# High Order Hierarchic Finite Elements and Their Application on Eddy Current Losses in Permanent Magnets of Synchronous Machines 

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#### Abstract

The paper presents electromagnetic finite element analyses for the eddy current losses in permanent magnets of synchronous machines. The main focus lies on high order element formulations and the comparison of the respective results against an analytical evaluation of these losses for both linear and rotational arrangements. Therefore, the representation of skin depth and wave length with the numerical analyses can be discussed in detail.


Index Terms - Eddy currents, Permanent magnets, Permanent magnet machines, High order elements, Finite element methods.

## I. Introduction

NOWADAYS, a rated apparent power of permanent magnet excited electrical machines in the range up to 50 MVA is considered as a realisable trend of development. Due to sub- and superharmonics of the air-gap field, the eddy current losses generated in the permanent magnets of such machines may always lead to an excessive heating [1]-[5]. In particular with surface mounted permanent magnets, this can cause the magnets to get partially or even fully demagnetised [6]-[9]. Thus, the precalculation of these eddy current losses caused by the harmonics of the air-gap field is a matter of interest with the design process of such electrical machines. On one hand by using very fast evaluation methods for the standard design procedures, on the other hand by using highly accurate calculation methods for reference purposes [10]-[12].

As depicted in Fig. 1 and Fig. 2, linear as well as rotational arrangements are considered. Both arrangements are described with few parameters, such as air-gap $\delta$, ratio of pole pitch and air-gap $\tau_{p} / \delta$, ratio of magnet height and air-gap $h_{M} / \delta$ as well as the pole coverage as ratio of magnet width and pole pitch $b_{M} / \tau_{p}$. With the same parameters and an increasing ordinal number of the harmonic waves in circumferential direction, it is expected that the difference between both arrangements will disappear.

In order to compare the various approximation orders of the finite element analyses, an analytical calculation will be used for the reference results. Both calculation methods use an excitation with a surface current sheet along the circumferential direction at the
inner stator boundary which can cover for any harmonic order generated from either PWM modulated stator currents, the slotting as well as the saturation. This surface current flow in axial direction $K_{z}(x, t)$ perpendicular to the cross section of the conducting region can be expressed by a travelling wave as

$$
\begin{equation*}
K_{z}(x, t)=\hat{K}_{z} \operatorname{Re}\left(\mathrm{e}^{j \omega t} \mathrm{e}^{-j \nu \pi x / \tau_{p}}\right) \tag{1}
\end{equation*}
$$

where $\omega=2 \pi f$ denotes the exciting circular frequency with respect to the moving region, $\nu$ the harmonic order and $-1 \leq x / \tau_{p} \leq 1$ being the region of two pole pitches along the circumferential direction, respectively. Referring to the total eddy current losses, there is no interaction between waves with different harmonic orders as well as different frequencies. Consequently, each travelling wave can be discussed separately.


Fig. 1: Simplified geometry of a pole pitch with a linear arrangement.


Fig. 2: Simplified geometry of a pole pitch with a rotational arrangement.

## II. Analytical Analysis

The analytical calculation is based on Laplacian and Helmholtz equations of a magnetic vector potential within the respective regions and uses a pole coverage of $b_{M} / \tau_{p}=1$, which occurs practically with Halbach arrays.

## A. Analytical Approach

The magnetic vector potential $A_{z}(\omega)$ is obtained from the Laplacian equation in the non-conducting regions of air-gap and rotor and the Helmholtz equation in the conducting region of the permanent magnets,

$$
\begin{equation*}
-\Delta A_{z}(\omega)=0, \quad\left(-\Delta+\frac{2 j}{d^{2}}\right) A_{z}(\omega)=0 \tag{2}
\end{equation*}
$$

as well as respective interface conditions between these regions, where

$$
\begin{equation*}
d=\sqrt{\frac{2}{\omega \mu_{M} \sigma_{M}}} \tag{3}
\end{equation*}
$$

denotes the skin depth of the eddy currents [13].
The total eddy current losses within the conducting areas are evaluated by using the Poynting theorem. Thereby, the apparent power per length $S^{\prime}(\omega)$ is obtained from the boundary $\partial \Gamma_{M}$ along the permanent magnets as

$$
\begin{equation*}
S^{\prime}(\omega)=\frac{j \omega}{2 \mu_{M}} \oint_{\partial \Gamma_{M}} A_{z}(\omega) \frac{\partial A_{z}^{*}(\omega)}{\partial n} d s \tag{4}
\end{equation*}
$$

Consequently, the total eddy current losses are always proportional to the square of the magnitude $\hat{K}_{z}$.

## B. Analytical Results

Fig. 3 and Fig. 4 depict the power losses of one NdFeB magnet in dependence of the exciting frequency and the ordinal number of the harmonics for a constant current sheet excitation of $\hat{K}_{z}=10^{4} \mathrm{~A} / \mathrm{m}$. Both arrangements show the data of air-gap $\delta=2 \mathrm{~mm}$, ratio of pole pitch and air-gap $\tau_{p} / \delta=60$, ratio of magnet height and air-gap $h_{M} / \delta=3$. Fig. 5 shows the respective ratio of the power losses between both arrangements modified in accordance to the different cross sections of the conducting areas within both arrangements.

Obviously, the total eddy current losses are quite similar between both arrangements with a deviation in the range $\pm 5 \%$ only. As mentioned in [13], there are different regions in dependence on both frequency $f$ and wave length $2 \tau_{p} / \nu$ of the excitation. With a ratio of wave length to skin depth $\left(2 \tau_{p}\right) /(\nu d) \ll 1$, the power losses versus frequency increase with a power of 2 . On the other hand with a ratio of wave length to skin depth $\left(2 \tau_{p}\right) /(\nu d) \gg 1$, the power losses versus frequency increase with a power of 0.5 only. However with very low ordinal numbers, there is a transitional region where the power losses are rather constant.


Fig. 3: Power losses of various harmonics versus frequency, linear arrangement, analytical results, pole coverage 1 .


Fig. 4: Power losses of various harmonics versus frequency, rotational arrangement, analytical results, pole coverage 1 .


Fig. 5: Ratio of power losses between rotational and linear arrangement, analytical results, pole coverage 1.

## III. Finite Element Analysis

The finite element analyses deal with a pole coverage of $b_{M} / \tau_{p}=1$ for the direct comparison of the analytical results with those from the numerical analyses. Further, the finite element analyses can examine very easily pole coverages within the practical range of $b_{M} / \tau_{p} \approx 2 / 3 \ldots 3 / 4$.

The finite element analyses carried out with different high order approximation functions utilise an iden-
tical discretisation with the minimum skin depth as approximately the half of the mesh size in radial direction and the mininum wave length as approximately 7.5 times the mesh size in circumferential direction.

## A. Higher Order Finite Elements

In the finite element context, any analytical function $u(\xi)$ gets approximated by a finite dimensional subset of interpolation functions defined on a finite element mesh. In local element coordinates, this reads as

$$
\begin{equation*}
u(\xi) \approx u^{h}(\xi)=\sum_{i=1}^{n_{e q}} u_{i} N_{i}(\xi) \tag{5}
\end{equation*}
$$

where $u^{h}(\xi)$ is the approximated function, with $N_{i}(\xi)$ being the shape functions, $u_{i}$ the related coefficients and $n_{e q}$ the number of unknown coefficients, respectively.


Fig. 6: Lagrange (left) and Legendre (right) based shape functions up to order $p=4$.

In the case of standard Lagrangian elements, the functions $N_{i}$ are defined by the corner coordinates and $u_{i}$ are the related values of the function $u^{h}(\xi)$ on these nodes. The shape functions of first order on the unit domain $\Omega[-1,1]$ are defined as

$$
\begin{equation*}
N_{1}(\xi)=\frac{1-\xi}{2}, \quad N_{2}(\xi)=\frac{1+\xi}{2} \tag{6}
\end{equation*}
$$

However, one disadvantage of the Lagrangian basis is that for each polynomial degree $p \geq 2$, a new set of shape functions as shown in Fig. 6 (left) is required, which prevents the efficient usage of different approximation orders within one finite element mesh.

In contrast, a set of hierarchic shape functions is defined in such a way that every basis of order $p$ is fully contained in the basis of order $p+1$ as shown in Fig. 6 (right). In this work, we make use of the Legendre based interpolation functions as

$$
\begin{equation*}
N_{k}(\xi)=l_{k-1}(\xi), \quad k=3,4, \ldots, p+1 \tag{7}
\end{equation*}
$$

where $l_{k}(\xi), k \geq 2$, denotes the integrated Legendre polynomials [14], [15]

$$
\begin{equation*}
l_{k}(\xi)=\sqrt{\frac{2 k-1}{2}} L_{k}(\xi), L_{k}(\xi)=\int_{-1}^{\xi} P_{k-1}(x) d x \tag{8}
\end{equation*}
$$

Therein, $P_{k}$ are the regular Legendre polynomials [16]

$$
\begin{equation*}
P_{k}(x)=\frac{1}{2^{k} k!} \frac{d^{k}}{d x^{k}}\left(x^{2}-1\right)^{k} \tag{9}
\end{equation*}
$$

the scaling factor arises from their orthogonality

$$
\begin{equation*}
\int_{-1}^{+1} P_{k}(x) P_{m}(x) d x=\frac{2}{2 k+1} \delta_{k m} \tag{10}
\end{equation*}
$$

Using the recursive formula of the regular Legendre polynomials $(k \geq 1)$

$$
\begin{align*}
& (k+1) P_{k+1}(x)= \\
& \quad=(2 k+1) x P_{k}(x)-k P_{k-1}(x) \tag{11}
\end{align*}
$$

yields the integrated Legendre polynomials

$$
\begin{equation*}
L_{k}(x)=\frac{P_{k}(x)-P_{k-2}(x)}{2 k-1}, \quad k \geq 2 \tag{12}
\end{equation*}
$$

and their recursive formula $(k \geq 2)$

$$
\begin{align*}
& (k+1) L_{k+1}(x)= \\
& \quad=(2 k-1) x L_{k}(x)-(k-2) L_{k-1}(x) \tag{13}
\end{align*}
$$

Due to the orthogonality of the Legendre polynomials $P_{k}$ along the unit interval $[-1,1]$, only the first two functions $N_{1}, N_{2}$ contribute to the value at the ends of the unit interval $[-1,1]$. All other functions $N_{k}$ of higher order $k>2$ give only a non-zero value within the interval. Therefore, they are also called internal modes or bubble modes.

On the other hand, the integrated Legendre polynomials $L_{k}$ fulfill the orthogonality

$$
\begin{equation*}
\int_{-1}^{+1} L_{k}(x) L_{m}(x) d x=0,|k-m|>2 \tag{14}
\end{equation*}
$$

Consequently, the sparsity of the matrices decreases only slightly with higher orders of these approximation functions.

Having this knowledge in mind, we can easily construct basis functions up to any order for both quadrilateral and hexahedral elements by applying a tensor product. The other element shapes can be constructed via the Duffy transformation.

## B. Numerical Results

Fig. 7 and Fig. 8 depict the respective ratio of the power losses between both arrangements modified in accordance to the different cross sections of the conducting areas within both arrangements. The numerical results are quite similar to the analytical results. Consequently, only the linear arrangement is discussed in more detail afterwards.


Fig. 7: Ratio of power losses between rotational and linear arrangement, numerical analyses, order $p=2$, pole coverage 1 .


Fig. 8: Ratio of power losses between rotational and linear arrangement, numerical analyses, order $p=2$, pole coverage $2 / 3$.

In order to study the effects of various pole coverages with different approximation orders, the ratio of the respective power losses is shown in Fig. 9 and Fig. 10. Obviously, the pole coverage only affects the power losses of the lower harmonics while the power losses of the higher harmonics are rather constant and directly proportional to the value of the pole coverage.

Finally, the relative error

$$
\begin{equation*}
\epsilon=P_{F E A} / P_{a n a}-1 \tag{15}
\end{equation*}
$$

between the power losses of finite element and analytical analyses with different approximation orders is shown in Fig. 11, Fig. 12, Fig. 13, Fig. 14. In addition, Fig. 15 and Fig. 16 depict this relative error for $1^{\text {st }}$ and $2^{\text {nd }}$ orders with the half mesh size in both directions. Table I and Table II list the respective data of these numerical analyses.


Fig. 9: Ratio of power losses between pole coverage $2 / 3$ and 1 , linear arrangement, numerical analyses, order $p=2$.


Fig. 10: Ratio of power losses between pole coverage $2 / 3$ and 1 , linear arrangement, numerical analyses, order $p=4$.

TABLE I
Matrix data in dependence on the order, DEFAULT MESH SIZE

| Order | Number of <br> Elements | Number of <br> Nodes | Unknowns | Non-zero <br> Entries |
| :---: | :---: | :---: | :---: | ---: |
| 1 | 1728 | 1843 | 1728 | 14976 |
| 2 | 1728 | 1843 | 5184 | 78544 |
| 3 | 1728 | 1843 | 8640 | 189132 |
| 4 | 1728 | 1843 | 13824 | 402696 |

TABLE II
Matrix data in dependence on the order, HALF MESH SIZE

| Order | Number of <br> Elements | Number of <br> Nodes | Unknowns | Non-zero <br> Entries |
| :---: | :---: | :---: | :---: | ---: |
| 1 | 5160 | 5272 | 5160 | 50712 |
| 2 | 5160 | 5272 | 15480 | 253800 |

As expected, $1^{\text {st }}$ order elements cannot encounter both for small skin depths as well as short wave lengths. $2^{\text {nd }}$ order elements are better with an exception of short wave lengths and very high frequencies. $3^{r d}$ and $4^{t h}$ order elements give the same results with a relative error less than $0.5 \%$ which means convergence with respect to the higher orders.

In comparison of the default mesh with the half size mesh, of course the results of $1^{\text {st }}$ and $2^{\text {nd }}$ order elements are better with the dense mesh. However, the results of $2^{\text {nd }}$ order elements with the dense mesh are still less accurate than the results of in particular $3^{\text {rd }}$ order elements with the default mesh. On the


Fig. 11: Relative error of power losses, linear arrangement, pole coverage 1, numerical analyses, order $p=1$.


Fig. 12: Relative error of power losses, linear arrangement, pole coverage 1, numerical analyses, order $p=2$.


Fig. 13: Relative error of power losses, linear arrangement, pole coverage 1, numerical analyses, order $p=3$.


Fig. 14: Relative error of power losses, linear arrangement, pole coverage 1, numerical analyses, order $p=4$.


Fig. 15: Relative error of power losses, linear arrangement, pole coverage 1, numerical analyses with half mesh size, order $p=1$.


Fig. 16: Relative error of power losses, linear arrangement, pole coverage 1, numerical analyses with half mesh size, or$\operatorname{der} p=2$.
other hand, the latter have approximately only the half number of unknowns.

Consequently, the usage of $3^{\text {rd }}$ or even higher order elements will be strongly suggested by evaluating eddy current losses. In particular with 3D meshes, the possibility of generating a relatively coarse mesh within the conducting regions shows explicit advantages against a dense mesh with $2^{n d}$ order elements.

## IV. Conclusion

The paper discusses both analytical and numerical calculation methods of eddy current losses in permanent magnets of electrical machines. Therein, the finite element analyses utilise different approximation orders with hierarchic shape functions in order to validate modelling of wave length as well as skin depth. Obviously, higher order elements with $p \geq 3$ can handle these parameters very well.

Further, linear and rotational arrangements are compared against their results by using identical geometry parameters and various pole coverages. With all harmonic orders along the entire frequency range, there is a deviation only in the range $\pm 5 \%$ between these two arrangements. It is shown that the pole coverage influences only the power losses of the lower harmonic waves while higher harmonic waves have
approximately constant power losses directly proportional to the value of the pole coverage.

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