CONSENSUS ALGORITHMS WITH STATE-DEPENDENT WEIGHTS

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ABSTRACT

We provide an analysis of a consensus-type algorithm with weights dependent only on the received data. Differently from previous approaches that require a global knowledge of the network, we consider general weights inferred only from local data which can be modified by local functions on each node. We provide convergence conditions of such algorithms for general weight functions and derive analytical steady states in some selected cases.

***Index Terms*—** consensus, weights, convergence

1. INTRODUCTION

Typically, in distributed consensus problems [1], the mixing weights are designed based on parameters that depend on the network topology, such as node degrees or network size [2].

Contribution and previous work. Although some work bypassing this requirement exists, e.g., [2–4], we here provide a novel general method for selecting weights based only on the (local) *received data*, without any (global) knowledge of the network. We prove that distributed algorithms with such weights converge to *a consensus* even for a time-variant, so-called switching topology, i.e., a topology where links may appear and disappear. We further conjecture (and provide an example) that under some conditions the weights may adopt negative values. Note that this conjecture is in agreement with findings from [5, 6] where negative weights are also allowed. The algorithm [2, 4] follows from our model as a special case.

Furthermore, our contribution includes an algorithm which converges to a harmonic mean of the initial values (unlike classical *average* consensus). Other special cases are presented by simulation. We conclude that the functions applied on the received data influence the steady state of the algorithm. This motivates applications for which, unlike classical *average* consensus algorithms [1–3] where the algorithm converges to the average of the initial values, the average is not required (or undesirable), but *an agreement* is sufficient [4, 7, 8].

Notation: The number of nodes (network size) is denoted by N. The set of neighbors of a node i is denoted as $\mathcal{N}_i$. If node i is included in its neighborhood (there is a self-loop), we write $\mathcal{N}_i^+ \equiv \mathcal{N}_i \cup \{i\}$. We always assume to have a network described by an undirected graph (bidirectional links). Matrix $\mathbf{I}$ is an identity matrix and $\mathbf{1}$ is a (column) vector of all ones. A row vector is denoted as $\mathbf{x}^\top$. $\mathbb{R}_0^+$ ($\mathbb{R}_0^-$) are non-negative real numbers (or non-positive, respectively).

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2. PROBLEM STATEMENT – GENERAL CASE

We here define a general consensus algorithm with weights dependent on the received data at a given time (iteration). The received data may be further modified by some functions. As we prove, for a broad class of general functions, the algorithm converges to a consensus. Note that e.g., in [9, 10] nonlinear functions were applied on the data, nevertheless, the weights remained fixed (or they were switching between two weight models) and dependent on the topology of the network.

Theorem 1. *Assuming a static network, then for any intial number $x_i(0) \in \mathbb{R}$, ($i = 1, 2, \ldots, N$), the update*

$$x_i(k) = w_{ii}(k-1)x_i(k-1) + \sum_{j \in \mathcal{N}_i} w_{ij}(k-1)x_j(k-1), \quad (1)$$

with weights

$$w_{ij}(k) = \frac{f_j(x_j(k); k)}{f_i(x_i(k); k) + \sum_{j \in \mathcal{N}_i} f_j(x_j(k); k)}, \quad (2)$$

for any function $f_i(\cdot; k) : \mathbb{R} \to \mathbb{R}_0^+(\mathbb{R}_0^-)$; with the convention that if for finite times $\{k_0\}$, $f_j(x_j(k_0); k_0) = 0, \forall j \in \mathcal{N}_i^+$, then $w_{ii}(k_0) = 1$; Eq. (1) *asymptotically converges to a consensus, i.e.,*

$$\lim_{k \to \infty} x_i(k) = c, \quad \forall i = 1, 2, \ldots, N. \quad (3)$$

Remark: Notice that if we violate the condition that $f_j(x_j(k_0); k_0) = 0, \forall j \in \mathcal{N}_i^+$ happens only *finite* many times (k_0 is from a finite set), then $w_{ii}(k) = 1$ and $x_i(k) = x_i(k_0)$, $\forall i; k > k_0$. In other words, the consensus is reached, in this case, only if $|x_i(k_0) - c| < \epsilon$, for all i and some $\epsilon > 0$, already holds (algorithm has already converged – within the ϵ limit – when we disconnect some nodes). If this happens only a finite number of times, the information can still spread over the network and a global consensus may be achieved.

Note that by a *static* network in Theorem 1 we imply that its adjacency matrix $\mathbf{A}$ does not vary. However, as noted in the remark, allowing some weights to become zero at a finite amount of time is equivalent to a network with switching topology (links between the nodes may vary).

Still having in mind that the functions at the nodes $f_i(\cdot)$ may change in time k, for the sake of clarity, we drop the additional argument k in the following notation, i.e., $f_i(\cdot)$, where this property is not explicitly required.

Proof of Theorem 1. Since all functions $f_i(\cdot):\mathbb{R}\to\mathbb{R}_0^+(\mathbb{R}_0^-)$ $(i=1,2,\dots,N)$, we conclude that $0\le w_{ij}(k)\le 1, \forall i,j$ and from a global (network) point of view, we observe that the weight matrix (2), i.e.,

$$\mathbf{W}(k)=[\mathbf{W}(k)]_{ij}=w_{ij}(k),$$

has the following properties:

1. $\mathbf{W}(k)\mathbf{1}=\mathbf{1}$ for all k, ($\mathbf{W}(k)$ is row-stochastic)
2. eigenvalue $\lambda_{\max}(k)=1$, with corresponding right eigenvector $\mathbf{v}_{\max}(k)=\mathbf{1}$,
3. left eigenvector corresponding to the eigenvalue $\lambda_{\max}(k)$, $\mathbf{u}_{\max}^\top(k)=(u_1(k),\dots,u_i(k),\dots,u_N(k))$ with

$$u_i(k)=\frac{f_i(x_i(k))\Big(f_i(x_i(k))+\sum\limits_{j\in\mathcal{N}_i}f_j(x_j(k))\Big)}{\sum\limits_{i'=1}^{N}f_{i'}(x_{i'}(k))\left(f_{i'}(x_{i'}(k))+\sum_{j'\in\mathcal{N}_{i'}}f_{j'}(x_{j'}(k))\right)}.$$

The choice of the weights (2) thus guarantees the well-known neccessary conditions for convergence [5, 11] of Eq. (1), i.e., $\lim_{k\to\infty}\mathbf{x}(k)=\lim_{K\to\infty}\prod_{k=0}^{K}\mathbf{W}(k)\mathbf{x}(0)$ may exist.

Furthermore, notice that matrix $\mathbf{W}(k)$ can be decomposed into

$$\mathbf{W}(k)=\mathbf{D}_1(k)(\mathbf{I}+\mathbf{A})\mathbf{D}_2(k), \tag{4}$$

where $\mathbf{D}_1(k)=\begin{pmatrix}\frac{1}{\sum_{j\in\mathcal{N}_1^+}f_j(x_j(k))} & & \\ & \frac{1}{\sum_{j\in\mathcal{N}_2^+}f_j(x_j(k))} & \\ & & \ddots\end{pmatrix}$, $\mathbf{D}_2(k)=\begin{pmatrix}f_1(x_1(k)) & & \\ & f_2(x_2(k)) & \\ & & \ddots\end{pmatrix}$, and $\mathbf{A}$ is the adjacency matrix of the network. Thus, in case matrix $\mathbf{A}$ is symmetric (undirected graph), any matrix of form (4) can be regarded as a transition matrix of a *reversible* (homogeneous) Markov chain [12, 13]. Thus, if $\mathbf{W}(k)$ is constant for all $k>0$, we find that $\lim_{K\to\infty}(\mathbf{W}(k))^K=\overline{\mathbf{W}}(k)=\mathbf{v}_{\max}(k)\mathbf{u}_{\max}^\top(k)$.

However, in general, we are interested in the convergence of the time-varying matrices $\mathbf{W}(k)$, i.e.,

$$\lim_{K\to\infty}\mathbf{W}(K)\mathbf{W}(K-1)\dots\mathbf{W}(0)=\lim_{K\to\infty}\prod_{k=0}^{K}\mathbf{W}(k). \tag{5}$$

From the theory of Markov chains, we call a sequence of matrices (strongly) *ergodic*, if the product of transition matrices $\mathbf{W}(k)$ converges to a matrix with identical rows[1].

To prove Theorem 1, we follow standard mathematical proofs and utilize the notion of *coefficient of ergodicity* $\mu(\mathbf{W})$ as proposed in [16, 17], i.e.,

$$\mu(\mathbf{W})\triangleq\min_{i,i'}\|\min(w_{ij},w_{i'j})\|_1=\min_{i,i'}\sum_j\min(w_{ij},w_{i'j}).$$

Thus, this coefficient "measures" the similarity between the rows of a matrix $\mathbf{W}$. Clearly, $\mu(\mathbf{W})=1$ if and only if all the rows of a row-stochastic matrix $\mathbf{W}$ are identical. To simplify the proof, we introduce yet another coefficient of ergodicity [16, 17], i.e.,

$$\delta(\mathbf{W})\triangleq\max_{i,i'}\|\mathbf{w}_i^\top-\mathbf{w}_{i'}^\top\|_\infty=\max_j\max_{i,i'}|w_{ij}-w_{i'j}|,$$

where $\mathbf{w}_i^\top$ is the i-th row of matrix $\mathbf{W}$. Similarly to $\mu(\mathbf{W})$, $\delta(\mathbf{W})$ describes the difference between the rows of a matrix $\mathbf{W}$. It can be observed that $\delta(\mathbf{W})=0$ if and only if all the rows of $\mathbf{W}$ are identical.

From [16, Lemma 3], we know that for a product of two row-stochastic matrices $\mathbf{A}$ and $\mathbf{B}$, i.e., $\mathbf{C}=\mathbf{AB}$, it holds that

$$\delta(\mathbf{C})\le(1-\mu(\mathbf{A}))\delta(\mathbf{B}).$$

By extending this to an infinite product of row-stochastic matrices, we obtain [16, Theorem 2]

$$\delta\left(\prod_{k=0}^{\infty}\mathbf{W}(k)\right)\le\prod_{k=0}^{\infty}(1-\mu(\mathbf{W}(k))).$$

For the proof we further require the notion of so-called *scrambling* matrices as defined in [16]. A scrambling matrix is a matrix where for every pair of rows, say (i,i'), there exists at least one column, say j, such that both $w_{ij}>0$ and $w_{i'j}>0$. Alternatively, we can say that no two rows of matrix $\mathbf{W}$ are orthogonal[18]. Note that a scrambling matrix does not imply an irreducible matrix (connected network; see Case 3 ahead), nor an irreducible matrix imply a scrambling. Also note that any matrix with a positive column is a scrambling matrix.

It can be observed that for any scrambling matrix $\mathbf{W}$, $0<\mu(\mathbf{W})<1$. And, $\mu(\mathbf{W})=0$, if and only if $\mathbf{W}$ is not a scrambling matrix. Also, as mentioned before $\mu(\mathbf{W})=1$ if and only if all the rows of a row-stochastic matrix $\mathbf{W}$ are identical. It thus suffices to show that there are infinitely many scrambling matrices in the product $\prod_{k=0}^{\infty}\mathbf{W}(k)$.

To prove Theorem 1, for the general case $f_i(\cdot)\ge 0$ (non-negative), we have to distinguish the following three cases:

Case 1) If all $f_i(x_i(k))>0$ (strictly positive) (or all $f_i(\cdot)$ are strictly negative), we observe that matrix $\mathbf{W}(k)$ is a *primitive* matrix (irreducible since it remains connected; and aperiodic since at least one self-loop will be present) [19]. We further observe from decomposition (4) that *any product* of $\mathbf{W}(k)\mathbf{W}(k-1)$ is again a primitive matrix[2]. Thus, for any positive diagonal matrix $\tilde{\mathbf{D}}(k)=\mathrm{diag}(\frac{f_1(x_1(k))}{\sum_{j\in\mathcal{N}_1^+}f_j(x_j(k-1))},\dots)$, matrix $(\mathbf{I}+\mathbf{A})\tilde{\mathbf{D}}(k)(\mathbf{I}+\mathbf{A})$ eventually becomes a scrambling matrix (information from some node spreads to all other nodes after some iterations k_1, i.e., there will be a positive column). Such "sub-products" of scrambling matrices appear in (5) infinitely many times.

Case 2) If we at some times $\{k_0\}$ disconnect all neighboring nodes of node i, node i stops receiving data (becomes iso-

[1]Note that unlike non-homogeneous Markov chains where the *right* product of transition matrices is considered, we are interested in the *left* product of the matrices. The theorems and tools used in the proof here, however, do not depend on the order of the matrices [14, 15].

[2]From [20, Lemma 4] we know that if all (long-enough) "sub-products" of product (5) are primitive matrices, then the product converges to a matrix with identical rows. Note that the condition, that also the product of matrices is a primitive matrix, is a crucial condition. In general, even if every matrix $\mathbf{W}(k)$ is primitive, the *product* of matrices need not to be a primitive matrix and thus the convergence is not satisfied [16].

lated), i.e., $f_j(x_j(k_0)) = 0, \forall j \in \mathcal{N}_i^+$, then $w_{ii}(k) = 1$, and the value at node i remains constant, i.e., $x_i(k_0+1) = x_i(k_0)$. In this case $\mu(\mathbf{W}(k_0)) = 0$. Thus, clearly, if we did not have the condition that this case happens only finite many times, $\lim_{K\to\infty} \delta\left(\prod_{k=0}^{K} \mathbf{W}(k)\right)$ could be larger than 0, and thus a consensus might not be reached.

Case 3) We disconnect at some times $\{k_1\}$ some (at least one, but not all) of neighboring nodes of node i, i.e., $f_j(x_j(k_1)) = 0$, for some $j \in \mathcal{N}_i^+$. This means that the j-th columns of $\mathbf{W}(k_1)$ will be equal to $\mathbf{0}$. Matrix $\mathbf{W}(k_1)$ nevertheless remains row-stochastic, with a positive off-diagonal element in every row. Note that in this case the diagonal element $w_{ii}(k_1) = 0$, thus this generalizes the conditions on convergence as proposed in [4, 21]. Also note, that this case may happen infinitely many times ($\forall k$).

Case 3 represents the case when some nodes at times $\{k_1\}$ do not transmit anything, only receive (recall, that they are not isolated), and thus the network is not (strongly) connected anymore. Similarly to Case 1, after some (finite) steps (depending on the connectivity of the network) there will be a strictly positive column in the matrix $\overline{\mathbf{W}}(k_1) \triangleq \prod_{k\in\{k_1\}} \mathbf{W}(k)$, thus $\overline{\mathbf{W}}(k_1)$ will be a scrambling matrix. For all (infinitely many) k_1, $\mu(\overline{\mathbf{W}}(k_1)) \leq 1$, and thus

$$\delta\left(\prod_{k=0}^{\infty} \mathbf{W}(k)\right) \leq \prod_{k=0}^{\infty}(1-\mu(\mathbf{W}(k))) = \prod_{k\in\{k_0\}} \underbrace{(1-\mu(\mathbf{W}(k)))}_{=1} \prod_{k\in\{k_1\}} \underbrace{(1-\mu(\overline{\mathbf{W}}(k)))}_{<1} = 0.$$

This concludes the proof of Theorem 1. ■

3. SPECIAL CASES

In this section we provide examples of weights with specific functions and derive their steady states.

First, let us recall the properties of the weight consensus matrix as proposed in [1, 22].

Lemma 1. *Having a static connected network described by an adjacency matrix* $\mathbf{A}$ *with a degree matrix* $\mathbf{D} = \mathrm{diag}(d_1, d_2, \ldots, d_N)$*, then weight matrix*

$$\mathbf{W} = (\mathbf{I}+\mathbf{D})^{-1}(\mathbf{I}+\mathbf{A}) \tag{6}$$

has the following properties:

1. $\mathbf{W}\mathbf{1} = \mathbf{1}$,
2. *maximum eigenvalue* $\lambda_{\max} \equiv \max_i |\lambda_i| = 1$*, with corresponding right eigenvector* $\mathbf{v}_{\max} = \mathbf{1}$,
3. *left eigenvector corresponding to* $\lambda_{\max}$, $\mathbf{u}_{\max}^\top = \frac{1}{\sum_{i=1}^N 1+d_i}(1+d_1, 1+d_2, \ldots, 1+d_N)$.

Thus,

$$\lim_{k\to\infty} \mathbf{W}^k = \mathbf{v}_{\max}\mathbf{u}_{\max}^\top.$$

Proof. See [1]. ■

Note that a consensus algorithm with weights (6) does not, in general, lead to an *average* consensus algorithm [1], unless the network has a regular topology, or a combination of two algorithms is performed[3] [4]. Also note, that in this case, the decomposition (4), takes the form, $\mathbf{D}_1(k) = \mathrm{diag}(\frac{1}{1+d_i}, \ldots,)$, $\mathbf{D}_2(k) = \mathbf{I}$, which is the case of the following theorem.

Theorem 2. *Assuming a static connected network, then for any initial number* $x_i(0) \in \mathbb{R}$ *($\forall i = 1, 2, \ldots, N$), the update algorithm* (1) *with functions in the weights (cf. Eq.* (2)*)*

$$f_i(x_i(k)) = 1, \quad i = 1, 2, \ldots, N, \tag{7}$$

asymptotically converges to the consensus

$$\lim_{k\to\infty} x_i(k) = \sum_{i=1}^{N} \frac{1+d_i}{\sum_j (1+d_j)} x_i(0). \tag{8}$$

Note that weights (7) are simply selected only according to the *number* of received messages at time k. A deeper analysis of this algorithm, including its convergence speed, can be found in [4].

Proof. The convergence of the algorithm to *a* consensus follows from Theorem 1. Moreover, from a global point of view,

$$\begin{aligned}\mathbf{x}(k) &= \mathbf{W}(k-1)\mathbf{x}(k-1) = (\mathbf{I}+\mathbf{D})^{-1}(\mathbf{I}+\mathbf{A})\mathbf{x}(k-1)\\ &= \left((\mathbf{I}+\mathbf{D})^{-1}(\mathbf{I}+\mathbf{A})\right)^k \mathbf{x}(0).\end{aligned}$$

Taking the results of Lemma 1 concludes the proof. ■

Conjecture 1. *Theorem 1 holds also for (appropriate) mixed positive/negative functions* $f_i(\cdot\,; k)$*, as long as the weights at each node sum to* 1 *(and having non-zero denominator in* (2)*), or for specific topologies, i.e., if* $f_i(\cdot\,; k) : \mathbb{R} \to \mathbb{R}$*. Then the update algorithm* (1) *with weights* (2)*, converges to a consensus.*

Remark: If $f_i(x_i(k)) < 0$ for *all* $i \in \{1, 2, \ldots, N\}$ and k, then $0 < w_{ij} < 1$, $\forall i, j$, and the algorithm is equivalent to case of $f_i(x_i(k)) > 0$ for all i, k (see the Case 1 of the proof of Theorem 1 for strictly positive case). For a more general case, see Theorem 3 ahead.

To support Conjecture 1, we define the following algorithm, which can take also negative values, but which, nevertheless, converges to a consensus.

Theorem 3. *For any number* $x_i(0) \in \mathbb{R}\backslash\{0\}$ *($i = 1, 2, \ldots, N$), the algorithm* (1) *with functions in the weights (cf. Eq.* (2)*)*

$$f_i(x_i(k)) = \frac{1}{x_i(k)} \tag{9}$$

asymptotically converges to the consensus, i.e.,

$$\lim_{k\to\infty} x_i(k) = \frac{\sum_{i=1}^N 1+d_i}{\sum_{i=1}^N \frac{1+d_i}{x_i(0)}} \quad, \forall i = 1, 2, \ldots, N. \tag{10}$$

[3] Unlike the case considered here, in [4] the network is considered to be without self-loops, i.e., $\tilde{\mathbf{W}} = \mathbf{D}^{-1}\mathbf{A}$. Nevertheless, our approach with weight matrix (6) is equivalent to that of [4], as shown in [1].

Corollary 1. *In case of a regular network (same degree d_i of every node),*

$$\lim_{k\to\infty} x_i(k) = \frac{N}{\sum_{i=1}^{N} \frac{1}{x_i(0)}},$$

thus, the algorithm from Theorem (3) *with weights* (9) *converges to a steady state equal to the harmonic mean of the initial values.*

Proof of Theorem 3. By plugging the weights (9) into (2), Eq. (1) yields

$$x_i(k) = \frac{1}{\frac{1}{x_i(k-1)} + \sum_{j'\in\mathcal{N}_i} \frac{1}{x_{j'}(k-1)}} + \sum_{j\in\mathcal{N}_i} \frac{1}{\frac{1}{x_i(k-1)} + \sum_{j'\in\mathcal{N}_i} \frac{1}{x_{j'}(k-1)}},$$

which can be rearranged as

$$\frac{1}{x_i(k-1)} + \sum_{j'\in\mathcal{N}_i} \frac{1}{x_{j'}(k-1)} = \frac{1+d_i}{x_i(k)}.$$

From a global (network) point of view, we can write

$$(\mathbf{I}+\mathbf{A})\hat{\mathbf{x}}(k-1) = (\mathbf{I}+\mathbf{D})\hat{\mathbf{x}}(k),$$

where $\hat{\mathbf{x}}(k) = (\frac{1}{x_1(k)}, \frac{1}{x_2(k)}, \ldots, \frac{1}{x_N(k)})^\top$.

Thus, we obtain

$$\begin{aligned}\hat{\mathbf{x}}(k) &= (\mathbf{I}+\mathbf{D})^{-1}(\mathbf{I}+\mathbf{A})\hat{\mathbf{x}}(k-1),\\ &= \left((\mathbf{I}+\mathbf{D})^{-1}(\mathbf{I}+\mathbf{A})\right)^k \hat{\mathbf{x}}(0)\end{aligned}$$

and using Lemma 1 we find

$$\lim_{k\to\infty} \mathbf{x}(k) = \frac{\sum_{i=1}^{N} 1+d_i}{\sum_{i=1}^{N} \frac{1+d_i}{x_i(0)}}\mathbf{1}.$$

Corollary 1 follows, if $d_i = d, \forall i$. ■

Due to space constraints we omit here other examples, which we mention in the simulation section and for which the steady states (or bounds on the steady states) can be also derived.

4. SIMULATIONS

We simulate random geometric networks (network with nodes communicating only with neighbors within some radius) with number of nodes $N = 20$.

In Fig. 1 we show an example of Theorem 1 for the weight functions $f_i(x_i(k)) = x_i(k)$, with initialization $x_i(0) = \{1, 2, \ldots, 20\}$, thus $\bar{x}(0) = 10.5$. We simulate the case when the two nodes are disconnected for iterations $\{k_0\}=\{3,4,\ldots,40\}\cup\{45,46,\ldots,60\}$, i.e., $f_j(x_j(k_0))=0$, $\forall j \in \{\mathcal{N}_1^+ \cup \mathcal{N}_7^+\}$. We observe that the algorithm still reaches a consensus as expected from Theorem 1. We compare these weights with so-called Metropolis weights [2] (dash-dotted lines) whose weights require the knowledge of the node degrees in the network. We observe that in that case the convergence is slightly slower ($k = 120$ vs. $k = 100$), and due to the disconnected nodes, the states also do not converge to the average of the initial values ($\bar{x}(k) \approx 10.7$).

In Fig. 2 we simulate seven different weights with following functions: $f_i^{(1)}(x_i(k))=x_i^2(k)$, $f_i^{(2)}(x_i(k))=\tan(x_i(k))$, $f_i^{(3)}(x_i(k)) = x_i(k)$, $f_i^{(4)}(x_i(k)) = \arctan(x_i(k))$, $f_i^{(5)}(x_i(k))=\sqrt{x_i(k)}$, $f_i^{(6)}(x_i(k))=1$, $f_i^{(7)}(x_i(k))=\frac{1}{x_i(k)}$. The simulations have been performed for 100 random initializations $x_i(0) \in (0,1), i = 1,2,\ldots,N$, for a fixed randomly selected geometric network. The depicted results are the averaged values over the initializations and nodes. We observe that in all cases the states converge to a consensus and that the slope in the transition phase (i.e. convergence speed) is (on average) comparable, independent from the weight function.

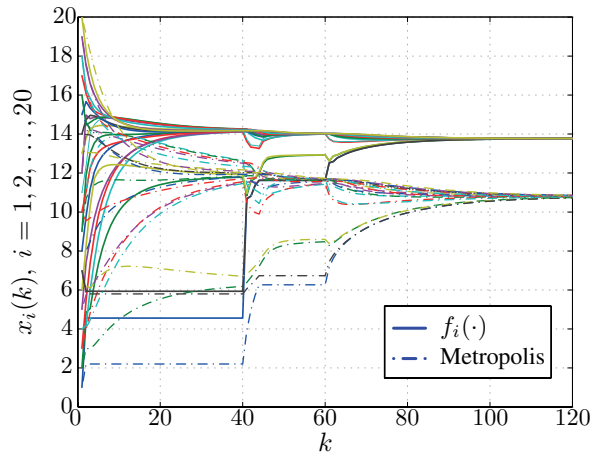

Fig. 1. Example of Alg. 1 with weights as $f_i(x_i(k)) = x_i(k)$ and two disconnected nodes during a finite number of iterations. Dash-dotted lines are Metropolis weights [2].

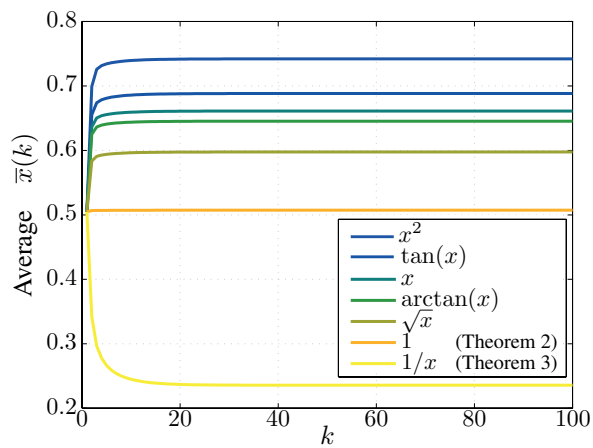

Fig. 2. Various weight functions. Averaged over 100 random initializations and over 20 nodes; $x_i(0) \in (0,1)$.

5. CONCLUSION

We provided a novel approach for designing weights for distributed consensus algorithms, without *any* knowledge about the network, based only on the received data. We derived steady states for some specific weight functions. A generalized proof for negative weights as proposed by the conjecture remains an open issue. Also a deeper analysis of convergence speed and time with respect to the weight function is worth looking into. The converse problem, i.e., to find appropriate functions for a desired steady state remains a challenging open question for future research.

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