On Modularity of Termination Properties of Rewriting under Strategies

Bernhard Gramlich and Klaus Györgyfalvay

TU Wien, Austria
{gramlich,klausgy}@logic.at

Abstract

The modularity of termination and confluence properties of term rewriting systems has been extensively studied, for disjoint unions and other more types of combinations. However, for rewriting under strategies the theory is less well explored. Here we extend the modularity analysis of termination properties systematically to (variants of) innermost and outermost rewriting. It turns out — as expected — that in essence innermost rewriting behaves nicely w.r.t. modularity of termination properties, whereas this is not at all the case for outermost rewriting, at least not without further assumptions.

Keywords and phrases Modularity, Preservation under Signature Extensions, Termination Properties, Rewriting under Strategies

1 Introduction and Overview

Whereas most known modularity results refer to unrestricted rewriting, cf. e.g. [2]–[16], in applications and programming language contexts one very often has restrictions imposed on the evaluation mechanism like (position-based) strategies. For instance, *innermost* rewriting closely corresponds to *eager* evaluation and *call-by-value* whereas *outermost* rewriting is close to *lazy* evaluation and *call-by-name*. Here we will study the modularity behaviour of normalization and termination of (different versions of) innermost and outermost rewriting. It will turn out that in this regard innermost rewriting has nice properties (which is not very surprising) whereas outermost rewriting is highly non-modular.

We will entirely focus here on the case of *disjoint* unions, cf. e.g. [14, 13]. Most results easily extend to slightly more general combinations like (at most) *constructor sharing* or *composable* systems, cf. e.g.[10]. The interference of the usual modularity analysis, taking into account the *layered* structure of *mixed* terms with strategy-based restrictions of rewriting steps is in general (highly) non-trivial, especially for the case of outermost rewriting.

The remainder of this extended abstract is structured as follows. In Section 2 we will very briefly recall some notions and notations. In the main Section 3 we first review what is known and then study the modularity of weak and strong termination properties of (variants of) innermost and outermost rewriting. As a next step we analyze under which additional assumptions modularity can be recovered (for outermost rewriting). Then, motivated by the negative results, we study a very special case of modularity, namely preservation under signature extensions. Finally, we briefly discuss directions for further research.

Due to lack of space we omit any proofs. Yet, for some negative results we give concrete counterexamples.

2 Preliminaries

We assume familiarity with the basics of term rewriting and of modularity in term rewriting (cf. e.g. [2], [4], [11]).

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We will deal with *(term) rewriting systems* (TRS) $\mathcal{R}^{\mathcal{F}} = (\mathcal{F}, R)$ consisting of (a signature $\mathcal{F}$ and) rules $l \to r$ over some signature $\mathcal{F}$ and set of variables $\mathcal{V}$. The rules $l \to r$ satisfy two conditions: The left-hand side l must not be a variable, and every variable appearing in r also appears in l. The *rewrite relation* induced by a TRS $\mathcal{R}$ is denoted by $\to_{\mathcal{R}}$ or $\to$ if $\mathcal{R}$ is clear from the context or irrelevant. We sometimes use notations like $s \to_q t$ or $s \to_{>\epsilon} t$ to indicate that the position of the redex contraction is q or is strictly below the root, repectively.

Innermost rewriting $\to_i$ is defined as follows, slightly abusing notation: $s \to_i t$ if t is obtained from s by contracting an *innermost redex*, i.e., a subterm of s which is reduced at the root such that all its proper subterms are in normal form. *Leftmost innermost rewriting* is defined by $s \to_{li} t$ if $s \to_i t$ such that in this step a leftmost innermost redex is contracted. *(Maximal) parallel innermost rewriting* $\to_{pi}$ is given by $s \to_{pi} t$ if $s = C[s_1, \ldots, s_n]_{p_1,\ldots,p_n} \to_i^* C[t_1, \ldots, t_n]_{p_1,\ldots,p_n} = t$ such that s_{p_i}, $1 \leq i \leq n$ are all the innermost redexes of s and $s_i \to_{i,\epsilon} t_i$ for all i. Analogously, the relations *outermost rewriting* $\to_o$, *leftmost outermost rewriting* $\to_{lo}$ and *(maximal) parallel outermost rewriting* $\to_{po}$ are defined. Note that whereas a parallel innermost step can always be sequentialized into a sequence of ordinary innermost steps, the analogous property does not hold in general for parallel outermost rewriting.

An orthogonal TRS is *left-normal* if in every rule $l \to r$ the constant and function symbols in the left-hand side precede (in the linear term notation) the variables.

Two TRSs $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ are *disjoint* if $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ (which then also implies $R_1 \cap R_1 = \emptyset$).

Finally, a *modular reduction* step $s \rightsquigarrow t$ means normalization of s in one system (i.e., reduction in one system to normal form w.r.t. that system), cf. [9].

3 Modularity of Termination Properties of Rewriting under Strategies

We will consider innermost and outermost rewriting as well as variants thereof, namely *leftmost* and *(maximal) parallel* versions of both, as well as weak termination and termination, also known as weak normalization (WN) and strong normalization (SN), respectively.

First let us recall in Table 1 the main basic results that are known concerning modularity of WN and SN, without mentioning the many advanced results about (non-)modularity of termination of standard rewriting.

property	is modular?	reason/reference
SN	−	[14, 13]
WN	+	[15, 16], [5], [3], [9]
SN($\rightsquigarrow$)	+	[9]
SIN	+	[6]
WIN	+	[6]

Table 1 Some known modularity results for termination properties of standard rewriting

From this table it is clear that — apart from termination properties of general rewriting — only (modularity of termination of) innermost rewriting has been studied to some extent, but outermost rewriting not at all, to the best of our knowledge. In the sequel we will investigate modularity of both termination (SN) and weak termination (WN) for standard, leftmost and (maximal) parallel innermost as well as outermost rewriting. For brevity we use the following abbreviations: WIN = WN($\to_i$), WLIN = WN($\to_{li}$), WPIN = WN($\to_{pi}$), SIN = SN($\to_i$), SLIN = SN($\to_{li}$), SPIN = SN($\to_{pi}$).

For a better understanding of the following tables let us mention that normalization and termination, respectively, of innermost rewriting remain invariant, when the variants $\to_{li}$ or $\to_{pi}$ are used instead of $\to_i$.

▶ Fact 3.1 (*selection invariance* for innermost rewriting). WIN $\iff$ WLIN $\iff$ WPIN and SIN $\iff$ SLIN $\iff$ SPIN.

This property, called *selection invariance* in [8] (cf. also [7]), seems to be 'folklore knowledge' in rewriting. A formal proof of a particular case (namely the equivalence of SLIN and SIN) is given in [8, Theorem]. Table 2 below shows which of the termination properties of innermost and outermost rewriting are modular, and which are not in general.

Table 3 then exhibits which of the negative results even hold for orthogonal TRSs (and which positive results hold for orthogonal systems). Observe that in left-normal normalizing TRSs leftmost-outermost reduction is normalizing, hence terminating (cf. e.g. [4]. Furthermore, in orthogonal systems we clearly have WLON $\iff$ SLON.

property	is modular?	reason/reference
SIN,SLIN,SPIN	+	Table 1, Fact 3.1
WIN,WLIN,WPIN	+	Table 1, Fact 3.1
SON	−	Table 4
WON	−	Table 4
SLON	−	Table 4
WLON	−	Table 4
SPON	−	Table 4
WPON	−	Table 4

Table 2 Modularity of innermost and outermost termination properties

property	is modular?
SON	−
WON, WPON, SPON[1]	+
WLON, SLON	− (but holds for left-normal TRSs)

Table 3 Modularity of outermost termination properties for orthogonal TRSs

Further easy positive results are possible by requiring non-collapsingness of the TRSs involved, which we do not detail here. Another question is, whether the negative results of Table 2 turn into positive ones, at least for the very special case of signature extensions. But as shown below in Table 4, this is only the case for left-linear TRSs. Observe that the positive preservation results in Table 4 for left-linear TRSs crucially rely on the property of left-linear systems that in a term $s = s[l\sigma]$, the redex $s|_p = l\sigma$ (for some rule $l \to r$) is still a redex after reducing in the 'substitution part' of $l\sigma$.

In the following we present a few counterexamples supporting some of the previous negative claims.

[1] Note for left-normal orthogonal TRSs any outermost rewriting strategy is well-known to be normalizing, not only parallel outermost. Thus, in this case the properties WON, WPON and SONcoincide.

property	is preserved under signature extensions?
SON	– (but holds for left-linear TRSs)
WON	– (but holds for left-linear TRSs)
SLON	– (but holds for left-linear TRSs)
WLON	– (but holds for left-linear TRSs)
SPON	– (but holds for left-linear TRSs)
WPON	– (but holds for left-linear TRSs)

Table 4 Preservation of outermost termination properties under signature extensions

▶ **Example 1** (counterexample to preservation of SON, WON, SLON, WLON under signature extensions)**.** Consider the TRS $\mathcal{R}$ over the signature $\mathcal{F} = \{f_1, f_2, g, c, d_1, d_2\}$:

$$\begin{aligned} g(f_1(x,y,y)) &\to g(f_2(x,x,y)) & \qquad g(f_2(*,x,y)) &\to c \\ g(f_1(y,x,y)) &\to g(f_2(x,x,y)) & g(f_2(x,*,y)) &\to c \\ f_2(x,x,y) &\to f_1(x,x,y) & g(f_2(x,y,*)) &\to c \\ d_1 &\to d_2 & g(f_2(x,x,x)) &\to c \end{aligned}$$

Here, the "$*$"-pattern notation in 3 of the rules is to be interpreted as follows: For $l \to r$ of shape $C[*] \to r$, the rule stands for the whole family of rules $C[*] \to r$ where $*$ is sucessively replaced by all most general f-patterns, for all $f \in \mathcal{F}$, i.e., by $f(x_1, \ldots, x_{ar(f)})$, such that the x_i are distinct fresh (w.r.t. $C[.]$) variables. With some effort one can show that $\mathcal{R}$ is SON, hence also WON, SLON and WLON. However, if we add a fresh unary function symbol H, all these properties get lost. To wit, consider $s = g(f_1(H(d_1), H(d_1), H(d_2)))$ which initiates the (only) outermost derivations (the contracted outermost redexes are underlined)

$$\begin{aligned} s \;&=\; g(f_1(\underline{H(d_1)}, H(d_1), H(d_2))) \to_o \underline{g(f_1(H(d_2), H(d_1), H(d_2)))} \\ &\to_o\; g(\underline{f_2(H(d_1), H(d_1), H(d_2))}) \to_o s \to \cdots \quad \text{and} \end{aligned}$$

$$\begin{aligned} s \;&=\; g(f_1(H(d_1), \underline{H(d_1)}, H(d_2))) \to_o \underline{g(f_1(H(d_1), H(d_2), H(d_2)))} \\ &\to_o\; g(\underline{f_2(H(d_1), H(d_1), H(d_2))}) \to_o s \to \cdots \end{aligned}$$

▶ **Example 2** (counterexample to to preservation of SPON, WPON under signature extensions)**.** Consider the TRS $\mathcal{R}$ over the signature $\mathcal{F} = \{f_1, f_2, g_1, g_2, c, d_1, d_2\}$:

$$\begin{aligned} g_1(f_1(x,y),x) &\to g_2(f_1(x,y),x) & \qquad g_2(f_1(*,x),y) &\to c \\ f_1(x,y) &\to f_2(x,y) & g_2(f_1(x,*),y) &\to c \\ g_2(f_2(x,y),y) &\to g_1(f_1(x,y),x) & g_2(f_1(x,y),*) &\to c \\ d_1 &\to d_2 & g_2(f_1(x,x),x) &\to c \end{aligned}$$

Again, with some effort one can show that $\mathcal{R}$ is SPON, hence also WPON. However, if we add the fresh unary function symbol H, these properties get lost. To wit, consider $s = g_2(f_1(H(d_1), H(d_2)), H(d_1))$ which initiates the (only) parallel outermost derivation (the contracted parallel outermost redexes are underlined)

$$\begin{aligned} s \;&=\; g_2(\underline{f_1(H(d_1), H(d_2))}, H(\underline{d_1})) \to_{po} \underline{g_2(f_2(H(d_1), H(d_2)), H(d_2))} \\ &\to_{po}\; \underline{g_1(f_1(H(d_1), H(d_2)), H(d_1))} \to_{po} s \to \cdots . \end{aligned}$$

Concerning future work, it is quite natural to ask how the situation looks like for strategies other than innermost and outermost and for restrictions of rewriting like *context-sensitivity* or *forbidden patterns*. Furthermore more general combinations of TRSs like *constructor sharing* or *composable* ([10]) ones are of interest, too. Another line of research is to take into account typing, i.e. to ask whether imposing a type discipline may facilitate the verification of termination properties of rewriting under strategies, cf. e.g. [17, 1]. On a more technical level it appears interesting to investigate relationships to other settings and approaches where non-left-linearity causes major problems, e.g., in (automatically) proving outermost termination and in dependency pair based termination proofs where signature extensions play a major role ([12]).

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