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Optimal cyclic exploitation of renewable resources*

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Abstract

The paper contributes to the topic of optimal utilization of spatially distributed renewable resources. Namely, a problem of "sustainable" optimal cyclic exploitation of a renewable resource with logistic law of recovery is investigated. The resource is distributed on a circle and is collected by a single harvester moving along the circle. The recovering and harvesting rates are position-dependent, and the latter depends also on the speed of the harvester, which is considered as a control. Existence of an optimal solution is proved, as well as necessary optimality conditions for the velocity of the harvester. On this base, a numerical approach is proposed, and some qualitative properties of the optimal solutions are established. The results are illustrated by numerical examples, which reveal some economically meaningful features of the optimal harvesting.

Keywords:

1 Introduction

The issue of optimal extraction of spatially distributed renewable resources is gaining a considerable and growing attention during the past decade (see e.g. [5, 4, 6, 7]). On the other hand, it poses analytical challenges, especially in models where the space is heterogeneous, meaning that some parameters (then also the optimal polices) are position-dependent, as in [5, 4, 6] and in the present paper.

Paper [7] considers an agent moving around a circle and harvesting renewable natural resource at her location. The speed of the motion and the amount of harvest

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at each time are chosen to maximize the profit (revenue minus expenses for harvesting and speed) discounted over the finite time. Paper [5] is devoted to the stability of patterns on a circle, created by the state variable in an distributed infinite-horizon optimal control problem. In [4] a similar special-diffusion optimal control problem is considered with different practical examples including distributed (on a circle) fishing around an island. Paper [6] studies a system with two state variables distributed on a circle with diffusion and harvesting by distributed agents. Along with a decentralized case a social planner problem is analyzed, where discounted aggregated utilities of all agents are maximized over infinite time.

The present paper studies the problem of optimal harvesting where the space is a circle. The resource grows according to the logistic dynamic low with parameters depending on the position on the circle. Resource is collected by a harvester at a single point at each moment of time. The harvesting intensity is assumed to be determined by the speed of the harvester at its current position and the detection/extraction (acquisition) rate at this position. The latter is a given (space-heterogeneous) data, while the former is viewed as a decision variable (control), which is assumed time-invariant. That is, the control—the speed of the harvester—depends only on the position of the harvester on the circle, and is kept the same on each round. In other words, an optimal *cyclic* harvesting is sought, where the optimality criterion is formulated in terms of the long run economic revenue from the harvesting, being in this way consistent with the concept of sustainable utilization of the resource. Formally, the objective function represents the limit, with the time going to infinity, of the revenue per unit of time. Thus the considered problem belongs to the class of averaged infinite-horizon problems []. **VV: Alexey, please give a reference.**

The model considered in this paper differs from [7] in that (i) homogeneity of the space is assumed in [7], (ii) the dynamics of the resource is simpler in [7] (exponential growth, rather than a logistic one), (iii) the objective is to maximize the total discounted revenue, while the averaged revenue is maximized here.

On the other hand, papers [5, 4, 6] consider a more complicated resource dynamics involving diffusion, which is reasonable for "moving" resources. We consider resources growing "on the spot", which is a simplification allowing to obtain more analytic results. Moreover, the problem considered in the present paper is different from those in [5, 4, 6], since (i) we maximize the averaged revenue rather than the discounted one as in these papers, and (ii) we consider extraction by a moving harvester, in contrast to the distributed harvesting in [5, 4, 6].

Below we describe the main results, together with the plan of the paper.

In Section 2 we give a precise formulation of the considered problem. Then in Section 3 we prove that when a cyclic control is applied, the amount of harvested resource stabilizes at every point on the circle to an explicitly determined limit value. This allows to reformulate the original averaged infinite-horizon problem as a (non-smooth) static distributed optimization problem on the circle.

In Section 4 we prove existence of an optimal control, which is not a standard task, as far as the static (maximization) problem into question turns out to be non-concave with respect to the control. The auxiliary properties obtained in this section are used also in the subsequent considerations.

In Section 5 we prove necessary optimality conditions that result in an approach for solving the problem numerically. A conceptual algorithm for that is presented, which reduce the original distributed problem to a scalar one.

Using the obtained optimality conditions we prove in Section 6 some qualitative properties of the optimal solution, which give some information about the possibility that the optimal harvesting involves periods of recovery without harvesting or not. Numerical experiments are also presented in this section. They illustrate and support the theoretical results and show some properties that are interesting from economic point of view. In particular, it is shown that technological progress (represented by the acquisition rate) leads, as expected, to a higher revenue, but as a byproduct it may lead to larger deserted areas (without any resource left in the long run) and to longer periods of no-harvesting.

Finally, in Section 7 we discuss some reasonable extensions that may be a subject of a future investigation.

2 Statement of the problem, assumptions and discussions

We begin with a short informal description of our harvesting scenario. The spatial domain into consideration is a one-dimensional closed curve, which we identify with the unit circle S^1. Assume that some renewable resource grows at each point $x \in S^1$. The low of growth can be specific for each point x, thus the space may be heterogeneous. A "harvesting machine" moves counterclockwise around the circle and at any time t picks a fraction of the available recourse at its current position $x(t)$. The harvested fraction depends on the speed of the machine at the current point. The latter is used as a control variable. Moreover, the harvesting machine starts at time $t = 0$ from a point $O \in S^1$ and we assume that after each round the machine can stop and stay for awhile at O, while it moves with a strictly positive speed on $S^1 \setminus \{O\}$.

In the next lines we give a formal description of the above scenario. For $t \geq 0$ and $x \in S^1$ we denote by $p(t, x)$ the amount of resource available at position x at time

t. Strictly speaking, $p : [0, \infty) \times S^1$ is absolutely continuous in t and measurable and bounded in x. The dynamics of the resource at point x (if not harvested) is described by the equation

$$\dot{p}(t,x) = (a(x) - b(x)p(t,x))\, p(t,x), \quad p(0,x) = p_0(x), \tag{1}$$

where (as everywhere in this paper) an upper dot "$\dot{}$" denotes differentiation with respect to the time t, $p_0(x)$ is an initial value, $a(x)$ and $b(x)$ are position-dependent parameters. As usual in this logistic equation, the parameters $a(x)$ and $b(x)$ characterize recovery and competition processes involved in the evolution of renewable resources.

Let $x : [0, \infty) \to S^1$ be an absolutely continuous function representing the position of the harvesting machine on S^1. Then $\dot{x}(t)$ is the (tangential) velocity of the machine at the point $x(t)$.

If at time t the machine crosses point $x \in S^1$ with a speed $v(x) > 0$ (that is, $x = x(t)$ and $v(x) = \dot{x}(t)$), then a fraction

$$1 - e^{-\gamma(x) r(x)}$$

of the available resource at x is harvested. Here $r(x) = 1/v(x)$ and $\gamma(x)$ is a resource acquisition parameter, which characterizes the ability to detect/extract resource at x. The above expression has its foundation in the retrieval theory. The higher is the speed $v(x)$ the lower is the value $r(x)$, hence the harvested fraction. The function $r : S^1 \to (0, \infty)$ is called *harvesting effort density* (see e.g. [1], [2]).

The harvesting leads to a jump-down of the resource at the point $x = x(t)$ which the harvesting machine crosses at time t. If $r(x)$ is the harvesting effort at this point, then

$$\begin{aligned} p(t+0,x) &= p(t-0,x) - \left(1 - e^{-\gamma(x)r(x)}\right) p(t-0,x) \\ &= e^{-\gamma(x)r(x)}\, p(t-0,x). \end{aligned} \tag{2}$$

In the sequel the effort density r will be used as a control function, therefore we impose bounds, which are position-specific:

$$r_1(x) \le r(x) \le r_2(x), \quad x \in S^1, \tag{3}$$

where $0 < r_1(x) \le r_2(x)$ are given numbers. Every measurable function $r : S^1 \to [0, \infty)$ satisfying the above inequalities is called *admissible effort.* In terms of the velocity, the above constraints mean that the harvesting machine cannot move with a speed higher than $1/r_1(x)$ and lower than $1/r_2(x)$ when crossing point x.

The following assumptions are needed for the formulation of the problem that we address, and for the further analysis.

Standing assumptions: The functions a, b, p_0, γ, r_1 and r_2 (all defined on S^1 and having real values) are non-negative, (Lebesgue) measurable and bounded, $p_0(x)$ is strictly positive. There exists a subset $S \subset S^1$ of positive measure where both $a(x)$ and $\gamma(x)$ are strictly positive. For some strictly positive constants b_0 and $\bar{r}$ it holds that

$$b(x) \geq b_0, \;\; 0 < r_1(x) \leq r_2(x) \leq \bar{r} \quad \text{for every } x \in S^1.$$

Given an admissible harvesting effort r, the point $x(t)$ (this is the harvesting machine) that starts at time $t = 0$ from the origin O and moves with speed $v(x) = 1/r(x)$ at position x will return to O at time

$$T_h(r) := \int_{S^1} r(x)\,\mathrm{d}x$$

(it is assumed that the length of S^1 equals 1 in the chosen measurement units). This follows from the identity $\tau = \int_0^\tau \mathrm{d}t$ in which one can change the variable t from $x(t) = s$ and use that $\mathrm{d}t = r(s)\,\mathrm{d}s$.[1] Being back at the origin O, the harvesting machine can stay there for a certain time $T_0 \geq 0$. Then at time $T = T_h(r) + T_0$. the harvesting machine moves again around the circle with the same harvesting effort $r(x)$ (and corresponding speed $v(x) = 1/r(x)$), followed by a pause T_0 at the origin O. The same motion is repeated further. In this way any admissible effort density r and time T satisfying

$$\int_{S^1} r(x)\,\mathrm{d}x \leq T \tag{4}$$

define a periodic motion $x(\cdot)$ with period T around the circle. The corresponding stopping time is $T - T_h(r)$.

Every pair (r, T) of a measurable function r and $T > 0$ satisfying (3) and (4) will be called *admissible harvesting policy.* The total revenue from the admissible

[1] The justification of this change of variables, as well as the one that appears below, requires attention. First of all, within one round the point $x(t)$ can be identified with the distance along the circle to the origin. Since we have $v(x) \geq 1/\bar{r}$, $x(t)$ is a strictly monotone absolutely continuous function satisfying $\dot{x}(t) = 1/r(x(t))$, $x(0) = 0$. Notice that the superposition $r(x(t))$ is measurable thanks to the strict monotonicity of $x(\cdot)$. For the change of the variable t from $s = x(t)$, where $x(t)$ appears as an argument of a measurable function, one may use [8, Theorem I.4.43].

harvesting policy (r, T) in the time period $[kT, (k+1)T]$ will be

$$\begin{aligned} J_k(r,T) \quad &:= \quad \int_{kT}^{(k+1)T} p(t, x(t)) \left(1 - e^{-\gamma(x(t))r(x(t))}\right) \mathrm{d}x(t) \\ &= \quad \int_{kT}^{kT+T_h(r)} p(t, x(t)) \left(1 - e^{-\gamma(x(t))r(x(t))}\right) \frac{1}{r(x(t))} \,\mathrm{d}t. \end{aligned}$$

Here $p(t, x)$ is defined by equation (1), regarding the jump condition (2). To make this precise, denote by $\tau(x)$ the time-instant at which $x(\tau) = x$ during the first rotation of the harvesting machine. Notice that $x(t) = x$ at any time $t = \tau(x) + kT$, that is, $\tau(x) + kT$ are the harvesting instances at place x. Then for every x the function $p(\cdot, x)$ is absolutely continuous, satisfies (1) on each interval $(kT+\tau(x), (k+1)T+\tau(x))$, and jumps down at $kT+\tau(x)$ according to (2). Notice that the solution $p(\cdot, x)$ solution exists on $[0, \infty)$ and is non-negative and bounded from above by $\bar{p} = \max\{\|p_0\|_{L_\infty}, \|a\|_{L_\infty}/b_0\}$. Moreover, p is piece-wise absolutely continuous in t and measurable and bounded in x. The superpositions $r(x(t))$ and $p(t, x(t))$ are measurable due the strict monotonicity of $x(\cdot)$ (see footnote 1). We consider $p(\cdot, x)$ as continuous from the left at the jump points, so that $p(t, x) = p(t-0, x)$ at any t.

Changing the variable t with $s = x(t)$ (see again footnote 1) we represent

$$J_k(r, T) = \int_{S^1} p(kT + \tau(x), x) \left(1 - e^{-\gamma(x)r(x)}\right) \mathrm{d}x.$$

This paper is devoted to the problem of maximization of the average long run revenue of harvesting:

$$\max_{r,T} \lim_{k \to +\infty} \frac{J_k(r, T)}{T}, \tag{5}$$

where the maximization is carried out on the set of all admissible harvesting policies (r, T), that is, subject to the constraints (3) and (4). Below we discuss the problem formulation and the assumptions.

Since $J_k(r, T)$ is the harvesting revenue in $[kT, (k+1)T]$ resulting from policy (r, T), the functional to be maximized is the efficiency of the harvesting in the k-th period (revenue per unit of time). This means that our problem is to find an admissible harvesting policy (which by definition is time-invariant) which has maximal efficiency in the long run.

The objective functional in (5) is not the only reasonable one to be maximized. Indeed: (i) it involves a priory only periodic harvesting policies; (ii) it takes into account only the long run performance. In particular, the amount of initial resource $p_0(x)$ turns out to be irrelevant (as it will be show below). These are essential shortcomings of our model, but on the other hand, features (i) and (ii) have clear counterparts in forestry, fishery, and rotational utilization of agricultural land.

In the presented model the harvesting effort is assumed to be inversely proportional to the speed of the harvesting machine. This assumption is plausible in many cases, but clearly it is a simplification. An alternative could be to include the intensity of harvesting as an additional position-dependent control function. However, this would bring some redundancy in the control capacity, which leads to technical inconveniences. The pause T_0 that the harvesting machine may enjoy at the origin is a "compensation" for that the harvesting effort cannot be set independent of the speed. It allows the resource to recover, similarly as lower than maximal possible harvesting effort would do. There is no harvesting during the pause, the harvesting machine just waits to start a next rotation. Such a feature could be observed, for example, in seasonal cycles of various nature (REFERENCES???)**Alexey**. We mention also that the requirement that the harvesting machine may stop only at the origin O is not essential for the results below (not counting the numerical illustrations).

Next, we mention that the restriction $r(x) \in [r_1(x), r_2(x)]$ and the respective requirements in the standing assumptions are natural, since they translate into restrictions for the speed of rotation, which are physically plausible.

The assumption that the initial resource density $p_0(x)$ is everywhere positive is technically convenient, but not restrictive at all. Indeed, if $p_0(x) = 0$ on a set of positive measure, then we may redefine it as a positive constant and set $\gamma(x) = 0$ for these x. Then there will be no harvesting at x.

Finally, we mention that a function $a(x)$ in (1) taking negative values might be meaningful, in general, if we deal with a self-distracting resource. However, in our long run optimization problem, $a(x) < 0$ is equivalent to $a(x) = 0$, since the resource at such locations x will asymptotically vanish.

3 Reformulation of the problem

In this section we obtain an explicit representation of the limit in (5), which will allow to reduce the problem to a static one.

For a fixed admissible harvesting policy (r, T) we denote $p_k(x) = p(kT+\tau(x), x)$. In view of (2), $p_{k+1}(x)$ is the solution of equation (1) on a time interval with length T (notice that the equation is stationary) with initial value $p(kT + \tau(x) + 0, x) = e^{-\gamma(x)r(x)}p_k(x)$. Since (1) is a Riccati equation, we find explicitly that

$$p_{k+1}(x) = \frac{p_k(x)}{e^{\gamma(x)r(x)-aT} + \frac{b(x)}{a(x)}\left(1 - e^{-a(x)T}\right)p_k(x)} \quad \text{if } a(x) > 0$$

and

$$p_{k+1}(x) = \frac{p_k(x)}{e^{\gamma(x)r(x)} + b(x)p_k(x)\,T} \quad \text{if } a(x) = 0.$$

In the second case, as well as in the case $p_0(x) = 0$ (which, however, is excluded by assumption) we obviously have $p_\infty(x) = 0$. In the first case $p_k(x)$ satisfies the recurrence

$$p_{k+1} = \frac{p_k}{\alpha + \beta p_k}, \quad \text{where } \alpha := e^{\gamma(x)r(x)-a(x)T} > 0, \quad \beta := \frac{b(x)}{a(x)}\left(1 - e^{-a(x)T}\right) > 0.$$

If $\alpha \geq 1$, then $p_{k+1} \leq p_k/(1 + \beta p_k)$ and apparently $p_k \longrightarrow 0$. Now let $\alpha < 1$ and $p_0 > 0$. The mapping $p \mapsto p/(\alpha + \beta p)$ is monotone increasing and has two fixed points: $p = 0$ and $p = p^* := (1 - \alpha)/\beta$. If for some k we have $p_k \geq p^*$, then

$$p_k = \frac{p_k}{\alpha + \beta p^*} \geq \frac{p_k}{\alpha + \beta p_k} = p_{k+1} = \frac{p_k}{\alpha + \beta p_k} \geq \frac{p^*}{\alpha + \beta p*} = p^*.$$

This means that p_k is monotonically convergent to p^*. If $p_k \leq p^*$, then the inverse chain of inequalities holds, hence p_k converges to p^* as well. Summarizing we obtain that for every $x \in S^1$

$$\lim_{k \to +\infty} p(kT + \tau(x), x) = \begin{cases} 0 & \text{if } \gamma(x)r(x) \geq a(x)T, \\ \frac{1-e^{\gamma(x)r(x)-a(x)T}}{\frac{b(x)}{a(x)}\left(1-e^{-a(x)T}\right)} & \text{if } \gamma(x)r(x) < a(x)T, \end{cases} \tag{6}$$

Thanks to the uniform boundedness of $p(kT + \tau(x), x)$ (an upper bound $\bar{p}$ was introduced in Section 2) and the above convergence, we can pass to the limit under the integral in (5) due to the Lebesgue dominated convergence theorem. Then problem (5) can be reformulated as

$$\max_{r,T} \frac{1}{T} \int_{S^1} p_\infty(x, r(x), T)\left(1 - e^{-\gamma(x)r(x)}\right) \, \mathrm{d}x, \tag{7}$$

where $p_\infty(x, r, T)$ results from the right-hand side of (6):

$$p_\infty(x, r, T) := \max\left\{0, \; \frac{a(x)}{b(x)}\,\frac{e^{a(x)T} - e^{\gamma(x)r}}{e^{a(x)T} - 1}\right\}.$$

In the special case $a(x) = 0$ (in which the above right-hand side is indefinite) we have $p_\infty(x, r, T) = 0$, according to (6). The maximization in (7) is carried out on the set of all admissible harvesting polices (r, T).

Further on we analyze problem (7), (4), (3).

4 Existence of an optimal solution

We start the investigation of problem (7), (4), (3) by proving that an optimal solution (r, T) does exist. This is not a routine task since, as it will be seen below, we deal with a maximization of a non-concave (in r) objective functional, therefore some specific feature of the problem have to be used in order to prove existence. In the same time we shall obtain some properties of the problem that will be useful for the further analysis.

To shorten the notations we define

$$f(x, r, T) := p_\infty(x, t, T) \left(1 - e^{-\gamma(x) r}\right) = \max\{0,\, g(x, r, T)\}, \tag{8}$$

where the function g can be written as follows

$$g(x, r, T) := \frac{a(x)}{b(x)} \frac{e^{a(x)T} - e^{\gamma(x) r}}{e^{a(x)T} - 1} \left(1 - e^{-\gamma(x) r}\right). \tag{9}$$

Then (7) becomes

$$\max_{r,T} \frac{1}{T} \int_{S^1} f(x, r(x), T)\,\mathrm{d}x \qquad \left(=: \max_{r,T} I(r, T)\right). \tag{10}$$

Since for every admissible policy (r, T) we have $T \geq \int_{S^1} r(x)\,\mathrm{d}x \geq \int_{S^1} r_1(x)\,\mathrm{d}x =: T^0 > 0$, we consider the function f on the domain $S^1 \times [0, \bar{r}] \times [T^0, \infty)$.

Lemma 1 *The function $f : S^1 \times [0, \bar{r}] \times \mathbb{R}_+$ has the following properties:*

(i) f is measurable in x and bounded from above by $\|a\|_{L_\infty}/b_0$;

(ii) $f(x, \cdot, T)$ is strictly positive and strictly concave on $\left(0, r^0(x, T)\right)$ and equals zero, $f(x, r, T) = 0$, for $r = 0$ and $r \geq r^0(x, T)$, where

$$r^0(x, T) = \begin{cases} \frac{a(x)T}{\gamma(x)} & \text{if } \gamma(x) > 0, \\ 0 & \text{if } \gamma(x) = 0. \end{cases}$$

(iii) f is Lipschitz continuous in $(r, T) \in [0, \bar{r}] \times [T^0, \infty)$ uniformly with respect to $x \in S^1$, that is, there exists a number L such that

$$|f(x, r, T) - f(x, r', T')| \leq L \left(|r - r'| + |T - T'|\right) \quad \text{for all } r,\, r' \in [0, \bar{r}],\ T,\, T' \in [T^0, \infty).$$

Proof. The measurability of g and (8) imply the measurability of f with respect to x. Moreover, from the definition of g in (9) we have for $r \geq 0$

$$g(x, r, T) \leq \frac{\|a\|_{L_\infty}}{b_0},$$

which proves property (i), since $f(x,r,T) = g(x,r,T)$ for $r \in \big[0, r^0(x,T)\big]$.

Obviously $g(x,r,T) \geq 0$ for $r \in \big[0, r^0(x,T)\big]$, hence $f(x,r,T) = g(x,r,T)$. In order to prove properties (ii) we calculate

$$\frac{\partial g}{\partial r}(x,r,T) = \frac{a(x)\gamma(x)}{b(x)} \, \frac{e^{a(x)T-\gamma(x)r} - e^{\gamma(x)r}}{e^{a(x)T} - 1}. \tag{11}$$

If $a(x)\gamma(x) = 0$ for some x the above derivative equals zero, but in this case $r^0(x,T) = 0$ and (ii) is trivial. In the alternative case $\frac{\partial g}{\partial r}(x,r,T)$ is strictly monotone decreasing in r, therefore g is strictly concave with respect to r. This proves properties (ii).

To prove property (iii) we notice that the maximum in (8) is achieved at g if and only if $a(x)T \geq \gamma(x)r$. In this area on the (r,T)-plane we have (skipping the argument x)

$$\begin{aligned}
\left|\frac{\partial g}{\partial T}(x,r,T)\right| &= \frac{a^2}{b}\, e^{aT} \frac{e^{\gamma r} - 1}{\left(e^{aT} - 1\right)^2} \left(1 - e^{-\gamma r}\right) \leq \frac{a^2}{b}\, e^{aT} \frac{e^{aT} - 1}{\left(e^{aT} - 1\right)^2} \left(1 - e^{-aT}\right) \\
&= \frac{\|a\|^2_{L_\infty}}{b_0} =: L_T.
\end{aligned}$$

Moreover, from (11) we have

$$\left|\frac{\partial g}{\partial r}(x,r,T)\right| \leq \frac{a(x)\gamma(x)}{b(x)} \leq \frac{\|a\|_{L_\infty}\|\gamma\|_{L_\infty}}{b_0} =: L_r.$$

Then we obtain property (iii) with $L = \max\{L_T, L_r\}$. Q.E.D.

Let $r^*(x,T)$ be the leftmost maximizer of $f(x,\cdot,T)$ on $[r_1(x), r_2(x)]$, that is,

$$r^*(x,T) := \min\{r^* : \; f(x,r^*,T) \geq f(x,r,T) \;\; \forall\, r \in [r_1(x), r_2(x)]\},$$

see Fig. 1. Notice that if $\gamma(x) > 0$ and $a(x) > 0$, then the function $f(x,\cdot,T)$ has a unique global maximizer $r = a(x)T/2\gamma(x)$, but it does not need to belong to $[r_1(x), r_2(x)]$. In the latter case $r^*(x,T)$ coincides with one of the bounds, Fig. 1 *b*), *c*), *d*). If $\gamma(x) = 0$ or $a(x) = 0$, then $r^*(x,T) = r_1(x)$. Moreover, $r^*(x,T)$ is measurable in x and continuous in T.

Lemma 2 *The supremum, $\hat{I}$, of the objective value in (7) subject to (3), (4) is finite and positive.*

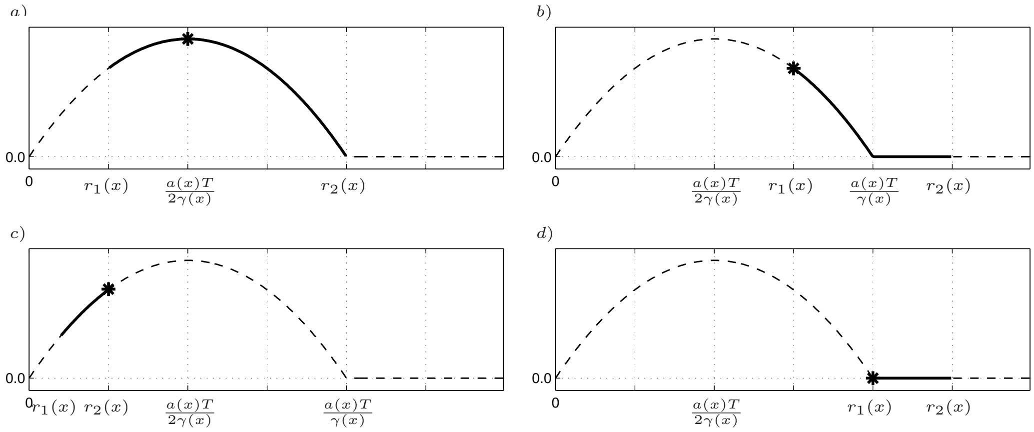

Figure 1: The function f (dashed line) and several configurations of the leftmost maximizer $r^*(x,T)$ of the restriction of f to the interval $[r_1(x), r_2(x)]$ (solid line).

Proof. Let (T, x) be an admissible policy. According to property (i) in Lemma 1 we have

$$I(r,T) \leq \frac{1}{T}\frac{\|a\|_{L_\infty}}{b_0} \leq \frac{\|a\|_{L_\infty}}{T^0 b_0},$$

thus $\hat{I} < \infty$.

According to the standing assumptions there is a subset $S_0 \subset S^1$ with positive measure and a number $\varepsilon > 0$ such that $a(x) \geq \varepsilon$ and $\gamma(x) \geq \varepsilon$ for $x \in S_0$. Let us fix T satisfying the inequalities $T \geq \int_{S^1} r_2(x)\,\mathrm{d}x$ and $\varepsilon T \geq 2\|r_2\gamma\|_{L_\infty}$. We have

$$\begin{aligned} I(r_2,T) &= \frac{1}{T}\int_{S^1} f(x, r_2(x), T)\,\mathrm{d}x \geq \frac{1}{T}\int_{S_0} f(x, r_2(x), T)\,\mathrm{d}x \\ &= \frac{1}{T}\int_{S_0} \frac{a(x)}{b(x)}\,\frac{e^{a(x)T} - e^{\gamma(x)r_2(x)}}{e^{a(x)T} - 1}\left(1 - e^{-\gamma(x)r_2(x)}\right)\mathrm{d}x, \end{aligned}$$

which is strictly positive due to the choice of T and the inequalities $a(x), \gamma(x) \geq \varepsilon$ for $x \in S_0$. Q.E.D.

Proposition 1 *Problem (7), (3), (4) has a solution.*

Proof. Let (r_k, T_k) be a maximizing sequence of admissible policies. Then the sequence T_k is bounded. Indeed, if for a subsequence $T_k \longrightarrow +\infty$, then, according to property (i) in Lemma 1,

$$I(r_k, T_k) = \frac{1}{T_k} \int_{S^1} f(x, r(x), T)\, \mathrm{d}x \leq \frac{1}{T_k} \frac{\|a\|_{L_\infty}}{b_0} \longrightarrow 0,$$

which contradicts Lemma 2. Thus we can extract a convergent subsequence (we shall use the same index k) $T_k \longrightarrow T \geq T^0$. Now define

$$\tilde{r}_k(x) = \min\{r_k(x), r^*(x, T)\}.$$

The pair $(\tilde{r}_k, T)$ does not need to be an admissible policy, but apparently $\tilde{r}_k$ is an admissible harvesting density. The set of admissible harvesting densities is convex, closed and bounded in $L_2(S^1)$, hence it is weakly compact due the lemma of Mazur and the theorem of Banach-Alaoglu [?]. **VV: more direct citation??** Then from the sequence $\tilde{r}_k$ one can extract a subsequence (we shall use again the same index k) which is L_2-weakly convergent to some admissible density r. Passing to the limit in the inequality

$$\int_{S^1} \tilde{r}_k(x)\, \mathrm{d}x \leq \int_{S^1} r_k(x)\, \mathrm{d}x \leq T_k$$

we obtain that (r, T) is an admissible policy.

Since the function $f(x, \cdot, T)$ is concave on $[0, r^0(x, T)] \supset [r_1(x), r^*(x, T)] \ni \tilde{r}_k(x)$, (see Lemma 1 (ii)) the theorem for upper semi-continuity of integral functionals with concave integrands [?] implies that

$$\limsup_{k \to +\infty} \int_{S^1} f(x, \tilde{r}_k(x), T)\, \mathrm{d}x \leq \int_{S^1} f(x, r(x), T)\, \mathrm{d}x.$$

Then

$$\begin{aligned} I(r, T) &= \frac{1}{T} \int_{S^1} f(x, r(x), T)\, \mathrm{d}x \geq \limsup_{k \to +\infty} \frac{1}{T_k} \int_{S^1} f(x, \tilde{r}_k(x), T)\, \mathrm{d}x \\ &\geq \limsup_{k \to +\infty} \frac{1}{T_k} \int_{S^1} f(x, r_k(x), T)\, \mathrm{d}x \\ &\geq \limsup_{k \to +\infty} \left[\frac{1}{T_k} \int_{S^1} f(x, r_k(x), T_k)\, \mathrm{d}x + \frac{L(T+1)}{T_k} |T_k - T| \right] = \hat{I}, \end{aligned}$$

where we use Lemma 1 (iii) in the last inequality. Hence, (r, T) is an optimal solution. Q.E.D.

We mention that the optimal harvesting polices is not unique for some data configurations. This point will be discussed in the next section.

5 Optimality conditions and a numerical approach

In order to characterize the solution(s) of problem (7), (3), (4) we split this problem into an *inner problem*, where T is fixed and the maximization is carried out only with respect to r, and *outer problem*, where (7) is maximized with respect to T, with the solution of the inner problem inserted.

Let us fix $T \geq T^0$ and consider the maximization problem

$$\max_r \frac{1}{T} \int_{S^1} f(x, r(x), T) \, \mathrm{d}x, \tag{12}$$

subject to (3), (4). Since $T \geq T^0$, the set of admissible densities r in this problem is non-empty and it has a solution $\hat{r}(\cdot, T)$, by a similar argument as in Proposition 1. If $T = T^0$, then the considered problem is trivial, since it has a unique admissible density $r = r_1$. We shall obtain necessary optimality conditions for the above problem with $T > T^0$, utilizing an elaborated form of the Lagrange principle given e.g. in [3, Section 4.2.2].

Theorem 1 *Let $T > T^0$ be fixed and let $\hat{r}(\cdot, T)$ be an optimal solution of problem (12), (3), (4). Then there exists a number (Lagrange multiplier) $\hat{\lambda} \geq 0$ such that*

(a) for a.e. $x \in S^1$ the number $\hat{r}(x, T)$ solves the problem of maximization of the point-wise Lagrangian,

$$\max_{r \in [r_1(x), r_2(x)]} \left\{ \frac{1}{T} f(x, r, T) - \hat{\lambda} \, r \right\}; \tag{13}$$

(b) the complementary slackness condition holds:

$$\hat{\lambda} \left(\int_{S^1} \hat{r}(x, T) \, \mathrm{d}x - T \right) = 0. \tag{14}$$

Proof. The theorem in [3, Section 4.2.2] claims in our rather special case that there exist non-negative Lagrange multipliers λ_0, λ such that $\lambda_0 + \lambda > 0$, (14) holds, and for a.e. $x \in S^1$ the value $\hat{r}(x, T)$ maximizes the pointwise Lagrange function

$$\frac{\lambda_0}{T} f(x, r, T) - \lambda r \tag{15}$$

on the set $[r_1(x), r_2(x)]$.

We shall prove that one can always take $\lambda_0 = 1$. If $\lambda_0 = 0$ then $\lambda > 0$. In that case the only solution of problem (15) is $r(x) = r_1(x)$. Then condition (14) yields

$\int_{S^1} r_1(x)\,\mathrm{d}x = T$, which means $T = T^0$ and contradicts the assumed $T > T^0$. Thus $\lambda_0 > 0$ and we can take in (15) $\hat{\lambda} = \lambda/\lambda_0$ instead of λ and 1 instead of λ_0.

Q.E.D.

Remark 1 Using the theorem in [3, Section 4.2.2] gives a substantial advantage compared with standard Lagrange and Karush-Kuhn-Tacker theorems, since the solution is claimed to be a global maximizer of the Lagrange function, although the function f is not concave in r (the so-called *hidden convexification*, known from the Pontryagin maximum principle). This avoids the need of using the derivative of the Lagrange function (notice that f is not differentiable in r!).

For a fixed $T \geq T^0$ and $\lambda \geq 0$ we consider the problem that arises in (13):

$$\max_{r \in [r_1(x), r_2(x)]} \left\{ \frac{1}{T} f(x, r, T) - \lambda\, r \right\}. \tag{16}$$

Taking into account the properties of the function $f(x, \cdot, T)$ in Lemma 1 (see also Fig 1) we consider the following cases.

Case 1: $\lambda > 0$. In this case for every $x \in S^1$ there is a unique maximizer $r^*(x, T; \lambda)$ in (16) and

$$r^*(x, T; \lambda) = \min\{r_2(x),\, \max\{r_1(x), \bar{r}(x, T; \lambda)\}\}\,, \tag{17}$$

where $\bar{r}(x, T; \lambda)$ is the unique maximizer of $f(x, r, T) - \lambda\, rT$ on $r \in [0, \infty)$. If $a(x)\,\gamma(x) = 0$ for some x, then $\bar{r}(x, T; \lambda) = 0$, otherwise $\bar{r}(x, T; \lambda)$ can be determined by solving with respect to r the equation

$$\frac{\partial g}{\partial r}(x, r, T) - \lambda\, T = 0,$$

where $\frac{\partial g}{\partial r}(x, r, T)$ is given in (11). Simple calculations give the following expression:

$$\bar{r}(x, T; \lambda) = \frac{a(x)T}{2\gamma(x)} - \frac{1}{\gamma(x)} \ln\left(m + \sqrt{m^2 + 1} \right), \tag{18}$$

where

$$m = \lambda\, T \frac{b(x)}{2a(x)\gamma(x)} \left(e^{\frac{a(x)T}{2}} - e^{-\frac{a(x)T}{2}} \right).$$

An equivalent representation in terms of hyperbolic functions reads as

$$\bar{r}(x, T; \lambda) = \frac{a(x)T}{2\gamma(x)} - \frac{1}{\gamma(x)} \operatorname{arsinh}\left(\lambda T \frac{b(x) \sinh(\frac{a(x)T}{2})}{a(x)\gamma(x)} \right), \tag{19}$$

Case 2: $\lambda = 0$, $a(x)\,\gamma(x) > 0$, and $r_1(x) < r^0(x,T)$. In this case the objective function in (16) also has a unique maximizer and it is given by (17) and (18) or (19) as in Case 1.

Case 3: $\lambda = 0$ and either $a(x)\,\gamma(x) = 0$ or $r_1(x) \geq r^0(x,T)$. In this case either $f(x,\cdot,T)$ is identically zero, or it is zero at least on $[r_1(x), r_2(x)]$. Thus any element of $[r_1(x), r_2(x)]$ solves (16). In this case it is reasonable to define $r^*(x,T;\lambda) = r_1(x)$. Indeed, with this definition the function $r^*(\cdot,T;\lambda)$ is the minimal solution of (16) and hence, it the is most favorable from the point of view of constraint (4).

Combining the above analysis with Theorem 1 we obtain the following corollary.

Corollary 1 *Let $\hat{r}$ be a solution of problem (12), (3), (4) with a fixed $T \geq T^0$. Then there exists $\hat{\lambda} \geq 0$ such that $\hat{r}(x) \geq r^*(x,T;\hat{\lambda})$ for every $x \in S^1$,*

$$\hat{r}(x) = r^*(x,T;\hat{\lambda}),$$

for all $x \in S^1$ for which Case 1 or Case 2 takes place, and the complementary slackness condition (14) holds.

Now we present an approach for determining the Lagrange multiplier $\hat{\lambda}$ needed to solve the inner problem (12), (3), (4). In the numerical procedure we target finding some solution, rather than all solutions, in the special cases where the optimal solution is not unique.

It is important to observe that the function $\lambda \to \bar{r}(x,T;\lambda)$, and hence also $\lambda \to r^*(x,T;\lambda)$, is monotone non-increasing in λ. This is evident from each of the representations (18) and (19) of $\bar{r}(x,T;\lambda)$. From (18) or (19) it is also obvious that

$$\lim_{\lambda\to\infty} \int_{S^1} r^*(x,T;\lambda)\,\mathrm{d}x = \int_{S^1} \lim_{\lambda\to\infty} r^*(x,T;\lambda)\,\mathrm{d}x = \int_{S^1} r_1(x)\,\mathrm{d}x = T^0.$$

This means that when λ runs increasingly on $(0,\infty)$ the value $\int_{S^1} r^*(x,T;\lambda)\,\mathrm{d}x$ covers continuously the interval $\big(T^0, \int_{S^1} r^*(x,T;0)\,\mathrm{d}x\big)$ from right to left. In addition, the function $\lambda \mapsto f(x, r^*(x,T;\lambda), T)$ is monotone non-increasing.

These observations allow to formulate the following conceptual algorithm for calculation of an optimal solution $\hat{r}(\cdot,T)$ of problem (12), (3), (4) for every $T \geq T^0$.

Conceptual algorithm:

1. If $T = T^0$, set $\hat{r}(\cdot,T) = r_1(\cdot)$.
2. If $T \in \big(T^0, \int_{S^1} r^*(x,T;0)\,\mathrm{d}x\big)$, then increase λ starting from $\lambda = 0$ until reaching the minimal $\hat{\lambda} = \hat{\lambda}(T) > 0$ such that $\int_{S^1} r^*(x,T;\hat{\lambda})\,\mathrm{d}x = T$. Then set $\hat{r}(\cdot,T) = r^*(\cdot,T;\hat{\lambda})$.
3. If $T \geq \int_{S^1} r^*(x,T;0)\,\mathrm{d}x$, set $\hat{r}(\cdot,T) = r^*(\cdot,T;0)$.

Proposition 2 *For every $T \geq T^0$ the function $\hat{r}(\cdot, T)$ defined in the algorithm is an optimal solution of problem (12), (3), (4).*

Proof. Point 1 is clear. Let T, $\hat{\lambda}$, and $r^*(\cdot, T)$ be as in point 2. Let $r^\sharp$ be an optimal solution of problem 12), (3), (4) for this T. Then, according to Corollary 1, there exists $\lambda^\sharp \geq 0$ such that $r^\sharp(x) = r^*(x, T; \lambda^\sharp)$ for all $x \in S^1$ for which Case 1 or Case 2 takes place.

Due to the minimality of $\hat{\lambda}$ and we have

$$\int_{S^1} r^*(x, T; \lambda)\, \mathrm{d}x > T$$

for every $\lambda < \hat{\lambda}$. Since $(r^\sharp, T) = (r^*(x, T; \lambda^\sharp), T)$ is an admissible policy, we obtain that $\lambda^\sharp \geq \hat{\lambda}$.

We consider the following three cases.

(i) If $\hat{\lambda} < \lambda^\sharp$, then

$$f(x, r^\sharp(x)) = f(x, r^*(x, T; \lambda^\sharp), T) \leq f(x, r^*(x, T; \hat{\lambda}), T) = f(x, \hat{r}(x, T), T),$$

where the first equality holds due to Corollary 1, since $\lambda^\sharp > 0$ and only Case 1 is relevant. Then $\hat{I}(T) = I(r^\sharp, T) \leq I(\hat{r}(\cdot, T), T) \leq \hat{I}(T)$. This implies the optimality of $\hat{r}(\cdot, T)$.

(ii) If $\lambda^\sharp = \hat{\lambda}$, then Case 1 takes place since $\hat{\lambda} > 0$ and $r^*(\cdot, T; \hat{\lambda})$ is uniquely determined, hence $r^\sharp(\cdot) = \hat{r}(\cdot, T)$.

Let us consider point 3 of the algorithm. Let $r^\sharp$ be an optimal solution of problem (12), (3), (4) and let $\lambda^\sharp \geq 0$ be the Lagrange multiplier from Corollary 1, so that $r^\sharp(x) = r^*(x, T; \lambda^\sharp)$ for those $x \in S^1$ for which Case 1 or Case 2 takes place.

If $\lambda^\sharp = 0$, then $r^\sharp(x) = r^*(x, T; \lambda^\sharp) = \hat{r}(x, T)$ for those x for which $a(x)\gamma(x) > 0$ and $r_1(x) < r^0(x, T)$ (Case 2). For x for which Case 3 take place we have $f(x, \hat{r}(x, T), T) = 0$ and $\hat{r}(x, T) = r_1(x) \leq r^\sharp(x)$, hence $f(x, r^\sharp(x), T) = 0$. Thus $I(\hat{r}(\cdot, T), T) = I(r^\sharp(\cdot), T)$ and $\hat{r}(\cdot, T)$ is optimal.

If $\lambda^\sharp > 0$, we obtain the optimality of $\hat{r}(\cdot, T)$ in the same way as in (i) above. Q.E.D.

Being able to determine an optimal solution $\hat{r}(\cdot, T)$ of problem (12), (3), (4) and to calculate the corresponding optimal value $\hat{I}(T) := I(\hat{r}(\cdot, T), T)$ it remains to solve numerically the outer problem

$$\max_{T \in [T^0, \infty)} \hat{I}(T). \tag{20}$$

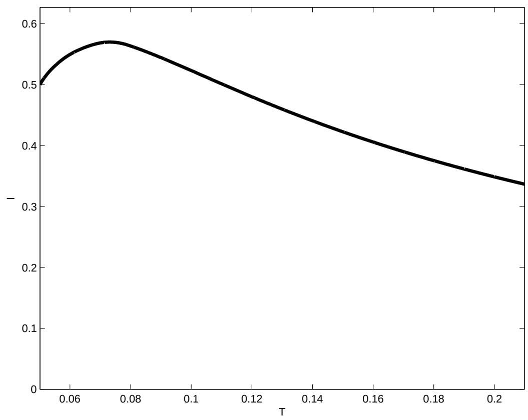

Figure 2: Objective function $I(T)$ with harvesting intensity $\gamma \equiv 0.45$, $r_1 = 0.05$, $r_2 = 0.1$, $b(x) \equiv 0.5$, $a(x) = 1 + 0.6\cos(4\pi x)$.

Plot of the function $\hat{I}(T)$ is given in Fig 2. We mention that in our numerical experiments the function $\hat{I}(T)$ is not monotone in general and even not concave. Thus, the uniqueness of the optimal period $\hat{T}$ is questionable.

We mention that the above two-stage numerical procedure involves two nested cycles – an outer one for T and an inner one for λ. However, a cleverer implementation substantially shortens one of them and practically reduces the calculations to a single loop.

The following corollary combines the results of this section with the existence of an optimal solution from the previous section.

Corollary 2 *Problem (7), (3), (4) has a solution in the form* $(r^*(x, \hat{T}; \hat{\lambda}), \hat{T})$*, where according to the conceptual algorithm*

$$\hat{\lambda} = \min_{\lambda \geq 0} \left\{ \lambda : \int_{S^1} r^*(x, \hat{T}; \lambda)\, \mathrm{d}x \geq \hat{T} \right\} \tag{21}$$

and

$$r^*(x, \hat{T}; \lambda) := \begin{cases} \min\left\{ r_2(x),\, \max\left\{ r_1(x), \bar{r}(x, \hat{T}; \lambda) \right\} \right\} & \text{if } \gamma(x)a(x) > 0, \\ r_1(x) & \text{if } \gamma(x)a(x) = 0, \end{cases} \tag{22}$$

where $\bar{r}(x, \hat{T}; \lambda)$ *is given by (18) or (19).*

6 Some qualitative properties and numerical results

In the first part of this section we prove two propositions which investigate the effect of the parameters $a(x)$ and $\gamma(x)$ on the optimal harvesting policy. We also provide sufficient conditions for the upper bound r_2 to be not active. i.e. when velocity is always higher than its lower bound $1/r_2$. In doing this we shall use the notations and the results from the last section.

Proposition 3 *There exists a measurable function $\bar{r}_2(x) \geq r_1(x)$ such that the following claim holds true: for every measurable upper bound $r_2(x) \geq \bar{r}_2(x)$, if*

$$\int_{S^1 \setminus \{x\colon \gamma(x)=0\}} \frac{a(x)}{\gamma(x)} \,\mathrm{d}x > 2, \tag{23}$$

then for every solution $(\hat{r}, \hat{T})$ of problem (7), (3), (4) there are no stops, i.e. $\int_{S^1} \hat{r}(x)\,\mathrm{d}x = \hat{T}$, and for a.e. x where $\gamma(x) > 0$ and $r_1(x) < \frac{\hat{T}a(x)}{2\gamma(x)}$ we have

$$\hat{r}(x) < \frac{\hat{T}}{2}\frac{a(x)}{\gamma(x)}. \tag{24}$$

Proof. According to Lemma 2 the supremum $\hat{I}$ is finite and positive. Let $(\hat{r}, \hat{T})$ be an optimal solution of problem (5), (3), (4). Due to Lemma 1(i) we have

$$\hat{I} = \frac{1}{\hat{T}} \int_{S^1} \hat{r}(x)\,\mathrm{d}x \leq \frac{\|a\|_{L_\infty}}{\hat{T} b_0},$$

hence $\hat{T} \leq \frac{\|a\|_{L_\infty}}{\hat{I}\, b_0} := \bar{T}$.

Define

$$\bar{r}_2(x) = \begin{cases} \max\left\{ r_1(x), \frac{a(x)\bar{T}}{2\gamma(x)} \right\} & \text{if } a(x)\,\gamma(x) > 0, \\ r_1(x) & \text{if } a(x)\,\gamma(x) = 0, \end{cases}$$

and let r_2 be an arbitrary measurable function satisfying $r_2(x) \geq \bar{r}_2(x)$. Then whenever $\gamma(x) > 0$ we have

$$r^*(x, \hat{T}; 0) = \max\left\{ r_1(x), \min\left\{ r_2(x), \frac{\hat{T}a(x)}{2\gamma(x)} \right\} \right\} = \max\left\{ r_1(x), \frac{\bar{T}a(x)}{2\gamma(x)} \right\} \geq \frac{\hat{T}a(x)}{2\gamma(x)}.$$

Using (23) we have

$$\int_{S^1} r^*(x, \hat{T}; 0)\,\mathrm{d}x \geq \int_{S^1} \frac{\hat{T}a(x)}{2\gamma(x)}\,\mathrm{d}x > \hat{T}. \tag{25}$$

Clearly, $\hat{r}$ solves problem (12), (3), (4) with $T = \hat{T}$. Let $\hat{\lambda}$ be the Lagrange multiplier from Corollary 1 for $T = \hat{T}$. Then $\hat{r}(x) \geq r^*(x, \hat{T}; \hat{\lambda})$, according to Corollary 1. Thus $\int_{S^1} r^*(x, \hat{T}; \hat{\lambda})\, \mathrm{d}x \leq \hat{T}$, which compared with (25) implies that $\hat{\lambda} > 0$. Then (4) is satisfied as equality due to the complementary slackness condition (14).

In order to prove (24) we notice that for x such that $\gamma(x) > 0$ and $r_1(x) < \frac{\hat{T}a(x)}{2\gamma(x)}$, it holds that $a(x)\,\gamma(x) > 0$. Since $\hat{\lambda} > 0$, Case 1 (see the previous section) takes place. Then

$$\hat{r}(x) = r^*(x, \hat{T}; \hat{\lambda}) = \max\left\{ r_1(x), \frac{\hat{T}a(x)}{2\gamma(x)} - \beta \right\},$$

where β is the second summand in the right-hand side in (18) or (19) for $T = \hat{T}$ and $\lambda = \hat{\lambda}$. Since $a(x)\,\gamma(x) > 0$ and $\hat{\lambda} > 0$ the number β is positive, which together with the inequality $r_1(x) < \frac{\hat{T}a(x)}{2\gamma(x)}$ implies (24). Q.E.D.

Proposition 4 *If $\gamma(x) > 0$ for almost all x and*

$$\int_{S^1} \frac{a(x)}{\gamma(x)}\, \mathrm{d}x < 2, \tag{26}$$

then there exists an optimal solution $(\hat{r}, \hat{T})$ of problem (7), (3), (4) such that:

(i) the lower bound is active, that is, $\hat{r}(x) = r_1(x)$ for x of positive measure,

or

(ii) there are stops: $\int_{S^1} \hat{r}(x)\, \mathrm{d}x < \hat{T}$.

Proof. Let us take the optimal solution $(\hat{r}, \hat{T})$ defined in Corollary 2, where $\hat{r}(x) = r^*(x, \hat{T}; \hat{\lambda})$. The function $r^*(x, \hat{T}; \cdot)$ is not increasing, so we have the following inequalities when $\hat{r}(x) > r_1(x)$ for all x:

$$\hat{r}(x) \leq r^*(x, \hat{T}; 0) = \min\left\{ \frac{a(x)\hat{T}}{2\gamma(x)}, r_2(x) \right\} \leq \frac{a(x)\hat{T}}{2\gamma(x)}.$$

Integrating and taking into account (26) we obtain

$$\int_{S^1} \hat{r}(x)\, \mathrm{d}x \leq \int_{S^1} \frac{a(x)\hat{T}}{2\gamma(x)}\, \mathrm{d}x < \hat{T}.$$

Q.E.D.

In our numerical experiments (e.g. see Figs. 3 and 4 below) the lower bound r_1 is always active, that is, possibility (i) in Proposition 4 holds. The reason may be that if r_1 is not active for some admissible pair (r, T) then an admissible solution with a smaller T exists, which gives a better objective value. It is proved in the next Proposition on some additional assumptions for the involved data.

Proposition 5 *Assume that the functions a, γ and r_1 are continuous, that $\gamma(x) > 0$ on S^1, and that there exists a scalar $\bar{T} > 0$ such that for all $x \in S^1$*

$$r_1(x) \le \frac{a(x)\bar{T}}{2\gamma(x)}, \quad r_2(x) > \frac{a(x)\bar{T}}{2\gamma(x)}, \tag{27}$$

and that

$$\int\limits_{S^1} \frac{a(x)}{\gamma(x)} \, \mathrm{d}x < 2. \tag{28}$$

Then there exists the solution $(\hat{r}, \hat{T})$ of problem (7), (3), (4) such that the lower bound is active (that is, $\hat{r}(x) = r_1(x)$ for x of positive measure), the upper bound r_2 is not active (that is, $\hat{r}(x) < r_2(x)$ for a.e. x), and $\hat{T} \le \bar{T}$.

Proof. We take the optimal solution $\hat{r}(x) = r^*(x, \hat{T}; \hat{\lambda})$ as stated in Corollary 2. First we prove that r_2 is not active for this solution. For any given T we introduce the value

$$I\Big(r^\sharp(\cdot, T), T\Big) = \frac{1}{T} \int_{S^1} \frac{a(x)}{b(x)} \, \frac{e^{a(x)T/4} - e^{-a(x)T/4}}{e^{a(x)T/4} + e^{-a(x)T/4}} \, \mathrm{d}x, \tag{29}$$

where $r^\sharp(x, T) = \frac{a(x)T}{2\gamma(x)}$. This is the maximal value of the objective functional in (10), where the maximization is carried out with respect to r, regardless of the pointwise constraints (3). Then $\hat{I} = I(\hat{r}, \hat{T}) \le I(r^*(\cdot, \hat{T}), \hat{T})$. The total derivative of (29) w.r.t. T is strictly negative,

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}T} &= \frac{1}{T^2} \int_{S^1} \frac{a(x)}{b(x)} \left(\frac{a(x)T}{\big(e^{a(x)T/4} + e^{-a(x)T/4}\big)^2} - \frac{e^{a(x)T/4} - e^{-a(x)T/4}}{e^{a(x)T/4} + e^{-a(x)T/4}} \right) \mathrm{d}x \\ &= \frac{1}{T^2} \int_{S^1} \frac{a(x)}{b(x)} \frac{a(x)T - e^{a(x)T/2} + e^{-a(x)T/2}}{\big(e^{a(x)T/4} + e^{-a(x)T/4}\big)^2} \, \mathrm{d}x < 0, \end{aligned} \tag{30}$$

because in the numerator of the integrand for $a(x)T > 0$ we have $a(x)T - e^{a(x)T/2} + e^{-a(x)T/2} < 0$. If we assume that $\hat{T} > \bar{T}$, then

$$\hat{I} \le I(r^\sharp(\cdot, \hat{T}), \hat{T}) < I(r^\sharp(\cdot, \bar{T}), \bar{T}). \tag{31}$$

Notice that due to (27), the control $r^\sharp(x,\bar T)$ satisfies the pointwise constraints (3), and due to (28), the pair $(r^\sharp(x,\bar T),\bar T)$ satisfies also the integral constraint (4). Thus $(r^\sharp(\cdot,\bar T),\bar T)$ is an admissible policy, and (31) contradicts the optimality of $(\hat r,\hat T)$. This contradiction implies that $\hat T\le\bar T$.

Since the function $r^*(x,\hat T;\cdot)$ is not increasing, using (22) and the second inequality in (27) we obtain that

$$\begin{aligned}\hat r(x) &= r^*(x,\hat T;\hat\lambda)\le r^*(x,\hat T;0)\le\max\left\{r_1(x),\frac{a(x)\hat T}{2\gamma(x)}\right\}\le\max\left\{r_1(x),\frac{a(x)\bar T}{2\gamma(x)}\right\}\\ &= \frac{a(x)\bar T}{2\gamma(x)}<r_2(x),\end{aligned}$$

so that the upper bound r_2 is not active.

Now assume that the lower bound r_1 is also non-active. We shall prove that $(r^\sharp(x,\hat T-\varepsilon),\hat T-\varepsilon)$ is an admissible policy for all sufficiently small $\varepsilon>0$. Due to (28), this pair satisfies the integral constraint (4). The second inequality in (27) together with $\hat T-\varepsilon<\bar T$ implies $r^\sharp(x,\hat T-\varepsilon)\le r_2(x)$. From $r_1(x)<\hat r(x)<r_2(x)$, (22) and (18) it follows that

$$\hat r(x)=r^*(x,\hat T;\hat\lambda)=\bar r(x,\hat T;\hat\lambda)\le\bar r(x,\hat T;0)=r^\sharp(x,\hat T).$$

Since r_1 is non-active, $r_1(x)<r^\sharp(x,\hat T)$ for all $x\in S^1$. Due to the continuity of r_1 and $r^\sharp$, and the compactness of S^1, this implies that $r_1(x)<r^\sharp(x,\hat T-\varepsilon)$ if $\varepsilon>0$ is sufficiently small. Thus $(r^\sharp(x,\hat T-\varepsilon),\hat T-\varepsilon)$ is admissible. Due to (30),

$$I(r^\sharp(\cdot,T-\varepsilon),\hat T-\varepsilon)>I(r^\sharp(\cdot,\hat T),\hat T)\ge I(\hat r,\hat T),$$

which contradicts the optimality of $(\hat r,\hat T)$. Thus r_1 is active. Q.E.D.

The above propositions exhibit the role of the regeneration/acquisition ratio $a(x)/\gamma(x)$ and its integral,

$$A=\int_{S^1\setminus\{x:\,\gamma(x)>0\}}\frac{a(x)}{\gamma(x)}\,\mathrm{d}x.$$

VV: If you like, you may move this to an earlier point, but do not use it in the formulations of the propositions and do not emphasize it too much. I do not like introducing terminology on so minor and wage ground. Proposition 3 states that if $A>2$, then the optimal cycling has no stops, which is seen on the left plots in Fig. 3: a), c). That is, when the rate of recovery is, in average, high enough relative to the acquisition rate, harvesting takes place all the time. If $A < 2$, then stops can occur, as stated by Proposition 4, see the right plots in Fig. 3: b), d). In this case the resource regenerates so slowly that the harvester may have to wait. But whether stops really appear or not, this heavily depends on other properties of the data. In Fig. 3 b), d), for example, there are stops, while in Fig. 4 b), d), where the only difference is that the lower bound r_1 is higher, stops disappear.

Fig. 3 compares the results for two different values of γ: $\gamma = 0.45$ and $\gamma = 0.55$. We observe that a larger acquisition rate γ leads to appearance of stops and to a larger period T (although the harvesting time $\int_{S^1} r(x)\,\mathrm{d}x$ is smaller due to the higher velocity $1/r(x)$). Fig. 4 presents a similar comparison of the results for $\gamma = 0.45$ and $\gamma = 0.55$, but with a higher value of the lower bound r_1 than that in Fig. 3. Here, a larger acquisition rate γ leads to a smaller harvesting period T, and in both cases there are no stops. It is remarkable that for the larger γ it is optimal to exhaust the resource in certain areas with low regeneration rate $a(x)$. That is, an improvement of the detection/acquisition technology may result in larger exhausted areas.

Notice that the example in Fig. 3 b), d) involves stops, $\int_{S^1} r(x)\,\mathrm{d}x < T$, when $\lambda_1 = 0$ and our solution (22) is the minimal maximizer $r^*(x, T; 0)$ that was used in the proof of existence in Section 4. But the optimal solution can be non-unique in presence of stops and exhausted areas. Indeed, if the harvester spends a little more time in the exhausted areas, taking somewhat slower speed there (somewhat larger $r(x)$), and reduces the time of stops at the same amount, then the integral constraint (4) still holds, and the objective functional does not change.

7 Discussion

The paper investigates a new model of periodic harvesting in a spatial domain, which exhibits some interesting properties in terms of harvesting period, period of no harvesting, and appearance of exhausted areas. In particular, it is shown that an increase in the acquisition parameter γ leads to an increase in the optimal revenue and in the optimal velocity, which in turn, leads to reduction of the harvesting time. As a result, exhausted areas could appear or grow during the optimal motion. Stops can also appear thanks to improvement of the resource acquisition technology, γ. The period of harvesting T can both increase of decrease, depending on the presence of stops between harvesting circles. It seems that the appearance of stops and exhausted areas is caused by the assumption of strict inverse proportionality of the harvesting density and the velocity. A relaxation of this assumption is an interesting topic of future research. Another (complementary) extension involves harvesting in a moving spatial sub-domain (rather than a point), as it is the case with rotational use of agricultural land.

References

[1] Koopman, B.O., 1957. The theory of search. Part III. The optimal distribution of searching efforts. Operations Research 5(5), 613–626.

[2] Stone, L.D., 1975. Theory of optimal search. Academic Press, New York.

[3] Alekseev, V.M., Tikhomirov, V.M., Fomin, S.V., 1987. Optimal Control. Consultants Bureau, New York.

[4] Martin D. Smith, M.D., Sanchirico, J.N., Wilen, J.E., 2009. The economics of spatial–dynamic processes: Applications to renewable resources. Journal of Environmental Economics and Management, 57, 104–121.

[5] Brock, W., Xepapadeas, A., 2008. Diffusion-induced instability and pattern formation in infinite horizon recursive optimal control. Journal of Economic Dynamics and Control, 32, 2745–2787.

[6] Brock, W., Xepapadeas, A., 2010. Pattern formation, spatial externalities and regulation in coupled economic–ecological systems. Journal of Environmental Economics and Management 59, 149–164.

[7] Behringer, S., Upmann, T., 2013. Optimal Harvesting of a Spatial Renewable Resource, CESifo Working Paper No. 4019.

[8] Warga, J, 1972. Optimal control of differential and functional equations. Academic Press.

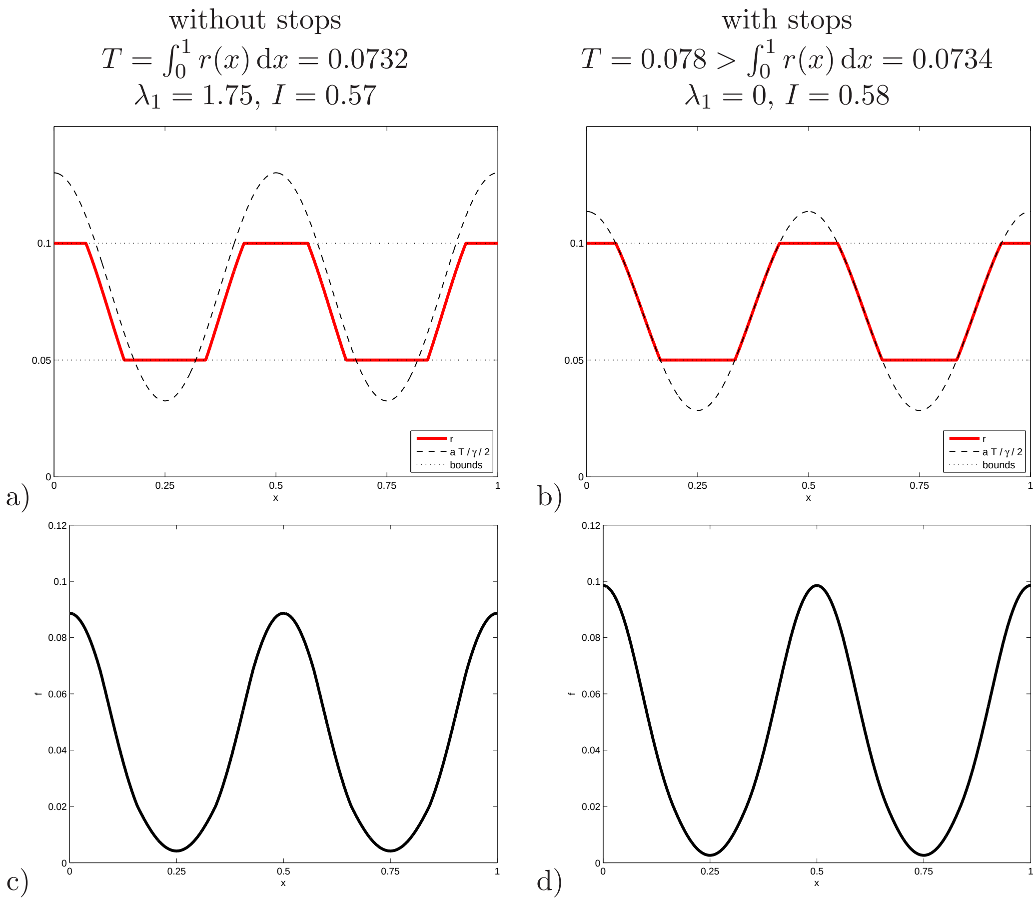

Figure 3: Left and right graphs correspondingly depict same computations for two different harvesting intensities $\gamma \equiv 0.45$ and $\gamma \equiv 0.55$. Graphs a) and b) show resulting controls r (bold line) in comparison with $\frac{Ta(x)}{2\gamma(x)}$ (dashed line). Constant bounds $r_1 = 0.05$ and $r_2 = 0.1$ are depicted with dotted lines. Parameters $b(x) \equiv 0.5$ and $a(x) = 1 + 0.6\cos(4\pi x)$. Graphs c) and d) show amount of harvested product at corresponding x.

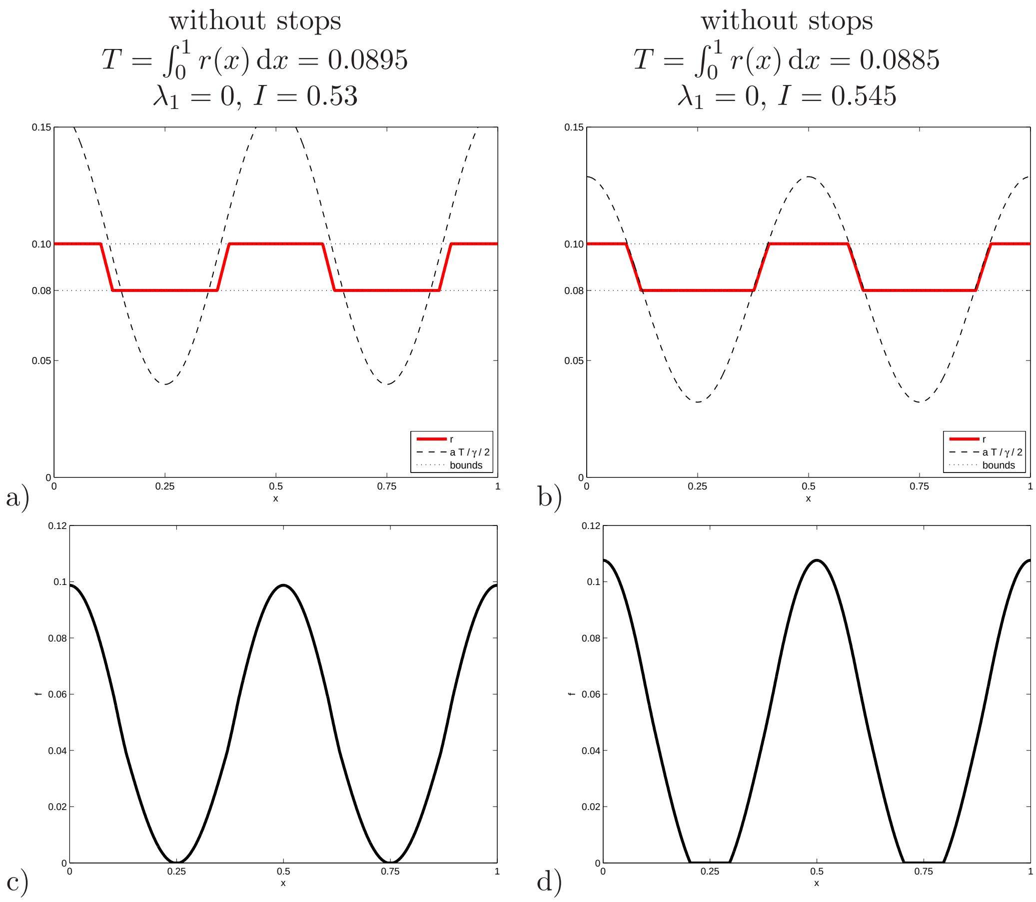

Figure 4: Left and right graphs correspondingly depict same computations for two different harvesting intensities $\gamma \equiv 0.45$ and $\gamma \equiv 0.55$. Graphs a) and b) show resulting controls r (bold line) in comparison with $\frac{Ta(x)}{2\gamma(x)}$ (dashed line). Constant bounds $r_1 = 0.08$ and $r_2 = 0.1$ are depicted with dotted lines. Parameters $b(x) \equiv 0.5$ and $a(x) = 1 + 0.6\cos(4\pi x)$. Graphs c) and d) show amount of harvested product at corresponding x. There are two small exhausted areas in c) and larger exhausted areas d) around $x = 0.25$ and $x = 0.75$, where the amount of harvested product is zero.