On the Euler Stage of Turbulent Separation near the Trailing Edge of a Bluff Body

Bernhard Scheichl

Institute of Fluid Mechanics and Heat Transfer, Vienna University of Technology, Resselgasse 3/E322, 1040 Vienna, Austria

AC²T research GmbH, Viktor-Kaplan-Straße 2D, 2700 Wiener Neustadt, Lower Austria, Austria

Abstract. A novel self-consistent description of time-mean two-dimensional turbulent-boundary-layer flow separating from a bluff body at arbitrarily large globally formed Reynolds numbers is presented. Contrasting with previous approaches, the theory deals with a sufficient delay of flow detachment or, correspondingly, increase of the turbulence intensity so as to both settle the question of the actual position of separation and trigger a turbulent boundary layer exhibiting a large relative streamwise velocity deficit. At separation, a generic variation of the velocity profile close to the body surface with the one-third power of the distance from it is detected. The Euler stage resulting from the breakdown of the incident boundary layer and governed by its vorticity is envisaged in detail. Specifically, an analytical solution to the central linear vortex-flow problem could be established. This represents the essential ingredient for the understanding of the multi-layered substructure of the flow more close to the surface, which completes the picture of gross separation at the Euler scale. Most important, the analysis does not resort to any specific turbulence closure. Concerning the canonical situation of circular-cylinder flow, a first comparison between the predicted and publicly available experimentally obtained values of the separation angle is encouraging.

Keywords: Brillouin–Villat singularity, Gross separation, Interacting boundary layers, Matched asymptotic expansions, Turbulence
PACS: 47.15.St, 47.27.E-, 47.27.-i, 47.27.nb, 47.32.C

MOTIVATION AND INTRODUCTION

Consider the incompressible, isoviscous flow around a rigid obstacle having an impervious, closed cylindrical surface aligned with the spanwise direction in an otherwise unbounded domain. Let the Reynolds number *Re* formed with the speed of the undisturbed uniform flow infinitely far upstream of the body, a typical value of the radius of its surface curvature, and the (constant) kinematic viscosity, take on arbitrarily large values, so that we are concerned with the formation of a nominally steady and two-dimensional turbulent boundary layer (BL) along the body surface, which finally undergoes break-away separation (provoking the complex picture of separated/reversed flow and the associated wake further downstream). A most complete rigorous description of the local separation process, in particular the prediction of its location, on the basis of the Reynolds-averaged Navier–Stokes equations in the limit $Re \to \infty$ still poses one of the most challenging problems in theoretical hydrodynamics. As achieving seminal progress towards its solution, the work by Neish & Smith [1] must be viewed as a milestone since it was not until its advent that the regularization of the inevitable Brillouin–Villat (B–V) singularity forming at detachment of the imposed potential flow, which then includes a dead-water cavity further downstream, was addressed correctly within the framework of triple-deck theory. Here the well-established picture of strictly laminar (steady) flow served as a starting point.

To be more specific, we refer to the configuration sketched in Figure 1(a). Let all velocity components and lengths be non-dimensional with the above reference scales and the local pressure difference with respect to flow stagnation with the corresponding dynamic head. Furthermore, let s, n be natural coordinates in streamwise direction along and normal from the contour of the body, respectively, having their origin in the stagnation point at its leading edge, u the streamwise velocity component, and $u = u_s(s;k)$ the surface speed exerted by that potential flow, parametrized by the so-called B–V parameter k. This is defined by the strength of the B–V singularity immediately upstream of the position of inviscid-flow detachment, $s = s_d(k)$: note the classical result $u_s(s;k)/u_s(s_d;k) \sim 1 + 2k\sqrt{s_d - s} + O(s_d - s)$ as $s_d - s \to 0_+$, cf. [3]. Also, k measures the downstream delay of separation as it increases monotonically for increasing values of s_d and, as a most appealing finding, for an increasing level of turbulence intensity in the BL. Finally, it is seen that in the singular limit $k \to \infty$ the form of the potential flow assumes that of the strictly attached one, which exhibits a second stagnation point defining the trailing edge at $s = s_t := s_d(\infty)$. Denotes d the distance $s_t - s_d(k)$, one conveniently addresses this situation by considering the singular limit $d \to 0_+$. It can be shown by conformal-mapping

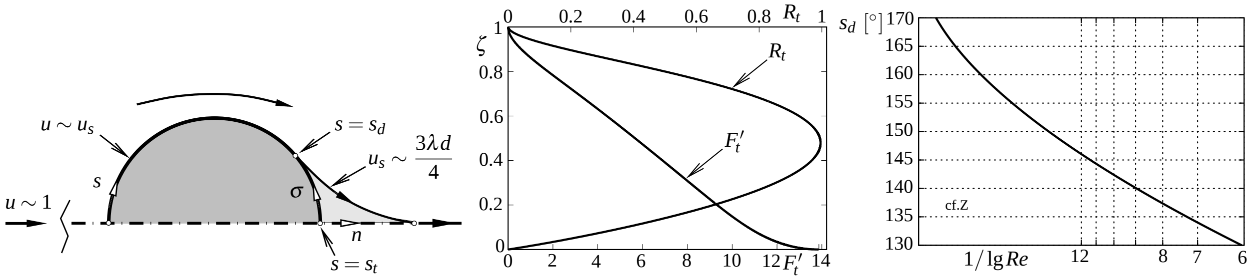

FIGURE 1. Flow configuration (a), shape functions according to (2) of stagnant-flow BL (b), separation angle vs. *Re* (c)

techniques that the aforementioned cavity then becomes cusp-shaped and asymptotically small, i.e. has an extent of $O(d)$, as it encompasses the trailing edge at $(s,n) = (s_t, 0)$. At the body scale, we have $u_s(s;k) \sim u_s(s,\infty) + O(d^4)$ (with $u_s(s;\infty) = 2\sin(s)$, $s_t = \pi$, in the canonical case of a circular obstacle) and $u_s(s,\infty) \sim \lambda(s - s_t) + O\big((s-s_t)^3\big)$ as $s - s_t \to 0_+$, with the constant $\lambda > 0$ depending on the overall body shape. We envisage the resulting subregion of extent d around the trailing edge by introducing stretched (locally Cartesian) coordinates $(x,y) := (s - s_t, n)/d$. In leading order, the potential flow here above the surface given by $y = 0$ is symmetric with respect to $x = 0$, matches the stagnating flow outside, and gives rise to the B–V singularity now located at $(x,y) = (-1,0)$ and the stagnant-flow cusp confined by the separated streamlines (separatrices). It can be expressed analytically by virtue of the aforementioned conformal-mapping strategy. The separatrices are represented by the half of the tetracuspid $x^{2/3} + y^{2/3} = 1$ with $y > 0$, for instance. Most important, we have $u_s(x;k)/d(k) \sim U_s(x) + O(d)$, where the expansion of the rescaled surface slip $U_s(x)$ immediately upstream of the B–V singularity in terms of powers of $\sqrt{S}$ with $S := -x - 1$ recovers its original form stated above, applying to $u_s(s;k)$ for $s_d - s \to 0_+$ and $k = O(1)$, up to third order since then the latter is not affected by the local surface curvature. Finally, the relation $k \sim 1/\sqrt{6d}$ ensues by comparison, and we have for

$$x \to -\infty: \quad U_s \sim -x + O\big(x^{-3}\big), \qquad S \to 0_+: \quad 4U_s/3 \sim g(S/6) + O\big(S^{3/2}\big) \quad \text{with} \quad g(t) := 1 + 2\sqrt{t} + 10t/3. \qquad \text{(1a,b)}$$

The laminar-flow situation is known to apply if $k = O(Re^{-1/16})$, and a considerable level of turbulence intensity in the BL requires $k = O(1)$. The elaborate analysis of the last case was put forward in [2], the limit $k = \infty$ associated with a perfectly attached external flow and a fully developed turbulent BL considered first in [1]. Two central shortcomings of the existing asymptotic theory for $Re \to \infty$ deserve to be highlighted:

(i) neither the correct values of k and, hence, s_d could be determined so far under the premise that $k = O(1)$,

(ii) nor a self-consistent flow picture could be completed in the formal limit $k = \infty$; for a critical analysis see [4].

The extended theory presented here aims at bridging the gap between the cases addressed in (i) and (ii) as it gives rise a fully self-consistent flow description in terms of a suitable distinguished limit $k \to \infty$ as $Re \to \infty$. Amongst others, this then allows for the prediction of the position of separation at the highest level of turbulence intensity possible.

INCIDENT BOUNDARY LAYER FLOW

We commence the analysis by considering the oncoming BL that splits in the common manner in the viscous surface layer and the outer main layer, the latter characterized by an asymptotically small turbulent velocity scale εu_s, where $\varepsilon := \kappa/\log Re$, with κ denoting the von Kármán constant, cf. [1, 2]. To leading order, the streamwise velocity deficit with respect to the imposed potential flow, the Reynolds shear stress, and the BL thickness, δ, then are given by, respectively, $\varepsilon F_\zeta(s,\zeta;d)$, $(\varepsilon u_s)^2 T R(x,\zeta;d)$, and $\varepsilon T \Delta(s;d)$, with the stream-function F, further shape functions R and Δ, the so-called turbulence intensity gauge parameter T, $0 < T \lesssim 1$, and the BL coordinate $\zeta := n/\delta \, (\leq 1)$. In the limit $d = 0$ of an attached potential flow, they grow according to $F \sim 3/(4\sigma^2\sqrt{-\log\sigma})F_t(\zeta)$, $R \sim 9/(-16\sigma^4 \log\sigma)R_t(\zeta)$, $\Delta \sim 3\sqrt{-\log\sigma}/[\sigma F_t(1)]$ as $\sigma := s_t - s \to 0_+$, cf. [1]. Thus, they admit a universal self-similar behavior, i.e. independent of their specific upstream history, as

$$2\zeta F_t'(\zeta) = F_t(1)\,R_t(\zeta)\,, \quad F_t'(1) = R_t(1) = R_t(0) = 0\,. \qquad (2)$$

That is, the original BL problem reduces to the homogeneous boundary value problem (2) to leading order in the limit $\sigma \to 0_+$. Supplemented with an asymptotically correct turbulence closure that accounts for Prandtl's mixing length by the match with the viscous sublayer in the form $R \sim R_t \sim (\kappa\zeta)^2$ as $\zeta \to 0$, (2) allows for a non-trivial solution. However, the growth of the velocity defect towards a moderately large one yields a two-tiered splitting of the sublayer, not accounted for in [1], connected via the celebrated logarithmic velocity law of the wall, which is prevalent for $\zeta \to 0$ if $\sigma = O(1)$. Consequently, here it is superseded by the half-power law known to otherwise characterize a mildly separating turbulent BL: $F_t' \sim F_t'(0) - (2/\kappa)[2\zeta F_t'(0)/F_t(1)]^{1/2} + o(\zeta^{1/2})$ as $\zeta \to 0$. Remarkably, (2) also applies in the entirely different context of an equilibrium BL having an accordingly enlarged velocity defect: see [5].

As recognized first in [4], the main tier of the boundary layer, described by (2), is transformed into a predominantly inviscid (time-mean) flow region where σ and δ have decreased and increased towards orders of magnitude measured by $\tau := (-2/\log\varepsilon)^{1/4}\varepsilon^{1/2}$ and $\mu := T(-\log\varepsilon)^{3/4}\varepsilon^{1/2}$, respectively. Here our concern is with the least-degenerate, distinguished limit expressed by the similarity parameter $D := d/\tau = O(1)$, which allows for a match with both the cases considered in [1] ($k \gg 1$, i.e. $D \ll 1$) and [2] ($k = O(1)$, i.e. $D \gg 1$): cf. issues (i) and (ii) above. Consequently, we then encounter a large relative velocity deficit expressed in the form $u/(\tau\lambda) \sim \hat{U}(x,Y;D) + o(1)$ in dependence of Cartesian coordinates x and Y, the latter resulting from suitably stretching y. Simultaneously, the Reynolds shear stress is of $O(T\tau^2)$ there. The further analysis of this flow region appears to be more appealing and alleviated as susceptible to substantial analytical progress if we take it a as yet slender at this stage or, equivalently, the ratio $\delta/\tau = O(-T\log\varepsilon)$ as asymptotically small, which again allows for a BL approximation. We indeed have this freedom as we bound the order of magnitude of the turbulence intensity gauge parameter T from above by the condition $T \ll -1/\log\varepsilon$. This is considerably relaxed in comparison to the, in sharp contrast, essentially algebraic dependence of T on Re required in the case $k = O(1)$: there $T = O(Re^{-4/9}/\varepsilon^2)$ [1, 2]. In the present situation, the description of the separation process on the scale of the resultant large-deficit BL is seen to be crucial, so we here disregard its impact on the near-surface flow.

Due to the scaling pursued here, the betoken shear layer is located at the base of the aforementioned subregion of the external potential flow. Thus, it is driven by the surface slip U_s and governed by Bernoulli's law in leading order. Upon introduction of a rescaled streamfunction $\hat{\Psi}(x,Y;D)$ and a correspondingly rescaled BL thickness $\hat{\Delta}(x;D)$, satisfying $\hat{\Psi}\big(x,\hat{\Delta}(x)\big) = 1$, and the match with the small-deficit flow far upstream, we write this in the von Mises form

$$\hat{U}(x,Y;D) = \hat{\Psi}_Y = \sqrt{U_{sD}(x)^2 + 2B(\hat{\Psi})}\,, \quad U_{sD} := DU_s\,, \quad B(\hat{\Psi}) := -3F_t'(\hat{\Psi})/4\,, \quad \hat{\Psi}(x,0) = 0 \quad (0 \le \hat{\Psi} \le 1)\,. \tag{3}$$

Hence, $\hat{\Psi}$ and $\hat{\Delta}$ are calculated readily by (numerical) integration. As a remarkable result, solely by assuming that $U_{sD} \to +\infty$ far upstream but without resorting to a specific form of the Bernoulli function $B(\hat{\Psi})$ one finds that a such described BL there is of universal small-deficit type as $\hat{\Psi}_Y \sim U_{sD} + U_{sD}^{-1}B(U_{sD}\hat{\Delta}\zeta) + O\big(U_{sD}^{-3}\big)$ and $\hat{\Delta} \sim U_{sD}^{-1} + O\big(U_{sD}^{-3}\big)$, with ζ, now redefined as $Y/\hat{\Delta}$, taken as of $O(1)$. Specifically, we have $\hat{\Delta} \sim U_{sD}^{-1} + 3U_{sD}^{-3}F_t(1)/4 + O\big(U_{sD}^{-5}\big)$ as $-X \gg 1$, and matching with the oncoming small-deficit BL is confirmed by virtue of (1a).

Since separation is triggered by the B–V singularity, in the inviscid limit considered here its position $x = -1$ must coincide with that of incipient flow reversal at the base of the BL. That is, $U_s(x) - 3/4$ and the rescaled surface slip $\hat{U}_s := \hat{U}(x,0;D)$ vanish simultaneously. Hence, (3) sorts out the correct value D^* of the similarity parameter D, which finally fixes the desired dependence of d on Re to leading order in terms of a first separation criterion:

$$D = D^* = [8F_t'(0)/3]^{1/2}\,, \quad d(Re) \sim 2^{7/4}[\varepsilon F_t'(0)/3]^{1/2}/(-\log\varepsilon)^{1/4}\,. \tag{4}$$

Together with the already mentioned result $k \sim 1/\sqrt{6d}$, these relationships represent a major step forward in the description of turbulent break-away separation in view of issue (i) above. One has to concede, however, that the actual value of the slip deficit given by $F_t'(0)$ depends (sensitively) on the specific choice of the shear stress closure used to solve (2). Here a mixing-length closure is preferred so as to account for the rather abrupt outer edge $y \sim \delta$ of the BL. Its specific algebraic form adopted here gives $F_t'(0) \doteq 13.8681$, hence $D^* \doteq 6.0812$: see Figure 1(b). For the circular-cylinder flow, the separation angles $s_d = \pi - d$ via (4), see Figure 1(c), agree nicely with measured ones: cf. [6].

The relationship (4) crucially affects the analysis of $\hat{\Psi}$ in the limit $x \to 0_-$ envisaged next as it induces the formation of a sublayer: we infer from (3) subject to (1b) and (4) the following behaviors for

$$S \to 0_+:\ \hat{\Psi} \sim \hat{\Psi}_0(Y) + 3\sqrt{S}F_t'(0)\,\hat{\Psi}_0'(Y)\int_0^Y \frac{\mathrm{d}t}{\hat{\Psi}_0'(t)^2} + O(S)\,,\ \ Y \to 0:\ \hat{\Psi}_0'(Y) \sim \frac{3}{\kappa^{2/3}}\left[\frac{F_t'(0)}{F_t(1)}\frac{Y}{2}\right]^{1/3}\big[1 + o(1)\big]\,. \tag{5a,b}$$

The novel one-third-power law (5b) characterizes the velocity profile at separation, $U(Y) := \hat{\Psi}_0'(Y)$, and the base of the outer main tier of the entire BL. The expansion (5a) indicates its breakdown at a streamwise scale measured by the BL thickness δ, which is of $O(\mu)$. Its advent is accompanied by a splitting of this tier due to the apparent non-uniformity of the expansion (5a). This gives rise to a further sublayer close to the surface where Bernoulli's law is retained in full whereas it makes way for an Euler flow linearized about the parallel flow given by $u \sim \tau\lambda U(Y)$ in the arising square region having an extent μ. This problem represents a cornerstone in the current theory of separation.

THE EULER FLOW PROBLEM AND BEYOND

We introduce an accordingly stretched streamwise coordinate X, the leading-order perturbation $\mu^{1/2}\Psi_Y(X,Y)$ of the separation profile $U(Y)$, and the accordingly scaled pressure $\mu^{1/2}P(X,Y)$ to arrive at the leading-order approximations

$$U(Y)\Psi_{YX} - U'(Y)\Psi_X = -P_X\,, \quad -U(Y)\Psi_{XX} = -P_Y \tag{6a,b}$$

of the momentum equations, governing the linearized Euler (Rayleigh) stage. By eliminating either Ψ or P, one obtains

$$P_{XX} + P_{YY} = 2(U'/U)(Y)P_Y\,, \quad \Psi_{XX} + \Psi_{YY} = (U''/U)(Y)\Psi\,. \tag{7a,b}$$

In (7b) a potentially Y-dependent additive function arising from integration with respect to X is set to zero as a consequence of the conditions of matching with the flow upstream. The inviscid perturbed flow described by (6a,b), (7a,b) exhibits a vorticity given by the negative right-hand side of (7b). The subsequent discussion of this stage will prove it as advantageously based on (7a) rather than on (7b) as (7a) contains only derivatives of the governed quantity.

At first, it is noted that (7a) is equivalent to the system of Beltrami equations $[P_X, P_Y] = U(Y)^2[V_Y, -V_X]$, where $V(X,Y) := -\Psi_X/U(Y)$ by (6a,b). Those are known to be invariant against a conformal mapping, $X + \mathrm{i}Y \mapsto \xi + \mathrm{i}\eta$, say, such that $[\Pi, \Upsilon](\xi,\eta) := [P,V](X,Y)$ and $\beta(\xi,\eta) := U(Y)$: $[\Pi_\xi, \Pi_\eta] = \beta^2[\Upsilon_\eta, -\Upsilon_\xi]$. In turn, (7a) transforms into $\Pi_{\xi\xi} + \Pi_{\eta\eta} = (2/\beta)(\beta_\xi\Pi_\xi + \beta_\eta\Pi_\eta)$. Upon taking the characteristic variable $\xi + \mathrm{i}\eta$ and η as the independent quantities, one finds that this generalized form of (7a) has the general solution $\Pi = H(\xi,\eta)\beta(\xi,\eta)$ with an arbitrary harmonic function H provided $1/\beta$ is also harmonic in ξ and η. When we conveniently choose $\beta = 1/[\eta + 1/U_e]$, here U_e denotes the value $3D^*/4$ of $U(Y)$ at the local BL edge, $Y = Y_e$, say, the conformal mapping is specified as

$$[\xi + \mathrm{i}\eta](X,Y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\xi(0,t)+\mathrm{i}\eta(0,t)}{X+\mathrm{i}(Y-t)}\,\mathrm{d}t\,,\ \ \eta(0,Y) := \frac{1}{U(Y)} - \frac{1}{U_e}\,,\ \text{ with }\ [\xi,\eta](0,Y) = \frac{1}{\pi}⨍_{-\infty}^{\infty}\frac{[\eta,-\xi](t)}{Y-t}\,\mathrm{d}t \tag{8}$$

forming a Hilbert pair. Here $U(Y)$ is continued skew-symmetrically for negative values of Y and $|U(Y)|$ set equal to $|U_e|$ for $|Y| > Y_e$. Then $\xi(Y)$ is an even function, and the X-axis is mapped onto the ξ-axis. By this strategy we achieve that $H(\xi,\eta)$ can be constructed by the conditions of matching with the external potential flow in terms of the B–V singularity, $P \sim -2\Im(X + \mathrm{i}Y)$ as $X \to -\infty$, and the inviscid sublayer in the limit $Y \to 0$. The latter are found to require a regular expansion having the form $P(X,Y) \sim P_0(X) - 3P_0''(X)Y^2/2 + O(Y^4)$ for $X < 0$ and an irregular one, predicting that $P(X,Y) = O(Y^{5/3})$, for $X > 0$. At the scale considered, here the separation point is set to $X = 0$.

Scrutinizing the Euler stage in more depths and breadths, specifically the behavior of P near $X = Y = 0$, under way initiates the analysis of separation at the scale of the near-wall layers, so as to e.g. refine the separation criterion (4).

ACKNOWLEDGMENTS

Special thanks go to the Austrian Research Promotion Agency (FFG; grant no. 824187, project acronym *XTribology*).

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