# A general integrator for the Landau-Lifshitz-Gilbert Equation


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## Landau-Lifshitz-Gilbert equation

Find $\boldsymbol{m} : (0,T) \times \Omega \to \mathbb{R}^3$ with $|\boldsymbol{m}| = 1$ almost everywhere such that

$$\boldsymbol{m}_t = -\frac{1}{1+\alpha^2}\boldsymbol{m} \times \boldsymbol{H}_{\text{eff}} - \frac{\alpha}{1+\alpha^2}\boldsymbol{m} \times (\boldsymbol{m} \times \boldsymbol{H}_{\text{eff}}),$$

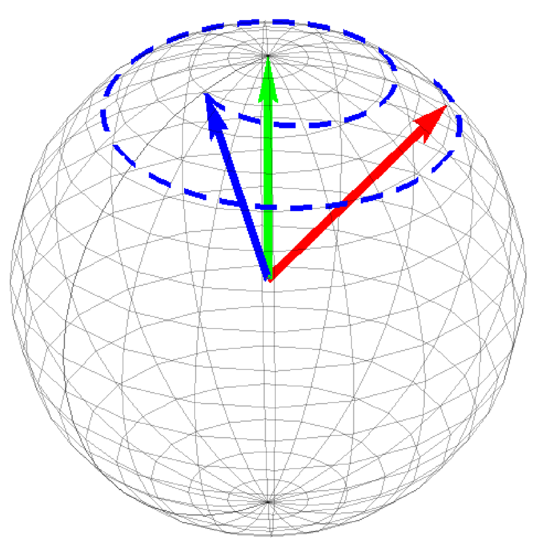
where effective field reads

$$\boldsymbol{H}_{\text{eff}} = \Delta \boldsymbol{m} + \boldsymbol{\pi}.$$

The operator $\boldsymbol{\pi}$ denotes some general (nonlinear) field contribution.

**Challenges for numerical analysis:**

- efficient treatment of nonlinearities
- side constraint $|\boldsymbol{m}| = 1$
- efficient computation of field contributions
- efficient coupling with other PDEs



## General time integrator

Based on equivalent formulation of LLG

$$\alpha\, \boldsymbol{m}_t + \boldsymbol{m} \times \boldsymbol{m}_t = \boldsymbol{H}_{\text{eff}} - (\boldsymbol{m} \cdot \boldsymbol{H}_{\text{eff}})\,\boldsymbol{m} \quad \text{and} \quad |\boldsymbol{m}| = 1 \text{ a.e.}$$

- extends integrator proposed by Alouges 2008
- $\boldsymbol{v} = \boldsymbol{m}_t$ belongs to tangent space, i.e., $\boldsymbol{v} \cdot \boldsymbol{m} = \frac{1}{2}\partial_t |\boldsymbol{m}|^2 = 0$

**Time-marching scheme for general effective field $\boldsymbol{H}_{\text{eff}}$**

- semi-implicit scheme to approximate $\boldsymbol{v}(t_j) \approx \boldsymbol{v}_h^j$ in discrete tangent space $\subset \mathcal{S}^1(\mathcal{T}_h)$
- for all time steps $t_j$:
  - $\alpha\,(\boldsymbol{v}_h^j, \boldsymbol{\phi}_h) + (\boldsymbol{m}_h^j \times \boldsymbol{v}_h^j, \boldsymbol{\phi}_h) = -(\nabla(\boldsymbol{m}_h^j + \theta k \boldsymbol{v}_h^j), \nabla \boldsymbol{\phi}_h) + (\boldsymbol{\pi}_h(\boldsymbol{m}_h^j), \boldsymbol{\phi}_h)$
  - $\boldsymbol{m}(t_{j+1}) \approx \boldsymbol{m}_h^{j+1} \in \mathcal{S}^1(\mathcal{T}_h)$ by nodewise projection of $\boldsymbol{m}_h^j + k\boldsymbol{v}_h^j$

**Convergence result:**

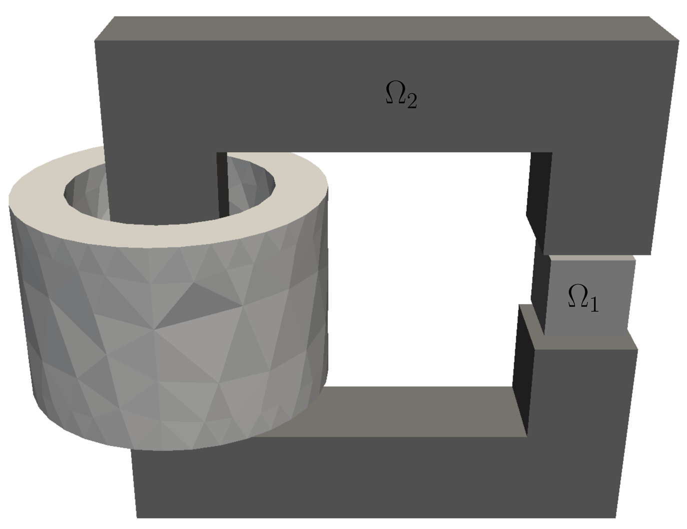
- unconditional weak subconvergence towards a weak solution provided
  - angle condition on triangulation $\mathcal{T}_h$
  - $\boldsymbol{\pi}_h(\cdot)$ uniformly bounded in $L^2(\Omega)$
  - certain weak convergence property of $\boldsymbol{\pi}_h(\cdot)$

## Multiscale modeling

**Collaboration partners:** Bruckner, Feischl, Führer, Goldenits, Suess (VUT, Vienna)

**Setting:**

- multiple domains of different scales
- consider LLG on microscopic domain $\Omega_1$
- consider material law $\boldsymbol{m} = \boldsymbol{\chi}(|\boldsymbol{H}|)\boldsymbol{H}$ on macroscopic domain $\Omega_2$ (nonlinear)
- yields uniformly monotone field operator
- fulfills above assumptions for convergence




- *Multiscale modeling in micromagnetics: well-posedness and numerical integration*
  Preprint available at arXiv:1209.554

## Maxwell's equations

**Full Maxwell system:**

$$\begin{aligned} \varepsilon_0\, \boldsymbol{E}_t - \nabla \times \boldsymbol{H} + \sigma\chi_\Omega\, \boldsymbol{E} &= -\boldsymbol{j} \\ \mu_0\, \boldsymbol{H}_t + \nabla \times \boldsymbol{E} &= -\mu_0\chi_\Omega\, \boldsymbol{m}_t \end{aligned}$$

**Eddy-current formulation:**

$$\mu_0\, \boldsymbol{H}_t + \tfrac{1}{\sigma}\nabla \times (\nabla \times \boldsymbol{H}) \quad = -\mu_0\chi_\Omega\, \boldsymbol{m}_t$$

## Coupling to Maxwell's equations

**Coupling to the full Maxwell system:**

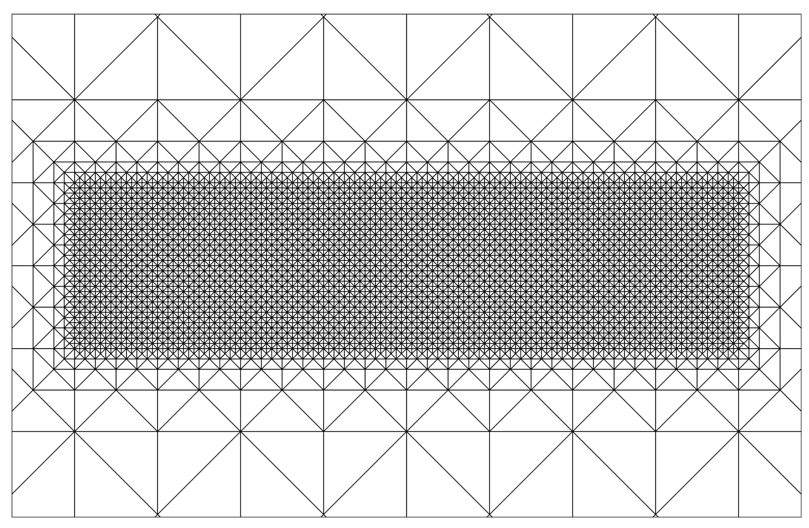
- algorithm decouples both equations
  - only two linear systems per timestep

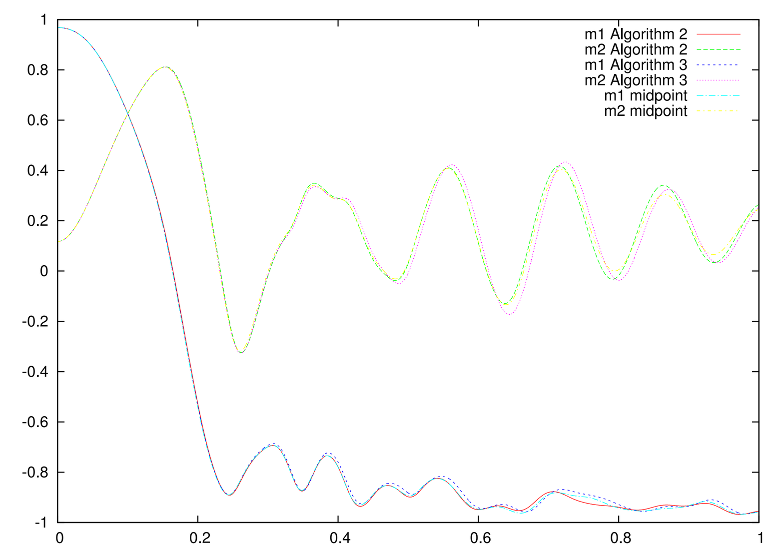
**Decoupled MLLG algorithm:**

- for all time steps $t_j$: Find $\boldsymbol{v}_h^j, \boldsymbol{E}_h^{j+1}, \boldsymbol{H}_h^{j+1}$, such that
  - $\alpha(\boldsymbol{v}_h^j, \boldsymbol{\phi}_h) + ((\boldsymbol{m}_h^j \times \boldsymbol{v}_h^j), \boldsymbol{\phi}_h) = -(\nabla(\boldsymbol{m}_h^j + \theta k \boldsymbol{v}_h^j), \nabla\boldsymbol{\phi}_h) + (\boldsymbol{H}_h^j, \boldsymbol{\phi}_h) + (\boldsymbol{\pi}_h(\boldsymbol{m}_h^j), \boldsymbol{\phi}_h)$
  - $\varepsilon_0(d_t\boldsymbol{E}_h^{j+1}, \boldsymbol{\psi}_h) - (\boldsymbol{H}_h^{j+1}, \nabla \times \boldsymbol{\psi}_h) + \sigma(\chi_\Omega \boldsymbol{E}_h^{j+1}, \boldsymbol{\psi}_h) = -(\boldsymbol{j}^j, \boldsymbol{\psi}_h)$
    $\mu_0(d_t\boldsymbol{H}_h^{j+1}, \boldsymbol{\zeta}_h) + (\nabla \times \boldsymbol{E}_h^{j+1}, \boldsymbol{\zeta}_h) = -\mu_0(\boldsymbol{v}_h^j, \boldsymbol{\zeta}_h)$
  - $\boldsymbol{m}(t_{j+1}) \approx \boldsymbol{m}_h^{j+1} \in \mathcal{S}^1(\mathcal{T}_h)$ by nodewise projection of $\boldsymbol{m}_h^j + k\boldsymbol{v}_h^j$

**Remarks:**

- same assumptions as before $\Rightarrow$ unconditional weak subconvergence
- analysis yields existence of weak solutions
- *A convergent linear finite element scheme for the Maxwell-Landau-Lifshitz-Gilbert equation*
  Preprint available at arXiv:1303.4009






**Coupling to eddy-current equation:**

- **Collaboration partners:** Tran, Kim-Ngan Le (UNSW, Sydney)
- fully decoupled algorithm (extends recent work of Tran, Kim-Ngan Le)
- same assumptions as before $\Rightarrow$ unconditional weak subconvergence (+ existence)
- *On a decoupled linear FEM integrator for eddy-current-LLG* (in progress)

## Including Magnetostriction

**Collaboration partners:** Rochat (EPFL, Lausanne)

- coupling to the conservation of momentum equation

$$\varrho \boldsymbol{u}_{tt} - \nabla \cdot \boldsymbol{\sigma} = 0$$

- field contribution $\boldsymbol{h} = \boldsymbol{h}(\boldsymbol{u}, \boldsymbol{m})$ depends on $\nabla \boldsymbol{u}$ and nonlinearly on $\boldsymbol{m}$
- algorithm decouples both equations

**Decoupled algorithm:**

- for all time steps $t_j$: Find $\boldsymbol{v}_h^j, \boldsymbol{u}_h^{j+1}(\mathcal{S}_0^1(\mathcal{T}_h))$, such that
  - $\alpha(\boldsymbol{v}_h^j, \boldsymbol{\phi}_h) + ((\boldsymbol{m}_h^j \times \boldsymbol{v}_h^j), \boldsymbol{\phi}_h) = -(\nabla(\boldsymbol{m}_h^j + \theta k \boldsymbol{v}_h^j), \nabla\boldsymbol{\phi}_h) + ((\boldsymbol{h}(\boldsymbol{u}_h^j, \boldsymbol{m}_h^j), \boldsymbol{\phi}_h) + (\boldsymbol{\pi}_h(\boldsymbol{m}_h^j), \boldsymbol{\phi}_h)$
  - $\boldsymbol{m}(t_{j+1}) \approx \boldsymbol{m}_h^{j+1} \in \mathcal{S}^1(\mathcal{T}_h)$ by nodewise projection of $\boldsymbol{m}_h^j + k\boldsymbol{v}_h^j$
  - $\varrho(d_t^2\boldsymbol{u}_h^{j+1}, \boldsymbol{\psi}_h) + (\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}(\boldsymbol{u}_h^{j+1}), \boldsymbol{\varepsilon}(\boldsymbol{\psi}_h)) = (\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}^m(\boldsymbol{m}_h^{j+1}), \boldsymbol{\varepsilon}(\boldsymbol{\psi}_h)).$

**Remarks:**

- only two linear systems despite nonlinear coupling of nonlinear PDEs
- convergence analysis more involved
- same assumptions as before $\Rightarrow$ unconditional weak subconvergence (+ existence)
- *On the Landau-Lifshitz-Gilbert equation with magnetostriction*
  Preprint available at arXiv:1303.4060