Sketch of proof:

where

\[ \alpha > 0 \rightarrow \text{Gilbert damping constant} \]

\[ h_{\text{eff}} = C_{\text{max}} \Delta \theta + \pi (m) + f \rightarrow \text{general effective field} \]

Initial and boundary conditions:

\[ m(0) = m_0, \quad \text{with } m_0 : \Omega \rightarrow \mathbb{R}^3, \quad |m_0| = 1 \]

where

\[ m \in \mathcal{M} \subset [0, T] \times \mathbb{R}^3 \]

Total magnetic Gibbs free energy:

\[ \mathcal{E}(m) = \frac{C_{\text{max}}}{2} \int_{\Omega} \| \nabla m \|^2 - \int_{\Omega} f \cdot m - \frac{1}{2} \int_{\Omega} \pi(m) \cdot m \]

Equivalent reformulations of LLG:

- Gilbert form: \( \rightarrow m - \alpha m \times m = h_{\text{eff}} \times m \)
- Alternative form: \( \rightarrow m + \alpha m \times m = h_{\text{eff}} - (m \cdot h_{\text{eff}}) m \) and \( |m| = 1 \)

Numerical challenges:

- Efficient treatment of nonlinearities
- Non-convex side constraint \( |m| = 1 \)
- Efficient computation of field contributions

Weak formulation of LLG

Find \( m \in H^1(\Omega) \), with \( |m| = 1 \) a.e. in \( \Omega \), s.t.

\[ \text{for all } \phi \in H^1(\Omega) \]

\[ (m_\Omega \cdot \phi)_b = -\alpha (m \times (m \cdot \phi))_b \]

\[ = -C_{\text{max}} (\nabla m \cdot \nabla \phi_b) + (\pi(m) \times m, \phi)_b + (f \times m, \phi)_b \]

\[ m(0) = m_0 \text{ in the sense of traces} \]

\[ \text{for a.e. } t \in (0, T) \]

\[ \mathcal{E}(m(t)) + \int_0^T \| m_i(t) \|^2_{L^2(\Omega)} dt \leq \mathcal{E}(m_0) \]

Standard tangent plane scheme

Predictor-corrector strategy: linear update + nodewise projection

Based on alternative form of LLG \( \rightarrow \) linear in \( \nabla \phi = m \)

\[ |m|^2 = 1 \Rightarrow m \cdot v = 0 \Rightarrow v \text{ belongs to tangent space} \]

Discretization of field contributions included in analysis

Set of admissible magnetizations:

\[ \mathcal{M}_K = \{ \phi \in H^1(\Omega) : |\phi(x)| = 1 \text{ for all } x \in N_\delta \} \]

Discrete tangent space:

\[ \mathcal{K}_m = \{ \psi \in H^1(\Omega) : \psi(x) = m(x) \text{ for all } x \in N_\delta \} \]

Time-marching scheme:

Let \( 0 \leq \theta \leq 1 \).

For \( 0 \leq s \leq N - 1 \) iterate:

\[ \text{compute } \psi_i = \phi_i \text{ on } \mathcal{K}_m \text{ s.t.} \]

\[ \alpha (\psi_i, \phi)_b + (m_i \times (m_i \cdot \phi), \phi)_b + C_{\text{max}} \| \nabla \phi \|_b \]

\[ = -C_{\text{max}} (\nabla m_i, \nabla \phi)_b + (\pi(m_i), \phi)_b + (f_i, \phi)_b \]

where

\[ \phi \in \mathcal{K}_m \]

\[ \text{define } m^{\text{t+1}}_i = m_i \text{ by } m^{\text{t+1}}_i(x) = \frac{m_i(x) + \kappa \psi_i(x)}{|m_i(x) + \kappa \psi_i(x)|} \text{ for all } x \in N_\delta \]

Convergence result:

Unconditional convergence (up to a subsequence) towards a weak solution of LLG provided

\[ 1/2 < \theta \leq 1, \quad \text{angle condition on triangulation } T_h \]

Uniform boundedness of \( \pi \)

Weak convergence properties of \( \pi \), \( f_i \), \( m_i \)

Sketch of proof:

- Boundedness of discrete energy
- Abstract arguments prove convergence subsequences
- Identify limit with weak solution of LLG

Projection-free tangent plane scheme

Nodewise projection step is removed \( \Rightarrow m^{\text{t+1}}_i = m_i + \kappa \psi_i \in S^1(T_h)^3 \)

Main features:

- Fully linear scheme for strongly nonlinear PDE
- Unconditional convergence result is preserved
- Avoid angle condition on triangulation \( T_h \)
- Unit-length constraint is violated at nodes of triangulation
- Violation is controlled by time-step size independently of number of iterations

Coupling with other PDEs

Effective field often comprises nonlocal field contributions \( \Rightarrow (nonlinear) \) coupling of LLG with other PDEs

Examples:

- Multiscale modeling
- Spin-polarized transport in ferromagnetic multilayers

Multiscale modeling

Setting:

- Multiple domains \( \Omega_1 \) and \( \Omega_2 \) of different scales
- Solve LLG on microscopic domain \( \Omega_1 \)
- Magnetostatic Maxwell equations and nonlinear material law on macroscopic domain \( \Omega_2 \)

Mathematical treatment:

- Strongly monotone operator
- Multiscale contribution discretized by Johnson-Nédélec FEM-BEM coupling
- Satisfies all assumptions for convergence of tangent plane scheme

Spin-polarized transport

Setting:

- Interaction between electric current and magnetization
- \( \Omega \) multilayer, \( \omega \subset \Omega \) ferromagnetic part
- LLG coupled with diffusion equation for spin accumulation field

\[ m_0 = -m \times (h_{\text{eff}} + f) + \alpha m \times m \]

\[ \text{in } \omega \]

\[ s_i = -\nabla \cdot (\beta m \otimes j + D_i (\nabla s - \beta m \otimes (\nabla \cdot m))) - D_i (s - D_i (s \times m)) \text{ in } \Omega \]

Remarks:

- Decoupled algorithm for coupled system
- Only two linear systems per time-step

References


Acknowledgment: This research has been supported by the Austrian Science Fund (FWF) under grant W1245

Doctoral Program Dissipation and Dispersion in Nonlinear PDEs

Der Wissenschaftsfonds.