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Brief paper

Properties of blocked linear systems[☆]

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ABSTRACT

This paper presents a systematic study on the properties of blocked linear systems that have resulted from blocking discrete-time linear time invariant systems. The main idea is to explore the relationship between the blocked and the unblocked systems. Existing results are reviewed and a number of important new results are derived. Focus is given particularly on the zero properties of the blocked system as no such study has been found in the literature.

1. Introduction

Blocking (or lifting) is an important technique that has been used in signal processing and multirate sampled-data systems (Chen & Francis, 1995; Meyer & Burrus, 1975).

In the literature, the blocking technique has most often been used to transform linear discrete-time periodic systems into linear time-invariant systems so that the well-established analysis and design tools in linear time-invariant systems can be extended to linear discrete-time periodic systems (Bolzern, Colaneri, & Scattolini, 1986; Colaneri & Kucera, 1997; Grasselli & Longhi, 1988; Grasselli, Longhi, & Tornambe, 1995; Meyer & Burrus, 1975). For example, the notions of poles and zeros of linear time-invariant systems have been extended to linear periodic systems in Bolzern et al. (1986) and Grasselli and Longhi (1988). The structural properties such as observability and reachability have been studied in Bittanti (1986), Grasselli and Longhi (1991), and Gohberg, Kaashoek, and Lerer (1992). The realization problem has been researched in Colaneri and Longhi (1995) and the related references listed in Bittanti and Colaneri (2009).

In this paper, a systematic study will be presented on the properties of the blocked systems resulting from blocking linear time-invariant systems. There are several motivations for doing this research. First, the blocked systems of linear time-invariant systems are useful in multirate sampled-data systems and in controller design as shown by Chen and Francis (1995) and Khargonekar, Poola, and Tannenbaum (1985). Second, it is not clear how the zeros of the blocked system relate to the zeros of the unblocked linear time-invariant system although it is well understood how the poles of the blocked system relate to those of the unblocked time-invariant system (Khargonekar et al., 1985). Lastly, the results developed for linear periodic systems are usually quite heavy in notation. The purpose here is to spell out their counterparts for linear time-invariant systems in a much simpler form.

The importance of studying the relationship between zeros of the unblocked and blocked arises from our recent research interest in econometrics modeling using generalized dynamic factor models (GDFMs), which have been used to model and forecast high-dimensional macroeconomic and financial time series (Deistler, Anderson, Filler, Zinner, & Chen, 2010; Forni, Hallin, Lippi, & Reichlin, 2000; Forni & Lippi, 2001; Stock & Watson, 2002a,b). In GDFMs, the latent variables are assumed to be stationary and are described as outputs of rational dynamic systems with tall matrix transfer functions (with more rows than columns). It has been shown by Anderson and Deistler (2008) that tall transfer functions do not have zeros generically. This means that the latent variables can be in general modeled as AR

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processes rather than ARMA processes. The advantage of using AR models is obvious. In our recent effort to deal with linear time-invariant systems with missing data (say some time series only have quarterly data, i.e. some monthly data are missing) using GDFMs, the blocking (or lifting) technique has been used as a main tool. Our aim is to show that the blocked system of a linear time-invariant system with missing data is generically zeroless and thus AR modeling approaches are sufficient in general. To achieve this goal, it is required to show that blocking a linear time-invariant system will not introduce new zeros. The relationship between the zeros of the unblocked and blocked systems established in this paper guarantees that blocking a linear time-invariant system does not introduce new zeros and thus paves the way to show that the blocked system of a linear time-invariant system with missing data is generically zero-free.

By making use of matrix fraction descriptions (MFDs), a number of important new results are provided. A relationship between the normal ranks of the transfer functions of the blocked system and the unblocked system is discovered, and more importantly, the relationship between the zeros of the blocked system and the unblocked system is established.

The paper is organized as follows. In Section 2, we introduce the unblocked system and its blocked version. In Section 3, we review some existing results and offer simpler proofs for some of them. Section 4 contains the major results and the last section concludes.

2. The unblocked and blocked systems

The unblocked system is defined by

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $y_k \in \mathbb{R}^p$ the output, and $u_k \in \mathbb{R}^m$ the input.

For the unblocked system, its transfer function is defined as

$$W(z) = [D + C(zI - A)^{-1}B], \tag{2}$$

where z is used as both a forward-shift operator and a complex number.

Throughout this paper, the following assumption, which is effectively just a full normal rank assumption, will be used. The dimensionality inequality (i.e. $p \geq m$) in the assumption is costless, since transposition captures its negation.

Assumption 1. The dimension of the output vector is not smaller than the dimension of the input vector, i.e. $p \geq m$, and the normal rank of $W(z)$ is m.

Define

$$Y_k = \begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+N-1} \end{bmatrix}, \quad U_k = \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+N-1} \end{bmatrix}, \quad k = 0, N, 2N, \ldots. \tag{3}$$

Then, the blocked system is defined by

$$\begin{aligned} x_{k+N} &= A_b x_k + B_b U_k \\ Y_k &= C_b x_k + D_b U_k \end{aligned} \tag{4}$$

where

$$A_b = A^N, \qquad B_b = \begin{bmatrix} A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix},$$

$$C_b = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix},$$

$$D_b = \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{N-2}B & CA^{N-3}B & \cdots & D \end{bmatrix}. \tag{5}$$

Define an operator Z such that it satisfies $Zx_k = x_{k+N}, ZY_k = Y_{k+N}, ZU_k = U_{k+N}$. Then the transfer function of the blocked system is given by

$$V(Z) = D_b + C_b(ZI - A_b)^{-1}B_b. \tag{6}$$

In this paper the relationship between the unblocked system (1) and the blocked system (4) will be investigated.

3. Existing results

In this section, the counterparts of some existing results for linear periodic systems will be spelled out for linear time-invariant systems.

3.1. Observability, reachability, and minimal realization

The concepts of observability, reachability, and minimal realization are defined as follows: The system (1) is said to be *reachable* if the matrix $[B \ AB \ \cdots \ A^{n-1}B]$ is of full row rank, and it is said to be *observable* if the matrix $[C' \ A'C' \ \cdots \ (A')^{n-1}C']'$ is of full column rank, where $'$ means transpose. The system (1) is said to be a *minimal realization* of a transfer function $W(z)$ if the system (1) is reachable and observable.

The results obtained in Bittanti (1986), Colaneri and Longhi (1995), Grasselli and Longhi (1991), and Gohberg et al. (1992) for linear periodic systems, when specialized to linear time-invariant systems, lead to the following theorem.

Theorem 1. *Consider the unblocked system (1) with transfer function $W(z)$ given by (2) and the blocked system (4) with transfer function $V(Z) = D_b + C_b(ZI - A_b)^{-1}B_b$, where A_b, B_b, C_b, D_b are defined by (5). Then*

- *The blocked system (4) is reachable if and only if the unblocked system (1) is reachable.*
- *The blocked system (4) is observable if and only if the unblocked system (1) is observable.*
- *The blocked system (4) is a minimal realization of $V(Z)$ if and only if the unblocked system (1) is a minimal realization of $W(z)$.*

3.2. Transfer functions of the blocked and unblocked systems

In this subsection, the relationship between $W(z)$ and $V(Z)$ will be reviewed. The following results were provided in Khargonekar et al. (1985) and were proved in Bittanti and Colaneri (2009).

Theorem 2. *Consider the unblocked system (1) with transfer function $W(z)$ and the blocked system (4) with transfer function $V(Z) = D_b + C_b(ZI - A_b)^{-1}B_b$, where A_b, B_b, C_b, D_b are defined by (5). Then*

$$V(Z) = \begin{bmatrix} V_1(Z) & Z^{-1}V_N(Z) & Z^{-1}V_{N-1}(Z) & \cdots & Z^{-1}V_2(Z) \\ V_2(Z) & V_1(Z) & Z^{-1}V_N(Z) & \ddots & Z^{-1}V_3(Z) \\ V_3(Z) & V_2(Z) & V_1(Z) & \ddots & Z^{-1}V_4(Z) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ V_N(Z) & V_{N-1}(Z) & V_{N-2}(Z) & \cdots & V_1(Z) \end{bmatrix}. \tag{7}$$

and

$$W(z) = V_1(z^N) + z^{-1}V_2(z^N) + \cdots + z^{-(N-1)}V_N(z^N) \tag{8}$$

where $V_1(Z) = D + C(ZI - A^N)^{-1}A^{N-1}B$ and $V_j(Z) = CA^{l-2}B + C(ZI - A^N)^{-1}A^{N+l-2}B$, $j = 2, \ldots, N$.

4. New results

In this section, a number of new results will be provided on the properties of the blocked system (4). The MFD of a transfer function will be used as the main tool to derive the main results.

Suppose the unblocked system (1) is a minimal realization of $W(z)$. Since the poles of the unblocked and blocked systems are the eigenvalues of A and A^N, it is obvious that Z_p is a pole of $V(Z)$ if and only if $W(z)$ has a pole at z_p with $z_p^N = Z_p$ for one or more of the Nth roots of Z_p (Khargonekar et al., 1985). As the relationship between the poles of the blocked and unblocked systems is now well understood, focus will be given particularly on system zeros in this paper. We can conjecture that zeros of blocked and unblocked systems may be related in a like way to poles. Indeed, for square systems, since zeros of a system are poles of the inverse, this should be no surprise. The result however is less obvious for nonsquare systems, and zeros at infinity are also of interest.

Throughout this paper, $rk(X)$ stands for the rank of a matrix X.

The definition of system zeros, especially finite zeros, is standard and can be found in Kailath (1980) and Rosenbrock (1970). Here, we quote the one used in Anderson and Deistler (2009) for convenience since it combines finite and infinite zeros in the one definition.

Definition 1. The finite zeros of the transfer function $W(z)$ with minimal realization $\{A, B, C, D\}$ are defined to be the finite values of z for which the rank of the following matrix falls below its normal rank

$$M(z) = \begin{bmatrix} zI - A & B \\ C & D \end{bmatrix}. \tag{9}$$

Further, $W(z)$ is said to have an infinite zero when $n + rk(D)$ is less than the normal rank of $M(z)$, or equivalently the rank of D is less than the normal rank of $W(z)$.

4.1. Left MFDs of the blocked and unblocked systems

Suppose the transfer function $W(z)$ of the unblocked system has a coprime left MFD as

$$W(z) = P^{-1}(z)Q(z) \tag{10}$$

with

$$\begin{aligned} P(z) &= P_0 + P_1 z + \cdots + P_\mu z^\mu, \\ Q(z) &= Q_0 + Q_1 z + \cdots + Q_\mu z^\mu \end{aligned} \tag{11}$$

where μ is defined so that P_μ and Q_μ are not both zero. By coprimeness, P_0 and Q_0 are not both zero.

For any coprime pair $(P(z), Q(z))$, it has been proved in Kailath (1980) and Wolovich (1974) that the *finite zeros* of $W(z)$ defined earlier can be equivalent computed as those values of z such that the numerator matrix $Q(z)$ has rank less than its normal rank. This fact will be used later in proving our major results.

It is easy to see that $y_k = W(z)u_k$ has a vector difference equation (VDE) representation

$$\begin{aligned} &P_0 y_{k+j} + P_1 y_{k+j+1} + \cdots + P_\mu y_{k+j+\mu} \\ &\quad = Q_0 u_{k+j} + Q_1 u_{k+j+1} + \cdots + Q_\mu u_{k+j+\mu}, \\ &\quad\quad j = 1, 2, \ldots, N-1. \end{aligned} \tag{12}$$

Let $\mu = \alpha N + \nu$, for fixed ν satisfying $0 \le \nu < N$, where $\alpha \in \{0, 1, 2, \ldots\}$. Then it follows from (12) and the definition of Y_k that

$$\begin{aligned} &\mathcal{A}_0 Y_k + \mathcal{A}_1 Y_{k+N} + \cdots + \mathcal{A}_\alpha Y_{k+\alpha N} + \mathcal{A}_{\alpha+1} Y_{k+(\alpha+1)N} \\ &\quad = \mathcal{B}_0 U_k + \mathcal{B}_1 U_{k+N} + \cdots + \mathcal{B}_\alpha U_{k+\alpha N} + \mathcal{B}_{\alpha+1} U_{k+(\alpha+1)N} \end{aligned} \tag{13}$$

where $k = 0, N, 2N, \ldots$, and

$$\mathcal{A}_i = \begin{bmatrix} P_{iN} & P_{iN+1} & \cdots & P_{(i+1)N-1} \\ P_{iN-1} & P_{iN} & \ddots & P_{(i+1)N-2} \\ \vdots & \ddots & \ddots & \vdots \\ P_{(i-1)N+1} & P_{(i-1)N+2} & \cdots & P_{iN} \end{bmatrix},$$

$$\mathcal{B}_i = \begin{bmatrix} Q_{iN} & Q_{iN+1} & \cdots & Q_{(i+1)N-1} \\ Q_{iN-1} & Q_{iN} & \ddots & Q_{(i+1)N-2} \\ \vdots & \ddots & \ddots & \vdots \\ Q_{(i-1)N+1} & Q_{(i-1)N+2} & \cdots & Q_{iN} \end{bmatrix} \tag{14}$$

where $i = 0, 1, \ldots, \alpha + 1$, $P_j = 0$ if $j > \mu$ or $j < 0$ and $Q_j = 0$ if $j > \mu$ or $j < 0$.

The transfer function of (13) is

$$V(Z) = \mathcal{A}^{-1}(Z)\mathcal{B}(Z) \tag{15}$$

with $ZU_k = U_{k+N}$ and

$$\begin{aligned} \mathcal{A}(Z) &= \mathcal{A}_0 + \mathcal{A}_1 Z + \cdots + \mathcal{A}_{\alpha+1} Z^{(\alpha+1)}, \\ \mathcal{B}(Z) &= \mathcal{B}_0 + \mathcal{B}_1 Z + \cdots + \mathcal{B}_{\alpha+1} Z^{(\alpha+1)}. \end{aligned} \tag{16}$$

The left MFD in (15) is called a blocked version of the left MFD given in (10).

Similar blocking techniques can be applied to right MFDs and all results obtained for left MFDs hold true for right MFDs, *mutatis mutandis*.

It should be pointed out that the polynomial blocking (or lifting) technique developed in Bittanti and Colaneri (2009), which was initially only for scalar polynomials, can be extended to polynomial matrices to derive the blocked left MFD given by (15). However, the polynomial blocking approach apparently requires the solving of matrix equations to obtain those matrices $\mathcal{A}_i$, $\mathcal{B}_i$, $i = 0, 1, \ldots, \alpha + 1$. The advantage of our approach is that those matrices are explicitly provided. Because the polynomial blocking approach actually leads to the same blocked MFD, it does not offer any advantage to our approach. As a consequence, all derivations of our major results in the later subsections are indeed necessary and the polynomial blocking approach cannot be used to avoid them.

4.2. Main results

In this section, we will only derive our main results based on left MFDs.

Two lemmas are needed.

Lemma 1. *Given N complex numbers λ_i, $i = 1, 2, \ldots, N$, which satisfy $\lambda_i \neq \lambda_j$ for $i \neq j$, and also N real $p \times m$ matrices Π_i, $i = 1, 2, \ldots, N$, which are all of full column rank, then the following matrix*

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 & \cdots & \Pi_N \\ \lambda_1 \Pi_1 & \lambda_2 \Pi_2 & \cdots & \lambda_N \Pi_N \\ \vdots & \ddots & \ddots & \vdots \\ \lambda_1^{N-1}\Pi_1 & \lambda_2^{N-1}\Pi_2 & \cdots & \lambda_N^{N-1}\Pi_N \end{bmatrix} \tag{17}$$

is of full column rank.

Proof. Rewrite Π as

$$\Pi = \begin{bmatrix} I_p & I_p & \cdots & I_p \\ \lambda_1 I_p & \lambda_2 I_p & \cdots & \lambda_N I_p \\ \vdots & \ddots & \ddots & \vdots \\ \lambda_1^{N-1} I_p & \lambda_2^{N-1} I_p & \cdots & \lambda_N^{N-1} I_p \end{bmatrix}$$

$$\times \begin{bmatrix} \Pi_1 & 0 & \cdots & 0 \\ 0 & \Pi_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \Pi_N \end{bmatrix}. \tag{18}$$

The first matrix on the right is a Kronecker product of a VanderMonde matrix with the identity matrix, and accordingly is nonsingular. Given the properties of the Πs, the conclusion follows immediately. □

Lemma 2. *For a nonzero complex number Z_0, let $z_i,\ i = 1, 2, \ldots, N$ be N distinct complex numbers such that $z_i^N = Z_0,\ i = 1, 2, \ldots, N$. Choose any $m \times m$ nonsingular matrix Ω and define*

$$\Upsilon = \begin{bmatrix} \Omega & \Omega & \cdots & \Omega \\ z_1\Omega & z_2\Omega & \cdots & z_N\Omega \\ \vdots & \ddots & \ddots & \vdots \\ z_1^{N-1}\Omega & z_2^{N-1}\Omega & \cdots & z_N^{N-1}\Omega \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} Q(z_1)\Omega & Q(z_2)\Omega & \cdots & Q(z_N)\Omega \\ z_1Q(z_1)\Omega & z_2Q(z_2)\Omega & \cdots & z_NQ(z_N)\Omega \\ \vdots & \ddots & \ddots & \vdots \\ z_1^{N-1}Q(z_1)\Omega & z_2^{N-1}Q(z_2)\Omega & \cdots & z_N^{N-1}Q(z_N)\Omega \end{bmatrix}. \tag{19}$$

Then

$$\mathcal{B}(Z_0)\Upsilon = \Lambda \tag{20}$$

with $\mathcal{B}(Z)$ from (16) and $Q(z)$ from (11).

Proof. Using the definitions of $\mathcal{B}_i$, $i = 0, 1, \ldots, \alpha + 1$, it is easy to check

$$\mathcal{B}(Z_0)\begin{bmatrix} \Omega \\ z_i\Omega \\ \vdots \\ z_i^{N-1}\Omega \end{bmatrix} = \begin{bmatrix} Q(z_i)\Omega \\ z_iQ(z_i)\Omega \\ \vdots \\ z_i^{N-1}Q(z_i)\Omega \end{bmatrix}. \tag{21}$$

Using the above equation, the conclusion of the lemma follows immediately. □

Regarding the relationship between the normal ranks of the transfer functions of the blocked and unblocked systems, the following result holds.

Theorem 3. *The normal rank of $V(Z)$ is mN if and only if the normal rank of $W(z)$ is m.*

Proof (*Sufficiency*)**.** The full normal rank of $W(z)$ implies that the normal rank of $Q(z)$ is m. The full normal rank of $Q(z)$ in turn implies that there exists a complex number $Z_0 \neq 0$ with N distinct roots z_i, $i = 1, 2, \ldots, N$ (i.e. $z_i^N = Z_0,\ i = 1, 2, \ldots, N$) such that $\det(\mathcal{A}(Z_0)) \neq 0$ and $rk(Q(z_i)) = m,\ i = 1, 2, \ldots, N$. Now choose any $m \times m$ nonsingular matrix Ω and define Υ and Λ as in (19), then it follows from Lemma 2 that $\mathcal{B}(Z_0)\Upsilon = \Lambda$. Noting that $z_i \neq z_j$ for $i \neq j$ and that Ω is nonsingular, it follows from Lemma 1 that Υ and Λ are of full column rank. Since Υ is a square matrix, it must be nonsingular, which implies that $\mathcal{B}(Z_0)$ is of full column rank, which in turn proves that the normal rank of $V(Z)$ is mN.

Necessity. Since the normal rank of $V(Z)$ is mN, there exists a complex number $Z_0 \neq 0$ such that $\det(\mathcal{A}(Z_0)) \neq 0$ and $\mathcal{B}(Z_0)$ is of full column rank. Now let $z_i,\ i = 1, 2, \ldots, N$ be the N distinct roots of Z_0. Using the same arguments as in the proof of the sufficiency part, it can be shown that Λ is of full column rank. It follows from the definition of Λ that $Q(z_i)\Omega,\ i = 1, 2, \ldots, N$ are of full column rank. Noting that Ω is nonsingular, it follows that $Q(z_i)$, $i = 1, 2, \ldots, N$ are of full column rank and thus that the normal rank of $W(z)$ is m. □

Although the relationship between poles of the blocked and unblocked systems is very simple, it is not clear whether such a simple relation still holds or not for system zeros. If such a simple relation holds also for zeros, how can it be proved? It turns out the relationship between zeros of the blocked and unblocked systems is highly nontrivial and is much harder to study. Because of this, we shall consider three cases separately, that is, (1) finite nonzero system zeros; (2) system zeros at infinity; and (3) system zeros at zero.

Two lemmas are needed.

Lemma 3. *Under Assumption 1, suppose also that $V(Z) = \mathcal{A}^{-1}(Z)\mathcal{B}(Z)$, where $\mathcal{A}(Z), \mathcal{B}(Z)$ are derived from a coprime MFD of $W(z)$ and are defined by (16). Suppose also that $V(Z)$ has a finite zero at $Z_0 \neq 0$. Then $rk(\mathcal{B}(Z_0)) < mN$.*

Proof. Let $R(Z)$ be the greatest left common divisor of $\mathcal{A}(Z), \mathcal{B}(Z)$ so that

$$\mathcal{A}(Z) = R(Z)\bar{\mathcal{A}}(Z), \qquad \mathcal{B}(Z) = R(Z)\bar{\mathcal{B}}(Z).$$

Then $V(Z)$ has a coprime MFD as

$$V(Z) = \bar{\mathcal{A}}^{-1}(Z)\bar{\mathcal{B}}(Z).$$

Since the normal rank of $W(z)$ is m, it follows from Theorem 3 that the normal rank of $V(Z)$ is mN. Since Z_0 is a finite zero of $V(Z)$ and $\bar{\mathcal{A}}^{-1}(Z)\bar{\mathcal{B}}(Z)$ is a coprime MFD of $V(Z)$, it follows that $rk(\bar{\mathcal{B}}(Z_0)) < mN$. This together with $\mathcal{B}(Z) = R(Z)\bar{\mathcal{B}}(Z)$ implies that $rk(\mathcal{B}(Z_0)) < mN$. □

Lemma 4. *There exists a finite complex number $Z_0 \neq 0$ such that $rk(\mathcal{B}(Z_0)) < mN$ if and only if there is a finite complex number $z_0 \neq 0$ such that $rk(Q(z_0)) < m$. In this case, there holds $z_0^N = Z_0$.*

Proof (*Sufficiency*)**.** Since $rk(Q(z_0)) < m$, there exists a nonzero vector β such that

$$Q(z_0)\beta = 0. \tag{22}$$

Define a nonzero vector as $\Psi = \begin{bmatrix}\beta' & z_0\beta' & \cdots & z_0^{N-1}\beta'\end{bmatrix}'$ and let $Z_0 = z_0^N$, then it is easy to check that

$$\mathcal{B}(Z_0)\Psi = \begin{bmatrix} Q(z_0)\beta \\ z_0Q(z_0)\beta \\ \vdots \\ z_0^{N-1}Q(z_0)\beta \end{bmatrix} = 0, \tag{23}$$

which means that $rk(\mathcal{B}(Z_0)) < mN$.

Necessity. Suppose there exists a complex number $Z_0 \neq 0$ such that $rk(\mathcal{B}(Z_0)) < mN$. Since $Z_0 \neq 0$, there exist N distinct complex numbers $z_i,\ i = 1, 2, \ldots, N$ such that $z_i^N = Z_0,\ i = 1, 2, \ldots, N$. If there exists a complex number z_{i_0}, $i_0 \in \{1, 2, \ldots, N\}$ such that $rk(Q(z_{i_0})) < m$, the necessity is proved.

Now, assume that $Q(z_i)$, $i = 1, 2, \ldots, N$ are all of full column rank. According to Lemmas 1 and 2, $\mathcal{B}(Z_0)$ would be of full column rank, which is a contradiction. This completes the proof. □

The first main result in this subsection is provided in the following theorem.

Theorem 4. *Consider the unblocked system (1) with transfer function $W(z)$ and the blocked system (4) with transfer function $V(Z) = D_b + C_b(ZI - A_b)^{-1}B_b$, where A_b, B_b, C_b, D_b are defined by (5). Under Assumption 1 and supposing that (A, B, C, D) is minimal, then $V(Z)$ has a finite zero at $Z_0 \neq 0$ if and only if $W(z)$ has a finite zero at $z_0 \neq 0$ with $z_0^N = Z_0$ for one or more of the Nth roots of Z_0.*

Proof (*Necessity*)**.** Since the normal rank of $W(z)$ is m, it follows from Theorem 3 that the normal rank of $V(Z)$ is mN. For the finite zero $Z_0 \neq 0$ of $V(Z)$, it follows from Lemma 3 that $rk(\mathcal{B}(Z_0)) < mN$. This according to Lemma 4 proves the necessity.

Sufficiency. Suppose that z_0 is a zero of the unblocked system, then for some nonzero $[x_0' \; u_0']'$ there holds

$$\begin{bmatrix} z_0 I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0.$$

Then, $Ax_0 = z_0 x_0 - Bu_0$. Using this equation repeatedly, it follows that for $1 \le i \le N-1$

$$A^i x_0 = z_0^i x_0 - \begin{bmatrix} A^{i-1}B & \cdots & B & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ z_0 u_0 \\ \vdots \\ z_0^{N-1} u_0 \end{bmatrix},$$

$$CA^i x_0 = -\begin{bmatrix} CA^{i-1}B & \cdots & CB & D & \cdots & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ z_0 u_0 \\ \vdots \\ z_0^{N-1} u_0 \end{bmatrix}. \tag{24}$$

It is immediate that

$$\begin{bmatrix} z_0^N I - A_b & -B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \\ z_0 u_0 \\ \vdots \\ z_0^{N-1} u_0 \end{bmatrix} = 0.$$

Since the normal rank of $W(z)$ is m, it follows from Theorem 3 that the normal rank of $V(Z)$ is mN. This fact together with the above equation proves that $V(Z)$ has a finite zero at $Z_0 = z_0^N \neq 0$. □

Remark 1. It follows from Section 6.4.1 and Remark 6.9 in Bittanti and Colaneri (2009) that

$$V(z^N) = M^{-1}(z) \begin{bmatrix} W(z) & 0 & \cdots & 0 \\ 0 & W(z\phi) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W(z\phi^{N-1}) \end{bmatrix} M(z) \tag{25}$$

where $1, \phi, \ldots, \phi^{N-1}$ are the N distinct roots of 1, and

$$M(z) = \begin{bmatrix} I & z^{-1}I & \cdots & z^{-(N-1)}I \\ I & (z\phi)^{-1}I & \cdots & (z\phi)^{-(N-1)}I \\ \vdots & \vdots & \ddots & \vdots \\ I & (z\phi^{N-1})^{-1}I & \cdots & (z\phi^{N-1})^{-(N-1)}I \end{bmatrix}. \tag{26}$$

It should be pointed out that an alternative proof for Theorems 3 and 4 can be given by making use of (25) and (26).[2]

For a zero at infinity we have the following result.

Theorem 5. *Consider the unblocked system (1) with transfer function $W(z)$ and the blocked system (4) with transfer function $V(Z) = D_b + C_b(ZI - A_b)^{-1}B_b$, where A_b, B_b, C_b, D_b are defined by (5). Under Assumption 1 and assuming that (A, B, C, D) is minimal, then $W(z)$ has a zero at $z = \infty$ if and only if $V(Z)$ has a zero at $Z = \infty$.*

Proof. Since the normal rank of $W(z)$ is m, it follows from Theorem 3 that the normal rank of $V(Z)$ is mN. Then, according to the definition of a zero at infinity, $W(z)$ has a zero at $z = \infty$ if and only if $rk(D) < m$ and $V(Z)$ has zero at $Z = \infty$ if and only if $rk(D_b) < mN$, where D_b is defined in (5). The theorem is proved by noting that $rk(D) < m$ if and only if $rk(D_b) < mN$. □

For the third case (zero at zero), the following result holds.

Theorem 6. *Consider the unblocked system (1) with transfer function $W(z)$ and the blocked system (4) with transfer function $V(Z) = D_b + C_b(ZI - A_b)^{-1}B_b$, where A_b, B_b, C_b, D_b are defined by (5). Under Assumption 1 and assuming that (A, B, C, D) is minimal, then $V(Z)$ has a zero at $Z = 0$ if and only if $W(z)$ has a zero at $z = 0$.*

Proof. The sufficiency can be proved the same way as the sufficiency part of Theorem 4 by replacing z_0 there with 0.

Necessity. Since the normal rank of $W(z)$ is m, it follows from Theorem 3 that the normal rank of $V(Z)$ is mN. Then the fact that $V(Z)$ has zero at $Z = 0$ implies that there exists a nonzero vector $\begin{bmatrix} x_0' & u_0' & u_1' & \cdots & u_{N-1}' \end{bmatrix}'$ such that

$$\begin{bmatrix} -A_b & -B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} = 0 \tag{27}$$

where the matrices A_b, B_b, C_b, D_b are defined in (5).

Suppose that $W(z)$ does not have a zero at $z = 0$. Then $\begin{bmatrix} -A & -B \\ C & D \end{bmatrix}$ is of full column rank because the normal rank of $W(z)$ is m. For the initial state x_0 and the control sequence $u_0, u_1, \ldots, u_{N-1}$, denote the corresponding state sequence of system (1) as $x_1, x_2, \ldots, x_N$.

Using (1), (27) and (5), it is easy to check that

$$x_i = A^i x_0 + A^{i-1}Bu_0 + \cdots + ABu_{i-2} + Bu_{i-1},$$
$$x_N = A^N x_0 + A^{N-1}Bu_0 + \cdots + ABu_{N-2} + Bu_{N-1} = 0,$$
$$\begin{aligned} Cx_i + Du_i \;=\; & CAx_0 + CA^{i-1}Bu_0 \\ & + \cdots + CABu_{i-2} + CBu_{i-1} + Du_i = 0 \end{aligned} \tag{28}$$

where $i = 1, \ldots, N-1$.

Consider the sequence (x_i, u_i), $i = 0, 1, \ldots, N-1$. Then there must exist a pair (x_{i_0}, u_{i_0}), $i_0 \in \{0, 1, \ldots, N-1\}$ such that $\begin{bmatrix} x_{i_0} \\ u_{i_0} \end{bmatrix} \neq 0$ (otherwise, we would have $\begin{bmatrix} x_0' & u_0' & u_1' & \cdots & u_{N-1}' \end{bmatrix}' = 0$). This together with the fact $\begin{bmatrix} -A & -B \\ C & D \end{bmatrix}$ is of full column rank implies that $\begin{bmatrix} -A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_{i_0} \\ u_{i_0} \end{bmatrix} \neq 0$. Since the third equation in (28) ensures that $Cx_{i_0} + Du_{i_0} = 0$, one must have $x_{i_0+1} = Ax_{i_0} + Bu_{i_0} \neq 0$. Noting that $\begin{bmatrix} x_{i_0+1} \\ u_{i_0+1} \end{bmatrix} \neq 0$ and repeating the argument, one must have $x_{i_0+2} \neq 0$. Continuing in the same way, one will have $x_N \neq 0$, which contradicts the second equation in (28). Therefore, $W(z)$ must have a zero at $z = 0$. □

It has been shown in Anderson and Deistler (2008) that for generic A, B, C, D, the system (1) is zero free, in other words, it has neither finite zeros nor infinite zeros when the system is tall (i.e. $p > m$). One natural question is: when the system (1) is tall and the matrices A, B, C, D take generic values, is the blocked system (4) zero-free?

Without the results derived in the previous subsection, this question would be very difficult to answer. However, with the results presented in Theorems 4–6, the answer becomes almost trivial and is provided in the following corollary.

[2] We are grateful to a reviewer for suggesting this alternative proof.

Corollary 1. *Consider the unblocked system* (1) *with transfer function* $W(z)$ *and the blocked system* (4) *with transfer function* $V(Z) = D_b + C_b(ZI - A_b)^{-1}B_b$, *where* A_b, B_b, C_b, D_b *are defined by* (5). *Under* Assumption 1 *and that* (A, B, C, D) *is minimal. Assume further that the matrices* A, B, C, D *take generic values and* $p > m$. *Then the blocked system* (4) *is zero-free.*

Proof. It follows from Theorems 4–6 immediately. □

5. Conclusions

In this paper, the properties of the blocked system of a linear time invariant system have been studied through investigating the relationship between the blocked and unblocked systems. It has been shown that the transfer function of the blocked system is of full column normal rank if and only if the transfer function of the unblocked system is of full column normal rank. This new result has been found applicable to the study of the relationship between the zeros of the blocked and unblocked systems. With its help and under certain conditions, it has been demonstrated that there is a close relationship between the zeros of the blocked and unblocked systems. These results are appealing and important. One interesting future topic is to study the relation between the zero structures of the unblocked system and of the blocked system. Another important future topic is how to extend the obtained results to the blocked systems of linear periodic systems.

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