Theoretical and Computational Fluid Dynamics (2014) 28(3):377–384
(the final publication is available at `link.springer.com/article/10.1007/s00162-014-0322-9`)

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A note on the far-asymptotics of Helmholtz–Kirchhoff flows

Brief communication

Received: 22 November 2013 / Accepted: 4 March 2014 / Published online: 5 April 2014

Abstract In this sequel to a rather recent paper on the classical problem of Helmholtz–Kirchhoff flows by Vic. V. Sychev, *TsAGI Science Journal* **41**(5), 531–533 (2010), the representation of the flow far from the body and its specific implications discussed in that study are revisited. Here the concise derivation of these findings resorts to well-known Levi-Cività's method and, alternatively, only fundamental properties of analytic functions and thin-airfoil theory. As particularly of interest when the well-known Kirchhoff parabola degenerates to an infinitely long cusp, integration constants debated controversially so far and important for the understanding and computation of those flows are specified by the integral conservation of momentum. Also, the parametric modification towards flows encompassing stagnant-fluid regions of finite extent and the previously unnoticed impact of higher-order terms on the associated high-Reynolds-number flows are addressed.

Keywords Bluff-body flows · Free streamlines · Ideal-fluid flows · Kirchhoff flows · Levi-Cività's method

1 Problem formulation

We consider two-dimensional steady flow of an incompressible ideal fluid under the absence of surface tension and body forces past a rigid obstacle described by a closed single-connected region with an impervious smooth contour. The flow is taken as uniform infinitely far away from this. This situation establishes the well-known problem of (Helmholtz–)Kirchhoff (H–K) flows as long as the trailing stagnant-fluid cavity confined by the two free streamlines separating from the obstacle is (semi-)open, i.e. extends to infinity downstream of it.

The configuration is sketched in Fig. 1, where first only non-degenerate H–K flows (divergent cavity, shown in white) shall be of interest. Let all lengths and components of the flow velocity be non-dimensional with a typical curvature radius of the body, $\tilde{R}$, and the speed of the uniform flow, $\tilde{U}$, respectively. We then consider the problem in the complex z-plane, $z = x + \mathrm{i}y$, where the x-axis is directed along the unperturbed flow and the location of the origin, $\mathscr{O}$, relative to the contour of the obstacle, $\mathscr{C}^o$, chosen arbitrarily. The problem is made up of finding the complex flow potential, $w(z)$, or the conjugate-complex flow velocity, $w'(z)$, outside of and along the boundary confining the cavity. This is given by the front arc $\widehat{\mathscr{S}^+\mathscr{L}\mathscr{S}^-}$ on $\mathscr{C}^o$ with $\mathscr{S}^\pm$ denoting the two separation points where $x = x_s^\pm$, say, and the two tangentially continuing free streamlines $z = z_s^\pm(x) = x + \mathrm{i}y_s^\pm(x)$, say, with $x \geq x_s^\pm$, $y_s^- < y_s^+$. We fix an arbitrary constant shift of w advantageously by requiring $w = 0$ on the leading-edge stagnation point, $\mathscr{L}$, (on $\mathscr{C}^o$) where $w' = 0$. Assuming parallel oncoming

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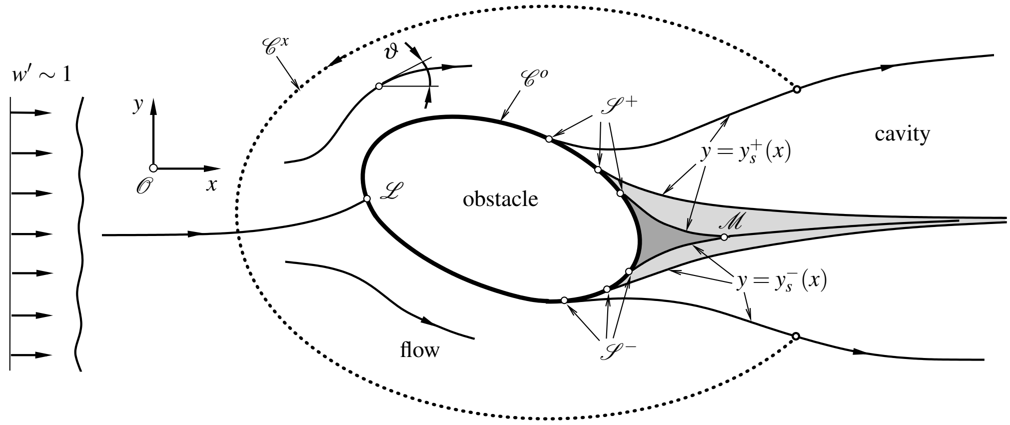

Fig. 1 Flow configuration in the physical z-plane

flow yields $w'(z) \to 1$ as $z \to \infty$ outside the cavity. Consequently, $w(z)$ is subject to two kinematic and one dynamic boundary condition: $\operatorname{Im} w = 0$ on the front arc,

$$\operatorname{Im} w\big(z_s^{\pm}(x)\big) = 0\,, \qquad \big|w'\big(z_s^{\pm}(x)\big)\big| = 1\,. \tag{1a,b}$$

The crucial dynamic condition (1b) represents the requirement of fluid at rest, i.e. constant pressure, inside the cavity, which allows for fixing $y_s^{\pm}(x)$ in the course of the construction of $w(z)$. Thus, the flow is governed by a nonlinear free-boundary problem. For the classical combined analytical–numerical methods of its solution we refer to the seminal textbook by Gurevich [1], chap. IV.

For $\mathscr{C}^o$ given, $w(z)$ represents a two-parameter family of flows, parametrised (uniquely) by the two prescribed positions of $\mathscr{S}^{\pm}$. Useful alternative parametrisations are provided by e.g. one of those and either the position of $\mathscr{L}$ or the circulation of the flow around the obstacle, Γ (invariant against a conformal mapping of z), or, as pursued in the following, the corresponding drag and lift coefficients, c_x and c_y, respectively:

$$\Gamma = \int_{\mathscr{C}^x} w'(z)\,\mathrm{d}z = w\big(z_s^-(x)\big) - w\big(z_s^+(x)\big)\,, \qquad c_x - \mathrm{i}c_y = \frac{\mathrm{i}}{2}\int_{\mathscr{C}^x}\big[w'(z)^2 + 1\big]\,\mathrm{d}z\,. \tag{2a,b}$$

Following [1], p. 99, here integration is carried out in counter-clockwise direction along any "proper" curve $\mathscr{C}^x$ in the single-connected flow region, i.e. with start-/endpoints $z = z_s^{\pm}(x)$ sharing a specific abscissa value $x \geq x_s$. The global conservation laws of mass and momentum are equivalent to, respectively, the zero imaginary part of Γ, representing the total flux out of the flow region separated by $\mathscr{C}^x$, owing to (1a) and the modified first Blasius formula (2b), where the front arc as the original integration path has been deformed to $\mathscr{C}^x$, cf. [1], pp. 95–96.

As in the preceding analysis by Sychev [2], our concern is with determining the asymptotic expansions of $w(z)$ as $z \to \infty$ and, in particular, of $z_s^{\pm}(x)$ as $x \to \infty$. Sychev [2] adopted their forms independently as provided in the classical literature, respectively, by Golubev [3] and in Birkhoff and Zarentonello's book [4], p. 68 ff. and chap. VI. There they originate in the representation of the problem by Levi-Cività (L-C), underlying his efficient original method [5] and (more advanced) related techniques, as outlined in [1], chap. IV. In agreement with the introductory considerations, the initially unknown coefficients in the those expansions are expressed in terms of c_x and c_y via (2b). In turn, Sychev [2] specified a novel relationship between the constants of integration by the global conservation of mass. Finally, his finding substantiates the degenerate form of H–K flows, namely that when the cavity assumes the shape of an infinitely long cusp (light-shaded in Fig. 1). As an expedient complement to his and previous related work (see [1] above), here we derive the open forms of those expansions in a concise manner and, simultaneously, together with his new result by taking up Levi-Cività's idea of obtaining the full solution to the problem. By presenting an appealing alternative, we even establish these expansions solely through (1a,b). Then we use (2a,b) by taking $\mathscr{C}^x$ at sufficiently large distances $r := |z|$ to restate and extend to higher orders the aforementioned dependences of the coefficients and constants by asymptotic analysis and confirm these closed results by quite basic and intuitive arguments.

We start by revisiting the L-C method [5, 1] in brief. Its central ingredient is the conformal mapping of the flow region in the z-plane onto the inner of the upper half of the unit circle in the τ-plane, say, such that the

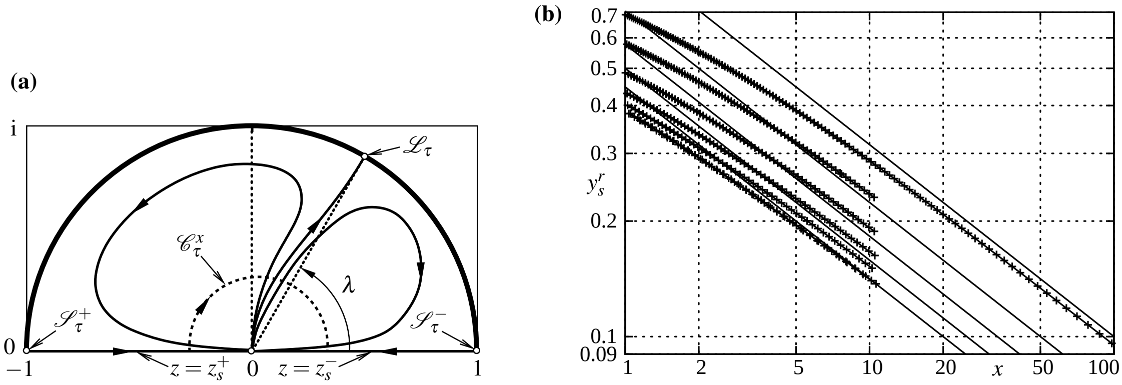

Fig. 2 **a** Flow configuration in the τ-plane according to (3): angle λ chosen as $\pi/3$, inner streamlines open for $\mathrm{Im}\, w(z) = 0$ or $\mathrm{Im}\,\tau = [\mathrm{Re}\tau/(2\cos\lambda - \mathrm{Re}\tau) - (\mathrm{Re}\tau)^2]^{1/2}$, closed otherwise (two sketched); **b** log-log plot of $y_s^r(x)$ (*curves* interpolate *data markers* computed numerically, from bottom to top: $c_x\,[\pi - \theta_s] \doteq 0.4986\,[55.042°$, B–V angle$]$, $0.4669\,[67.919°]$, $0.3270\,[86.032°]$, $0.2435\,[93.855°]$, $0.1429\,[102.958°]$, $0.0425\,[113.725°]$, $\ll 10^{-8}\,[124.202°]$) vs. $y = (cx)^{-1/2}$ (*straight lines*, $c = 1, 2, 3, 4, 5$)

front arc mentioned above and the free streamlines are mapped onto the circular arc and the bounding section of the abscissa, respectively, see Fig. 2a. Here the subscript τ refers to the origin of the corresponding objects in the z-plane. In the hence most simple representation

$$w(z) = \alpha(\tau + 1/\tau - 2\beta)^2, \quad \alpha > 0, \quad \beta = \cos\lambda, \quad 0 < \lambda < \pi, \tag{3}$$

the two control parameters α and β are taken as functions of c_x and c_y; the flow is symmetric iff $c_y = \beta = 0$. The images of all streamlines touch in the branching point $\tau = 0$ of (3), the image of $z = \infty$. The method seeks the (numerical) representations of the inverse mapping $\tau \mapsto z$ and of $\omega(\tau) := \mathrm{i}\ln\bigl(w'(z)\bigr) = \vartheta + \mathrm{i}\ln|w'|$ – for the flow angle ϑ see Fig. 1 – analytic in the closed area of interest apart from the second branching point, $\mathscr{L}_\tau$.

2 Asymptotic state of Kirchhoff flows reappraised

In the present context, only the first three terms of the Taylor expansion $\omega \sim -\gamma\tau + \delta\tau^2 + \sigma\tau^3 + O(\tau^4)$ or of

$$w'(z) \sim 1 + \mathrm{i}\gamma\tau - (\gamma^2/2 + \mathrm{i}\delta)\tau^2 + (\gamma\delta - \mathrm{i}\sigma)\tau^3 + O(\tau^4), \quad [\gamma, \delta, \sigma](\alpha, \beta) \in \mathbb{R}, \quad \tau \to 0, \tag{4}$$

are of relevance. Herein the – due to (1b) real – coefficients govern the essential topology of the H–K flow for $z \to \infty$, namely non-degeneracy ($\gamma > 0$), asymmetry (as $\beta = \delta = 0$ for symmetric flows), and the leading-order description of a both symmetric and degenerate flow ($\beta = \gamma = 0$, $\sigma \neq 0$). Eliminating τ in (4) via (3) (i.e. by $\tau \sim -(\alpha/w)^{1/2} + O(1/w)$ as $w \to \infty$) and integration with respect to w yields Golubev's interesting relationship [3], taken up in [1], p. 99, and [2], eq. (2), in straightforward manner, refined by one further term:

$$z \sim w + 2\mathrm{i}\gamma(\alpha w)^{1/2} - \kappa^* \ln w + \mu + \mathrm{i}\nu + \chi^* w^{-1/2} + O\bigl(w^{-3/2}\bigr), \quad w \to \infty, \tag{5}$$

$$[\kappa^*, \chi^*] := \bigl[\alpha\gamma^2/2 - \mathrm{i}\alpha(2\beta\gamma + \delta), -2\alpha^{1/2}\,\mathrm{Im}\,\kappa^* - 2\mathrm{i}\alpha^{3/2}(4\beta^2\gamma + 4\beta\delta + \gamma^3 + \gamma - \sigma)\bigr], \quad \mu, \nu \in \mathbb{R}. \tag{6}$$

Expansion (5) is found to be purely of multi-pole type. The arbitrary constant of integration, $\mu + \mathrm{i}\nu$, forms the third independent parameter as it reflects the unrestricted freedom of positioning the origin $\mathscr{O}$ relatively to $\mathscr{C}^o$. We evaluate (5) for $z = z_s^\pm(x)$ and eliminate w by virtue of (1a), where $\ln w$ changes by $2\mathrm{i}\pi$ when we turn from the case $\mathrm{Im}\, w = 0_+$ ($z = z_s^+$) to the alternative one $\mathrm{Im}\, w = 0_-$ ($z = z_s^-$). Also, we introduce the useful replacements $a_0 := 2\alpha^{1/2}\gamma$, $b_0 := -\mathrm{Im}\,\kappa^*$. This reproduces the expansion of $y_s^\pm(x)$, cf. [4] and [1], p. 99, in eq. (1) in [2], and, by inversion of (5), eq. (3) in [2] with enhanced accuracy:

$$y_s^\pm(x) \sim \pm a_0 x^{1/2} + b_0 \ln x + \nu - (a_0^2\pi/8)(1 \mp 1) \pm (a_0^3/16)x^{-1/2}\ln x + O\bigl(x^{-1/2}\bigr), \quad x \to \infty, \tag{7}$$

$$w(z) \sim z - \mathrm{i}a_0 z^{1/2} + (a_0^2/8 - \mathrm{i}b_0)\bigl[1 - \mathrm{i}a_0 z^{-1/2}/2\bigr]\ln z - (a_0^2/2 + \mu + \mathrm{i}\nu) + O\bigl(z^{-1/2}\bigr), \quad z \to \infty. \tag{8}$$

The dominant terms proportional to a_0 in these relationships give rise to the K-parabola in the expansion (7) (self-evidently not affected by μ). Also, Sychev's core results is here confirmed without his explicit use of the integral mass balance, namely a term $a_0^2\pi/4$ contributing to the divergence of $y_s^+ - y_s^-$ as $x \to \infty$ and thus disproving the finite width of the cavity as $x \to \infty$ in the degenerate case $a_0 = 0$ as asserted in [4], p. 71. Such a situation is related to the Oseen flow studied by Gustafsson and Protas [6], also see the references therein.

Evaluation of (2a) upon substitution of (7) into (8) demonstrates the identity of Γ and the strength $2\pi b_0$ of the individual vortex that arises by (8), indicating the dominant deviation from symmetry of the flow. Simultaneously, the constant term in (7) cancels a virtual imaginary contribution to Γ and, as becomes evident below, an extra drag term, caused by the individual source of strength $a_0^2\pi/4$ in (8). This interplay can be interpreted as an additional displacement effect by the far-downstream part of a Rankine body, "superimposed" to the K-parabola but without causing drag. The evaluation of (2b), now eased when the integrand is rewritten as $2w' + (w'-1)^2$, immediately recovers the Kutta–Zhukovsky theorem in the form $c_y = -\Gamma$ and relates a_0 to the drag as $c_x = a_0^2\pi/4$, cf. [1], p. 99, and eq. (1) in [2]. [1] Hence, c_x and c_y are independent key quantities. The identical cancellation of higher-order contributions to the integrals in (2a,b) finally confirms the structure of (7) and (8).

The $x^{-1/2}$- and $z^{-1/2}$-contributions to (7), (8) are also consistent with the accuracy implied by (4) and can be made explicit via (6). Their coefficients include contributions owing to the aforementioned shift of $\mathscr{O}$. In general, those expansions proceed purely in terms of half-integral powers of the arguments, multiplied with positive integer powers of their logarithms and gauge functions obtained by repeated differentiation of the terms stated explicitly, depending on a_0, b_0 solely, and here sufficing our needs. The important class of symmetric flows is distinguished by $b_0 = 0$, $\nu = a_0^2\pi/8$ (where the non-zero real part of the coefficient $\mathrm{i}a_0^3/8 - \mathrm{i}a_0\kappa^* - \chi^* + \mathrm{i}a_0(\mu + \mathrm{i}\nu)/2$ of the $z^{-1/2}$-contribution to (8) is noticed).

Having constructed the full solution to a specific flow problem numerically on the basis of the L-C method, one calculates the positions of the streamlines ex post conveniently by integration of $\mathrm{d}z/\mathrm{d}\tau = (\mathrm{d}w/\mathrm{d}\tau)/w'(z)$. Here the nominator is given by (3) and the denominator by the already determined Maclaurin series of $\omega(\tau)$. Then (4) and (6) furnish the Laurent expansion

$$\mathrm{d}z/\mathrm{d}\tau \sim -2\alpha\tau^{-3} + 2\alpha(2\beta + \mathrm{i}\gamma)\tau^{-2} + 2\kappa^*\tau^{-1} - 2\alpha[2\beta + \beta\gamma^2 + \delta\gamma - \mathrm{i}(2\beta\delta - \sigma)] + O(\tau)\,, \quad \tau \to 0\,. \tag{9}$$

The vanishing real part of the integral of (9) along $\mathscr{C}_\tau^x$, see Fig. 2a, yields the distance of the endpoints of $\mathscr{C}_\tau^x$, seen to be of $O(\beta x^{-1/2})$, and its negative imaginary part $y_s^+ - y_s^-$, in agreement with (7) and the constant contribution $a_0^2\pi/4$ verified by the negative half of the residual of (9). Moreover, since by the Riemann mapping theorem any symmetric H–K flow is equivalent to the canonical one about the unit cylinder, the full solution to this problem has attracted much interest. Its systematic treatment dates back to that by Schmieden [7,8] and was taken up recently in advanced form by Scheichl et al. [9,10]. The data taken from the latter studies displayed in Fig. 2b confirm the results (7), (8) numerically: y_s^r denotes the remainder term of $O(x^{-1/2})$ in (7) for y_s^+, $\pi - \theta_s$ measures the separation angle in clockwise direction from $\mathscr{L}$ as we set $\theta_s := \arg z$ in $\mathscr{S}^+$, and the monotonic decrease of c_x is controlled by the entire range of geometrically admissible separation angles. These vary from the well-known Brillouin–Villat (B–V) angle up to that characterising the degenerate limit of H–K flows where we have $c_x = a_0 = 0$. The interval $[-1:1)$ of integration over τ gives values in x ranging from $\cos\theta_s$ up to some maximum, which is enlarged distinctly in the degenerate case.

3 Kirchhfoff flows viewed from slender-airfoil theory

According to Van Dyke [11], the flows considered here describe the perturbation of parallel flow by a slender obstacle, formed by the cavity, with semi-infinite extent and a shape that has the speed on its surface undisturbed. The obstacle itself has now shrunken to the origin $\mathscr{O}$, and the free streamlines give rise to a branch cut in the z-plane along the positive x-axis. At this scale, the entire upstream history of the flow or, equivalently, the shape of the front arc $\widehat{\mathscr{S}^+\mathscr{L}\mathscr{S}^-}$, interpreted as the leading edge of that slender body and controlling the form of $w(z)$ where $r = O(1)$, is condensed in the values of the two principal parameters a_0 and b_0, i.e. c_x and c_y.

In the light of this analogy, it is tempting to derive the far-field representations (7), (8) exclusively on the basis of classical thin-airfoil theory. Write $z = r\mathrm{e}^{\mathrm{i}\theta}$ with $0 < \theta < 2\pi$, $w^e(z)$ for the dominant part of the

[1] note that $2\tilde{R}$ is the correct reference length in (2b), rather than $\tilde{R}$.

far-field excess of $w-z$ such that the effect of the non-trivial "shape functions" $y_s^{\pm}(x)$ are neglected at this stage of approximation, and $u^e(r,\theta)$ for the respective streamwise velocity component $\mathrm{Re}\,(\mathrm{d}w^e/\mathrm{d}z)$. One then infers from (1b) that

$$u^e \to 0 \quad \text{as} \quad \theta \to 0_+\,,\ \ \theta \to 2\pi_-\,. \tag{10}$$

We first focus on the determination of $w^e(z)$ subject to these boundary conditions.

Inspection of Laplace's equation, $r^{-1}\partial_r(r\,\partial_r u^e) + r^{-2}\partial_\theta^2 u^e = 0$, reveals that the expression for u^e in the limit $r \to \infty$ consists of terms of the form $\gamma_n(r)\,r^{-n/2} f_n(\theta)$, $f_n(\theta) = \sin(n\,\theta/2)$, $n = 1,2,\dots$ with some gauge functions $\gamma_n(r)$ of sub-algebraic variation, so that $\ln\gamma_n = o(\ln r)$, $\gamma_n'(r) = o(\gamma_n/r)$, $\gamma_n''(r) = o(\gamma_n/r^2)$. Note that any non-vanishing γ_n' is essentially of algebraic variation, otherwise $\gamma_n \sim r\gamma_n'$ by integration, in contradiction to the above estimates. Moreover, this variation must be given by r^{-1} as any other power of r would again lead to an algebraic variation of γ_n. This implies $\gamma_n' \sim \chi_n(r)/r$, $\gamma_n'' \sim -\chi_n(r)/r^2$ where $\chi_n = o(\gamma_n)$ now indicates the variation of sub-algebraic strength. From the Laplacian it is readily deduced then that these derivatives evoke further contributions to u^e of the form $\chi(r)\,r^{-n/2} g_n(\theta)$ with g_n satisfying $g_n'' + (n/2)^2 g_n = n f_n$. We decompose $g_n = g_n^h + g_n^p$ where the homogeneous solution g_n^h of the former equation proportional to f_n again agrees with the homogeneous boundary conditions (10) and $g_n^p(\theta) = -\theta\cos(n\,\theta/2)$ is its particular one. Since the latter is in conflict with those, only the trivial result $\chi_n = 0$, i.e. $\gamma_n = const$, applies. We thus arrive at the complete representation of $\mathrm{d}w^e/\mathrm{d}z$ in terms of powers of $z^{-n/2}$ and, via integration, the far-field eigenform of the H–K flow problem,

$$w^e(z) = -\mu - \mathrm{i}\Big[a_0 z^{1/2} + b_0 \ln z + \nu + \sum\nolimits_{n=1}^{\infty} \big(a_n z^{1/2} + b_n\big) z^{-n}\Big], \quad [a_n, b_n](a_0, b_0, \mu, \nu) \in \mathbb{R}, \quad z \to \infty. \tag{11}$$

It is noted that the foregoing derivation of (11) is of the type of the (higher-order) analysis of the well-known potential-flow singularity in $\mathscr{S}^{\pm}$ as presented in [10] (where only the upper half of the z-plane is of relevance). Alternatively, expressing $\mathrm{d}w^e/\mathrm{d}z$ as a Maclaurin series in $1/\zeta$, $\zeta := z^{1/2}$, $[\xi,\eta] := [\mathrm{Re}\,\zeta, \mathrm{Im}\,\zeta]$, as $\zeta \to \infty$ ensues in more concise manner from the following rationale. Assume $\eta \ge 0$ and consider $\mathrm{d}w^e/\mathrm{d}z$ as the asymptotic representation of some function, $h(\zeta)$, which is analytic in ζ and vanishes as $\zeta \to \infty$, and its real part, $u^h(\xi,\eta)$, as the according extension of u^e for finite values of r. We then have $u^h(\xi,0) = 0$ for all $\xi > M$ with some $M > 0$. By Cauchy's integral formula, integrating $(\mathrm{i}/\pi)\,u^h(t,0)/(\zeta - t)$ over all $|t| \le M$ (where singularities of $u(t)$ are adequately excluded) for $\eta > 0$ gives $h(\zeta)$ and taking the limit $\zeta \to \infty$ the pure multi-pole expansion.

The terms in (11) represent a complete set of eigenfunctions, also showing up in (8), but with the dependences of a_n and b_n for $n \ge 1$ attributed to a_0, b_0, $\mu + \mathrm{i}\nu$ being the only independent flow parameters. Furthermore, $w^e(z)$ describes a flow pattern symmetric with respect to the x-axis, where the terms containing half-integral powers of z govern symmetric/antisymmetric x-/y-components of the flow velocity, i.e. a symmetric flow caused by a virtual source distribution along the x-axis representing the cavity, and those regular in z^{-1} the reverse case. Full symmetry requires $b_n(a_0, 0, c, d) = 0$.

Let the subscript i, $i = 0,1,\dots$, indicate the asymptotic far-field expansion in z or x of any flow quantity up to i non-trivial terms; here terms exhibiting logarithms are treated fully autonomously. We furthermore express the complete sought expansion (8) by $w \sim z + w^e(z) + w^p(z)$, requiring $w^p = o(w^e)$ for the particular contribution $w^p(z)$ that accounts for non-trivial values of $y_s^{\pm}(x)$. Then (7), (8) are derived simultaneously term by term (with arbitrary accuracy) from (11) by virtue of the following iteration instruction, based upon expanding (1a,b) about $w = z$ accordingly and initialised by $i = 1$ (and $w_0^p(z) = y_{s,0}(x) = 0$):

(i) from $w_{i+1}(z) = z + w_i^e(z) + w_{i-1}^p(z)$, $z_{s,i}^{\pm} = x + \mathrm{i}y_{s,i-1}^{\pm}(x)$ calculate the updates $y_{s,i}^{\pm}(x) = -\big[\mathrm{Im}\,w_{i+1}(z_{s,i}^{\pm})\big]_i$;
(ii) define $\Delta w_i^p := w_i^p(z) - w_{i-1}^p(z)$, $u_i^p(x,y) := \mathrm{Re}(\mathrm{d}\Delta w_i^p/\mathrm{d}z)$ and calculate (the principal value of) Δw_i^p subject to $u_i^p(x,0_\pm) = -\big[|w_i'(z_{s,i+1}^{\pm})| - 1\big]_i/2$, merely by inspection, and hence the update $w_i^p(z)$;
(iii) increase i by 1 and go to (i).

Three issues deserve to be highlighted: at first, the boundary condition for the perturbation u_i^p of the streamwise velocity component, referring to the i-th term Δw_i^p in the expansion of $w^p(z)$, formulated in step (ii) is due to the trivial zeroth-order approximation of $y_s^{\pm}(x)$; secondly, any non-unique contribution to u_i^p satisfies a homogeneous boundary condition complying with (10) and is thus already absorbed by $w^e(z)$; third, the procedure (i)–(iii) accepts a passive breakdown of the expansions of the left-hand sides of (1a,b) near $z = z_s^{\pm}(x)$ as $x \to \infty$, whereas those of $w(z)$ and $w'(z)$ in the complex argument z are uniformly valid for $0 < \theta < 2\pi$. We skip the details of the manipulations the steps (i)–(iii) involve. Up to $i = 4$, they recover (7) and (8).

We close the analysis of H–K flows by a descriptive prediction of the K-parabola and an associated body drag independent of lift, essentially based upon dimensional arguments. Let x_r denote some sufficiently large reference value and define $[\hat{x},\hat{z}] := [x,z]/x_r$ as $O(1)$-quantities accordingly. Slender-airfoil theory implies

$$(y_s^+ - y_s^-)/x_r \sim \varepsilon_-(x_r)\,\hat{q}(\hat{x})\,, \quad (y_s^+ + y_s^- - 2\nu)/x_r \sim 2\varepsilon_+(x_r)\,(\ln\hat{x} + \ln x_r)\,, \tag{12a,b}$$

cf. (7) with the constant shift ν of the K-parabola in the y-direction, and $w'(z) \sim 1 - \varepsilon_- \mathrm{i}\,\hat{q}'(\hat{z})/2 - \varepsilon_+ \mathrm{i}/\hat{z} + O(\varepsilon_-^2, \varepsilon_+^2, \varepsilon_-\varepsilon_+)$, cf. (8): the function $\hat{q}(\hat{z})$ is initially unknown, $\varepsilon_-\hat{q}'(\hat{x})$ and $2\varepsilon_+/\hat{z}$ are viewed as dominant local source and vortex strengths where the for $x_r \gg 1$ small thickness and camber ratios, ε_- and $\varepsilon_+ = b_0/x_r$, typical of the cavity shape accommodate the symmetric and antisymmetric flow contributions, respectively. Since the drag cannot be affected by the linear terms in the last expansion, see (2a,b), we have $c_x/x_r = O(\varepsilon_-^2)$. Also, c_x must be independent of x_r and, accordingly, the right-hand side of (12a) of $O(x_r^{-1})$. Hence, we obtain $\varepsilon_- = O(x_r^{-1/2})$ and identify $\hat{q}$ with $\hat{z}^{1/2}$, which gives the above prediction.

4 Closed to vanishingly small cavities

In the limit $\gamma \to 0_+$ or, equivalently, $a_0 \to 0$, regions of non-uniformity of the expansions (4) and (7), (8) where $\tau = O(\gamma^{1/2})$ and $x, z = O(a_0^{-1})$, respectively, are detected. There $y_s^+ - y_s^-$ is of $O(a_0^{1/2})$ as the K-parabola is shifted further downstream. It finally disappears when $a_0 = 0$.

We briefly consider the natural extension of H–K flows when the cavity has closed up (dark-shaded in Fig. 1) as the merging point of the free streamlines has in turn moved from infinity to a finite position, $\mathscr{M}$, so that $y_s^- = y_s^+$ on and downstream of $\mathscr{M}$. Without going into the details of the systematic calculation of $w(z)$ for such flows by means of the L-C method, resulting from an accordingly extended form of (3), we stress to be again concerned with a two-parametric family of flows, e.g. governed by the chosen positions of $\mathscr{S}^\pm$. However, the requirement of 2π-periodicity of $w(z)$ in θ for sufficiently large values of r superseding (10) renders the physical flow region double-connected, so that in (8), (11) the half-integral powers of z vanish in favour of non-zero real parts of the coefficients in (11). This is found to form a constraint in the respective τ-plane which determines the the value of $\rho := |w'(z_s^\pm)|\ (<1)$ or, equivalently, the position of $\mathscr{M}$, whereas $a_0 = 0$ and, hence, $c_x = 0$ holds throughout (D'Alembert's paradox).

It is instructive and (computationally advantageous) to consider ρ as varied whereas c_y (measuring the asymmetry of the flow) is kept constant. One finds that l defined by one of the lengths $\overline{\mathscr{S}^+\mathscr{M}}$ and $\overline{\mathscr{S}^-\mathscr{M}}$ depends (uniquely and) continuously on ρ, with $\mathrm{d}l/\mathrm{d}\rho > 0$ and the singular limits $l = \infty$ for $\rho = 1$ and $l = \rho = 0$. In the latter case, $\mathscr{M}$ and $\mathscr{S}^\pm$ collapse to finally represent a rear stagnation point, which recovers the situation of a fully attached flow. For $l \to \infty$, the expansions (7), (8), (11) cease to be valid where $r = O(l)$. The description of $w(z)$ on this scale elucidates the cuspidal trailing edge of the cavity, located at $x = x_{\mathscr{M}}$, say, and includes the aforementioned replacements of (8) and (11) as $r/l \to \infty$. The main result $y_s^+ - y_s^- = O\big((x_{\mathscr{M}} - x)^{3/2}\big)$ as $x_{\mathscr{M}} - x \to 0_+$ is also inferred directly from the singular form $z_s^\pm(x)$ has immediately downstream of $\mathscr{S}^\pm$.

5 Further outlook: from laminar to turbulent asymmetric high-Reynolds-number flows

A comprehensive numerical investigation of closed-cavity flows, by systematically varying ρ and c_y and focussing on non-uniqueness, is desirable. Also, the generalisation of the current basic analysis towards the parametrisation and topology of the far field given an arrangement of several obstacles, i.e. holes in the z-plane, would be of interest. Recently, Crowdy [12] studied full solutions to such problems in a noteworthy manner. The lasting importance a most comprehensive analysis of potential flows with free streamlines has for the – yet incomplete – understanding of their "real" counterparts deserves the following digression.

In a specific sense, such potential flows represent the limiting states of viscous ones once the Reynolds number $Re := \tilde{U}\tilde{R}/\tilde{\nu}$ with $\tilde{\nu}$ denoting the now non-vanishing uniform kinematic viscosity of the fluid assumes arbitrarily large values. The corresponding real viscous flows then are parametrised by Re, where c_x, c_y are functions of Re. Currently, a complete flow description in the limit $Re \to \infty$ exists only for a steady flow about a single obstacle that generates laminar boundary layers along $\mathscr{C}^o$. For an overview of this well-established asymptotic theory we refer to Sychev et al. [13] and the associated references therein. Here the B–V condition of an asymptotically weak separation singularity as $Re \to \infty$ sorts out the limiting potential flow, i.e. the positions of $\mathscr{S}^{\pm}$. The K-parabola provides the transition of the flow on the body scale towards the reversed-flow eddy and the wake further downstream. The first is seen to take place where x scales with Re as it is generally associated with a finite limiting value of c_x, which, however, differs from that found by potential-flow theory. That transition can be accomplished by a direct match where the eddy is found as slender, having an extent in the x- and y-directions of, respectively, $O(Re)$ and $O(Re^{1/2})$, where to leading order again the requirement of vanishing velocity perturbations along the dividing streamlines applies. In the spirit of Van Dyke [11], the K-parabola then forms the upstream apex of an approximatively elliptic eddy. However, such a slender eddy was found to be incompatible with a self-consistent picture of a globally steady flow by Sychev [14]. Consequently, this possibility is ruled out in favour of a more complex transition process over a region where $x = O(Re^{1/2})$, $y = O(Re^{1/4})$ that finally matches the leading-edge cusp of a Saddovski vortex with its diameter measured by Re. As finally shown by Chernyshenko [15], consistency of this asymptotic structure of a thick backflow eddy as $Re \to \infty$ singles out the particular member of the one-parametric class of such eddies that implies a c_x-value vanishing at a rate of $O(Re^{-1})$.

In connection with this theory, three topics lacking appreciation so far deserve attention as follows.

(A) A higher-order theory that takes into account even the asymmetric distortions of the K-parabola due to the logarithmic terms in (7) and (8) might aim at a correct prediction also of c_y and its possible deviation of the potential-flow limit, stated in (7).

(B) For an obstacle symmetric with respect to a parallel to the x-axis, such an asymmetry of the potential flow lacks a sound physical background so far except for the case of a perfectly symmetric obstacle, i.e. a circular cylinder, where steadiness of the potential flow is preserved even for a non-zero time-independent angular speed about its axis, $\tilde{\Omega}$. Then the viscous flow and thus c_x and c_y are parametrised not only by Re but also the Strouhal number $St := \tilde{\Omega}\tilde{R}/\tilde{U}$. In the limit $Re \to \infty$, the intriguing question arises whether and how, at least for asymptotically small values of St, the symmetric steady-flow structure ($c_y = 0$) can be perturbed such that the disturbed potential flow complies with the boundary layer undergoing separation. A prediction of accordingly small lift (and drag) coefficients in a suitably distinguished double limit $Re \to \infty$, $St \to 0$ would provide a rigorous and also quantitative clarification of the famous Magnus effect for stationary flow. Here the thorough analysis of the unsteady (boundary layer) flow (about a cylinder having impulsively initiated constant spin), see Degani et al. [16] and the references therein, is definitely of interest.

(C) Attempts to describe the last effect as well as grossly separated flow as steady-state phenomena, in the latter case rather surprisingly involving zero body drag [15], are often doubted to be reasonable if one takes a more physically motivated point of view, cf. [13], p. 255. As pointed out there and taken up recently by Sychev [17, 18], allowing the flow to become turbulent at the onset of the wake that emanates from the recirculation region might finally give preference to a slender separated-flow structure, still at a distance $x = O(Re)$ relevant for purely laminar flow. Also, the status quo of the asymptotic theory of blunt-body separation that stipulates an already turbulent boundary layer and its empirical support, see [10, 9], Neish and Smith [19], and the survey by Smith et al. [20], tend to support a slender eddy in the time-mean flow, at least qualitatively. However, here the positions of $\mathscr{S}^{\pm}$, thus the actual downstream structure of the flow on the body scale, and the associated crucial question whether the drag remains finite in the high-Re-limit still pose a challenge. Hence, this can be mastered comprehensively only once it is clarified whether the potential flow admits the H–K form or the cavity closes up. Settling these issues unambiguously also underpins the importance of a most complete understanding of such flows, in particular, either remote from the obstacle or in the vicinity of the merging point $\mathscr{M}$.

Investigations in the directions of research put forward by (A)–(C) are under way.

Acknowledgements This research was funded by the Austrian Research Promotion Agency (FFG) within the Austrian Competence Centers for Excellent Technologies (COMET) K2 Programme (FFG grant no.: 824187, project acronym: *XTribology*).

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