Metric Regularity and Stability of Optimal Control Problems for Linear Systems

Marc Quincampoix and Vladimir Veliov

Metric Regularity and Stability of Optimal Control Problems for Linear Systems[*]

M. Quincampoix[1]   V.M. Veliov[2]

[1] Laboratoire de Mathématiques de Bretagne Atlantique, unité CNRS UMR6205, Université de Brest, France, `marc.quincampoix@univ-brest.fr`
[2] Institute of Mathematical Methods in Economics, Vienna University of Technology, Austria, `veliov@tuwien.ac.at`

Abstract

This paper studies stability properties of the solutions of optimal control problems for linear systems. The analysis is based on an adapted concept of metric regularity, the so-called *strong bi-metric regularity*, which is introduced and investigated in the paper. It allows to give a more precise description of the effect of perturbations on the optimal solutions in terms of a Hölder-type estimation, and to investigate the robustness of this estimation. The Hölder exponent depends on a natural number k, which is known as the *controllability index* of the reference solution. An inverse function theorem for strongly bi-metrically regular mappings is obtained, which is used in the case $k = 1$ for proving stability of the solution of the considered optimal control problem under small non-linear perturbations. Moreover, a new stability result with respect to perturbations in the matrices of the system is proved in the general case $k \geq 1$.

Keywords: optimal control, linear control systems, metric regularity, inverse function theorem

MSC Classification: 49K40, 90C31, 49N05, 93C05, 47J07, 54C60

1 Introduction

We investigate regularity and stability properties of the solution of the following optimal control problem:

$$\min g(x(T)) \tag{1}$$

subject to the linear dynamics

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) + d(t), \quad x(0) = x_0, \tag{2}$$
$$u(t) \in U. \tag{3}$$

Here $x \in \mathbf{R}^n$, $u \in U \subset \mathbf{R}^r$, the time interval $[0, T]$ is fixed, $g : \mathbf{R}^n \to \mathbf{R}$ is smooth and convex, A and B are smooth matrix functions with appropriate dimensions. The initial state x_0 is given. The control constraining set $U \subset \mathbf{R}^r$ is a convex compact polyhedron. As usual, a dot above a symbol denoting a function of the time t means the time-derivative.

Optimal control problems for linear systems have been profoundly studied since the early days of the optimal control theory but there are issues of interest that are recent research topics or are still open. In particular, this concerns the stability analysis of the optimal solution, which is burdened by the fact that the optimal control is discontinuous (bang-bang). This may be the case also for optimal control problems that are non-linear, but affine with respect the control. The "bang-bang" structure of the optimal control brings a challenge also for numerical approximations. We refer to the recent papers [8, 9, 10] on stability analyses and to [1, 2, 3, 15] about error analyses for problems with bang-bang solutions.

We analyze the stability of the control problem (1)–(3) through the following necessary optimality conditions (which are, in fact, sufficient under the suppositions made in Section 3): any optimal pair $(\hat{x}, \hat{u})$ together with a corresponding absolutely continuous function $\hat{p} : [0, T] \to \mathbf{R}^n$ satisfies the following (generalized) equations:

$$(4) \qquad 0 = \dot{x}(t) - A(t)\,x(t) - B(t)\,u(t) - d(t), \quad x(0) = x_0,$$

$$(5) \qquad 0 = \dot{p}(t) + A^\top(t)\,p(t),$$

$$(6) \qquad 0 \in B^\top(t)\,p(t) + N_U(u(t)),$$

$$(7) \qquad 0 = p(T) - \nabla g(x(T)),$$

where $N_U(u)$ is the normal cone to U defined as

$$N_U(u) = \begin{cases} \emptyset & \text{if } u \notin U, \\ \{l \in \mathbf{R}^r : \langle l, v - u \rangle \leq 0 \ \ \forall\, v \in U\} & \text{if } u \in U. \end{cases}$$

(Note that (6) is equivalent to $u(t) \in \underset{w \in U}{\operatorname{Argmin}} \, \langle B^\top(t)\,p(t), w \rangle$.)

Then the following question is relevant for the stability of the solution of problem (1)–(3): if the left-hand side of (4)–(7) is replaced with a vector $y = (\xi, \pi, \rho, \nu)$, does the resulting perturbed version of (4)–(7) still have a solution (x, p, u), and how far is it from the solution $(\hat{x}, \hat{p}, \hat{u})$ of the original system (4)–(7).

The answer of the first question is apparently positive, while one of the main results in this paper gives a Hölder estimation for the solution(s) (x, p, u) corresponding to disturbance y in a neighborhood of zero:

$$(8) \qquad \operatorname{dist}\left((\hat{x}, \hat{p}, \hat{u}), (x, p, u)\right) \leq c\, \|y\|^{1/k}.$$

One of the aims of this paper is to correctly define the meaning of the "neighborhood", the norm $\|\cdot\|$, the metric "dist" (and the respective spaces), and the number k for which the estimation (8) holds.

A related question is whether the estimation (8) is stable with respect to perturbations itself. It turns out that in the context of system (4)–(7) the stability of estimation (8) is valid for perturbations that are small in a substantially stronger norm, $\|\cdot\|_\sim \geq \|\cdot\|$, than the one in the right-hand side of (8)[1]. We grasp this phenomenon in general, by defining

---

[1] A similar situation is encountered also in [5].

the so-called *strong bi-metric Hölder regularity*. An inverse function theorem is proved for strongly bi-metrically regular mappings in the Lipschitz case $k = 1$.

For our particular system (4)–(7) we give a sufficient condition for strong bi-metric Hölder regularity, where the natural number k is the so-called controllability index of the solution $(\hat{x}, \hat{u})$ of the original problem (1)–(3). The metric "dist" in which we compare the controls $\hat{u}$ and u, in particular, is defined (in view of the bang-bang structure of $\hat{u}$) as the measure of the set where $u(t) \neq \hat{u}(t)$. Using the proved inverse function theorem we obtain that in the Lipschitz case $k = 1$ the strong metric bi-regularity of (4)–(7) is preserved under sufficiently "small" perturbations that can be non-linear in x.

As a byproduct we obtain the (somewhat surprising) fact that the nonlinear optimal control problem resulting from such perturbations has no "singular arcs" (i.e. optimal arcs which are not uniquely determined by the Pontryagin system).

In the general case $k \geq 1$ we also provide a stability result of system (4)–(7) (and the underlying problem (1)–(3)) with respect to perturbations in the matrices A and B which are small in suitable norms.

We mention also that in this paper, the bi-metric regularity is only used in relation to the stability of linear optimal control problem, which is the main purpose of the paper. The authors intend to provide a exhaustive study and analysis of this notion in another paper.

The paper is organized as follows. Section 2 is devoted to preliminaries on strong metric regularity and to the statement and the proof of the inverse function theorem for strongly bi-metric regular mappings. Section 3 contains the main results of the paper concerning stability and bi-metric regularity of optimal control problems for linear systems. Section 4 deals with a perturbation analysis with respect either to non-linear additive disturbances (in the case $k = 1$), or to disturbances in the matrices of the linear system (and $k \geq 1$).

2 Preliminaries on Metric Regularity

The concept of metric regularity developed in the past decades pays an important role in the contemporary optimization theory. A comprehensive exposition is given in [6]. In the present paper we need an extension of this concept that is presented below in this section

Let X and Y be two metric spaces with distances d_X and d_Y, respectively. Denote by $\mathcal{B}_{d_X}(x; \alpha)$ the closed ball with radius $\alpha > 0$ centered at $x \in X$ and by $\mathcal{B}_{d_Y}(y; \alpha)$ the respective closed ball in Y.

Definition 1 A set-valued map $F : X \rightsquigarrow Y$ is said to be *strongly (Hölder) metrically regular of order $k \geq 1$* at $(\bar{x}, \bar{y}) \in \operatorname{Graph} F$ if there exist numbers $\varsigma \geq 0$, $a > 0$ and $b > 0$ such that the mapping $\mathcal{B}_{d_Y}(\bar{y}; b) \ni y \to F^{-1}(y) \cap \mathcal{B}_{d_X}(\bar{x}, a)$ is single-valued and Hölder continuous with exponent $1/k$ and constant ς:

$$d_X\left(F^{-1}(y) \cap \mathcal{B}_{d_X}(\bar{x}, a),\, F^{-1}(y') \cap \mathcal{B}_{d_X}(\bar{x}; a)\right) \leq \varsigma\, d_Y(y, y')^{1/k} \ \text{ for all } y, y' \in \mathcal{B}_{d_Y}(\bar{y}; b).$$

This definition is an extension of the standard one (see e.g. [6]) and is introduced in [12]. A variational characterization of the Hölder metrically regularity is implied by the results in [11].

The analyses in the next section requires a more delicate notion which involves two distances in the space Y: one defining the neighborhood in which F^{-1} is locally single-valued and

3

Lipschitz, and another one, with respect to which we have the Lipschitz continuity. Namely, let X be as above and Y be equipped with two distances, d_Y and $\tilde{d}_Y$, with $d_Y \leq \tilde{d}_Y$. Denote by $\mathcal{B}_{\tilde{d}_Y}(y;\alpha)$ the ball in the metric $\tilde{d}$ with radius $\alpha > 0$ centered at $y \in Y$.

Definition 2 A set-valued map $F : X \mapsto Y$ is *strongly bi-metrically regular of order $k \geq 1$* at $(\bar{x}, \bar{y}) \in \operatorname{Graph} F$ if there exist numbers $\varsigma \geq 0$, $a > 0$ and $b > 0$ such that the mapping $\mathcal{B}_{\tilde{d}_Y}(\bar{y}; b) \ni y \to F^{-1}(y) \cap \mathcal{B}_{d_X}(\bar{x}, a)$ is single-valued and Hölder continuous in the metric d_Y with exponent $1/k$ and constant ς, that is

$$d_X\left(F^{-1}(y) \cap \mathcal{B}_{d_X}(\bar{x}, a),\, F^{-1}(y') \cap \mathcal{B}_{d_X}(\bar{x}; a)\right) \leq \varsigma\, d_Y(y, y')^{1/k} \ \text{ for all } y, y' \in \mathcal{B}_{\tilde{d}_Y}(\bar{y}; b).$$

Of course, the strong bi-metric regularity implies the usual strong metric regularity with respect to the metric $\tilde{d}_Y$ in Y. However, the latter property may be essentially weaker than the bi-metric one, as it is the case for the applications to linear control discussed in this paper. On the other hand, using only the metric d_Y makes the regularity property too strong in our context. We mention that a similar situation, where using two norms (in a specific problem in linear spaces and with $k = 1$) is encountered also in [5].

The following inverse function theorem extends those in [6, Theorem 3G.3] and [7, Theorem 3]. The latter theorems apply to the usual (single-metric) strong regularity with $k = 1$.

Theorem 1 *Let X be a complete metric space and let Y be a linear space equipped with two metrics: d_Y and $\tilde{d}_Y$, where $d_Y \leq \tilde{d}_Y$ and both metrics are shift-invariant. Let the set-valued map $F : X \rightsquigarrow Y$ be strongly bi-metrically regular of order $k = 1$ at $(\bar{x}, 0)$ with constants ς, a, b. Let $\mu > 0$ and ς' be such that $\varsigma\mu < 1$ and $\varsigma' < \varsigma/(1 - \varsigma\mu)$. Then for every positive constants a' and b' satisfying*

$$2a' \leq a, \quad 3b' + a'\mu \leq b, \quad b'\varsigma' \leq a',$$

for every function $f : X \to Y$, and every points $\tilde{x} \in \mathcal{B}_{d_X}(\bar{x}; a')$ and $\tilde{y} \in \mathcal{B}_{\tilde{d}_Y}(0; b')$ satisfying

$$(9) \qquad\qquad \tilde{y} \in f(\tilde{x}) + F(\tilde{x}), \quad \tilde{d}_Y(f(\tilde{x}), 0) \leq b',$$

and

$$(10) \qquad\qquad \tilde{d}_Y(f(x), f(x')) \leq \mu\, d_X(x, x') \ \ \forall x, x' \in \mathcal{B}_{d_X}(\bar{x}, a'),$$

we have that the mapping $y \mapsto (f + F)^{-1}(y) \cap \mathcal{B}_{d_X}(\tilde{x}, a')$ is single-valued and Lipschitz continuous with constant ς' (in the metric d_Y in Y) on $\mathcal{B}_{\tilde{d}_Y}(\tilde{y}; b')$, that is, $f + F$ is strongly bi-metrically regular at $(\tilde{x}, \tilde{y})$ with constants ς', a', b'.

The proof of this theorem follows, essentially, that of [7, Theorem 3]. However, numerous small changes are necessary due to the more general spaces that we need in the present paper and due to the bi-metric version of the strong regularity. Therefore we present a detailed proof.

Proof of Theorem 1. Let us fix a', b' and ς' as in the theorem. Take an arbitrary function $f : X \to Y$ and $\tilde{x} \in \mathcal{B}_{d_X}(\bar{x}; a')$, $\tilde{y} \in \mathcal{B}_{\tilde{d}_Y}(0; b')$ such that (9) and (10) are fulfilled.

By assumption $y \mapsto s(y) := F^{-1}(y) \cap \mathcal{B}_{d_X}(\bar{x}; a)$ is a Lipschitz continuous function (in the metric d_Y in Y) on $\mathcal{B}_{\tilde{d}_Y}(\bar{y}; b)$ with constant ς. Then inclusion $y \in f(x) + F(x)$ with

4

$x \in \mathcal{B}_{d_X}(\tilde{x}; a')$ is equivalent to $x = s(y - f(x))$, provided that $y - f(x) \in \mathcal{B}_{\tilde{d}_Y}(0; b)$. Let us take arbitrary $x \in \mathcal{B}_{d_X}(\tilde{x}; a')$, $y \in \mathcal{B}_{\tilde{d}_Y}(\tilde{y}; b')$. We have

$$d_X(x, \bar{x}) \le d_X(x, \tilde{x}) + d_X(\tilde{x}, \bar{x}) \le a' + a' \le a,$$

thus $x \in \mathcal{B}_{d_X}(\bar{x}; a)$. Moreover,

$$\begin{aligned}
\tilde{d}_Y(y - f(x), 0) &= \tilde{d}_Y(y, f(x)) \le \tilde{d}_Y(y, \tilde{y}) + \tilde{d}_Y(\tilde{y}, 0) + \tilde{d}_Y(0, f(\tilde{x})) + \tilde{d}_Y(f(\tilde{x}), f(x)) \\
&\le b' + b' + b' + \mu\, d_X(\tilde{x}, x) \le 3b' + \mu a' \le b.
\end{aligned}$$

Thus for $y \in \mathcal{B}_{\tilde{d}_Y}(\tilde{y}; b')$ the inclusion $x \in (f + F)^{-1}(y) \cap \mathcal{B}_{d_X}(\tilde{x}; a')$ is equivalent to $x = s(y - f(x))$. We shall prove that this equation has a unique solution in $\mathcal{B}_{d_X}(\tilde{x}; a')$.

For a fixed $y \in \mathcal{B}_{\tilde{d}_Y}(\tilde{y}; b')$ let us denote $Z(x) := s(y - f(x))$, $x \in \mathcal{B}_{d_X}(\tilde{x}; a')$. We have

$$\begin{aligned}
d_X(\tilde{x}, Z(\tilde{x})) &= d_X(s(\tilde{y} - f(\tilde{x})), s(y - f(\tilde{x}))) \le \varsigma\, d_Y(\tilde{y} - f(\tilde{x}), y - f(\tilde{x})) \\
&= \varsigma\, d_Y(\tilde{y}, y) \le \varsigma\, \tilde{d}_Y(\tilde{y}, y) \le \varsigma b' \le \varsigma' b'(1 - \varsigma\mu).
\end{aligned}$$
(11)

Moreover, for every $x, x' \in \mathcal{B}_{d_X}(\tilde{x}; a')$

$$\begin{aligned}
d_X(Z(x), Z(x')) &= d_X(s(y - f(x)), s(y - f(x'))) \le \varsigma d_Y(f(x), f(x')) \\
&\le \varsigma \tilde{d}_Y(f(x), f(x')) \le \varsigma\mu\, d_X(x, x').
\end{aligned}$$
(12)

Due to (11), (12) and $\varsigma\mu < 1$ we can apply the classical Banach fixed point theorem: the mapping Z has a unique fixed point in $\mathcal{B}_{d_X}(\tilde{x}; a')$. Since it depends on the fixed y we denote it by $\tilde{s}(y)$. Thus for $y \in \mathcal{B}_{\tilde{d}_Y}(\tilde{y}; b')$ the set $x \in (f + F)^{-1}(y) \cap \mathcal{B}_{d_X}(\tilde{x}; a')$ consists of the single point $\tilde{s}(y)$. It remains to prove that $\tilde{s}$ is Lipschitz continuous with constant ς in $\mathcal{B}_{\tilde{d}_Y}(\tilde{y}; b')$ (with respect to the metric $\tilde{d}_Y$). For $y, y' \in \mathcal{B}_{\tilde{d}_Y}(\tilde{y}; b')$ we have

$$\begin{aligned}
d_X(\tilde{s}(y), \tilde{s}(y')) &= d_X(Z(\tilde{s}(y)), Z(\tilde{s}(y'))) = d_X(s(y - f(\tilde{s}(y))), s(y' - f(\tilde{s}(y')))) \\
&\le \varsigma\, d_Y(y - f(\tilde{s}(y)), y' - f(\tilde{s}(y'))) \le \varsigma\, d_Y(y, y') + \varsigma\, d_Y(f(\tilde{s}(y)), f(\tilde{s}(y'))) \\
&\le \varsigma\, d_Y(y, y') + \varsigma\, \tilde{d}_Y(f(\tilde{s}(y)), f(\tilde{s}(y'))) \le \varsigma\, d_Y(y, y') + \varsigma\mu\, d_X(\tilde{s}(y), \tilde{s}(y')).
\end{aligned}$$

Then $(1 - \varsigma\mu)d_X(\tilde{s}(y), \tilde{s}(y')) \le \varsigma\, d_Y(y, y')$, hence $d_X(\tilde{s}(y), \tilde{s}(y')) \le \varsigma'\, d_Y(y, y')$. $\qquad$ Q.E.D.

The above theorem has no clear counterpart for $k > 1$. However, the following is true.

Proposition 1 *Let X be a complete metric space and let Y be a linear space equipped with two metrics: d_Y and $\tilde{d}_Y$, where $d_Y \le \tilde{d}_Y$ and both metrics are shift-invariant. Let the set-valued map $F : X \rightsquigarrow Y$ be strongly bi-metrically regular of order k (a natural number) at $(\bar{x}, 0)$ with constants ς, a. Then for every function $f : X \to Y$ and for every solution $\tilde{x}$ of the inclusion $0 \in f(x) + F(x)$ for which $d_X(\tilde{x}, \bar{x}) \le a$ and $\tilde{d}_Y(f(\tilde{x}), 0) \le b$ it holds that*

$$d_X(\tilde{x}, \bar{x}) \le \varsigma\, d_Y(f(\tilde{x}), 0)^{1/k}.$$

Proof. If is enough to notice that $\tilde{x}$ solves the inclusion $\tilde{y} \in F(x)$ with $\tilde{y} = -f(\tilde{x})$ and apply Definition 2. $\qquad$ Q.E.D.

3 Hölder metric regularity of linear optimal control problems

The main issue of this paper is to investigate properties of regularity, as introduced in the previous section, and stability of the solution(s) of optimal control problem (1)–(3).

We begin with some assumptions.

Assumption (A1): The functions $A : [0,T] \to \mathbf{R}^{n \times n}$ and $B : [0,T] \to \mathbf{R}^{n \times r}$ are $\bar{k}$ times, respectively $\bar{k} + 1$ times, continuously differentiable (for some natural number $\bar{k}$); $d \in L^1(0,T)$. Moreover, $g : \mathbf{R}^n \to \mathbf{R}$ is convex and differentiable with a locally Lipschitz derivative.

Admissible control functions in the above problem are all measurable selections of U. Denote the set of admissible controls by $\mathcal{U}$. For $u \in \mathcal{U}$ equation (2) has a unique absolutely continuous solution $x[u](\cdot)$ on $[0,T]$. The reachable set $R = \{x[u](T) : u \in \mathcal{U}\}$ is a convex and compact subset of $\mathbf{R}^n$, hence problem (1)–(3) has at least one solution $(\hat{x}, \hat{u})$.

Define the sequence of matrices

$$(13) \qquad B_0(t) = B(t), \quad B_{i+1}(t) = -A(t)B_i(t) + \dot{B}_i(t), \quad i = 0, \ldots, \bar{k} - 1.$$

Moreover, denote by E the set of all (non-degenerate) edges of U, and by $\bar{E}$ – the set of all vectors $u_2 - u_1$, where $[u_1, u_2] \in E$.

Assumption (A2): $\mathrm{rank}[B_0(t)\, e, \ldots, B_{\bar{k}}(t)\, e] = n$ for every $e \in \bar{E}$ and $t \in [0,T]$. Moreover, $\nabla g(x) \neq 0$ for every $x \in R$ (with $\nabla g(x)$ denoting the gradient of g at x).

The rank condition in the above assumption is the well-known *general position hypotheses* [13], which is an extension of the Kalman condition to non-autonomous linear control systems and a general polyhedral set U. The second part of the assumption makes the problem meaningful, since it rules out the possibility of infinitely many solutions.

The Pontryagin maximum principle claims that any optimal pair $(\hat{x}, \hat{u})$ together with a corresponding absolutely continuous function $\hat{p} : [0,T] \to \mathbf{R}^n$ satisfies the equations (4)–(7).

The following lemma is well-known (see e.g. [15]).

Lemma 1 *Let the matrices A and B be measurable and essentially bounded g is differentiable and convex. Then $(\hat{x}, \hat{u})$ is a solution of problem (1)–(3) if and only if the triple $(\hat{x}, \hat{p}, \hat{u})$ (with an absolutely continuous $\hat{p}$) is a solution of system (4)–(7). If (A1) and (A2) hold, then the solution $(\hat{x}, \hat{u})$ of (1)–(3) is unique, hence that of (4)–(7) is also unique. Moreover, $\hat{u}(t)$ is a vertex of U for a.e. $t \in [0,T]$.*

Let $(\hat{x}, \hat{p}, \hat{u})$ be a solution of system (4)–(7) (and then $(\hat{x}, \hat{u})$ is a solution of (1)–(3)).

Definition 3 *Controllability index* of the solution $(\hat{x}, \hat{p}, \hat{u})$ of system (4)–(7) is the minimal number k such that for every $t \in [0,T]$ and for every $e \in \bar{E}$ at least one of the numbers $\left\langle B_i^\top(t)\, \hat{p}(t), e \right\rangle$, $i = 0, \ldots, k$, is not equal to zero.

Clearly, if (A2) is fulfilled, then $k \leq \bar{k}$. Indeed, for every t we have $\hat{p}(t) \neq 0$ (due to $\nabla g(\hat{x}(T)) \neq 0$). Then from (A2) at least one of the numbers $\langle B_i(t) e, \hat{p}(t) \rangle$, $i = 0, \ldots, \bar{k}$, is non-zero, thus $k \leq \bar{k}$.

If k is the controllability index of $(\hat{x}, \hat{p}, \hat{u})$, then due to the continuity of B_i and $\hat{p}$ there exists a positive number m_0 such that

$$(14) \qquad \sum_{i=0}^{k} \left| \left\langle B_i^\top(t) \hat{p}(t), e \right\rangle \right| \geq m_0 \quad \forall e \in \bar{E}, \ \ \forall t \in [0, T].$$

This inequality, with some $m_0 > 0$, is the key property to be used in the theorems below.

Remark 1 Inequality (14) simplifies in the special case of a box-like set U. Namely, if $U = [-1, 1]^r$, then (14) reads in the following way: for every component $[B_i^\top(t) \hat{p}(t)]_j$, $j = 1, \ldots, r$, of the vector $B_i^\top(t) \hat{p}(t)$ it holds that

$$\sum_{i=0}^{k} \left| \left[B_i^\top(t) \hat{p}(t) \right]_j \right| \geq m_0 \quad \forall t \in [0, T].$$

We mention that the function $[B^\top(t) \hat{p}(t)]_j$ is known in the literature as *switching function* for the j-th control component (cf. [8]). Clearly, $[B_i^\top(t) \hat{p}(t)]_j$ is the i-th derivative of the j-th switching function.

The generalized equations (4)–(7) can be written in the form $0 \in F(x, p, u)$, where

$$(15) \qquad F(x, p, u) := \begin{pmatrix} \dot{x} - A x - B u - d \\ \dot{p} + A^\top p \\ B^\top p + N_U(u) \\ p(T) - \nabla g(x(T)) \end{pmatrix}.$$

Thus the inclusion $0 \in F(x, p, u)$ is equivalent to our original problem (1)–(3). The main goal in this section is to investigate the stability of the solutions of this inclusion with respect to perturbations, and the metric regularity of the mapping $F : \mathcal{X}_k \to \mathcal{Y}_k$, where the spaces $\mathcal{X}_k$ and $\mathcal{Y}_k$ are introduced in the next paragraphs.

The norms in $L^1(0, T)$ and $L^\infty(0, T)$ are denoted by $\| \cdot \|_1$ and $\| \cdot \|_\infty$, respectively. The notation $W^{m,\infty} = W^{m,\infty}([0, T]; \mathbf{R}^m)$ (or $W^{m,1}$) is used for the space of all functions $x : [0, T] \to \mathbf{R}^n$ with absolutely continuous $(m-1)$-st derivative and with the m-th derivative belonging to $L^\infty(0, T)$ (or to $L^1(0, T)$), respectively. The norm is $\|x\|_{m,\infty} := \sum_{i=0}^{m} \|x^{(i)}\|_\infty$, where $x^{(0)} = x$ and $x^{(i)}$ is the i-th derivative of x. The notations $\|A\|_{k+1,\infty}$ and $\|B\|_{k,\infty}$ have the same meaning with the operator norm of the involved matrices.

The set of admissible controls $\mathcal{U}$ is viewed as a subset of $L^\infty(0, T)$ equipped with the metric

$$d^\#(u_1, u_2) = \text{meas} \{ t \in [0, T] : u_1(t) \neq u_2(t) \},$$

where "meas" stands for the Lebesgue measure in $[0, T]$. This metric is shift-invariant and we shall shorten $d^\#(u_1, u_2) = d^\#(u_1 - u_2, 0) =: d^\#(u_1 - u_2)$.

Then the triple (x, p, u) is considered as an element of the (affine) space

$$\mathcal{X}_k = W_{x_0}^{1,1} \times W^{k+1,\infty} \times \mathcal{U},$$

where $W_{x_0}^{1,1} = \{x \in W^{1,1} : x(0) = x_0\}$. We endow $\mathcal{X}_k$ with the (shift-invariant) metric

$$d_{\mathcal{X}}(x, p, u) = \|x\|_{1,1} + \|p\|_{k+1,\infty} + d^{\#}(u).$$

The image space $\mathcal{Y}_k$, $k \geq 1$, will be

$$\mathcal{Y}_k = L^1 \times W^{k,\infty} \times W^{k+1,\infty} \times \mathbf{R}^n,$$

where we shall use the following two norms

$$
\begin{aligned}
\|y\| = \|(\xi, \pi, \rho, \nu)\| &:= \|\xi\|_1 + \|\pi\|_\infty + \|\rho\|_\infty + |\nu|, \\
\|y\|_\sim = \|(\xi, \pi, \rho, \nu)\|_\sim &:= \|\xi\|_1 + \|\pi\|_{k,\infty} + \|\rho\|_{k+1,\infty} + |\nu|.
\end{aligned}
$$

generating the metrics $d_{\mathcal{Y}}$ and $\tilde{d}_{\mathcal{Y}}$, respectively. Also we define the space $\mathcal{Y}_0 = L^1 \times L^\infty \times L^\infty \times \mathbf{R}^n$ with the above norm $\|\cdot\|$.

Notice that due to (A1) we have that $(\hat{x}, \hat{p}, \hat{u}) \in \mathcal{X}_k$. In order to ensure that F maps $\mathcal{X}_k$ into $\mathcal{Y}_k$ we need to interpret the set $N_U(u)$ in (15) as $\{\xi \in W^{k+1,\infty} : \xi(t) \in N_U(u(t)) \ \forall \, t \in [0, T]\}$ (strictly speaking, we should use the notation $N_{\mathcal{U}}(u)$ instead of the point-wise $N_U(u)$, but the overload of the latter would not lead to confusions).

The main results in this section follow.

Theorem 2 *Let assumptions (A1) and (A2) be fulfilled, let $(\hat{x}, \hat{p}, \hat{u})$ be a solution of the generalized equation $0 \in F(x, p, u)$ (with F given in (15)) and let k be its controllability index. Then for every number $b > 0$ there exists a number c such that for every $s \in \{0, \ldots, k\}$ and every $y = (\xi, \pi, \rho, \nu) \in \mathcal{Y}_s$ with $\|y\| \leq b$ there exists (x, p, u) such that $y \in F(x, p, u)$ and*

$$(16) \qquad \|x - \hat{x}\|_{1,1} + \|p - \hat{p}\|_{s+1,\infty} + d^{\#}(u - \hat{u}) \leq c \, \|y\|^{\frac{1}{k}}.$$

Moreover, for every solution of $y \in F(x, p, u)$ it holds that

$$(17) \qquad \|x - \hat{x}\|_{1,1} + \|p - \hat{p}\|_{s+1,\infty} + \|u - \hat{u}\|_1 \leq c \, \|y\|^{\frac{1}{k}}.$$

As formulated, the above theorem applies only to individual problems of the type (1)–(3), as far as the constant c may be specific for each problem. It turns out that the constant c depends on the data of the problem only through certain norms and therefore is the same for large well-defined families of problems. This result may be useful in the error analysis of discrete approximations, but the main reason for formulating and proving it in the next proposition is that it will be substantially used in the proof of Theorem 3 below.

For a function $g : \mathbf{R}^n \to \mathbf{R}$ which is differentiable with a locally Lipschitz derivative denote

$$
\begin{aligned}
\Gamma[\nabla g](\alpha) \;=\; \inf\{\gamma : \; &|\nabla g(x)| \leq \gamma \text{ and } \nabla g \text{ is Lipschitz continuous} \\
&\text{with constant } \gamma \text{ in the ball } |x| \leq \alpha \}.
\end{aligned}
$$

Clearly, the function $\Gamma[\nabla g](\cdot)$ is finite and monotone increasing.

Proposition 2 *Let the natural numbers n, r, $\bar{k}$ and $k \leq \bar{k}$, and the compact convex polyhedral set $U \subset \mathbf{R}^r$ be fixed. Then for every triple of positive numbers K, b and μ and a function $\gamma : [0, \infty) \to \mathbf{R}$ there exists a number $c = c(K, b, \mu, \gamma(\cdot))$ with the following property.*

Let the time-horizon T satisfies $T \leq K$. Let the $(n \times n)$-matrix function $A(t)$, the $(n \times r)$-matrix function $B(t)$, both defined on $[0, T]$, and $g : \mathbf{R}^n \to \mathbf{R}$ be such that assumptions (A1) and (A2) be fulfilled and

$$\text{(18)} \qquad |x_0|, \ \|A\|_{k,\infty}, \ \|B\|_{k+1,\infty}, \ \|d\|_1 \ \leq \ K, \ \ \Gamma[\nabla g](\alpha) \leq \gamma(\alpha), \ \ \forall \alpha > 0.$$

Let $(\hat{x}, \hat{p}, \hat{u})$ be a solution of the generalized equation $0 \in F(x, p, u)$ (with F given in (15)) and let (14) be fulfilled with $m_0 \geq \mu$.

Then for every $s \in \{0, \dots, k\}$ and every $y = (\xi, \pi, \rho, \nu) \in \mathcal{Y}_s$ with $\|y\| \leq b$ the inclusion $y \in F(x, p, u)$ has a solution such that (16) holds. Moreover, for every solution of $y \in F(x, p, u)$ the estimation (17) holds.

Theorem 3 *Let assumptions (A1) and (A2) be fulfilled, let $(\hat{x}, \hat{p}, \hat{u})$ be a solution of the generalized equation $0 \in F(x, p, u)$ (with F given in (15)) and let k be its controllability index. Then the mapping F is strongly bi-metrically regular of order k at $((\hat{x}, \hat{p}, \hat{u}), 0)$ with respect to the metric $d_{\mathcal{X}}$ in $\mathcal{X}_k$ and the metrics $d_{\mathcal{Y}}$ and $\tilde{d}_{\mathcal{Y}}$ in $\mathcal{Y}_k$. Moreover, the number a in the definition of strong bi-metrically regularity (Definition 2) can be taken to be $+\infty$, that is, F^{-1} is Hölder continuous in the metric $d_{\mathcal{Y}}$ in some $\tilde{d}_Y$-neighborhood of the origin.*

Remark 2 Theorem 2 reveals a certain *stability property* of the system of necessary optimality conditions (4)–(7), which is equivalent to problem (1)–(3): if the left-hand side of the inclusion $0 \in F(x, p, u)$ is disturbed by a "small" (in the metric d_Y) perturbation y, then a solution of $y \in F(x, p, u)$ still exists and estimations of order $\|y\|^{1/k}$ hold for appropriate distances of such solutions to $(\hat{x}, \hat{p}, \hat{u})$. On the other hand, this stability property can be destroyed by an arbitrarily small (in the metric d_Y) perturbation y, as simple examples show. In contrast, Theorem 3 implies that perturbations y which are sufficiently small in the metric $\tilde{d}_Y$ do not destroy the stability and preserve the uniqueness.

We start the proofs with two lemmas. Denote by V the set of all vertices of U.

Lemma 2 *Let N be the number of vertices of U and δ be the maximal length of an edge. Then for every $z \in \mathbf{R}^r$, $u \in V \cap \underset{w \in U}{\mathrm{Argmin}} \langle z, w \rangle$ and $v \in U$ there exists $v' \in V$ such that $[v', u] \in \bar{E}$ and*

$$\langle z, v - u \rangle \geq \frac{1}{N\delta} |v - u| |\langle z, v' - u \rangle|.$$

Proof. Let $\{v_i\}_{i=1,\dots,s} \subset V$ be the set of all neighboring to u vertices. Since $u \in V$, it is a routine task to prove that the cone generated by the vectors $\{v_i - u\}$ contains $U - u$. In particular we have

$$v - u = \sum_{i=1}^{s} \alpha_i \frac{v_i - u}{|v_i - u|},$$

9

where $\alpha_i \geq 0$. Then $|v - u| \leq \sum_{i=1}^{s} \alpha_i$, hence there exists j such that

$$\alpha_j \geq \frac{1}{s}|v - u| \geq \frac{1}{N}|v - u|.$$

Set $v' = v_j$. Using that $u \in V \cap \operatorname*{Argmin}_{w \in U}\langle z, w \rangle$ we have

$$
\begin{aligned}
\langle z, v - u \rangle &= \sum_{i=1}^{s} \alpha_i \left\langle z, \frac{v_i - u}{|v_i - u|} \right\rangle = \sum_{i=1}^{s} \alpha_i \left| \left\langle z, \frac{v_i - u}{|v_i - u|} \right\rangle \right| \\
&\geq \alpha_j \left| \left\langle z, \frac{v_j - u}{|v_j - u|} \right\rangle \right| \geq \frac{1}{N|v' - u|}|v - u|\,|\langle z, v' - u \rangle| \\
&\geq \frac{1}{N\delta}|v - u|\,|\langle z, v' - u \rangle|.
\end{aligned}
$$

Q.E.D.

For a natural number $k \geq 0$ and reals $L, m \geq 0$ let us define the class of functions

$$\mathcal{F}_k(M, m) := \left\{ l \in W^{k+1,\infty}([0, T]; \mathbf{R}) : \|l\|_{k+1,\infty} \leq M, \ \sum_{i=0}^{k} |l^{(i)}(t)| \geq m \ \forall\, t \in [0, T] \right\}.$$

The following lemma is a particular cases of [14, Corollary 2.2] (somewhat reformulated).

Lemma 3 *For every pair of positive numbers K and μ and a natural number k there exists a constant $c_0 = c_0(K, \mu, k)$ such that whatever are the numbers $T, C, M \in (0, K]$ and $m \geq \mu$, the inequality*

$$\int_0^T |l(t)|\,|\varphi(t)|\,\mathrm{d}t \ \geq \ c_0\,\|\varphi\|_1^{k+1}$$

holds for every $l \in \mathcal{F}_k(M, m)$ and every measurable function $\varphi : [0, T] \to \mathbf{R}$ satisfying $|\varphi(t)| \leq C$ for a.e. t.

Proof of Proposition 2 and Theorem 2.

We shall prove Proposition 2. Then also Theorem 2 will be true, since as explained above, assumption (A2) implies (14) with some $m_0 > 0$.

Let us fix arbitrarily the positive numbers K, b, μ and a function $\gamma[0, \infty) \to \mathbf{R}$. The number c in Proposition 2 will be specified later in the proof. Let x_0, A, B, d and g be as in the formulation of the theorem. Let $(\hat{x}, \hat{p}, \hat{u})$ be a solution of $0 \in F(x, p, u)$, that is, of (4)–(7). Since (A1) and (A2) are fulfilled for the chosen configuration of data, according to Lemma 1 this solution is unique, $(\hat{x}, \hat{u})$ is the unique solution of problem (1)–(3), and $\hat{u}(t) \in V$ for a.e. $t \in [0, T]$.

Take an arbitrary $y = (\xi, \pi, \rho, \nu) \in \mathcal{Y}_0$ with $\|y\| \leq b$ and consider the following "disturbed" system for (x, p, u):

$$
\begin{aligned}
(19) \qquad && 0 &= \dot{x}(t) - A(t)\,x(t) - B(t)\,u(t) - d(t) - \xi(t), \\
(20) \qquad && 0 &= \dot{p}(t) + A^\top(t)\,p(t) - \pi(t), \\
(21) \qquad && 0 &\in B^\top(t)\,p(t) - \rho(t) + N_U(u(t)), \\
(22) \qquad && 0 &= p(T) - \nabla g(x(T)) - \nu.
\end{aligned}
$$

Notice that the above system is a necessary and sufficient optimality condition for the problem

$$(23) \qquad \min \left\{ g(x(T)) + \langle \nu, x(T) \rangle - \int_0^T [\langle \pi(t), x(t) \rangle + \langle \rho(t), u(t) \rangle] \, \mathrm{d}t \right\}$$

subject to

$$(24) \qquad \dot{x}(t) = A(t)\, x(t) + B(t)\, u(t) + d(t) + \xi(t), \quad x(0) = x_0,$$

$$(25) \qquad u(t) \in U.$$

This follows from the first part of Lemma 1 after reformulation of the last problem as a terminal problem for a $(n+1)$-dimensional linear system (this standard reformulation will be used later in the proof of Theorem 3). Moreover, problem (23)–(25) has a solution, than (19)–(22) has a solution, too, and let $(\tilde{x}, \tilde{p}, \tilde{u})$ be an arbitrary solution. Denote

$$(26) \qquad q(t) = \int_0^t \Phi(t, s)\, \xi(s) \, \mathrm{d}s,$$

where $\Phi(t, \tau)$ is the fundamental matrix solution of the linear system $\dot{x}(t) = A(t)\, x(t)$ normalized at $t = \tau$. Then from the Cauchy formula for (2) and (24) we see that $\hat{x}(t) + q(t)$ is the solution of (24) for control $\hat{u}$. Since $(\tilde{x}, \tilde{u})$ is an optimal solution of (23)–(25), we have

$$
\begin{aligned}
0 \geq\ & g(\tilde{x}(T)) + \langle \nu, \tilde{x}(T) \rangle - \int_0^T [\langle \pi(t), \tilde{x}(t) \rangle + \langle \rho(t), \tilde{u}(t) \rangle] \, \mathrm{d}t \\
& - g(\hat{x}(T) + q(T)) - \langle \nu, \hat{x}(T) + q(T) \rangle + \int_0^T [\langle \pi(t), \hat{x}(t) + q(t) \rangle + \langle \rho(t), \hat{u}(t) \rangle] \, \mathrm{d}t.
\end{aligned}
$$

Moreover, $\bar{x}(t) := \tilde{x}(t) - q(t)$ is the solution of (2) with control $\tilde{u}$. Then the above inequality becomes

$$
\begin{aligned}
0 \geq\ & g(\bar{x}(T) + q(T)) - g(\hat{x}(T) + q(T)) + \langle \nu, \bar{x}(T) - \hat{x}(T) \rangle \\
& - \int_0^T [\langle \pi(t), \bar{x}(t) - \hat{x}(t) \rangle + \langle \rho(t), \tilde{u}(t) - \hat{u}(t) \rangle] \, \mathrm{d}t,
\end{aligned}
$$

and since g is convex and differentiable

$$0 \geq \langle \nabla g(\hat{x}(T) + q(T)) + \nu, \bar{x}(T) - \hat{x}(T) \rangle - \int_0^T [\langle \pi(t), \bar{x}(t) - \hat{x}(t) \rangle + \langle \rho(t), \tilde{u}(t) - \hat{u}(t) \rangle] \, \mathrm{d}t.$$

Obviously one can estimate $|q(T)| \leq c_1 \|\xi\|_1$, where (due to (18)) c_1 depends only on K. Moreover, due to (18) again, $\hat{x}(T)$ is contained in a sufficiently large ball at zero with radius β depending only on K. Then $\gamma := \gamma(\beta + c_1 b)$ is an upper estimate of both ∇g and its Lipschitz constant in the ball $|x| \leq \beta + c_1 b$, which depends only on K and b. Then we may rewrite the above inequality as

$$(27) \quad 0 \geq \langle \nabla g(\hat{x}(T)) + \zeta, \bar{x}(T) - \hat{x}(T) \rangle - \int_0^T [\pi(t)(\bar{x}(t) - \hat{x}(t)) + \rho(t)(\tilde{u}(t) - \hat{u}(t))] \, \mathrm{d}t,$$

where

$$|\zeta| \leq \gamma c_1 \|\xi\|_1 + |\nu| \leq c_2(\|\xi\|_1 + |\nu|) \leq c_2\, b,$$

11

and c_2 depends only on K, b and the function $\gamma(\cdot)$. Using the Cauchy formula for (2) and the expression $\hat{p}(t) = \Phi^\top(T, t) \nabla g(\hat{x}(T)))$ (which follows from (5) and (7)) we obtain the following relation and estimations:

$$(28) \qquad \langle \nabla g(\hat{x}(T)), \bar{x}(T) - \hat{x}(T) \rangle = \int_0^T \langle \hat{\sigma}(t), \tilde{u}(t) - \hat{u}(t) \rangle \, \mathrm{d}t,$$

$$(29) \qquad |\bar{x}(T) - \hat{x}(T)| \leq c_3 \|\tilde{u} - \hat{u}\|_1, \qquad \int_0^T |\bar{x}(t) - \hat{x}(t)| \, \mathrm{d}t \leq c_3 \|\tilde{u} - \hat{u}\|_1,$$

where

$$\hat{\sigma}(t) = B^\top(t) \hat{p}(t)$$

and c_3 is a constant, which in view of (18) may be taken as depending only on K. Using (28) in (27) we obtain that

$$0 \geq \int_0^T \langle \hat{\sigma}(t), \tilde{u}(t) - \hat{u}(t) \rangle \, \mathrm{d}t + \langle \zeta, \bar{x}(T) - \hat{x}(T) \rangle - \int_0^T [\langle \pi(t), \bar{x}(t) - \hat{x}(t) \rangle + \langle \rho(t), \tilde{u}(t) - \hat{u}(t) \rangle] \, \mathrm{d}t.$$

Since due to (6) $\hat{u}(t) \in \operatorname*{Argmin}_{w \in U} \langle \hat{\sigma}(t), w \rangle$ for a.e. t, the first term is non-negative. Then

$$\left| \langle \zeta, \bar{x}(T) - \hat{x}(T) \rangle - \int_0^T [\langle \pi(t), \bar{x}(t) - \hat{x}(t) \rangle + \langle \rho(t), \tilde{u}(t) - \hat{u}(t) \rangle] \, \mathrm{d}t \right| \geq \int_0^T \langle \hat{\sigma}(t), \tilde{u}(t) - \hat{u}(t) \rangle \, \mathrm{d}t.$$

Using (29) and the estimation for $|\zeta|$ we obtain

$$(30) \qquad c_4 \|y\| \, \|\tilde{u} - \hat{u}\|_1 \geq \int_0^T \langle \hat{\sigma}(t), \tilde{u}(t) - \hat{u}(t) \rangle \, \mathrm{d}t,$$

with another constant c_4 depending only on K, b and $\gamma(\cdot)$.

Since we know that $\hat{u}(t) \in V$ almost everywhere, according to Lemma 2 and $\hat{u}(t) \in \operatorname{Argmin} \langle \hat{\sigma}(t), w \rangle$ we have that the set

$$W(t) := \left\{ v' \in V : v' - \hat{u}(t) \in \bar{E}, \ \langle \hat{\sigma}(t), \tilde{u}(t) - \hat{u}(u) \rangle \geq \frac{1}{N\delta} |\tilde{u}(t) - \hat{u}(t)| \, |\langle \hat{\sigma}(t), v' - \hat{u}(t) \rangle| \right\}$$

is non-empty for a.e. t. Since V is a finite set, the mapping $t \mapsto W(t)$ is closed-valued and Theorem 8.2.9 in [4] implies that it is measurable. Then it has a measurable selection $u^*(t) \in W(t)$. From (30)

$$(31) \qquad \frac{1}{N\delta} \int_0^T |\tilde{u}(t) - \hat{u}(t)| \langle \hat{\sigma}(t), u^*(t) - \hat{u}(t) \rangle \, \mathrm{d}t \leq c_4 \|y\| \, \|\tilde{u} - \hat{u}\|_1.$$

Since $e(t) := u^*(t) - \hat{u}(t) \in \bar{E}$ and the last set is finite, one can split $[0, T]$ into a finite number of measurable sets Δ_j such that $e(t) = e_j$ is constant for $t \in \Delta_j$. Denote $\varphi_j(t) = \chi_j(t)|\tilde{u}(t) - \hat{u}(t)|$, where χ_j is the characteristic function of Δ_j.

One can directly verify that

$$\frac{\mathrm{d}^i}{(\mathrm{d}t)^i}(B^\top(t) \hat{p}(t)) = B_i^\top(t) \hat{p}(t), \quad i = 0, \ldots, \bar{k},$$

Thus $\hat{\sigma}^{(i)}(t) = B_i^\top(t)\,\hat{p}(t)$. Denote $l_j(t) = \langle \hat{\sigma}(t), e_j\rangle$. Then (A1) and (14) imply that $l_j \in \mathcal{F}_k(M, m_0)$, where M is an appropriate constant depending only on $\|A\|_{W^{k,\infty}(0,T)}$, $\|B\|_{W^{k+1,\infty}(0,T)}$ and $|\nabla g(\hat{x}(T))|$, hence M depends only on K and $\gamma(\cdot)$. Then Lemma 3 (with $C = \mathrm{diam}(U)$) implies the inequality

$$\int_0^T |l_j(t)|\,|\varphi_j(t)|\,\mathrm{d}t \geq c_0\|\varphi_j\|_1^{k+1},$$

where $c_0 > 0$ is the constant in Lemma 3, depending only on k, K, b and μ. Thus

$$\int_0^T |\tilde{u}(t) - \hat{u}(t)|\,\langle \hat{\sigma}(t), u^*(t) - \hat{u}(t)\rangle\,\mathrm{d}t = \sum_{j=1}^N \int_0^T |l_j(t)|\,|\varphi_j(t)|\,\mathrm{d}t \geq c_0 \sum_{j=1}^N \|\varphi_j\|_1^{k+1}$$

Then using the Hölder inequality we obtain that

$$\int_0^T |\tilde{u}(t) - \hat{u}(t)|\,\langle \hat{\sigma}(t), u^*(t) - \hat{u}(t)\rangle\,\mathrm{d}t \geq c_0 \sum_{j=1}^N \|\varphi_j\|_1^{k+1} \geq \frac{c_0}{N^k}\left(\sum_{j=1}^N \|\varphi_j\|_1\right)^{k+1} = \frac{c_0}{N^k}\|\tilde{u} - \hat{u}\|_1^{k+1}.$$

Combining this with (31) we obtain

(32) $$\|\tilde{u} - \hat{u}\|_1^k \leq \frac{c_4}{c_0}\delta N^{k+1}\|y\| =: c\|y\|.$$

Thus we obtain for $\|\tilde{u} - \hat{u}\|_1$ the estimation in the second claim of Proposition 2. Notice that the constant c depends only on K, b, μ, and the function γ (besides the fixed n, r, k and U).

The estimation for $\|\tilde{x} - \hat{x}\|_1$ and $\|\tilde{p} - \hat{p}\|_{s+1,\infty}$, follows from (32) in an obvious way using the corresponding equations in (4)–(7) and (19)–(22).

By a standard argument, among the solutions of problem (23)–(25) there is at least one, $(\tilde{x}, \tilde{u})$, for which the values of $\tilde{u}$ are for a.e. t vertices of U. Let $(\tilde{x}, \tilde{p}, \tilde{u})$ be the corresponding solution of (19)–(22). Then (32) holds. Since $\tilde{u}(t), \hat{u}(t) \in V$ we have $|\tilde{u}(t) - \hat{u}(t)| \geq \eta$ whenever $\tilde{u}(t) \neq \hat{u}(t)$, where $\eta > 0$ is the minimal distance between different vertices of U. Then

$$\eta\,d^\#(\tilde{u} - \hat{u}) \leq \int_0^T |\tilde{u}(t) - \hat{u}(t)|\,\mathrm{d}t \leq c\,\|y\|^{\frac{1}{k}},$$

This proves the first claim of Proposition 2. Q.E.D.

Proof of Theorem 3. According to Definition 2 (applied with $a = +\infty$) it is enough to prove that there exist positive numbers β and ς such that $F^{-1}(y)$ is single-valued and

(33) $$d_{\mathcal{X}}(F^{-1}(y), F^{-1}(y')) \leq \varsigma\,d_{\mathcal{Y}}(y', y)^{\frac{1}{k}}$$

for all y, $y' \in \mathcal{Y}_k$ for which $\tilde{d}_{\mathcal{Y}}(y) \leq \beta$ and $\tilde{d}_{\mathcal{Y}}(y') \leq \beta$. The numbers $\beta > 0$ and ς will be fixed later in the proof.

We shall make use of Proposition 2 with $s = k$. For that we take $b = 1$, $\mu = m_0/2$, $\gamma(\alpha) := \Gamma[\nabla g](\alpha) + 2$, and a number K so large that

$$|x_0|,\ \|A\|_{k,\infty},\ \|B\|_{k+1,\infty},\ \|d\|_1\ \leq\ K - 1.$$

Let c be the constant in Proposition 2 corresponding to the natural numbers $n + 1$, r, $\bar{k}$ and k, and the above constants K, b, μ, and function $\gamma(\cdot)$.

Let us take an arbitrary $y \in \mathcal{Y}_k$ with $\|y\|_\sim \leq \beta$, where $\beta \leq b$ will be defined in the next paragraphs. Let $(\tilde{x}, \tilde{p}, \tilde{u})$ be a solution of the disturbed system (19)–(22). We can rewrite this system in an equivalent form as

$$
\begin{aligned}
0 &= \dot{x}^0(t) + \langle \pi(t), x(t) \rangle + \langle \rho(t), u(t) \rangle, \quad x^0(0) = 0, \\
0 &= \dot{x}(t) - A(t)\, x(t) - B(t)\, u(t) - d(t) - \xi(t), \\
0 &= \dot{p}^0(t), \\
0 &= \dot{p}(t) + A^\top(t)\, p(t) - \pi(t)\, p^0(t), \\
0 &\in B^\top(t)\, p(t) - \rho(t)\, p^0(t) + N_U(u(t)), \\
0 &= p^0(t) - 1, \\
0 &= p(T) - \nabla g(x(T)) - \nu.
\end{aligned}
$$

The above system is exactly in the form of (4)–(7) with A, B, d and g replaced with

$$
\tilde{A}(t) = \begin{pmatrix} 0 & -\pi^\top(t) \\ 0 & A(t) \end{pmatrix}, \qquad \tilde{B}(t) = \begin{pmatrix} -\rho^\top(t) \\ B(t) \end{pmatrix}, \qquad \tilde{d}(t) = \begin{pmatrix} 0 \\ d(t) + \xi(t) \end{pmatrix}
$$

and

$$
\tilde{g}(x^0, x) = g(x) + \langle \nu, x \rangle + x^0,
$$

respectively. That is, (x, p, u) is a solution of (19)–(22) if and only if the triple

$$
x^*(t) = \begin{pmatrix} -\int_0^t [\langle \pi(s), x(s) \rangle + \langle \rho(s), u(s) \rangle]\, \mathrm{d}s \\ x(t) \end{pmatrix}, \qquad p^*(t) = \begin{pmatrix} 1 \\ p(t) \end{pmatrix}, \qquad u(t)
$$

is a solution of the system

$$
\begin{aligned}
(34) \qquad\qquad 0 &= \dot{x}^*(t) - \tilde{A}(t)\, x^*(t) - \tilde{B}(t)\, u(t) - \tilde{d}(t), \\
(35) \qquad\qquad 0 &= \dot{p}^*(t) + \tilde{A}^\top(t)\, p^*(t), \\
(36) \qquad\qquad 0 &\in \tilde{B}^\top(t)\, p^*(t) + N_U(u(t)), \\
(37) \qquad\qquad 0 &= p^*(T) - \nabla \tilde{g}(x^*(T))
\end{aligned}
$$

in a space $\mathcal{X}_k$ defined as above, but the dimension of the functions x^* and p^* is $n + 1$ and the additional initial condition $x^0(0) = 0$ has to be included into the definition of the space.

Obviously assumptions (A1) are fulfilled for the above $\tilde{A}$, $\tilde{B}$, $\tilde{d}$ and $\tilde{g}$ with $\bar{k} = k$. We shall verify that also (14) is fulfilled for the solution $(\tilde{x}^*, \tilde{p}^*, \tilde{u})$ of (34)–(37) corresponding to $(\tilde{x}, \tilde{p}, \tilde{u})$.

The matrices $\tilde{B}_i$ corresponding to $\tilde{A}$, $\tilde{B}$, $\tilde{d}$ (see (13)) have the form

$$
\tilde{B}_i(t) = \begin{pmatrix} \zeta_i^\top(t) \\ B_i(t) \end{pmatrix}, \quad i = 0, \ldots, k - 1,
$$

where

$$
|\zeta_i(t)| \leq C_1 \|y\|_\sim, \quad i = 0, \ldots, k - 1,
$$

14

and the number C_1 depends on the norms $\|A\|_{W^{k-1,\infty}}$ and $\|B\|_{W^{k,\infty}}$, but not on y. Then taking $\beta > 0$ small enough we may ensure that whenever $\|y\|_\sim \leq \beta$, inequality (14) is fulfilled with $m_0/2$ (instead of m_0) for the matrices $\tilde{A}$, $\tilde{B}$. This implies, in particular, that $(\tilde{x}^*, \tilde{p}^*, \tilde{u})$ is the unique solution of system (34)–(37) (see Lemma 1).

Similarly as above one can estimate

$$\|\tilde{A}\|_{k,\infty} \leq \|A\|_{k,\infty} + C_2\|y\|_\sim, \quad \|\tilde{B}\|_{k,\infty} \leq \|B\|_{k,\infty} + C_2\|y\|_\sim, \quad \|d\|_1 \leq \|d\|_1 + C_2\|y\|_\sim,$$

where C_2 is independent of y. Then we require additionally for $\beta > 0$ that $C_2\beta < 1$ so that we have

$$\|\tilde{A}\|_{k+1,\infty}, \|\tilde{B}\|_{k,\infty}, \|\tilde{d}\|_1 \leq K \quad \text{for every } y \in \mathcal{Y}_k, \text{ for which } \|y\|_\sim \leq \beta.$$

Obviously we have also that $|(0, x_0)| \leq K$. Moreover, we have

$$\Gamma[\nabla \tilde{g}](\alpha) \leq \sup_{|x| \leq \alpha} \nabla g(x) + |\nu| + 1 \leq \Gamma[\nabla g](\alpha) + 2 \leq \gamma(\alpha).$$

In addition we require that β satisfies $2\beta \leq b$.

Now we can apply Proposition 2 (with $s = k$) for the system (34)–(37). For every $y' \in \mathcal{Y}_k$ for which $\tilde{d}_\mathcal{Y}(y') \leq \beta$ we have by the above argument (since y above was arbitrary with $\tilde{d}_\mathcal{Y}(y) \leq \beta$) that the solution of the inclusion $y' \in F(x, p, u)$ is unique in $\mathcal{X}_k$, call it (x', p', u').

We can rewrite the inclusion $y' \in F(x, p, u)$ as $y' - y \in F(x, p, u) - y$. Then the solution (x', p', u') will be a solution of system (34)–(37) with perturbed left-hand side $((0, \xi' - \xi), (0, \pi' - \pi), \rho' - \rho, (0, \nu' - \nu))$. Since $\tilde{d}_\mathcal{Y}(y' - y) \leq 2\beta \leq b$ we can apply the last statement of Proposition 2 to obtain the estimation (33) with $\varsigma = c$. Notice that the range of (x', p', u') is not restricted, which justifies the last statement of the theorem. The proof is complete. Q.E.D.

4 Perturbations in linear optimal control problems

Let assumption (A1) be fulfilled for the problem (1)–(3) and let us introduce non-linear disturbances in this problem. Namely, we consider the perturbed problem

$$(38) \qquad\qquad\qquad \min g(x(T)) + \gamma(x(T))$$

subject to

$$(39) \qquad \dot{x}(t) = A(t)\,x(t) + h(t, x(t)) + (B(t) + H(t))\,u(t) + d(t), \quad x(0) = x_0,$$
$$(40) \qquad\qquad u(t) \in U,$$

where $\gamma : \mathbf{R}^n \to \mathbf{R}$, $h : [0, T] \times \mathbf{R}^n \to \mathbf{R}^n$, $H : [0, T] \to \mathbf{R}^{n \times r}$ are sufficiently times differentiable functions, as specified below. The disturbances h and H are presumably "small" in a sense that will be clarified below, therefore we assume that all trajectories of (39) generated by admissible controls are contained in a compact set $D \subset \mathbf{R}^n$ whenever h and H are bounded by some constant $\varepsilon_0^0 > 0$.

Due to the convexity of the set of admissible velocities the reachable set of system (39), (40) is compact, hence problem (38)–(40) has a solution (x^*, u^*). The Pontryagin maximum principle asserts that the following system is fulfilled by (x^*, u^*) and an absolutely continuous function $p^* : [0, T] \to \mathbf{R}^n$:

$$(41) \quad 0 = \dot{x}(t) - A(t)\, x(t) - h(t, x(t)) - (B(t) + H(t))\, u(t) - d(t), \quad x(0) = x_0,$$

$$(42) \quad 0 = \dot{p}(t) + (A(t) + h_x(t, x(t)))^\top p(t),$$

$$(43) \quad 0 \in (B(t) + H(t))^\top p(t) + N_U(u(t)),$$

$$(44) \quad 0 = p(T) - \nabla g(x(T)) - \nabla\gamma(x(T)),$$

where h_x is derivative of h with respect to x.

The next theorem investigates the effect of the disturbance (h, H, γ) if the non-disturbed system is strongly bi-metrically regular with $k = 1$. In the case $k = 1$ we use the same metrics in the spaces $\mathcal{X}_1$ and $\mathcal{Y}_1$ as in the previous section, namely

$$d_\mathcal{X}(x, p, u) = \|x\|_{1,1} + \|p\|_{2,\infty} + d^\#(u),$$

in $\mathcal{X}_1$ and

$$\|y\| = \|(\xi, \pi, \rho, \nu)\| \quad := \quad \|\xi\|_1 + \|\pi\|_\infty + \|\rho\|_\infty + |\nu|,$$
$$\|y\|_\sim = \|(\xi, \pi, \rho, \nu)\|_\sim \quad := \quad \|\xi\|_1 + \|\pi\|_{1,\infty} + \|\rho\|_{2,\infty} + |\nu|.$$

in $\mathcal{Y}_1$.

Theorem 4 *Assume that (A1) is fulfilled with $\bar{k} = 1$ for the non-perturbed system (4)–(7) and that this system is strongly bi-metrically regular with $k = 1$ at $((\hat{x}, \hat{p}, \hat{u}), 0)$. Then there exist numbers $\varepsilon_0 > 0$, $\delta > 0$ and c with the following property.*

Let γ, h, H be (componentwise) twice continuously differentiable and these functions and all the derivatives up to second order (with respect to (t, x)), as well as the Lipschitz constant with respect to x of the second derivatives of h, are bounded by a number $\varepsilon \le \varepsilon_0$ in $C(D)$, $C([0, T] \times D)$ or $C([0, T])$, respectively. Then
(i) system (41)–(44) has a unique solution (x^*, p^*, u^*) in the δ-neighborhood of $(\hat{x}, \hat{p}, \hat{u})$ in $\mathcal{X}_1$ and

$$d_\mathcal{X}(x^* - \hat{x}, p^* - \hat{p}, u^* - \hat{u}) \le c\varepsilon;$$

(ii) system (41)–(44) is strongly bi-metrically regular with $k = 1$ at $((x^*, p^*, u^*), 0)$ with respect to the metric $d_\mathcal{X}$ in $\mathcal{X}_1$ and the metrics $d_\mathcal{Y}$ and $\tilde{d}_\mathcal{Y}$ in $\mathcal{Y}_1$.

Proof. First we notice that system (41)–(44) can be written in the form

$$0 \in f(x, p, u) + F(x, p, u),$$

where F (corresponding to the non–perturbed system) is given by (15) and

$$f(x, p, u) = \begin{pmatrix} -h(t, x) - H(t)u \\ h_x(t, x)^\top p \\ H(t)^\top p \\ -\nabla\gamma(x(T)) \end{pmatrix}.$$

16

Let ς, a, b be the numbers in the definition of strong bi-metric regularity of F at $((\hat{x}, \hat{p}, \hat{u}), 0)$, and let μ, ς', a', b' be the the numbers in Theorem 1. Below we shall define a constant $\bar{c}$ depending only on the data of the problem (1)–(3). With the help of these constants we define $\varepsilon_0 > 0$, $\alpha > 0$ and c as any numbers satisfying the relations

(45) $$\varepsilon_0 \leq \varepsilon_0^0, \quad \bar{c}\varepsilon_0 \leq b', \quad \bar{c}\varepsilon_0 \leq \mu, \quad \delta = a', \quad c = \bar{c}\varsigma', \quad c\varepsilon_0 \leq a'.$$

Now we shall prove the claims of the theorem with the so-defined constants.

We shall apply Theorem 1 for a points $(\tilde{x}, \tilde{p}, \tilde{u}) \in \mathcal{X}_1$ and $\tilde{y} = (\tilde{\xi}, \tilde{\pi}, \tilde{\rho}, \tilde{\nu}) \in \mathcal{Y}_1$ such that

(46) $\quad \tilde{y} \in f(\tilde{x}, \tilde{p}, \tilde{u}) + F(\tilde{x}, \tilde{p}, \tilde{u}), \quad d_{\mathcal{X}}(\tilde{x} - \hat{x}, \tilde{p} - \hat{p}, \tilde{u} - \hat{u}) \leq a' \quad \text{and} \quad \tilde{d}_{\mathcal{Y}}(\tilde{y}) \leq b'.$

Let us check the inequality in (9), which in our case reads as

(47) $\quad \tilde{d}_{\mathcal{Y}}(f(\tilde{x}, \tilde{p}, \tilde{u}), 0) = \|h(\cdot, \tilde{x}(\cdot))\|_1 + \|h_x(\cdot, \tilde{x}(\cdot))\|_{1,\infty} + \|H(\cdot)^\top \tilde{p}(\cdot)\|_{2,\infty} + |\nabla\gamma(\tilde{x}(T))| \leq b'.$

Since $\varepsilon_0 \leq \varepsilon_0^0$ we have $\tilde{x}(t) \in D$. Using also that $\|\tilde{p}\|_{2,\infty} + \|\tilde{u}\|_1 \leq d_{\mathcal{X}}(\hat{x}, \hat{p}, \hat{u}) + a'$, one can estimate the left-hand side of the desired inequality by the derivatives up to second order of h and H in $[0, T] \times D$. That is, with an appropriate constant $\bar{c}$ we estimate the left-hand side by $\bar{c}\varepsilon \leq \bar{c}\varepsilon_0 \leq b'$. For later use it is important to notice that in proving (47) we do not use the last inequality in (46).

Now let us verify (10). This is also a routine task since the Lipschitz constant of f is proportional to ε, say $\bar{c}\varepsilon$ and can be chosen smaller than μ. We skip these simple but cumbersome calculation, in which the second derivatives of h and H appear, since for evaluation of $\tilde{d}_{\mathcal{Y}}(f(x, p, u), f(x', p', u'))$ we have to involve the second derivatives of h and H (remember we have that $k = 1$). Therefore we need also the Lipschitz constant with respect to x of the second derivatives of h to be smaller than ε.

Thus we can apply Theorem 1, which claims that $f + F$ is strongly bi-metrically regular at $((\tilde{x}, \tilde{p}, \tilde{u}), \tilde{y})$ with constants ς', a', b' whenever (46) is satisfied.

We apply this result with $(\tilde{x}, \tilde{p}, \tilde{u}) = (\hat{x}, \hat{p}, \hat{u})$ and $\tilde{y} = f(\hat{x}, \hat{p}, \hat{u})$, which obviously satisfies the first two requirements in (46). The last inequality in (46) is a consequence of (47), applied with $(\tilde{x}, \tilde{p}, \tilde{u}) = (\hat{x}, \hat{p}, \hat{u})$, which was proved without using that $\tilde{d}_{\mathcal{Y}}(\tilde{y}) \leq b'$, as it was noticed there. (We proved even that $\tilde{d}_{\mathcal{Y}}(f(\hat{x}, \hat{p}, \hat{u})) \leq \bar{c}\varepsilon$, which will be used below.) Thus $f + F$ is strongly bi-metrically regular at $((\hat{x}, \hat{p}, \hat{u}), f(\hat{x}, \hat{p}, \hat{u}))$ with constants ς', a', b'.

Now we consider the inclusion $0 \in f + F$. Due to the last statement and the inequality $\tilde{d}_{\mathcal{Y}}(f(\hat{x}, \hat{p}, \hat{u})) \leq b'$, we obtain that there is a unique solution (x^*, p^*, u^*) of $0 \in f + F$ in the neighborhood of radius $a' = \delta$ and

(48) $$d_{\mathcal{X}}(x^* - \hat{x}, p^* - \hat{p}, u^* - \hat{u}) \leq \varsigma' d_{\mathcal{Y}}(f(\hat{x}, \hat{p}, \hat{u})) \leq \varsigma' \bar{c}\varepsilon = c\varepsilon.$$

This proves the first claim of the theorem.

To prove that $f + F$ is strongly bi-metrically regular at $((x*, p*, u*), 0)$ we have to verify only the second inequality in (46). It reads as $d_{\mathcal{X}}(x^* - \hat{x}, p^* - \hat{p}, u^* - \hat{u}) \leq a'$ and is implied by (48) and the last inequality in (45). $\hspace{2em}$ Q.E.D.

Remark 3 The question arises if Theorem 4 remains true if the disturbance H in the control matrix depends on x. Our proof does not work in this case and the question is open.

As a consequence of Theorem 4 we obtain the following result for the non-linearly "disturbed" optimal control problem (38)–(40).

Proposition 3 *Let the assumptions of Theorem 4 be fulfilled. Then there exist numbers $\varepsilon'_0 > 0$, $\alpha' > 0$ and c' such that if the size of the disturbance (h, H, γ) does not exceed a number $\varepsilon \le \varepsilon'_0$ in the sense of Theorem 4, then problem (38)–(40) has a unique solution (x^*, u^*) in the α'-neighborhood of $(\hat{x}, \hat{u})$ in the space $\mathcal{X}_1$ (projected on $p = 0$) and*

$$\|\hat{x} - x^*\|_{1,1} + d^{\#}(\hat{u} - u^*) \le c'\,\varepsilon.$$

Moreover, $u^*(t)$ is a vertex of U for a.e. $t \in [0, T]$.

Proof. Due to Pontryagin's maximum principle the triple (x^*, p^*, u^*) (with some absolutely continuous p^*) satisfies (41)–(44). Since (x^*, u^*) is in the α'-neighborhood of $(\hat{x}, \hat{u})$ one can ensure that (x^*, p^*, u^*) belongs to α-neighborhood of $(\hat{x}, \hat{p}, \hat{u})$ (in the notation of Theorem 4) by choosing α' sufficiently small. This is due to equations (42) and (44), where x^* appears only in the boundary condition (44). Then the first statement of the proposition follows from the first claim of Theorem 4.

Now let us prove the last statement of the proposition. We have that the solution (x^*, u^*) together with some p^* satisfies system (41)–(44) and claim (ii) of Theorem 4 holds for (x^*, p^*, u^*). That is, system (41)–(44) is strongly bi-metrically regular with $k = 1$ at $((x^*, p^*, u^*), 0)$. Let ς, a and b be the constants in Definition 2. If a perturbation $y = (\xi, \pi, \rho, \nu)$ has $\|y\|_\sim \le b$ then the solution $(\tilde{x}, \tilde{p}, \tilde{u})$ of the so-disturbed version of (41)–(44) is locally unique and satisfies, in particular,

$$d^{\#}(\tilde{u} - u^*) \le \varsigma\, \|(\xi, \pi, \rho, \nu)\|.$$

We shall disturb system (41)–(44) by a perturbation $(\xi, \pi, \rho, \nu) = (0, 0, \rho, 0)$ with $\|\rho\|_{2,\infty} \le b$. Then the above inequality becomes

(49)
$$d^{\#}(\tilde{u} - u^*) \le \varsigma\, \|\rho\|_\infty.$$

Now let us assume that $u^*(t)$ is not a vertex of U on a set of positive measure. Then for a.e. such t there exists a face of U containing $u^*(t)$ in its relative interior. Since the faces are finitely many, there is at least one containing $u^*(t)$ in its relative interior on a set of positive measure, Δ_0. Let E_0 be the set of edges belonging to this face and let $\bar{E}_0$ be the set of vectors $e = v_2 - v_1$ with $[v_1, v_2] \in E_0$. Due to (43) we have

$$\left\langle (B(t) + H(t))^\top p^*(t), e \right\rangle = 0 \quad \forall\, e \in \bar{E}_0,\ t \in \Delta_0.$$

Let us fix one $e \in \bar{E}_0$.

Let ρ be any function with $\|\rho\|_{2,\infty} \le b$. The corresponding $\tilde{p}$ satisfies the same equation (42) as p^*, only with possibly different end-point condition. Then the functions $\left\langle (B(t) + H(t))^\top \tilde{p}(t), e \right\rangle$ that may result from various ρ satisfying $\|\rho\|_{2,\infty} \le b$ is an (at most) n-dimensional affine subspace of $W^{2,\infty}$. Therefore, one can choose ρ with an arbitrarily small norm $\|\rho\|_{2,\infty}$ such that for the corresponding $\tilde{p}$ we have

$$\left\langle (B(t) + H(t))^\top \tilde{p}(t), e \right\rangle - \langle \rho(t), e \rangle \neq 0 \ \text{ for almost all } t \in \Delta_0.$$

18

This implies that $\tilde{u}(t) \neq u^*(t)$ for a.e. $t \in \Delta_0$, since $u^*(t)$ is not a minimizer of $\left\langle (B(t) + H(t))^\top \tilde{p}(t), v \right\rangle$ on $v \in U$. Hence, $d^{\#}(\tilde{u} - u^*) \geq \text{meas}(\Delta_0)$. This contradicts (49) and completes the proof. Q.E.D.

Remark 4 The claim that the values of the optimal control u^* of the non-linear control problem (38)–(40) are almost everywhere vertices of U deserves a comment. In fact, the proof of the proposition just shows that with the (unique) adjoint functions p^* the minimization condition (43) in the Pontryagin principle determines u^* uniquely. Thus "singular arcs" (that is, sets of positive measure where the minimization condition in the maximum principle does not uniquely determine the control) do not appear in problem (38)–(40).

It is an open question if an arbitrarily small (in the sense of Theorem 4) non-linear perturbation of a strongly bi-metrically regular linear system can lead to a singular solution if $k > 1$. We have some reasons to think that this is possible, but we have no example for that.

So far in this section we discussed state- and control-dependent perturbations to a linear system which is strongly bi-metrically regular with Hölder exponent $k = 1$. This analyses was facilitated by the inverse function Theorem 1, which has no known extension for $k > 1$. Nevertheless, it turns out that under assumptions (A1) and (A2) the solution of problem (1)–(3) exhibits a certain stability with respect to perturbations in the matrices A and B. Notice that such perturbations are state- and control-dependent, however, the dependence is linear.

Theorem 5 Let assumptions (A1) and (A2) be fulfilled for the system (4)–(7) and let k be the associated controllability index of the solution $(\hat{x}, \hat{p}, \hat{u})$. Then there exist numbers $\delta > 0$ and c such that for every pair of matrices $\tilde{A} \in W^{\bar{k},\infty}$, $\tilde{B} \in W^{\bar{k}+1,\infty}$ with $\|\tilde{A} - A\|_{k,\infty} + \|\tilde{B} - B\|_{k+1,\infty} \leq \delta$ the following is true:
(i) system (4)–(7) for matrices $\tilde{A}$ and $\tilde{B}$ (instead of A and B) has a unique solution $(\tilde{x}, \tilde{p}, \tilde{u}) \in \mathcal{X}_k$ and

$$(50) \qquad d_{\mathcal{X}}(\tilde{x} - \hat{x}, \tilde{p} - \hat{p}, \tilde{u} - \hat{u}) \leq c \left(\|\tilde{A} - A\|_1 + \|\tilde{B} - B\|_\infty \right);$$

(ii) the mapping in the right-hand side of system (4)–(7) for matrices $\tilde{A}$ and $\tilde{B}$ is strongly bi-metrically regular of order k at $((\tilde{x}, \tilde{p}, \tilde{u}), 0)$ with respect to the metric $d_{\mathcal{X}}$ in $\mathcal{X}_k$ and the metrics $d_{\mathcal{Y}}$ and $\tilde{d}_{\mathcal{Y}}$ in $\mathcal{Y}_k$.

Proof. First of all we notice that if $\delta > 0$ is chosen sufficiently small then (A1) and (A2) are fulfilled for $\tilde{A}$ and $\tilde{B}$. Then according to Lemma 1 system (4)–(7) for matrices $\tilde{A}$ and $\tilde{B}$ has a unique solution $(\tilde{x}, \tilde{p}, \tilde{u})$, and it obviously belongs to the space $\mathcal{X}_k$.

Similarly as in the proof of Theorem 4, system (4)–(7) for matrices $\tilde{A}$ and $\tilde{B}$ can be written in the form

$$0 \in f(x, p, u) + F(x, p, u) =: \tilde{F}(x, p, u),$$

where F (corresponding to matrices A and B) is given by (15) and

$$f(x,p,u) = \begin{pmatrix} (\tilde{A}-A)x + (\tilde{B}-B)u \\ (\tilde{A}-A)^\top p \\ (\tilde{B}-B)^\top p \\ 0 \end{pmatrix}.$$

We are going to apply Proposition 1. First, the mapping F is strongly bi-metrically regular with some constants ς, $a = +\infty$ and $b > 0$, according to Theorem 3. We need to verify that the inequalities $d_\mathcal{X}(\tilde{x}-\hat{x}, \tilde{p}-\hat{p}, \tilde{u}-\hat{u}) \le a$ and $\tilde{d}_\mathcal{Y}(f(\tilde{x},\tilde{p},\tilde{u})) \le b$ take place if δ is chosen sufficiently small. The first inequality is automatic, the second one is straightforward, since $\tilde{A}-A$ and $\tilde{B}-B$ are δ-small just in the suitable norms. Then claim (i) of the theorem follows from Proposition 1.

To prove claim (ii) we observe that due to (50) if δ is chosen sufficiently small, then $\|\tilde{p}-\hat{p}\|_{k,\infty}$ will be small enough, so that the controllability index of of $(\tilde{x},\tilde{p},\tilde{u})$ does not exceed k. Then the strong bi-metrically regularity of order k of $\tilde{F}(x,p,u)$ at $((\tilde{x},\tilde{p},\tilde{u}),0)$ follows from Theorem 3. Q.E.D.

Remark 5 In view of Lemma 1 the above theorem can be easily translated in terms of the solutions $(\hat{x},\hat{u})$ and $(\tilde{x},\tilde{u})$ of problem (1)–(3) with matrices (A,B) and $(\tilde{A},\tilde{B})$.

We need that the disturbances in the differential equation (2) are linear in order to ensure that the resulting disturbance $\tilde{y}$ in the proof belongs to the space $\mathcal{Y}_k$ (in order to apply Proposition 1) with $k > 1$. For $k = 1$ the linearity can be relaxed, as in Theorem 4, but in this case the result in this theorem is stronger than that of Theorem 5, anyway.

We mention that a result in the same spirit as Theorem 5 is proved in [8] in the case $k = 1$ and with $U = [-1,1]^r$. It concerns the stronger notion of *structural stability* and the proof relays on an inverse function theorem for the switching points of the optimal control, which has no counterpart in the case $k > 1$. Notice that in the case $k = 1$ the statement of the above theorem is much weaker than that of Theorem 4.

References

[1] W. Alt, R. Baier, M. Gerdts, F. Lempio. Error bounds for Euler approximations of linear-quadratic control problems with bang-bang solutions. *Numerical Algebra, Control and Optimization*, **2**(3):547-570, 2012.

[2] W. Alt and M. Seydenschwanz. An implicit discretization scheme for linear-quadratic control problems with bang-bang solutions. Submitted.

[3] W. Alt, R. Baier, F. Lempio, M. Gerdts. Approximations of linear control problems with bang-bang solutions. *Optimization*, **62**(1):9–32, 2013.

[4] J.-P. Aubin, H. Frankowska, *Set-valued analysis.*, Birkhäuser, 1990.

[5] A.L. Dontchev and W.W. Hager. Lipschitzian stability for state constrained nonlinear optimal control. *SIAM J. Control Optim.*, **36**(2):698–718, 1998.

[6] A. Dontchev, R.T Rockafellar. *Implicit functions and solution mappings.*, Springer, 2009.

[7] A. Dontchev, V. Veliov. Metric regularity under approximations. *Control&Cybernetics*, **38**(4B):1275-1282, 2009.

[8] U. Felgenhauer. On stability of bang-bang type controls. *SIAM J. of Control and Optimization*, **41**(6):1843-1867, 2003.

[9] U. Felgenhauer. Variational inequalities in bang-singular-bang control investigation. (to appear).

[10] U. Felgenhauer, L. Poggolini, G. Stefani. Optimality and stability result for bang-bang optimal controls with simple and double switch behavior. *Control&Cybernetics*, **38**(4B):1305-1325, 2009.

[11] H. Frankowska, Some inverse mapping theorems, Ann. Inst. Henri Poincaré Analyse Non Linéaire, **7** (1990), 183-234.

[12] H. Frankowska, M. Quincampoix. Hölder metric regularity for set-valued maps. *Math. programming, Ser. A*, **132**(1):333-354, 2012.

[13] L. S. Pontryagin, V. G. Boltyanskij, R. V. Gamkrelidze, E. F. Mishchenko, *The mathematical theory of optimal processes*, Fizmatgiz, Moscow, 1961 (Pergamon, Oxford, 1964).

[14] V.M. Veliov. On the convexity of integrals of multivalued mappings: application in control theory. *Journal of Optimization Theory and Applications*, **34**(3):541-563, 1987.

[15] V.M. Veliov. Error analysis of discrete Approximation to bang bang optimal control problems: the linear case. *Control&Cybernetics*, **34**(3):967-982, 2005.