x_t &= Fx_{t-1} + G\varepsilon_t \\ z_t &= Cx_t \end{aligned}$$

where $C = \left[\left(M^N\right)^T M^N \right]^{-1} \left(M^N\right)^T H^N$.

  - $k(z) = C(Iz^{-1} - F)^{-1}G$
  - $z_t = k(z)\varepsilon_t$ is the corresponding Wold decomposition

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Zeroless Transfer Function

Definition

A transfer function $w(z)$ has a zero at z_0 if $b(z)$ in an irreducible left-MFD $w(z) = a(z)^{-1}b(z)$ has not full rank at z_0.

- Since $\hat{y}_t = w(z)\varepsilon_t = Mz_t = Mk(z)\varepsilon_t$ holds,
  - $w(z)$ is zeroless if and only if $k(z)$ is zeroless

Generic AR Result

Theorem

Consider the set of all minimal state space realizations (F, G, C) for $k(z)$ for given $n \geq r > q$.
Then, the transfer functions are zeroless for generic values of (F, G, C).

(Compare Anderson and Deistler, 2008)

- E.g., consider the singular MA(1) system

$$\begin{pmatrix} z_t^{(1)} \\ z_t^{(2)} \end{pmatrix} = \begin{pmatrix} 1 - az \\ 1 - bz \end{pmatrix} \varepsilon_t .$$

Outline

Zeroless Spectral Factors and AR Processes

Theorem

The following statements are equivalent:

- *The stable miniphase spectral factors $k(z)$ of the spectral density f_z of $(z_t)_{t\in\mathbb{Z}}$ are zeroless.*
- *$(z_t)_{t\in\mathbb{Z}}$ is a solution of a stable AR-system, i.e.*

$$z_t = a_1 z_{t-1} + \cdots + a_p z_{t-p} + \nu_t$$

where $\det \underbrace{(I - a_1 z - \cdots - a_p z^p)}_{a(z)} \neq 0, |z| \leq 1$ *and* $rk(\Sigma_\nu) = q$, $\Sigma_\nu = \mathbb{E}(\nu_t \nu_t')$.

Compare Anderson and Deistler, CDC, 2008

Outline

Regular and Singular AR Systems

Consider an AR system (in the lag-operator z)

$$(I_r - a_1 z - \cdots a_p z^p)\, z_t = \nu_t = b\varepsilon_t\,,$$

where

- $a_i \in \mathbb{R}^{r\times r},\ i \in \{1,\dots,p\}$,
  - $(\varepsilon_t)_{t\in\mathbb{Z}}$ is white noise with $\mathbb{E}(\varepsilon_s\varepsilon_t') = \delta_{st} I_q$,
  - $\Sigma_\nu = bb',\ b \in \mathbb{R}^{r\times q}$,
  - $rk(b) = q$,
  - $\det(a(z)) \neq 0,\ |z| \leq 1$,

Such a system is called regular if $q = r$ holds and singular if $q < r$ holds.

Why are Singular AR Systems Interesting?

- Because they are models for latent variables in GDFMs for $r > q$

Difference to regular AR Systems

- Identifiability problem: Coefficients $a_i \in \mathbb{R}^{r \times r}$, $i \in \{1, \ldots, p\}$ not necessarily uniquely determined, $b \in \mathbb{R}^{r \times q}$ up to postmultiplication with orthogonal matrices.
- Coprimeness: Singular AR systems are not necessarily left coprime.

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Yule-Walker Equations for Singular AR Systems

- Yule-Walker equations are obtained by taking expectations

$$\mathbb{E}\Bigg(\overbrace{(a_1,\dots,a_p)\begin{pmatrix} y_{t-1}\\ \vdots \\ y_{t-p}\end{pmatrix}+\nu_t}^{=y_t}\Bigg)(y'_{t-1},\dots,y'_{t-p}) = \mathbb{E}y_t(y'_{t-1},\dots,y'_{t-p})$$

$$(a_1,\dots,a_p)\,\underbrace{\mathbb{E}\left(\begin{pmatrix} y_{t-1}\\ \vdots \\ y_{t-p}\end{pmatrix}(y'_{t-1},\dots,y'_{t-p})\right)}_{=\Gamma_p} = \mathbb{E}y_t(y'_{t-1},\dots,y'_{t-p})$$

$$\Sigma_\nu = \mathbb{E}(\nu_t y'_t) = \mathbb{E}y_t y'_t - \mathbb{E}\left((a_1,\dots,a_p)\begin{pmatrix} y_{t-1}\\ \vdots \\ y_{t-p}\end{pmatrix} y'_t\right)$$

- Second moments of $(y_t)_{t\in\mathbb{Z}}$ known, parameters $[(a_1,\dots,a_p),\Sigma_\nu]$ are unknown

Γ_p might be Singular and Γ_{p+1} is Singular for Singular AR(p) Systems

Possible singularity of Toeplitz matrix Γ_p of an AR(p) process

Equivalent to saying that the components of $y_{t-1}, \ldots, y_{t-p}$ are linearly dependent.

Γ_{p+1} is always singular for a singular AR(p) process

Non-Uniqueness of the Solutions of the Yule-Walker Equations

If Γ_p is singular, then the Yule-Walker Equations do not give a unique solution.
It can be shown that there is an unstable system in the solution set of the Yule-Walker equations if the Toeplitz matrix Γ_p is singular.

Extra conditions to obtain uniqueness:

Prescription of appropriate column degrees for $\bar{a}(z)$ corresponding to a first basis among the rows of Γ_p

Theorem

The system $(\bar{a}(z), b)$ is stable and left coprime if and only if the solution set of the Yule-Walker equations contains a stable solution.

- W. Chen, B.D.O. Anderson, M. Deistler, A. Filler, JTSA 2011
- M. Deistler, A. Filler, B. Funovits, CIS 2011

Outline

1. Structure Theory for GDFMs
2. Regular and Singular Multivariate AR Systems - The Single Frequency Case
3. Regular and Singular AR Systems - The Mixed Frequency Case
   - Identifiability of Regular and Singular Multivariate AR Systems from Mixed Frequency Data
   - Extended Yule Walker Equations
   - Generic Identifiability
   - Substitution Method for 2-dimensional AR(1) Case
   - Alternative Approach - Blocking
4. Exact Interpolation in Singular Mixed Frequency AR Systems

Outline

Motivation and Problems

- Many high dimensional time series only available at different sampling frequencies
  - E.g., quarterly GDP and monthly labour market data

Problems

1. Are the system and noise parameters of the underlying true (high frequency) system identifiable from mixed frequency data? Can we get consistent parameter estimators?
   - If yes, Kalman filter procedures and other linear least squares procedures can be applied for forecasting and interpolation.
2. Can we reconstruct the unobserved slow variables from the observed mixed frequency data?
   - If error covariance matrix is singular, error-free interpolation may be possible

Model

- Stable and left coprime AR(p) system with
  - n_f fast variables, observed for $t \in \mathbb{Z}$, n_s slow variables, observed for $t \in N\mathbb{Z}$

Fast underlying system

$$y_t = \begin{pmatrix} y_t^f \\ y_t^s \end{pmatrix} = \underbrace{\begin{pmatrix} a_{ff}(1) & a_{fs}(1) \\ a_{sf}(1) & a_{ss}(1) \end{pmatrix}}_{=a_1} \begin{pmatrix} y_{t-1}^f \\ y_{t-1}^s \end{pmatrix} + \cdots \underbrace{\begin{pmatrix} a_{ff}(p) & a_{fs}(p) \\ a_{sf}(p) & a_{ss}(p) \end{pmatrix}}_{=a_p} \begin{pmatrix} y_{t-p}^f \\ y_{t-p}^s \end{pmatrix} + \underbrace{\begin{pmatrix} b^f \\ b^s \end{pmatrix} \varepsilon_t}_{\nu_t}, \quad t \in \mathbb{Z},$$

where $\mathbb{E}(\varepsilon_t \varepsilon_t') = I_q$ and error covariance matrix $\Sigma_\nu = \begin{pmatrix} \Sigma_{ff} & \Sigma_{fs} \\ \Sigma_{sf} & \Sigma_{ss} \end{pmatrix}$ of rank q.

Parameter space $\Theta \subseteq \mathbb{R}^{p \cdot n^2 + nq - \frac{q(q-1)}{2}}$ for...

AR systems of order p and with innovation error covariance matrix of rank q satisfying the stability and coprimeness condition.

Outline

Extended Yule Walker Equations

Following the idea of B. Chen, P. Zadrozny, Advances in Econometrics 1998:
Postmultiplying y_t by np lagged values of the fast variables $\left((y^f_{t-1})^T, \dots, (y^f_{t-np})^T\right)$ ($n = n_f + n_s$, and the number np is a consequence of the Cayley-Hamilton theorem) and taking expectations

$$\mathbb{E}\left[y_t\left((y^f_{t-1})^T, \dots, (y^f_{t-np})^T\right)\right] = (a_1, \dots, a_p)\underbrace{\mathbb{E}\left[\begin{pmatrix} y_{t-1} \\ \vdots \\ y_{t-p} \end{pmatrix}\left((y^f_{t-1})^T, \dots, (y^f_{t-np})^T\right)\right]}_{=Z} \tag{1}$$

Observations

- Only those second moments are used which can be observed in principle.
- $(a_1, \dots, a_p)$ is identifiable if the matrix $Z \in \mathbb{R}^{n \times n_f \cdot p \cdot n}$ has full row rank.

Z has the Form of a Controllability Matrix

Write the AR(p) system in companion form:

$$\underbrace{\begin{pmatrix} y_t \\ \vdots \\ y_{t-p+1} \end{pmatrix}}_{=x_{t+1}} = \underbrace{\begin{pmatrix} a_1 & \cdots & a_{p-1} & a_p \\ I_n & & & \\ & \ddots & & \\ & & I_n & 0 \end{pmatrix}}_{\mathscr{A}} \underbrace{\begin{pmatrix} y_{t-1} \\ \vdots \\ y_{t-p} \end{pmatrix}}_{=x_t} + \underbrace{\begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{=\mathscr{B}} \varepsilon_t \text{.and define } K = \mathbb{E}\left(x_t x_t'\right) \begin{pmatrix} I_{n_f} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \Gamma_p \begin{pmatrix} I_{n_f} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Fact

It can be shown that

$$Z = \mathbb{E}\left[\begin{pmatrix} y_{t-1} \\ \vdots \\ y_{t-p} \end{pmatrix} \left((y^f_{t-1})^T, \dots, (y^f_{t-np})^T\right)\right] = \mathbb{E}\left[x_t \left((y^f_{t-1})^T, \dots, (y^f_{t-np})^T\right)\right] = \left(K, \mathscr{A}K, \dots, \mathscr{A}^{np-1}K\right)$$

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Main Theorem

Theorem (Generic Identifiability)

1. ***System Parameters*** $(a_1, \ldots, a_p)$**:**
   The matrix Z in the extended Yule Walker equations has full row rank $n \cdot p$ on a generic subset of the parameter space Θ.
2. ***Noise Parameters*** Σ_ν**:**
   Σ_ν is generically identifiable.

- Rank deficiency of Z is not sufficient for concluding that the parameters are not identifiable.

B.D.O. Anderson, M. Deistler, E. Felsenstein, B. Funovits, P. Zadrozny, M. Eichler, W. Chen M. Zamani, accepted for CDC 2012

Identifiability of Σ_ν

For given $(a_1, \ldots, a_p)$ generically Σ_ν is obtained from a vectorization of the equations

$$\Gamma_p = \mathscr{A}\Gamma_p\mathscr{A}^T + \mathscr{G}\Sigma\mathscr{G}^T$$
$$\gamma_0 = \mathscr{H}\Gamma_p\mathscr{H}^T$$

where $\mathscr{H} = (I, 0, \ldots, 0)$ and $\mathscr{G} = \mathscr{H}^T$
See B.D.O. Anderson, M. Deistler, E. Felsenstein, B. Funovits, P. Zadrozny, M. Eichler, W. Chen M. Zamani, accepted for CDC 2012

Second Order Asymptotics for Estimation of Mixed Frequency AR Systems.

Assumptions

- Parameters are identifiable
- Regular error covariance matrix (as a starting point)

Two ways for losing efficiency:

1. Missing data:
   - Compare mixed frequency data MLE to single frequency data MLE
2. Use of algorithms:
   - Compare extended YW estimator and a GMM estimator used in Chen and Zadrazny 1998 to the mixed frequency MLE.

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Discussion of the 2-dimensional AR(1) Case, Diagonal Error Covariance Matrix ($\Sigma_{sf} = 0$)

Theorem

The system and noise parameters $\begin{pmatrix} a_{ff} & a_{fs} \\ a_{sf} & a_{ss} \end{pmatrix}$ *,*Σ_{ff} *and* Σ_{ss} *are not identifiable if and only if* $(a_{fs} = 0) \wedge (a_{sf} = 0) \wedge (a_{ss} \neq 0)$.

Equivalence classes in non-identifiable case

- Slow and fast process are orthogonal.
    - Slow process is an AR(1) process on $t \in 2\mathbb{Z}$
- If $a_{ss}^2 \neq 0$, the equivalence classes of observational equivalence consist of two point, $+\sqrt{a_{ss}^2}$ and $-\sqrt{a_{ss}^2}$.
    - Solution set of the extended Yule Walker equations consist of affine subspaces.

Diagonal Error Covariance Matrix ($\Sigma_{sf} = 0$)

Observations

$rk(Z) = np$ not necessary for identifiability

- We have identifiability if $a_{fs} \neq 0$ or $a_{sf} \neq 0$ or $a_{ss} = 0$.
- Matrix Z is rank deficient for $a_{ss} = 0, \ a_{sf} = 0$ even if $a_{fs} \neq 0$ which shows that the condition that $rk(Z) = np$ hold, is not necessary for identifiability.

- Extended Yule Walker equations do not use the full information contained in the second moments which are in principle observed.

Outline

An Alternative Approach for the Mixed Frequency Case Blocking

Blocking

$\left(\begin{pmatrix} y_t^f \\ y_t^s \end{pmatrix}, t \in \mathbb{Z}\right)$ fast underlying process

$(y_t^f, t \in \mathbb{Z})$, $(y_t^s, t \in 2\mathbb{Z})$ observed processes

$\left(\begin{pmatrix} y_t^f \\ y_{t-1}^f \\ y_t^s \\ y_{t-1}^s \end{pmatrix}, t \in 2\mathbb{Z}\right)$ blocked fast underlying process

$\left(\begin{pmatrix} y_t^f \\ y_{t-1}^f \\ y_t^s \end{pmatrix}, t \in 2\mathbb{Z}\right)$ blocked observed process

Spectral Density of the Fast Underlying Blocked Process

$$f_u(\lambda) = \left(\begin{array}{ccc|c} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & \color{red}{f_{34}} \\ \hline f_{41} & f_{42} & \color{red}{f_{43}} & f_{44} \end{array}\right), \lambda \in [0, \tfrac{\pi}{2}]$$

Here only f_{34} and f_{43} cannot be observed. Under the assumption that

$$rk \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = rk \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

f_{43} can be shown to be uniquely determined from the known elements in f_u.

Here we don't restrict ourselves to the AR case.

Outline

Exact Interpolation in Singular Mixed Frequency AR Systems

- Interesting feature of singular multifrequency AR systems is that in some cases exact interpolation is possible.

Singular 2-dimensional Mixed Frequency Data AR(1) System as Example for Exact Interpolation

- Singular covariance matrix Σ_ν of rank one.
- One fast variable, observed for $t \in \mathbb{Z}$, and
- one slow variable, observed for $t \in 2\mathbb{Z}$.

Assume:

1. Parameters of the underlying system identifiable, i.e. we have all the coefficients in the equation

$$\begin{pmatrix} y_t^f \\ y_t^s \end{pmatrix} = \begin{pmatrix} a_{ff} & a_{fs} \\ a_{sf} & a_{ss} \end{pmatrix} \begin{pmatrix} y_{t-1}^f \\ y_{t-1}^s \end{pmatrix} + \begin{pmatrix} b^f \\ b^s \end{pmatrix} \varepsilon_t, \quad \mathbb{E}\left(\varepsilon_t^2\right) = 1, \quad t \in \mathbb{Z}.$$

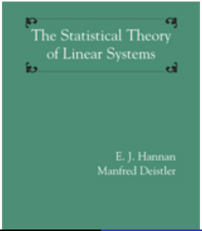
2. Noise ε_t can be recovered from fast variable

Exact interpolation of y_t^s for $t \in 2\mathbb{Z} - 1$

y_t^s linear combination of random variables y_{t-1}^f, y_{t-1}^s, and ε_t.

E.J. Hannan and Manfred Deistler: The Statistical Theory of Linear Systems

- This edition includes an extensive new introduction that
  - outlines central ideas and features of the subject matter, as well as
  - developments since the book's original publication, such as subspace identification, data-driven local coordinates, and the results on post-model-selection estimators.
  - It also provides a section of errata and an updated bibliography.



Questions?

Thank you!