LC Near Abelian Groups and Applications

W. Herfort, K.H. Hofmann and F.G. Russo

(Wien, Darmstadt and New Orleans, Palermo and IMPA - Brazil)

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Mimar Sinan Fine Arts University
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Sketching the thread

- (1944) Iwasawa describes groups whose subgroup lattice is Dedekind: $\langle A, B\rangle \cap C = \langle A, B \cap C\rangle$ for A a subgroup of C. In particular, finite such p-groups are near abelian (plus an extra condition if $p = 2$).
- He proved: Finite p-groups are Dedekind if and only if any two subgroups commute, i.e., are QUASIHAMILTONIAN.
- Certain gaps in Iwasawa's original proof have been corrected (e.g., M. Suzuki, R. Schmidt, Y. Berkovich).
- (1977) F. Kümmich, PHD-student of P. Plaumann, studied topologically quasihamiltonian groups.
- (1986) Y. Mukhin classified topological Dedekind groups.

Recent results by K.H. Hofmann and F.G. Russo

In their paper (2012, Forum Mathematicum) K.H. Hofmann and F.G. Russo introduced the notion of NEAR ABELIAN compact p-group and found for an odd prime p the following equivalent conditions for a compact p-group:

- G is near abelian;
- G is topologically quasihamiltonian;
- G is the strict inverse limit of finite near abelian groups.

When $p = 2$ then, in addition, the dihedral group D_8 must not be involved in G.

What to do? Where to go?

- Describe locally compact near abelian groups and study their properties.
- Apply this in order to describe topologically quasihamiltonian groups as Iwasawa has done for all such groups that are either locally finite or contain $\mathbb{Z}$.

Monothetic groups

- Recall that a MONOTHETIC GROUP is a locally compact group which contains a dense cyclic subgroup.
- Examples are
  - the discrete group of integers $\mathbb{Z}$; this is the only non-compact example;
  - the discrete cyclic groups C_n of order n;
  - the compact groups p-adic integers $\mathbb{Z}_p$;
  - the tori $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and their cartesian products $\mathbb{T}^k$ for $k \in \mathbb{N}$ or $k = \mathbb{N}$;

Inductively monothetic groups

- A locally compact group is an INDUCTIVELY MONOTHETIC GROUP (for short IMG) if every topologically finitely generated subgroup is monothetic. Such G is either
  1. discrete – then it is isomorphic to a subgroup of either $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$;
  2. infinite compact – then it is connected of dimension 1 or it is procyclic;
  3. a local product of its p-primary subgroups.

- The 2-dimensional torus $\mathbb{T} \times \mathbb{T}$ is monothetic but it is not IMG since it contains $C_p \times C_p$ which is not monothetic!

Scalar action of IMG-s

- A group H ACTS BY SCALARS on an LCA-group A if every element of H leaves all closed subgroups of A invariant.
- If an IMG Γ acts faithfully on an LCA-group by scalars then one of the following is true:
  1. Γ is trivial or Γ contains two elements; then Γ acts by inversion;
  2. Γ is isomorphic to a discrete proper subgroup of $\mathbb{Q}$;
  3. Γ is an infinite subgroup of $\mathbb{Q}/\mathbb{Z}$ each of whose primary components is finite;
  4. Γ is a local direct product as above with every factor compact.

Simple example: Every profinite abelian group A is naturally a $\hat{\mathbb{Z}}$-module. Thus every closed subgroup of $\hat{\mathbb{Z}}$ acts by scalars on A.

Near abelian groups

- A locally compact group G is NEAR ABELIAN if it is an extension of an abelian locally compact group A by an IMG H that acts by scalars on A.
- Simple examples:
  - The p-adic integers act by scalars on the quasicyclic p-group $\mathbb{Z}(p^\infty)$; hence give rise to extensions each a locally compact near abelian p-group;
  - Let C_2 act on any abelian group by inversion. Then $\mathbb{R} \rtimes C_2$ is near abelian.

Basic structure of a near abelian group

Let G be nonabelian near abelian group. One of the following is true:

1. G/A acts by inversion on A: Then $G/C_G(A) \cong \mathbb{Z}(2)$ and $G_0 \leq G' \leq A$. Moreover, $A \in \mathrm{SIN}_G$.
2. G/A does not act by inversion on A. Then
   - 2.1 A is periodic;
   - 2.2 G/A has rank 1 and exclusively is either
     - 2.2.1 torsion free and then $G = \varprojlim_K G/K$, $K \leq A$ compact; or;
     - 2.2.2 periodic; then

       its p-primary subgroups are compact.

       G is periodic and contains an inductively monothetic subgroup H with $G = AH$ (a supplement).

ad 2.2.1: Fine structure for G/A torsion free

For each $m \in \mathbb{N}$, there are primes p_m, elements $b_m \in G \setminus A$, $a_m = (a_{mp})_p \in A$ and, for every fixed prime p, units $r_{mp} \in R(A_p)^\times$, such that for $H_m = \langle b_m \rangle$ and $G_m = AH_m$ we have

(1) $(G_m : A) = p_1 \cdots p_m$, $m = 1, 2, \ldots$.

(2) $r_{p_{m+1}}^{p_m} = r_{p_m}$ in $R(A_p)^\times$,

   (i) $b_{m+1}^{p_m} = a_m b_m$,

   (ii) $(a_m b_m)^{b_{m+1}} = a_m b_m$,

   (iii) for all $a \in A_p$ we have $a^{b_m} = a^{r_{pm}}$.

The "converse" is true: data as above determine uniquely a near abelian group of type 2.2.1.

Topologically quasihamiltonian groups – for short tqh

F. Kümmich defined a group to be tqh if $\overline{XY}$ is a subgroup whenever X and Y are closed subgroups. Alternatively, if $\overline{XY} = \overline{YX}$.

Here is our description of the structure of such groups:

- G contains $\mathbb{Z}$ as a discrete subgroup: then G is as in 2.2.1. and D_8 is not involved.
- G is a p-group. Thus $G = AH$ with $H = \overline{\langle b \rangle}$ and for all $a \in A$ $b^{-1}ab = a^{1+p^s}$ with $s \geq 1$ ($s \geq 2$ if $p = 2$).
- G is a pq-group. Then it is Frobenius with elementary abelian kernel and complement of order $q \neq 2$.
- G is periodic. It is a local product of coprime groups that are either abelian, pq-, or p-groups.

Main ingredients of our proof – Comparison with relevant results of Y. Mukhin

- 1. Establishing structure theorems for near abelian groups;
  2. Every inductively near abelian p-group is near abelian;
  3. If G contains discrete $\mathbb{Z}$ an inverse limit argument is used;.
  4. Every tqh p-group is inductively near abelian;
  5. Studying pq-groups and assemblying details.
- (Y. Mukhin, 1986)
  1. defines Д-groups to satisfy the Dedekind law $\overline{\langle A, B\rangle} \cap C = \overline{\langle A, B \cap C\rangle}$ for $A \leq C$ only if A is a procyclic p-group for some p;
  2. Proves that every inductively compact Д-group is (in our sense) near abelian.
  3. Periodic tqh-groups are in Д and from this our classification of periodic tqh-groups follows.
- tqh-groups need not be Dedekind, even not if they are abelian.

Thank you for the attention!