11^{th} International Conference on Vibration Problems
Z. Dimitrovová *et.al.* (eds.)
Lisbon, Portugal, 9–12 September 2013

STABILIZATION OF THE RADIAL POSITION OF A TETHERED SATELLITE SYSTEM

Alois Steindl*[1]

[1]University of Technology, Vienna, Austria
Alois.Steindl@tuwien.ac.at

Keywords: Optimal control theory, Feedback stabilization, Tension control, Satellite dynamics.

Abstract. *We investigate the stabilization of the radial position of a tethered satellite by tension control with respect to in-plane and out-of-plane perturbations.*

In previous investigations ([1, 2]) we considered the deployment and retrieval of a satellite from the main station, assuming that the motion of the satellite is restricted to the orbital plane. Now we ask, whether it is possible to stabilize the stationary radial position for small deviations from this configuration in arbitrary directions by applying a tension force on the tether.

Due to the rotation of the system around the central mass the dynamics in the plane and transversally to it are significantly different: For the in-plane motion the variation of the tethers's length acts like an external force, and the motion can be extinguished in finite time. For the pure out-of-plane perturbation the length change rate acts as parametrical excitation. By pulling the tether periodically with the proper phase, this motion can only be controlled to decay algebraically. Even for this rather simple sub-problem we need to apply nonlinear bifurcation theory to obtain the center and stable manifolds, because due to the parametrical input, the linearization at the stationary solution fails to describe the dynamics properly. In a first attempt to find a control law to diminish both types of oscillations we use the length change rate as control parameter and apply Optimal Control theory to obtain the feedback law. Since the length change rate might not be a valid control variable, because it is impossible to push the tether and also the tension in the tether could be too large, in an improved model we use the tension as control variable with proper constraints and regard the length and its change rate as additional state variables. The expected solutions are obtained numerically ([3]) by solving a boundary value problem for the Hamiltonian differential equations with intermediate switching conditions, derived by Pontryagin's Maximum Principle.

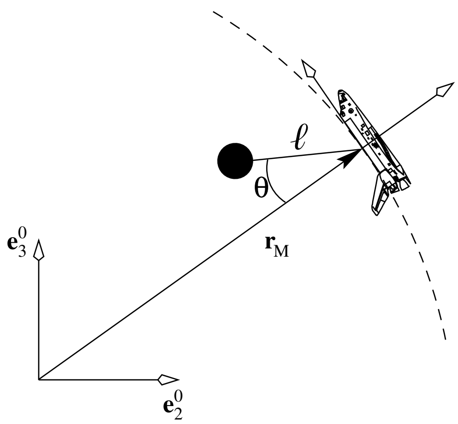

Figure 1: Simplified model of the tethered satellite. The satellite may swing in the orbital plane (θ) and also out of the plane (ψ).

1 INTRODUCTION

Tethered satellites are an already well established concept in space exploration. A satellite is connected to a space station on an Keplerian orbit by an extremely light and flexible cable. It can be used for example to perform measurements in the atmosphere, while the main station rotates on a significantly higher level to save energy.

Important tasks in the operation of these tethered satellites are the deployment from the space station and the retrieval back to the main station. Especially the retrieval process has to be carried out very cautiously, because the satellite could hit the space station and damage it seriously. Unless the satellite is equipped with some position control, it can only be steered by pulling the tether.

Another similar task is the stabilization of the straight downhanging relative equilibrium with respect to disturbances in the orbital plane and out of the plane. Especially during the deployment or retrieval process the satellite will dominantly oscillate in the orbital plane, but also disturbances out of the plane have to be taken into account.

In this talk we investigate this stabilization problem und the assumption, that the applied tension in the tether serves as control variable for both types of oscillations. While the stabilization for deviations in the orbital plane works very efficiently – usually small perturbations can be corrected in finite time –, the out-of-plane control takes infinitely long and the oscillations decay either exponentially with a small decay rate or even only algebraically. We formulate several Optimal Control problems to steer the satellite from a nearby initial position to the local vertical position, which is a relative equilibrium state for the tethered satellite. Since the problem for the full system leads to overly complex expressions, we will consider some subproblems, which help to comprehend the overall solution structure of the combined system.

In the mathematical treatment the linear dynamics of the out-of-plane oscillations, which can be described by an non-semisimple $1 : -1$ resonance, also known as "Hamiltonian Hopf" bifurcation, plays a central role. The nonlinear terms in this bifurcation problem govern the decay or growth of the out-of-plane oscillations. The linearized system for the in-plane oscillations and for the length of the tether shows hyperbolic behaviour: Along the stable directions these motions decay rapidly. Taking into account the nonlinear coupling between these subsystems, we have to determine at least the leading approximations for the Centre Manifold and the contributions of these entries to the slow dynamics on the Centre Manifold.

The optimal trajectories are also computed numerically by solving a 2-point boundary value problem on a long interval by an efficient shooting algorithm. Since we are mainly interested in the behaviour of the system for infinite planning horizon, for the numerical approach we state asymptotic boundary conditions at the end of the finite integration interval to ensure the decay of the perturbations. Also for these asymptotic boundary conditions we need the leading approximations for the invariant manifolds.

2 Mechanical Model and Optimal Control problem

For our mechanical model we state the following assumptions:

- The satellite is considered as a point mass, which is connected by an inextensible, massless tether with the main station.
- The main station moves along a circular Keplerian orbit; it's motion is not affected by the satellite.
- The length ℓ of the tether is much smaller than the orbital radius r_M of the main station. For the equations of motion we retain only the leading terms in ℓ/r_M. The motions of the statellite are controlled by varying the tether length.
- We consider only small deviations of the tether's position from the local vertical direction, also the variation of ℓ should remain small.

In dimensionless form the simplified equations of motion read

$$\ddot{\psi} = -4\psi - \frac{2\dot{\ell}}{\ell}\dot{\psi}, \tag{1}$$

$$\ddot{\theta} = -3\theta - \frac{2\dot{\ell}}{\ell}, \tag{2}$$

$$\ddot{\ell} = 3\ell - u. \tag{3}$$

Here ψ and θ are the angles of the out-of-plane and in-plane positions, respectively. The scaled length of the tether is denoted by ℓ, the control variable u denotes the tension force on the tether. Derivatives w.r.t. the scaled time t are denoted by $\dot{(\cdot)}$. All nonlinearities have been neglected, except the parametric action of the length rate change on the out-of-plane dynamics.

The hierarchical structure of these equations – the angular dynamics is driven by the term $2\dot{\ell}/\ell$, the length ℓ is controlled by the control force u – suggests to consider $v = 2\dot{\ell}/\ell$ as control variable in a simplified problem. This might of course yield a mechanically impossible solution, because u could possibly leave its allowed domain. Nevertheless we investigate this simplified problem first, because it nicely displays the solution structure. If we would disregard the variable ℓ at all, the length of the tether during the control process could deviate too far from its nominal value. Therefore we introduce the variable

$$w = 2\log\ell \quad \Rightarrow \quad \dot{w} = 2\dot{\ell}/\ell = v$$

and state the boundary conditions

$$w(0) = 0, \qquad \lim_{t\to\infty} = 0. \tag{4}$$

2.1 Simplified Optimal Control problem

We look for an optimal control $v(t)$, which minimizes the cost function

$$I = \int_0^\infty \frac{\psi_1^2 + \psi_2^2 + \theta_1^2 + \theta_2^2 + w^2 + v^2}{2} dt \tag{5}$$

subject to the first order system of differential equations

$$\dot{\psi}_1 = 2\psi_2, \tag{6}$$
$$\dot{\psi}_2 = -2\psi_1 - v\psi_2, \tag{7}$$
$$\dot{\theta}_1 = \omega\theta_2, \tag{8}$$
$$\dot{\theta}_2 = -\omega\theta_1 - v/\omega, \tag{9}$$
$$\dot{w} = v, \tag{10}$$

with $\omega = \sqrt{3}$.

In order to obtain a more efficient control strategy, one could vary the weights in the cost function (5), but we would like to derive simple expressions to illustrate the control structure. Especially it seems unnecessary to include the cost term $v^2/2$ in (5), but it helps to generate converging trajectories. If this term were not present in the simple model, one would have to introduce bounds for the control variable and investigate bang-bang solutions.

Applying Pontryagin's maximum principle [4] we build the Hamiltonian

$$\begin{aligned} H = &-\frac{\psi_1^2 + \psi_2^2 + \theta_1^2 + \theta_2^2 + w^2 + v^2}{2} + 2p_1\psi_2 - p_2(2\psi_1 + v\psi_2) \\ &+ \omega p_3\theta_2 - p_4(\omega\theta_1 + v/\omega) + p_5 v \end{aligned} \tag{11}$$

and calculate the optimal control

$$v^\star = \arg\max_v H = p_5 - p_4/\omega - p_2\psi_2. \tag{12}$$

The adjoint variables p_i satisfy the differential equations

$$\dot{p}_1 = -\frac{\partial H}{\partial \psi_1} = \psi_1 + 2p_2, \tag{13}$$
$$\dot{p}_2 = -\frac{\partial H}{\partial \psi_2} = \psi_2 - 2p_1 + p_2 v^\star, \tag{14}$$
$$\dot{p}_3 = -\frac{\partial H}{\partial \theta_1} = \theta_1 + \omega p_4, \tag{15}$$
$$\dot{p}_4 = -\frac{\partial H}{\partial \theta_2} = \theta_2 - \omega p_3, \tag{16}$$
$$\dot{p}_5 = -\frac{\partial H}{\partial w} = w. \tag{17}$$

At $t = 0$ we state the initial conditions

$$\begin{aligned} &\psi_1(0) = \psi_0, &&\theta_1(0) = \theta_0, &&w(0) = 0, \\ &\psi_2(0) = \dot{\psi}_0/2, &&\theta_2(0) = \dot{\theta}_0/2, && \end{aligned}$$

and for $t \to \infty$ we require

$$\lim_{t\to\infty} \|(\psi_1, \psi_2, \theta_1, \theta_2, w)\| = 0. \tag{18}$$

Apparently the trivial state $\boldsymbol{q} = (\psi_1, \psi_2, \theta_1, \theta_2, w) \equiv 0$, $\boldsymbol{p} \equiv 0$ solves the equations (6-10) and (13-17). With the reordered phase state vector

$$\boldsymbol{y} = (\psi_1, \psi_2, p_1, p_2, \theta_1, \theta_2, p_3, p_4, w, p_5)^T$$

the Jacobian at the trivial solution reads

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_c & 0 \\ 0 & \mathbf{A}_h \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega & 0 & 0 & 1/\omega^2 & 0 & -1/\omega \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/\omega & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{19}$$

It has a blockdiagonal structure, with the entries for the out-of-plane variables $\boldsymbol{y}_c = (\psi_1, \psi_2, p_1, p_2)^T$ decoupled from the remaining system.

2.1.1 Hamiltonian Hopf bifurcation

The linear dynamics for the variables $\boldsymbol{y}_c$ is governed by the matrix $\mathbf{A}_c$, which is already in Jordan Normal Form and has a non-semisimple pair of imaginary eigenvalues $\pm 2i$. Since the control variable u enters nonlinearly, the linear part just describes the uncontrolled out-of-plane oscillation. The influence of the nonlinear terms can be investigated by the bifurcation theory for the Hamiltonian Hopf bifurcation ([5]): This bifurcation occurs in Hamiltonian systems, when two pairs of imaginary eigenvalues coincide and split off the imaginary axis. In our case there doesn't exist a corresponding bifurcation parameter, therefore we simply have to investigate the dynamics directly at the bifurcation point.

In [5] a new set of variables is introduced,

$$X = \frac{\psi_1^2 + \psi_2^2}{2}, \qquad Y = \frac{p_1^2 + p_2^2}{2},$$
$$Z = p_1\psi_1 + p_2\psi_2, \qquad S = q_2 p_1 - q_1 p_2,$$

which are the invariants of the semi-simple part of $\mathbf{A}_c$ and satisfy the relation

$$S^2 + Z^2 = 4XY. \tag{20}$$

By using Normal Form theory it is possible to simplify the Hamiltonian for the out-of-plane subsystem to

$$H = -X + 2S + aZ^2, \tag{21}$$

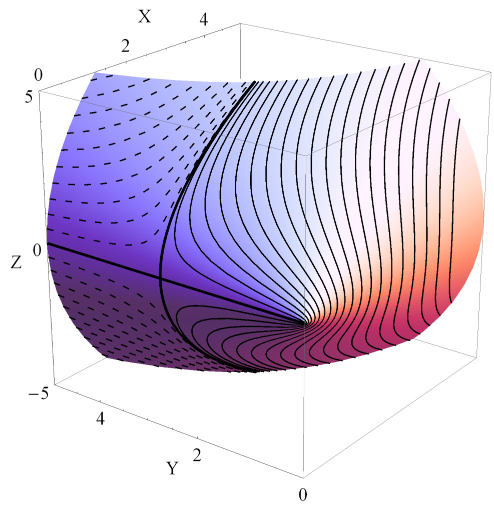

Figure 2: Trajectories of the flow generated by the Hamiltonian (22) on the cone $Z^2 = 4XY$. The thick lines indicate the invariant manifolds of the nontrivial equilibrium $(X, Y, Z) = (0, 1/(4a), 0)$.

where the coefficient a depends on the nonlinearities of the original system. In [5] it is shown, that also the quartic Z^2 could be eliminated and only a term bY^2 survives the transformation to Normal Form, but in our case b would be zero, because all nonlinear terms have a factor ψ_2. Therefore we keep the term aZ^2.

Since S is a first integral of the Hamiltonian (21) and we are interested in decaying oscillations, the candidates for optimal solutions have to lie on the invariant set $S = 0$, which according to (20) is a cone in (X, Y, Z)-space. With $S = 0$ and (20) the Hamiltonian (21) can be simplified further to

$$H = -X(1 - 4aY). \tag{22}$$

As can be seen in Figure 2, there exists a nontrivial relative equilibrium of the reduced system at the point $(X, Y, Z) = (0, 1/(4a), 0)$. It corresponds to a steady rotation of the adjoint variables. Along the parabolic $H = 0$ isocline through this point

$$Y = 1/(4a), \quad X = aZ^2$$

the trajectories converge to the saddle point for $Z < 0$ or deviate from it for $Z > 0$.

If we would only try to reduce the out-of-plane oscillations, without taking care of the in-plane motions or the length restriction (4), we could already determine the optimal control law: For a given initial state (ψ_1, ψ_2) we determine (p_1, p_2) from the conditions $4aY = 1$, $S = 0$ and $Z < 0$ and set $v = -p_2\psi_2$. For this restricted case the constant a equals $3/16$. The decay of the solution to the vertical position is of course very slow, due to the purely imaginary eigenvalues of the linearized subsystem.

2.1.2 Simultaneous control of the oscillations

The submatrix $\mathbf{A}_h$ in (19) is hyperbolic, it has a pair of real eigenvalues $\pm\lambda$ and a quadruple of complex eigenvalues $\pm\mu_R \pm \mu_I i$. Therefore the planar subsystem could be stabilized easily, because it has a 3-dimensional stable manifold. For every state vector (θ_1, θ_2, w) close to zero we could choose (p_3, p_4, p_5) such, that the trajectory decays exponentially.

For the simultaneous control we have to expect the following behaviour: The variables $\boldsymbol{y}_h = (\theta_1, \theta_2, w, p_3, p_4, p_5)^T$ decay quickly to the Center Manifold of the system, on which the slow decay to the origin takes place.

In our problem the Center manifold can be computed explicitely using Mathematica ([6]):

First we observe, that in the expression (12) for the control variable there is a quadratic contribution $-p_2 v_2$ from the out-of-plane variables $\boldsymbol{y}_c$. Therefore we assume, that the Centre Manifold starts at second order and use the ansatz

$$\boldsymbol{y}_h = \boldsymbol{h}(\boldsymbol{y}_c), \tag{23}$$

where the components of h_i are quadratic functions of $\boldsymbol{y}_c$. The holonomic equation for $\boldsymbol{h}$

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{y}_c} \mathbf{A}_c \boldsymbol{y}_c - \mathbf{A}_h \boldsymbol{h} = \boldsymbol{b}, \tag{24}$$

with

$$\boldsymbol{b} = (0, p_2\psi_2/\omega, -p_2\psi_2, 0, 0, 0)^T$$

is solved by comparing coefficients and yields the following entries

$$p_3 = \frac{-192(p_2\psi_1 + p_1\psi_2)}{8923} + \frac{1584336(\psi_1^2 - \psi_2^2)}{79619929}, \tag{25}$$

$$p_4 = \frac{152\sqrt{3}(p_1\psi_1 - p_2\psi_2)}{8923} + \frac{2044536\sqrt{3}\psi_1\psi_2}{79619929}, \tag{26}$$

$$p_5 = \frac{4208 p_1\psi_1 + 4715 p_2\psi_2}{8923} - \frac{2008656\psi_1\psi_2}{79619929} \tag{27}$$

for the adjoint variables and similar expressions for the state variables.

This result is now used in two ways: First the expressions for the variables p_4 and p_5 are inserted into the Hamiltonian for the out-of-plane system and give a new value for the coefficient a of Z^2 in (21). We obtain the modified value

$$a = \frac{3}{16} + \frac{18657}{71384}.$$

Second we use the expressions to specify proper boundary conditions for the numerical procedure: At the end of the integration interval the trajectories will be close to the Center Manifold, but still not very small. Therefore we require, that the values of $p_3 \ldots p_5$ assume the values given in (25-27) at the endpoint.

A typical phase portrait for the in-plane oscillations is shown in Figure 3: Starting with a small out-of-plane perturbation, the in-plane motion become quite large and slowly decay along the Center Manifold.

Conclusions

In order to solve the simultaneous Optimal Control Problem for in-plane and out-of-plane perturbations, we found the nonlinear behaviour of the out-of-plane dynamics, which is governed by a Hamiltonian Hopf scenario. We also determined the Center Manifold approximation for the full system, which determines the convergence of the system to the desired vertical position. This way we have confirmed, that the stabilization of the out-of-plane oscillation is possible, although it works on a very large time scale.

The calculations for the original problem, where the tension force is used as control variable, proceeds in the same way and has already been carried out.

In order to accelerate the decay, one shouldn't include the cost of the control effort in the cost function, but then we would obtain a bang-bang type solution, for which we would have to apply symplectic maps to study the nonlinear interaction and convergence behaviour.

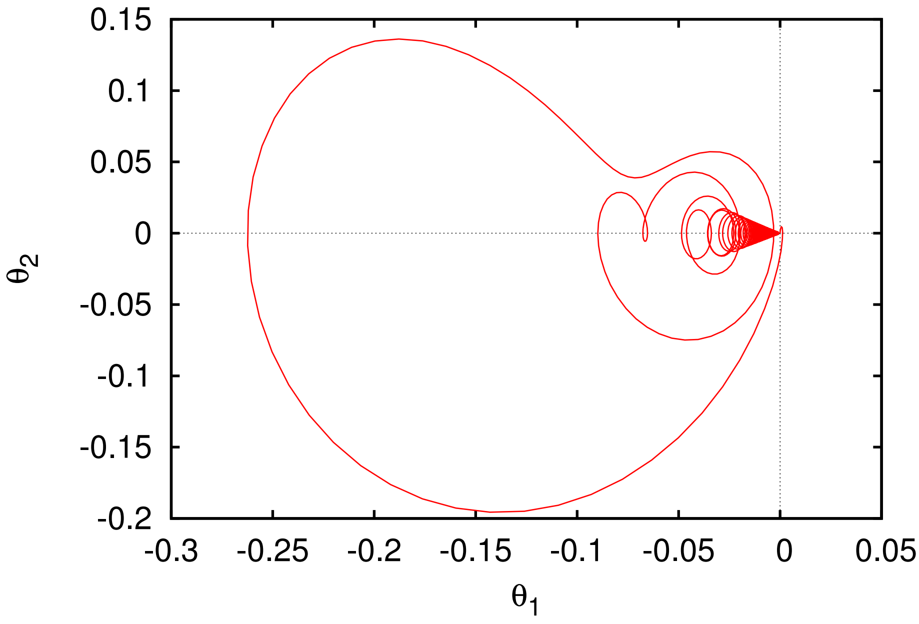

Figure 3: Phase plot of the in-plane oscillations.

Acknowledgement

This research project has been supported by "Österreichischer Austauschdienst" (OeAD), under the project UA12/2011.

REFERENCES

[1] A. Steindl, H. Troger: *Optimal Control of Deployment of a Tethered Subsatellite*; Nonlinear Dynamics, **31** (2003), 257 – 274.

[2] B. Barkow, A. Steindl, H. Troger, G. Wiedermann: *Various Methods of Controlling the Deployment of a Tethered Satellite*; Journal of Vibration and Control, **9** (2003), 187 – 208.

[3] H. J. Oberle, W. Grimm, and E. Berger. *BNDSCO, Rechenprogramm zur Lösung beschränkter optimaler Steuerungsprobleme*. Benutzeranleitung M 8509, Techn. Univ. München, 1985.

[4] G. Leitmann, An Introduction to Optimal Control, McGraw-Hill, New York, 1966.

[5] J. C. Van der Meer, *The Hamiltonian Hopf bifurcation*. Lecture Notes in Mathematics 1160, Springer-Verlag, 1986.

[6] Wolfram, Stephen. Mathematica: a system for doing mathematics by computer. Addison-Wesley Publishing Co., Redwood City, California.