

## WEAKLY NONLINEAR BORES IN LAMINAR HIGH REYNOLDS NUMBER TWO LAYER FLUIDS

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*Summary* The structure of weakly nonlinear bores forming in two layer fluids is found to be characterized by the nondimensional parameters measuring, respectively, the deviation from the critical value of a properly defined Froude number and the relative importance of dissipative and dispersive effects. Their influence on the flow properties and in particular the emergence of non-classical bores which violate the classical Lax shock inequalities are studied in detail.

### MOTIVATION

If the flow in a liquid layer is forced to adjust to a new downstream state a hydraulic jump or bore may form. In many cases of practical importance, including applications in civil, mechanical as well as chemical engineering, the streamwise extent of the adjustment region is small compared to a characteristic length of the global flow field. As a result the phenomenon can be treated as a jump discontinuity similar to gasdynamic shocks. In contrast to gasdynamic shocks, however, where the resulting jump conditions represent an exact solution of the governing equations if dissipative effects are neglected, the corresponding relationships for hydraulic jumps require the acceleration of the fluid in the direction normal to the solid boundary to be small and, therefore, are only approximate in this limit. Also, while the internal dissipative structure of gasdynamic shocks is well understood, progress to include viscous effects into hydraulic theory has been slow. This is of course not surprising if the flow in the liquid layer is turbulent. However, even in the case of laminar flow the phenomena associated with dissipation and surface tension are found to be surprisingly rich. In the case of single layer fluids significant progress has recently been made by among others Kluwick et al. [2] using asymptotic methods. This raises the question whether a similar approach is possible also for internal shocks (bores, hydraulic jumps) that arise in density stratified fluids and are prevalent in the ocean and the atmosphere.

### SHORT OUTLINE

Two layer fluids represent the simplest model of stratified flows with free surface and, therefore, have received considerable interest in the past. Studies carried out within the framework of hydraulic theory have shown that nearly resonant internal waves exhibit so-called mixed nonlinearity leading in turn to the possible formation of both positive and negative jump discontinuities. Also it was found that the admissibility of such hydraulic jumps can no longer be decided on the basis of the Oleinik criterion as in cases of strictly positive or negative nonlinearity but requires the investigation of their internal dissipative, dispersive structure. To this end Kluwick et al. [1] adopted a structure equation of the form of a modified BKdV equation which represents an ad hoc model as far as dissipation is concerned. In the present study, in contrast, we consider cases where hydraulic jump solutions of the full Navier–Stokes equations can be constructed using rigorous asymptotics. Specifically, it is assumed that a suitably defined Reynolds number is large so that viscous effects in the unperturbed state are essentially confined to a thin shear layer adjacent to the wall and near the interface between the two layer fluid layers where the flow is taken to be laminar, Fig. 1. Also, we require that the Froude number differs only slightly from its critical value for which the speed of upstram propagating internal layer waves vanishes which in turn allows for a weakly nonlinear analysis. It is then found that the flow is dominated by the dynamics inside a thin sublayer of the oncoming wall shear layer where the classical Prandtl equations hold in leading order. In contrast to classical boundary layer theory, however, the driving pressure disturbances are no longer imposed but arise from the viscous inviscid interaction between the flows inside and outside the sublayer. This in turn leads to the fundamental (lower deck) problem

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \quad U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = -\frac{dP}{dX} + \frac{\partial^2 U}{\partial Y^2},$$

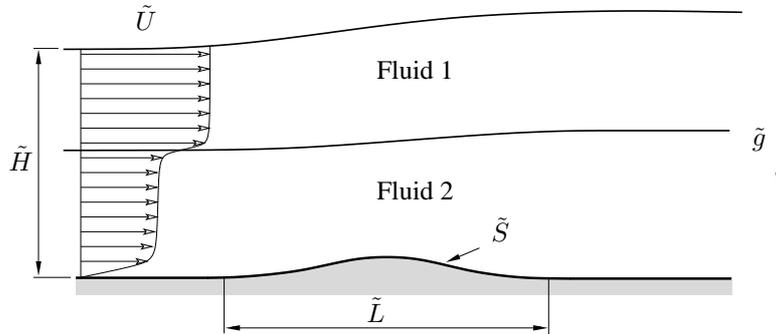
$$Y = 0 : U = V = 0, \quad Y \rightarrow \infty : U = Y + A(X), \quad X \rightarrow -\infty : U = Y,$$

$$\frac{dF(P)}{dX} = K_D \frac{d^3 P}{dX^3} + K_A \frac{d(A - S)}{dX}, \quad F(P) = \sigma_1 P^3 + \sigma_2 P^2 + KP.$$

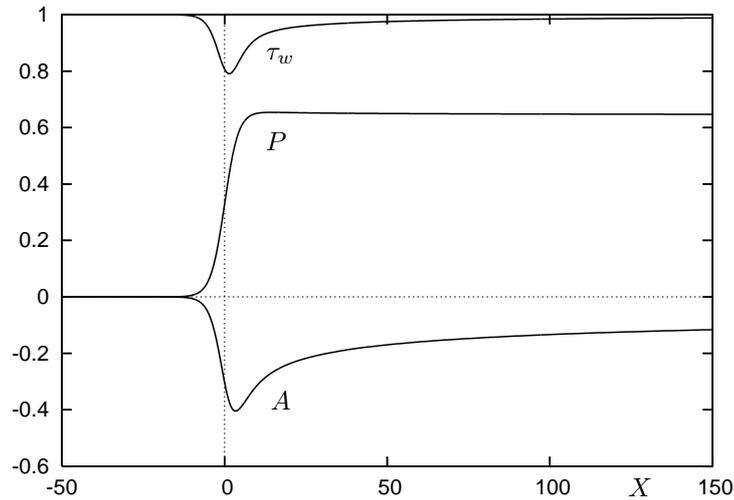
Here  $X, Y, U, V, P, A$  denote distances in streamwise and normal (measured from the solid wall with shape  $S(X)$ ) direction, the corresponding velocity components, the pressure and the displacement function. All quantities are suitably nondimensionalized and scaled.  $\sigma_1, \sigma_2$  are normalized to  $\pm 1$  and distinguish cases of positive and negative nonlinearity. The detuning parameter  $K$  measures the difference between the actual and critical value of the Froude number while the

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constants  $K_D$  and  $K_A$  characterize the relative importance of dispersive and dissipative effects. Representative solutions of the interaction problem will be discussed. Furthermore, it will be shown that, as in the case of the mBKdV equation, the novel structure problem admits so-called non-classical hydraulic jumps, i.e. jumps which transmit rather than absorb wave fronts and thus violate the Oleinik criterion. This raises the question how such jumps can form from smooth initial conditions which will be addressed in detail. As an example Fig. 2 displays the distributions of the wall pressure  $P$ , the displacement function  $A$ , and the wall shear stress  $\tau_w = \partial U / \partial Y|_{Y=0}$  for a nonclassical jump.



**Figure 1.** Schematic of a two layer fluid flow with an isolated obstacle of length  $\tilde{L}$  and height  $\tilde{S}$  on what is otherwise a flat bottom.  $\tilde{U}$  and  $\tilde{H}$  are characteristic values of the velocity and overall depth of the liquid layer.



**Figure 2.** Distributions of  $A$ ,  $P$  and  $\tau_w$  for a nonclassical shock:  $S(X) \equiv 0$ ,  $K_A = 0.01$ ,  $K_D = 1$ ,  $\sigma_1 = 1$ ,  $\sigma_2 = -1$ .

## References

- [1] Kluwick A., Scheichl S., Cox E.A.: Near critical hydraulic flows in two-layer fluids. *J. Fluid Mech.* **575**:20–36, 2007.
- [2] Kluwick A., Exner A., Cox E.A., Grinschgl, Ch.: On the internal structure of weakly nonlinear bores in laminar high Reynolds number flow. *Acta Mech.* **210**:135–157, 2010.