A relevant difference between (9) and least squares formulations is that the former can be used to *prove* k-explicit stability bounds on u, while the seconds *requires* these bounds to be well-posed.

We note that, using an appropriate operator $T : \mathcal{V} \to \mathcal{V}$, any well-posed formulation in the form (2) can be translated in a sign-definite one: $a_T(u, v) := a(u, Tv) = F(Tv) =: F_T(v)$. However, the operator T is often not explicit or its approximation by a Galerkin scheme requires some strict assumptions on the discretisation (see [2]).

References

[1] A. Buffa and P. Monk, *Error estimates for the ultra weak variational formulation of the Helmholtz equation*, M2AN, Math. Model. Numer. Anal., 42 (2008), pp. 925–940.

[2] P. Ciarlet Jr., *T-coercivity: Application to the discretization of Helmholtz-like problems*, Comput. Math. Appl., 64 (2012), pp. 22–34.

[3] P. Cummings and X. Feng, *Sharp regularity coefficient estimates for complex-valued acoustic and elastic Helmholtz equations*, Math. Models Methods Appl. Sci., 16 (2006), pp. 139–160.

[4] U. Hetmaniuk, *Stability estimates for a class of Helmholtz problems*, Commun. Math. Sci., 5 (2007), pp. 665–678.

[5] R. Hiptmair, A. Moiola, and I. Perugia, *Plane wave discontinuous Galerkin methods for the 2D Helmholtz equation: analysis of the p-version*, SIAM J. Numer. Anal., 49 (2011), pp. 264–284.

[6] J. M. Melenk, *On Generalized Finite Element Methods*, PhD thesis, University of Maryland, 1995.

[7] A. Moiola and E. A. Spence, *Is the Helmholtz equation really sign indefinite?*, Tech. Report Preprint Series MPS-2012-23, Department of Mathematics and Statistics, University of Reading, 2012. http://www.reading.ac.uk/maths-and-stats/research/maths-preprints.aspx

[8] C. S. Morawetz, *Notes on time decay and scattering for some hyperbolic problems*, Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1975.

[9] F. Rellich, *Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral*, Math. Z., 46 (1940), pp. 635–636.

[10] E.A. Spence, I.V. Kamotski, and V.P. Smyshlyaev, *Coercivity of combined boundary integral equations in high frequency scattering*. In preparation, 2013.

[11] E. A. Spence, S. N. Chandler-Wilde, I. G. Graham, and V. P. Smyshlyaev, *A new frequency-uniform coercive boundary integral equation for acoustic scattering*, Comm. Pure Appl. Math., 64 (2011), pp. 1384–1415.

Hardy space method for exterior Maxwell problems

Lothar Nannen

(joint work with Thorsten Hohage, Achim Schädle, Joachim Schöberl)

We consider scattering and resonance problems on connected, unbounded domains $\Omega \subset \mathbb{R}^3$, which are complements of compact sets. Scattering or source problems for the time-harmonic Maxwell's equations consist in finding an outgoing electric field $\mathbf{u} \in H_{\mathrm{loc}}(\mathbf{curl}; \Omega)$ satisfying

$$\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} - \omega^2 \varepsilon\, \mathbf{u} \cdot \mathbf{v}\, dx = l(\mathbf{v}) \qquad \text{for all } \mathbf{v} \in H_{\mathrm{c}}(\mathbf{curl}; \Omega). \tag{1}$$

for a given frequency $\omega > 0$, the local permittivity ε and a linear form $l(\mathbf{v}) = \int_\Omega \mathbf{g}\cdot\mathbf{v}\,dx + \int_{\partial\Omega} \mathbf{g}_{\partial\Omega}\cdot\mathbf{v}\,dx$ with a compactly supported source term $\mathbf{g}$ and boundary data $\mathbf{g}_{\partial\Omega}$. Here $H_{\mathrm{c}}(\mathbf{curl};\Omega)$ denotes the space of all vector fields $\mathbf{v}$ which are compactly supported in $\overline{\Omega}$ and are square integrable together with the curl $\mathbf{curl}\,\mathbf{v}$. $H_{\mathrm{loc}}(\mathbf{curl};\Omega)$ denotes the space of vector fields $\mathbf{v}$, which are square integrable on any compact subset $K \subset \overline{\Omega}$ together with $\mathbf{curl}\,\mathbf{v}$.

The radiation condition defining the term "outgoing" is typically formulated as Silver-Müller radiation condition. Then it is well-known that problem (1) is well-posed (see e.g. [1]). For $\omega > 0$ there exist other radiation conditions, which are equivalent for solutions $\mathbf{u}$ to (1):

(i) A series representation $\mathbf{u}$ in terms of Hankel functions of the first kind (e.g. [1]),
(ii) a boundary integral representation of $\mathbf{u}$ (e.g. [1]),
(iii) the condition that a holomorphic extension of $\mathbf{u}$ with respect to the radial variable (e.g. a complex scaling) is exponentially decreasing (e.g. [2, 3]) and
(iv) the so called pole condition, which characterizes outgoing solutions via the singularities of their Laplace transformed functions ([4] for Helmholtz problems and [5] for Maxwell problems).

Based on these radiation conditions there are several numerical methods to solve (1), e.g. classical infinite element methods [6], non-reflecting boundary conditions [7], boundary integral approaches [8], local high order approximations [9], complex scaling methods (known as perfectly matched layer methods) [10, 11, 12, 3] and Hardy space infinite elements [13, 14, 5].

Except for the two latter these methods depend non-linearly on the frequency ω, since this is the case for the radiation conditions (i) and (ii) on which they are based. This is a severe drawback for resonance problems, where we are looking for eigenpairs $(\mathbf{u},\omega^2)$ consisting of an outgoing resonance function $\mathbf{u} \in H_{\mathrm{loc}}(\mathbf{curl};\Omega)\setminus\{0\}$ and the square of a resonance $\omega \in \mathbb{C}$ such that

$$\int_\Omega \mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v} = \int_\Omega \omega^2\varepsilon\,\mathbf{u}\cdot\mathbf{v}\,dx \qquad \text{for all } \mathbf{v} \in H_{\mathrm{c}}(\mathbf{curl};\Omega). \tag{2}$$

The radiation conditions (iii) and (iv) are independent of the frequency ω. Therefore methods based on these radiation conditions, namely complex scaling methods and the Hardy space infinite element method, can be constructed such that they lead to a generalized matrix eigenvalue problem of the form

$$S\underline{\mathbf{u}} = \omega^2 M_\varepsilon \underline{\mathbf{u}} \tag{3}$$

with complex symmetric, non-hermitian matrices S and M_ε. This problem can be solved by a standard shift-and-invert Arnoldi method. Although it is possible to solve the non-linear eigenvalue problem resulting e.g. from a boundary element method ([15]), it is desirable to avoid it.

Therefore, currently complex scaling methods based on (iii) are the standard methods for solving resonance problems (see e.g. [16, 17]). Usually, due to the resulting exponential decay of the solution, the unbounded domain is truncated

to a bounded domain consisting of the computational domain and a perfectly matched layer with the artificial, anisotropic damping. The method can be easily implemented in standard finite element codes, since only the bilinear forms have to be changed.

Unfortunately, these methods give rise to spurious resonance modes. It is shown in [17], that the spurious resonance modes arise from a discretization of an essential spectrum. Moreover, several parameter of the complex scaling method like the type of scaling, the thickness of the layer and the underlying finite element method have to be optimized for each specific problem.

The Hardy space infinite element method also leads to spurious resonance modes, but less parameters have to be chosen by hand. Moreover, numerical tests indicate a super-algebraic convergence with respect to the number of degrees of freedom in radial direction. On the other hand, the method is a tensor product method of standard finite element basis functions with special infinite basis functions in the Hardy space of the complex unit disk. Therefore, a non-standard infinite element has to be included in a finite element code.

The numerical tests in [5] for Maxwell problems were made with the open source finite element package Netgen/NGSolve from Joachim Schöberl together with the open source module ngs-waves containing the routines for the infinite elements. Numerical tests comparing a complex scaling method with the Hardy space infinite element method can be found for Helmholtz problems in [14]. They indicate, that the Hardy space infinite element method needs less computational effort than the complex scaling method.

References

[1] D. Colton, R. Kress, *Inverse acoustic and electromagnetic scattering theory*, vol. 93 of Applied Mathematical Sciences, Springer-Verlag, Berlin, second ed., 1998.

[2] P. D. Hislop, I. M. Sigal, *Introduction to spectral theory*, vol. 113 of Applied Mathematical Sciences, Springer-Verlag, Berlin, second ed., 1996.

[3] F. Collino, P. Monk, *The perfectly matched layer in curvilinear coordinates*, SIAM J. Sci. Comput., 19 (1998), pp. 2061–2090.

[4] T. Hohage, F. Schmidt, L. Zschiedrich, *Solving time-harmonic scattering problems based on the pole condition. I. Theory*, SIAM J. Math. Anal., 35 (2003), 183–210.

[5] L. Nannen, T. Hohage, A. Schädle, J. Schöberl *Exact sequences of high order Hardy space infinite elements for exterior Maxwell problems*, accepted for publication in SIAM J. Sci. Comput. (2013).

[6] L. Demkowicz, M. Pal, *An infinite element for Maxwell's equations*, Comput. Methods Appl. Mech. Engrg., 164 (1998), pp. 77–94.

[7] M. J. Grote, J. B. Keller, *Nonreflecting boundary conditions for Maxwell's equation*, J. Comput. Phys., 139 (1998), pp. 327–342.

[8] A. Buffa, M. Costabel,C. Schwab, *Boundary element methods for Maxwell's equations on non-smooth domains*, Numer. Math., 92 (2002), pp. 679–710.

[9] D. Givoli, *High-order local non-reflecting boundary conditions: a review*, Wave Motion, 39 (2004), pp. 319–326.

[10] B. Simon, *The definition of molecular resonance curves by the method of exterior complex scaling*, Phys. Lett. A, 71A (1979).

[11] J.-P. Berenger, *A perfectly matched layer for the absorption of electromagnetic waves*, J. Comput. Phys., 114 (1994), pp. 185–200.

[12] W. C. Chew, W. H. Weedon, *A 3d perfectly matched medium from modified Maxwell's equations with stretched coordinates*, Microwave Optical Tech. Letters, 7 (1994), pp. 590–604.
[13] T. Hohage, L. Nannen, *Hardy space infinite elements for scattering and resonance problems*, SIAM J. Numer. Anal., 47 (2009), 972–996.
[14] L. Nannen, A. Schädle, *Hardy space infinite elements for Helmholtz-type problems with unbounded inhomogeneities*, Wave Motion, 48 (2010), pp. 116–129.
[15] O. Steinbach, G. Unger, *Convergence analysis of a Galerkin boundary element method for the Dirichlet Laplacian eigenvalue problem*, SIAM J. Numer. Anal. 50 (2012), pp. 710–728.
[16] S. Hein, T. Hohage, W. Koch, and J. Schöberl, *m Acoustic resonances in high lift configuration*, J. Fluid Mech., 582 (2007), pp. 179–202.
[17] S. Kim, J. E. Pasciak, *The computation of resonances in open systems using a perfectly matched layer*, Math. Comp., 78 (2009), pp. 1375–1398.

A parallel space-time multigrid method

Martin Neumüller
(joint work with Olaf Steinbach)

As a model problem we consider the heat equation in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ with boundary $\Gamma := \partial\Omega$ and a simulation interval $[0, T]$,

$$\begin{aligned} \partial_t u(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) &= f(\boldsymbol{x}, t) && \text{for } (\boldsymbol{x}, t) \in Q := \Omega \times (0, T), \\ u(\boldsymbol{x}, t) &= 0 && \text{for } (\boldsymbol{x}, t) \in \Sigma := \Gamma \times (0, T), \\ u(\boldsymbol{x}, 0) &= u_0(\boldsymbol{x}) && \text{for } (\boldsymbol{x}, t) \in \Sigma_0 := \Omega \times \{0\}. \end{aligned} \tag{1}$$

Subdividing the simulation interval $[0, T]$ in subintervals

$$0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T, \quad \text{with } t_n = n\,\tau \text{ and } \tau = \frac{T}{N},$$

and using a standard finite element discretization in space and a discontinuous Galerkin approximation in time, this leads to the linear algebraic equations

$$[K_\tau \otimes M_h + M_\tau \otimes K_h]\, \boldsymbol{u}^{n+1} = \boldsymbol{f}^{n+1} + N_\tau \otimes M_h \boldsymbol{u}^n. \tag{2}$$

Here, M_h is the standard mass matrix and K_h is the standard stiffness matrix

$$M_h[i, j] := \int_\Omega \varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \quad K_h[i, j] := \int_\Omega \nabla\varphi_j(\boldsymbol{x}) \cdot \nabla\varphi_i(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

for $i, j = 1, \ldots, N_x$. The matrices with respect to the time discretization, where a discontinuous Galerkin approximation is used, are given by

$$\begin{aligned} K_\tau[k, \ell] &:= -\int_0^\tau \psi_\ell(t)\partial_t\psi_k(t) \mathrm{d}t + \psi_\ell(\tau)\psi_k(\tau), \\ M_\tau[k, \ell] &:= \int_0^\tau \psi_\ell(t)\psi_k(t) \mathrm{d}t, \quad N_\tau[k, \ell] := \psi_\ell(\tau)\psi_k(0) \end{aligned}$$

for $k, \ell = 1, \ldots, N_t$. Moreover, the right hand side is given by

$$\boldsymbol{f}^{n+1}[\ell N_x + j] := \int_{t_n}^{t_{n+1}} \int_\Omega f(\boldsymbol{x}, t)\varphi_j(\boldsymbol{x})\psi_\ell(t) \mathrm{d}\boldsymbol{x} \mathrm{d}t$$