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Analyzing fuzzy and contextual approaches to vagueness by semantic games

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Erklärung zur Verfassung der Arbeit

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Abstract

How can natural language be 'understood' by computers? Or, more specifically, how can the semantics of a natural language statement be modeled by means of logic in order to facilitate formal reasoning? This perennial problem has many, partly intertwined facets; one of them being the pervasiveness of vagueness in all natural languages. Originally discarded by Frege as a 'defect' of ordinary language outside the scope of logic, vagueness nowadays has given rise to a multitude of approaches within logics, analytic philosophy, and linguistics.

In this thesis we aim to shed some new light on (i) how to justify certain models of vagueness by means of game-theoretic semantics and (ii) how such different approaches to vagueness can be related to each other.

Fuzzy logic is sometimes refered to as the 'logic of vagueness'. There are several ways to attach a suitable semantics to fuzzy logics; particularly Giles's semantic game for Łukasiewicz logic provides a game-theoretic semantics for this so-called t-norm based fuzzy logic. However none of these approaches has yet been extended to (semi-)fuzzy quantification. By introducing the notion of random witness selection we show how certain proportional semi-fuzzy quantifiers can be characterized within an extension of Giles's game. We also provide a game-based characterization of Stewart Shapiro's contextual account of vagueness in the tradition of Giles's game by introducing a third player called Nature.

We pick out Chris Barker's account of 'The Dynamics of Vagueness' as a representative for a scale-(or degree-)based linguistic approaches to vagueness. First, we show how t-norm based fuzzy logics can be recovered from Barker's account by measuring contexts. Although context sizes change in a non truth-functional manner, t-norms and co-t-norms emerge as limit cases.

We also investigate the delineation-based approach by the philosopher Stewart Shapiro. Both Barker's and Shapiro's approaches describe how context changes and evolves during a conversation, albeit by different means. We examine which kind of situations can be modeled in either of these approaches making the same assumptions and precisify what exactly it means for two models 'to make the same assumptions'. We observe how context updates proceed in both models for a series of vague statements. As it turns out, the resulting models using both approaches give rise to exactly the same first-order inferences under certain conditions.

Kurzfassung

Wie können Computer natürlichsprachlichen Text verstehen? Oder, präziser gefragt, wie kann die Semantik der natürlichen Sprache mittels Logik modelliert werden, um formales Schließen zu ermöglichen? Dieses grundlegende Problem besitzt viele, teilweise ineinander verflochtene Facetten; eine davon ist die Existenz von Vagheit in allen natürlichen Sprachen. Ursprünglich von Frege als „Defekt" der Alltagssprache verworfen und außerhalb der Zuständigkeit von Logik positioniert, gibt es heutzutage eine Vielzahl von Modellierungsansätzen innerhalb der Logik, der analytischen Philosophie und der Linguistik.

Im Rahmen dieser Arbeit wollen wir erörtern (i) wie bestimmte Modelle von Vagheit mittels spieltheoretischer Semantik motiviert werden können und (ii), wie solche verschiedene Ansätze zueinander in Beziehung gesetzt werden können.

Fuzzy-Logik wird manchmal als die „Logik der Vagheit" bezeichnet. Es gibt mehrere Möglichkeiten der Fuzzy-Logik eine geeignete Semantik zuzuordnen; insbesondere Giles Auswertungsspiel für Łukasiewicz Logik liefert eine spieltheoretische Semantik für diese sogenannte t-Norm basierte Fuzzy-Logik. Jedoch wurde noch keiner dieser Ansätze auf (semi-)fuzzy Quantifizierung erweitert. Durch die Einführung einer zufälligen Auswahl von Konstanten lassen sich bestimmte proportionale, semi-fuzzy Quantoren mittels erweiterter Giles-Spiele charakterisieren. Außerdem präsentieren wir eine spielbasierte Charakterisierung von Stewart Shapiros kontextbasiertem Modell der Vagheit in der Tradition von Giles Spiel. Hierzu ist es erforderlich, einen dritten Spieler, genannt „Nature", einzuführen.

Wir nehmen Chris Barkers Modell „The dynamics of vagueness" als Vertreter für einen skalen-(oder grad-)basierten linguistischen Ansatz zur Vagheit. Zuerst demonstrieren wir, wie t-Norm basierte Fuzzy-Logiken durch Anwendung einer geeigneten Maßfunktion auf Barkers Kontexte hervortreten. Obwohl sich die Größen der Kontexte in einer nicht wahrheitsfunktionalen Art und Weise verhalten, treten als Grenzfälle t-Normen und co-t-Normen auf.

Weiters greifen wir nocheinmal den kontextuellen Ansatz des Philosophen Stewart Shapiro auf. Sowohl Barker als auch Shapiro beschreiben detailliert, wie sich der Kontext während einer Konversation weiterentwickelt, wenn auch mittels verschiedener Arten von Formalismen. Wir prüfen, was für Situationen in beiden dieser Ansätze mit den gleichen Annahmen modelliert werden können und präzisieren, was es für zwei Modelle bedeutet, „die gleichen Annahmen zu treffen". Wir beobachten, wie sich der Kontext in beiden Modellen bei einer Reihe von vagen Aussagen ändert und, wie sich herausstellt, lassen sich aus den resultierenden Kontexten beider Modelle jeweils genau dieselben logischen Schlüsse ziehen.

Contents

CHAPTER 1

Introduction

1.1 Motivation

Vagueness is a pervasive phenomenon in all natural languages; it is indeed hard to come up with a passage of text used in everyday's life which does not involve a vague expression. Therefore, any system performing natural language processing and reasoning would benefit from being able to handle vague concepts. In this thesis we strive to employ the tools and methods provided by formal logics to model reasoning with vague information. As it turns out, classical logic alone is not sufficient for this undertaking as it has been tailored for an idealized, completely precise language such as mathematics, but not for natural languages as used by humans. Frege himself, one of the founders of modern logic, explicitly considered only sharply defined concepts which are independent of context and discarded the treatment of vagueness as a 'defect' of ordinary language.

Nowadays quite many software applications involving natural language processing [15, 88] simply ignore vagueness. There are also a few ones which acknowledge the ubiquity of vagueness [46, 93]. They frequently employ fuzzy logic motivated as 'the logic of vagueness' based on the idea that truth is not a crisp concept, but gradable. This means that a vague statement may receive a truth value other than 'true' or 'false', but somewhere in between. Formulas can be evaluated efficiently in fuzzy logic: the truth value of a complex statement is calculated solely from the truth values of the components it is composed of. However, fuzzy logics typically lack a well-founded interpretation of truth values—and thus it is a priori unclear how the results of such a fuzzy model should be justified with respect to linguistic data. While philosophers usually put much effort in grounding their approaches on first principles, fuzzy logicians often neglect this point in favour of desired logical and mathematical properties such as algebraic characterizations. One way to provide such an interpretation to truth values as well as to logical connectives and to quantifiers is game semantics. This approach is taken up in this thesis.

As suggested by other approaches to vagueness in linguistics and analytic philosophy a single truth value does not suffice to capture the semantic content of a vague statement. Instead, vague expressions are strongly dependent on the context within which they are used, and their use may alter this context. It is therefore crucial to explicitly model some notion of context in order to fruitfully describe reasoning with vague information.

When analyzing linguistic and philosophical approaches from a logical point of view, deep and subtle discrepancies stemming from different methodologies prevalent in these disciplines form an immediate obstacle. Whereas, for example, linguistic models of vagueness aim at devising *truth conditions* determining the acceptability or grammaticality of sentences, logical approaches to vagueness go beyond that and aim at modeling inference. Philosophical approaches to vagueness may disagree in what they postulate as the source of vagueness and thus arrive at quite different formalisms intended to describe the same phenomena of vagueness. From a pragmatic point of view the question arises whether these different kinds of approaches, regardless of their different motivation, will nevertheless allow one to draw the same inferences from a given situation. More than that, is it possible that all situations which can be expressed in one model can also be expressed in another model? Until now there is no systematic examination of such different approaches with respect to their ability to draw inferences from vague premises. This is however strongly needed for embedding features of these approaches into existing logical models like fuzzy logics because such an analysis is crucial to assess the practical benefit of any new computational model of vagueness.

1.2 What is Vagueness?

An immediate problem when comparing different approaches to vagueness is the lack of a common definition of what exactly vagueness actually is. Stewart Shapiro opens his monograph ‘Vagueness in Context’ [82] with admitting that there is no neutral definition of vagueness; he therefore resorts to the very preliminary characterization, that ‘A word is vague if it is relevantly similar to ‘bald’, ‘heap’, and ‘red’.’[1] Vagueness of these predicates can be viewed as uncertainty about their applicability. For example, a speaker might seem reluctant to judge a man of average height as ‘tall’ but also reluctant to judge him as ‘not tall’. What is the source of this uncertainty? Different approaches to vagueness give different answers to this question. Either the meaning of ‘tall’ itself is vague and can only be determined relative to a context, as argued by contextual approaches to vagueness. Or, as upheld by epistemic accounts of vagueness, there always exists

[1] [82], Chapter 1.1, Page 4

a precise threshold for tallness, however in most situations we are ignorant of that exact point. But do we maybe fail in some other way at evaluating the truth conditions for tallness?

Is there a difference between the vagueness of predicates like ‘empty’ and ‘at four o’clock’ and predicates such as ‘tall’? Emptiness and four o’clock arguably are precise concepts, which may be used in a ‘sloppy’ way. Tallness, however, does not possess such a precise underlying context. Moreover, vague expressions like ‘tall’ exhibit a much stronger dependence on a comparison class than expressions with an underlying precise concept as indicated by [6]. When uttering ‘at four o’clock’ the exact level of precision may still depend on the concrete situation, but hearing ‘at four o’clock’ out of context does not cause much confusion. There is much more space for interpretation when uttering ‘large’. Without a context one cannot even approximately determine which sizes are considered to be ‘large’ and which are not. Therefore linguists sometimes distinguish between imprecision and ‘genuine’ vagueness; this distinction is typically not made explicit in philosophical theories of vagueness and completely ignored in fuzzy logics.

Nevertheless, in spite of these discrepancies between different approaches to vagueness, focus is usually put on the following characteristics:

Borderline cases. Consider the predicate ‘tall’ as a paradigmatically vague predicate and consider a group of people. There might easily be some people which clearly count as tall, as well as some people who clearly are not tall. However, for some people the status of tallness might not be settled; it is rather indeterminate whether they are tall or not—the so-called *borderline cases* of tallness. This poses a challenge to classical two-valued logic. A speaker may be unable to decide whether a borderline tall person should be denoted as tall or not. Note that even if the speaker has perfect information about the exact heights of all the people this might not not help him to make a definite judgement.

Blurry boundaries. The existence of borderline cases, i.e., undefinedness of certain instances, is not sufficient to characterize vagueness. Another crucial property is the *lack of sharp boundaries*. As explained above, it may occur that some people are neither clearly tall nor clearly not tall, but it is still impossible to draw sharp boundaries between the borderline cases and these two groups. Relatedly, the principle of *tolerance* states that a very small, i.e., unnoticeable, difference in heights should not lead to a different judgement. How tolerance is defined exactly differs between between approaches to vagueness.

Sorites paradox. A very distinctive feature of vague predicates—if not *the* most striking one—is subjectability to the Sorites paradox [45]. Consider a group of 8001 men lined up and ordered by their height. The first man is only $150cm$ tall, the last man $230cm$, and there is common understanding that the first one is clearly not tall while the last one clearly is tall. Moreover, any two men standing next to each other only differ in their height by exactly $0.1mm$. Intuitively, a difference of $0.1mm$ cannot make a difference to whether a man counts as tall or not (by tolerance). Such a series is called a *Sorites series*. The Sorites paradox then employs two premisses: (i) *Man* $\#1$ *is not tall* and (ii) *If man* $\#n$ *is not tall then neither is man* $\#n + 1$. By starting at premiss (i) and repeatedly applying premiss (ii) one finally arrives at the conclusion that man $\#8001$ is not tall, which seems absurd. Similar Sorites series can be formulated also for all other vague predicates. Different accounts of vagueness often come to very different solutions to this paradox.

Fuzzy logicians, for example, argue that the first and the last statements do not have truth values 0 and 1, but values very close to that [38]. They use a fuzzy version of modus ponens where the truth value assigned with the conclusion is calculated from the truth values from its premisses. When repeatedly applying premiss (ii), the truth value of the statement *Man* $\#n$ *is tall* slowly gradually increases for growing n until it comes close to 1. Van Rooij, on the other hand, claims that transitivity fails with respect to tolerance and thus substitutes modus ponens by a variation which restricts the repeated application of premiss (ii) [90]. Contrastingly, Shapiro suggests that the process of drawing a Sorites inference dynamic changes the context and speakers will implicitly retract judgements about people's heights during that process. So in the end, a competent speaker will concur that man $\#8001$ is tall, but will retracting some of his judgements made in the reasoning process [82].

Phenomena related to vagueness. Finally, vagueness should be distinguished from other related phenomena:

A statement like 'Someone said something' may be called 'vague' in informal speech, since it does not give precise information about the speaker nor the content. Nevertheless, observing the above characteristics of vagueness, there are no borderline cases of 'someone' nor of 'something', no blurry boundaries, and no Sorites series. We follow the majority of researchers in treating such a statement as an example of *underspecificity* or *generality* rather than vagueness.

Ambiguity may arise when words have multiple senses like 'bank', which can be something you sit on or something you take your money to. Additionally, ambiguity may arise from sentences having multiple readings due to their syntactic structure such as 'John floated the boat between the rocks' [80]. The phrase 'between the rocks' could either describe where John floated the boat or which boat John floated. By the criteria above, ambiguity does not exhibit characteristics of vagueness and will therefore be treated separately from accounts of vagueness.

There is convincing evidence that vagueness and ambiguity are cognitively handled at different layers by humans by the so-called *ellipsis test* introduced by Lakoff [57]: Consider the sentence 'John went to the bank and Jane did too.' Despite the ambiguity of 'bank', intuitively the word is used used in the same sense for both occurrences, i.e., the concept of the 'bank' attended by John is inserted also for the second conjunct, where it is left open by the ellipsis. However, consider the sentence 'John is tall and his six year old son is too.' Contrarily to the previous example, it seems natural to use a different comparison class—and thus a different standard of tallness—for John and for his six year old son.

Context dependence is a general phenomenon in natural language not particular to vagueness, but models of vagueness typically agree that the context of utterance determines vague standards. For example, when calling someone 'tall', this can have a completely different meaning depending on the current topic: For basketball players the standard for tallness will be much higher than for children. Pinkal elaborates more extensively how the statement 'The Santa Maria was a fast ship' receives different truth values in a number of plausible situations [75]; plausible comparison classes could be ships in general, sailing ships, sailing ships in the fifteenth century, and so on. But even within a fixed comparison class, as argued by, e.g., Nic Smith [84], a Sorites series can easily be constructed. Thus Smith concludes that context dependence does not lie 'at the heart of vagueness'.

Nevertheless, as indicated by many philosophers and linguists (see, e.g., Lewis [61]) the context does not only affect vague statements by determining a fixed comparison class. Context is in fact dynamic, shaped by the use of vague language, possibly resulting in a shift of vague standards. Take as an example an assertion of 'Italy is boot-shaped'; this establishes a rather low standard of precision and an assertion of 'France is hexagonal' may be acceptable. However, if one initially denies that Italy is boot-shaped, one assumes a higher standard under which France will not count as hexagonal. Therefore modeling context is arguably still crucial to investigating vagueness in natural language.

Kyburg and Morreau [56] explain how vague standards are established by the use of definite descriptions. As a running example they consider a group of pigs of varying size. If someone starts talking about 'the fat pig' this only makes sense if there is exactly one fat pig in this group. Therefore an according standard will be established if there is a clear gap between the fattest pig and the others. Fernández [26] demonstrates how definite descriptions can be modeled by clustering algorithms. If there emerges a cluster consisting of one pig which is clearly the fattest, this one is selected, otherwise the use of the definite description is ambiguous and leads to confusion in this case.

Fuzzy logicians usually also acknowledge the significance of context, however they do not model it as part their approach. Rather one has to take into account context and comparison classes when constructing a fuzzy model in order to determine adequate truth values.

1.3 Vagueness in Analytic Philosophy

Vagueness has not always been a popular topic of formal investigation. As mentioned in Section 1.1 Frege discarded vagueness as a 'defect' of natural language [29]. Similarly, Russell on the one hand acknowledged the importance of vagueness in language, but on the other hand he upheld that logic assumes precision and therefore vagueness cannot be in the scope of logics [78]. However, since Frege and Russell a multitude of logical approaches to vagueness has emerged, many of them using non-classical logics. I will shortly discuss the most important ones and show how they try to solve the Sorites paradox. For a more in-depth general comparison of these approaches, refer to Keefe [49] or, as more recent overview, Smith [84]

Epistemicism. One way to resolve the Sorites paradox is to postulate that there are always sharp cut-off points and all predicates have well-defined extensions; there are no borderline cases. Thus, the inductive premiss (ii) is discarded. From this point of view vagueness does not differ in its semantic analysis from classical logic. Epistemicists, most notably Williamson [92], take this position—we just do not know where the cut-off points are and which objects exactly are in a vague predicate's extension. They see vagueness as a purely epistemic phenomenon; it surfaces just because of our ignorance of vague standards. So, borderline cases can only be observed at the level of language, but they are not part of the meaning of predicates. Williamson does not deny that the meaning of vague language is determined by its use, however this mechanism is complex and we do not know it.

Truth value gaps. Another very simple approach is to model truth value gaps directly as a third truth value 'indefinite'. Vague predicates may be *true*, *false*, or neither. Thus each predicate has a distinct extension and anti-extension, and these do not have be fully exhaustive on the domain; borderline cases will neither be in a predicate's extension nor in its anti-extension. Logical connectives can be defined by the so-called strong or weak Kleene truth tables. Figure 1.1a shows the strong Kleene truth tables, where u marks the third truth value 'indefinite'. The weak versions of these can easily be obtained by setting the truth value to u whenever at least one components is indefinite as depicted in Figure 1.1b. The truth value of a complex formula is then determined recursively as usual, i.e., one stipulates truth functionality.

Supervaluation. Supervaluation [27] discards this truth functional evaluation by introducing precisification spaces, but keeps the idea of partial interpretations. We start with the observation, that, when evaluating a compound statement, a simple truth value gap semantics treats all its indefinite propositions alike; this is dictated by truth functionality: consider 'John' as a borderline case of tallness. Then 'John is tall' is indefinite, just as is 'John is not tall'. Accordingly, both sentences are interchangeable when evaluating a compound statement. The sentence 'John is tall and John is tall' expectedly comes out as indefinite, just as (unexpectedly)

$\wedge$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

$\vee$	0	u	1
0	0	u	1
u	u	u	1
1	1	1	1

$\rightarrow$	0	u	1
0	1	u	0
u	1	u	u
1	1	1	1

$\neg$	
0	1
u	u
1	0

(a) Strong Kleene logic

$\wedge$	0	u	1
0	0	u	0
u	u	u	u
1	0	u	1

$\vee$	0	u	1
0	0	u	1
u	u	u	u
1	1	u	1

$\rightarrow$	0	u	1
0	1	u	0
u	u	u	u
1	1	u	1

$\neg$	
0	1
u	u
1	0

(b) Weak Kleene logic

Figure 1.1: The Kleene truth tables

does ‘John is tall and John is not tall’. Adopting a supervaluationist view of vagueness, the latter statement cannot be indefinite since it always turns out false as soon as we determine John’s status of tallness.

In a supervaluationist setup formulas are evaluated with respect to a set of classical (complete) interpretations which all extend the initial partial one: the precisification space. Extending a partial interpretation here means to agree on all objects which are already in the partial predicate’s extension and its anti-extension, and to differ only for objects which are indefinite. The precisification space contains all *admissible* ways of making the vague predicates precise. For example, a Sorites series as defined above could be modeled by a partial interpretation where the first few men are not considered as tall, the last few ones are considered as tall, and the remaining ones are indefinite. Then for each possible cut-off point between the tall and the not tall men, there exists one admissible precisification where all men up to this point are not tall and the others are tall. Supervaluation entertains the slogan ‘Truth is supertruth’ advocating that truth is relative to the whole space of precisifications, and a formula is considered *true* if and only if it is true for *all* precisifications.[2] So vagueness is not conceived of as ignorance, but as a deficiency of a predicate’s meaning which can be resolved by taking into account all ways of making the predicate precise. The second Sorites premiss is still invalidated, since it is false for each precisification, albeit at a different point. There is no sharp boundary between the tall and the non-tall men. However the sentence ‘There is a man $\#n$ who is not tall such that man $\#n+1$ is tall’ comes out supertrue which arguably is unintuitive, and thus considered an artifact of the model.

Contextualism. There is a common understanding already for the above theories that the

[2] There exists also the concept of *subvaluation* and *truth value gluts* advocating that borderline cases are not to be seen as neither true nor false, but as *both* true and false (see, e.g., [3]). Accordingly, the analogon of supertruth dictates that a formula is considered true if it is true for *some* precisification.

meaning of a vague predicate depends on the context of utterance. This context, for example, determines the relevant comparison class. As pointed out in Section 1.2 the predicate 'tall' may have different extensions when uttered in a conversation about basketball players or in a conversation about children. *Contextualist* theories of vagueness go beyond this static notion of context; instead they stipulate that the 'nature of vagueness' can only be captured by modeling how the context changes over time, i.e., how it evolves *during* a conversation. The dependence on a comparison class may be a common feature of vagueness; however, according to contextualists, it is not essential for explaining the nature of vagueness. Considering the predicate 'tall for a basketball player' we see that even after explicitly fixing a comparison class, it is still easy to form a Sorites series of basketball players. Instead, if basketball players are lined up according to their height, and a person is repeatedly asked whether the next man in line is 'tall for a basketball player', it is essential to model what is going on, i.e., how the context evolves, during this conversation. Soames [85] argues that such dynamic models of vagueness are indeed necessary for fruitfully modeling vagueness. In his widely cited paper 'Scorekeeping in a language game' [61] Lewis reviews from a high-level point of view the use of contexts for modeling natural language. Amongst others, he explains how the prevalent vague standards can be shifted upwards and downwards during a conversation, but also notes that these shifts are not symmetric: Intuitively, we are much less reluctant when raising the standards, i.e. when getting more precise, than when lowering them.

Contextualist theories employ a more fine-grained notion of context where not only comparison classes and other static content are fixed, but also vague interpretations prevalent at a point in time. Uttering vague expressions may result in a shift of vague standards, all while the overall comparison class stays the same. These vague standards may be modeled explicitly by some degree on an appropriate scale or implicitly by fixing (possibly partial) valuations. In principle, contextual approaches to vagueness are orthogonal to static approaches. Most often, the truth value gap approach as described above is used to model the state of affairs *at one point in a conversation*. However one may also choose a supervaluational or a degree-based approach for the underlying models. The contextual framework then has to specify how these models are updated during a conversation.

For this thesis I will mainly focus Stewart Shapiro's account of "Vagueness in Context"[3] [82, 83] in Chapter 5. Shapiro uses partial valuations to model the state of affairs at one point in a conversation; connectives are given by the strong Kleene truth tables. During a conversation, these partial valuations typically get precisified: The applicability of a vague predicate to ele-

[3]In fact, Shapiro insists that the *meaning* of vague predicates is invariant regardless of the current context, however their *extensions* change dynamically. Apart from this arguably rather philosophical distinction his framework essentially *is* contextual in the sense that Shapiro models how the information transported by vague expressions changes during a conversation.

ments formerly not in its extension or anti-extension may get decided, thus updating the current context, or *conversational record*. Crucially, at some point also elements may get removed from a predicate's extension or anti-extension. As long as a vague proposition is left undecided, a speaker may decide to (explicitly or implicitly) state it either as true or false without compromising her competence. Shapiro denotes this 'ability to go either way' as *open texture*. However, by uttering such a statement the speaker determines the state of this very proposition–and maybe implicitly even also the state of other propositions. If for example a speaker decides to judge someone as tall, implicitly other people taller than the person in question will be regarded as tall in the further conversation. So, unlike as e.g. for epistemic approaches to vagueness the extension of a vague predicate depends on decisions made by the people participating an the conversation. This principle is called *judgement dependence* by Shapiro. This contextualist approach is not restricted to vagueness; in formal semantics often *dynamic semantics* are used to model meaning in a context-dependent way, namely as a function on contexts—its so-called *context change potential*. This way, also other phenomena not related to vagueness can be modeled such as the binding of anaphoric relations or the projection of presuppositions [8, 36, 69]

1.4 Vagueness in Linguistics

Linguistic approaches to vagueness investigate the semantic and pragmatic questions arising from the ubiquitous use of vague expressions in natural language. Often there are close ties between linguistic and philosophical accounts of vagueness; for example Kyburg and Moreau [56] use supervaluation to model vagueness in language. However, other linguists tend to take a more fine-grained approach to vagueness. Solt [86] and Barker [6] distinguish between *vagueness* and *imprecision*[4]: The proposition 'It is four o'clock' is merely imprecise rather than vague, whereas 'John is tall' exhibits *genuine* vagueness. The basic idea is that the first sentence has an underlying precise concept—which is typically used in an imprecise way—pointing to a pragmatic phenomenon. In contrast, the second sentence has no such underlying precise concept (unless one adopts an epistemic view of vagueness, according to which all concepts are precise). Linguists typically agree that vagueness exhibits a strong dependence on context, linguistic theories differ however in what detail context—possibly including relevant comparison classes—is formalized.

In linguistics vague predicates are typically analyzed as *gradable* adjectives such as 'tall' allowing comparisons between objects like 'John is taller than Jane'. Adjectives which do not allow such comparisons such as 'dead' typically do not give rise to vagueness. Kennedy [51]

[4] The term 'imprecision' is also used within fuzzy logics as presented in the next section denoting the imprecision of a (fuzzy) truth value. The notion of imprecision as it is used here is typically not distinguished from vagueness by fuzzy logicians; they rather opt to model these two phenomena uniformly by fuzzy predicates.

provides an extensive analysis of the semantics of gradable predicates, note however that he avoids the term *vagueness*; Kennedy instead only refers to *gradable adjectives* without explicitly modeling context.

Linguistic models of vagueness can be roughly divided into two groups: Degree-based models, modeling vague adjectives as relations between objects and degrees, and delineation-based models, modeling vague adjectives as crisp predicates with context-dependent extensions.

Scale-based approaches. According to scale-based approaches to vagueness [5, 14, 51, 81] objects possess a gradable property *to some degree*. This degree refers to a scale associated with the adjective; e.g. plausible degrees for tallness are lengths measured in metres or centimeters. Kennedy [51] surveys how different scale structures give rise to different conclusions following from a vague proposition. Consider for example 'tall' with an (upwards) open scale and 'full' which arguably has a closed scale. Then from 'John is taller than Jane' one cannot conclude whether either John or Jane are considered tall; however from 'That glass is fuller than this one' one may infer that this glass cannot be full. Moreover, other gradable adjectives such as 'stupid' may also require more complex non-linear scale structures.

Usually also called 'degree-based approaches' in the literature (see, e.g. [89]) we avoid this terminology here in order to not confuse them with logical approaches to graded truth like fuzzy logic as explained in the next section. In fact that 'degree-based' does not necessarily mean that also truth is gradable; this approach should therefore is therefore distinct from fuzzy logic for modeling vagueness. Rather, 'degree-based' denotes a degree of applicability of a vague (gradable) predicate. Take for example the sentence 'John is tall'. The predicate 'tall' may apply to John *to some degree*—namely John's height. However, the sentence as a whole is still be considered crisp, i.e. *true* or *false* depending on the context. In the most simple case, John will be considered tall, if his degree of tallness exceeds some contextually determined standard of tallness. Note that the sentence 'John is taller than Jane' does not refer to this standard, evaluating the sentence amounts to comparing John's and Jane's degrees of tallness.

Chapter 4 discusses Barker's dynamic approach to vagueness [5], which is degree-based and can be seen as a contextual model of vagueness. Barker models contexts as sets of possible worlds, and the meaning of vague predicates as filters on contexts: each possible world determines a threshold value for each vague predicate under consideration and, moreover, to what degree the objects under consideration possess these properties. Thus Barker's approach to vagueness is two-fold: there may be ignorance about the exact current standard for tallness, but also about John's height. Determining whether a simple proposition like 'John is tall' is true at a given possible world is straight-forward by comparing the two according degrees. The meaning of 'John is tall' then removes all possible worlds from the context where John is not tall. Barker also describes how to model *predicate modifiers* like 'very', 'definitely', 'clearly', and combinations thereof. The evaluation of such complex predicates in a possible world may depend on the whole context; e.g. testing whether 'John is definitely tall' is true at a possible world may involve also other possible worlds in the current context. Chapter 4 explains Barker's formalism in more detail, points out some technical problems in his presentation, and shows how it can be combined with other approaches to vagueness.

Delineation-based approaches. Delineation-based approaches do not resort to degrees in order to model gradable adjectives, but model them as crisp predicates with context-dependent extensions [48, 54, 56, 60]. Ignorance of the current standard of a vague predicate is again modeled by possible worlds; each possible world induces a (possibly partial) valuation of the vague predicates in question. Contrasting degree-based approaches, that standard is determined only implicitly by the objects possessing or not possessing that predicate in a possible world. Similarly, comparisons like 'John is taller than Jane' cannot be defined in terms of comparing heights; instead this sentence is true if and only if the set of possible worlds where John is tall is a superset of the set of possible worlds where Jane is tall.

Lewis [60] advocates the use of supervaluation to model truth in a context. Kyburg and Morreau [56] present a more modern contextual model of vague predicates, also using supervaluation, but describing how the context changes during a conversation. They uphold that 'Just as a home handyman can fit an adjustable wrench to a nut [...] a speaker can adjust the extension of a vague expression to suit his needs'[5]. As long as new information is not in conflict with the already established facts, the context is updated by making it more precise, analogously to Barker's approach. However, if conflicting information has to be incorporated, they use belief revision theory [2] to model what Lewis calls a 'shift of standards' [61]. Chapter 5 shows how this approach can be seen as a refinement of Shapiro's contextual account of vagueness outlined above. Also, it is discussed to what extent this approach and Barker's approach provide equally expressive models of vagueness.

1.5 Fuzzy Logics

The term *fuzzy logic* was coined by Lotfi A. Zadeh in the 1960s in the context of fuzzy set theory, a generalization of classical set theory, as an account of reasoning with approximate information [94]. Unlike for classical sets, the membership degree of an element in a *fuzzy set* may take any value in the real unit interval $[0, 1]$.

A central feature of fuzzy logics is truth functionality: The truth value of a compound proposition can be calculated solely from the truth values of its constituents. This property supports the efficient evaluation of formulas with respect to given models.

[5] [56], Section 1, page 577

Zadeh distinguishes between *fuzzy logic in a broad sense* including his own work on natural language semantics and approximate reasoning and *fuzzy logic in a narrow sense* considering formal systems of mathematical logic. The latter branch was considerably advanced by Petr Hájek and his t-norm based approach to many-valued logics in the 1990s. He combined fuzzy logic with a more traditional approach to mathematical logic focusing on axiomatic systems, proof systems, algebraic characterizations, soundness, completeness and complexity results—tools which aid us to logically model reasoning. The monograph [37] marks what is now usually called *mathematical fuzzy logic*; the Handbook of Mathematical Fuzzy Logic [13] provides a more recent overview of relevant work in this area. Some logics already formerly known such as Łukasiewicz or Gödel-Dummet infinite-valued logic turn out to play an important role in Hájek's framework, that is based on the following design choices:

- **Truth values:** The real unit interval $[0, 1]$ is taken as the set of truth values with 0 and 1 denoting absolute falsity and absolute truth, respectively. Truth values are totally ordered by the standard ordering $\leq$.

- **Truth functionality:** For an n-ary connective $\circ_n$ and formulas $\phi_1, \ldots, \phi_n$ the truth value assigned to $\circ_n(\phi_1, \ldots, \phi_n)$ depends only on the truth values assigned to $\phi_1, \ldots, \phi_n$. Therefore, a logical connective $\circ_n$ is specified by its *truth function* $f_{\circ_n} : [0, 1]^n \to [0, 1]$. Truth functionality separates fuzzy logics from approaches to probabilistic reasoning, as the probability that two events both obtain cannot be computed from their individual probability values alone (unless they denote independent events), instead they depend also on their joint probability.

- **Conservativity:** All fuzzy logics are conservative generalizations of classical logic, i.e., for the truth values 0 and 1 all truth functions must behave classically. Hence, all tautologies in a fuzzy logic are necessarily also classical tautologies.

- **Conjunction:** The truth function for conjunction is stipulated to be non-decreasing in both arguments, commutative, associative, continuous, with 1 as its unit element and 0 as its zero element. Continuity ensures that small variations in the truth values assigned to two formulas cannot cause a 'jump' in the truth value assigned to their conjunction. The other requirements ensure that fuzzy logics are a generalization of classical (bivalent) logic. These properties are all met by so-called *continuous t-norms* as defined below. Hájek in fact defines two types of conjunction: *strong* and *weak* conjunction, denoted by the connectives $\&$ and $\wedge$, respectively. The t-norm $*$ is taken as truth function for the former, while the latter is defined as below.

- **Other connectives:** The remaining propositional connectives are derived from strong conjunction and its truth function: Implication is defined via the t-norm's *residuum*[6] $f_{\Rightarrow_*}(x,y) =_{df} \sup\{z \mid z * x \leq y\}$ which amounts to 1 if and only if $x \leq y$. Weak conjunction $\wedge$ is expressed via $\phi \wedge \psi \equiv \phi \& (\phi \rightarrow \psi)$. Disjunction $\vee$ is expressed via $\phi \vee \psi \equiv ((\phi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \phi) \rightarrow \phi)$ and negation $\neg$ via $\neg\phi \equiv \phi \rightarrow \perp$ where the truth constant $\perp$ always evaluates to 0. By these definitions the truth functions for weak conjunction and disjunction always amount to $f_{\wedge}(x,y) = \min(x,y)$ and $f_{\vee}(x,y) = \max(x,y)$ regardless of the continuous t-norm used.

Definition 1 (Continuous t-norm)**.** A continuous t-norm is a function $* : [0,1] \times [0,1] \rightarrow [0,1]$ satisfying the following properties:

Commutativity: $x * y = y * x$,
Monotonicity: $x * y \leq w * v$ if $x \leq w$ and $y \leq v$,
Associativity: $x * (y * z) = (x * y) * z$,
Identity: $x * 1 = x$, and
is continuous in both arguments.

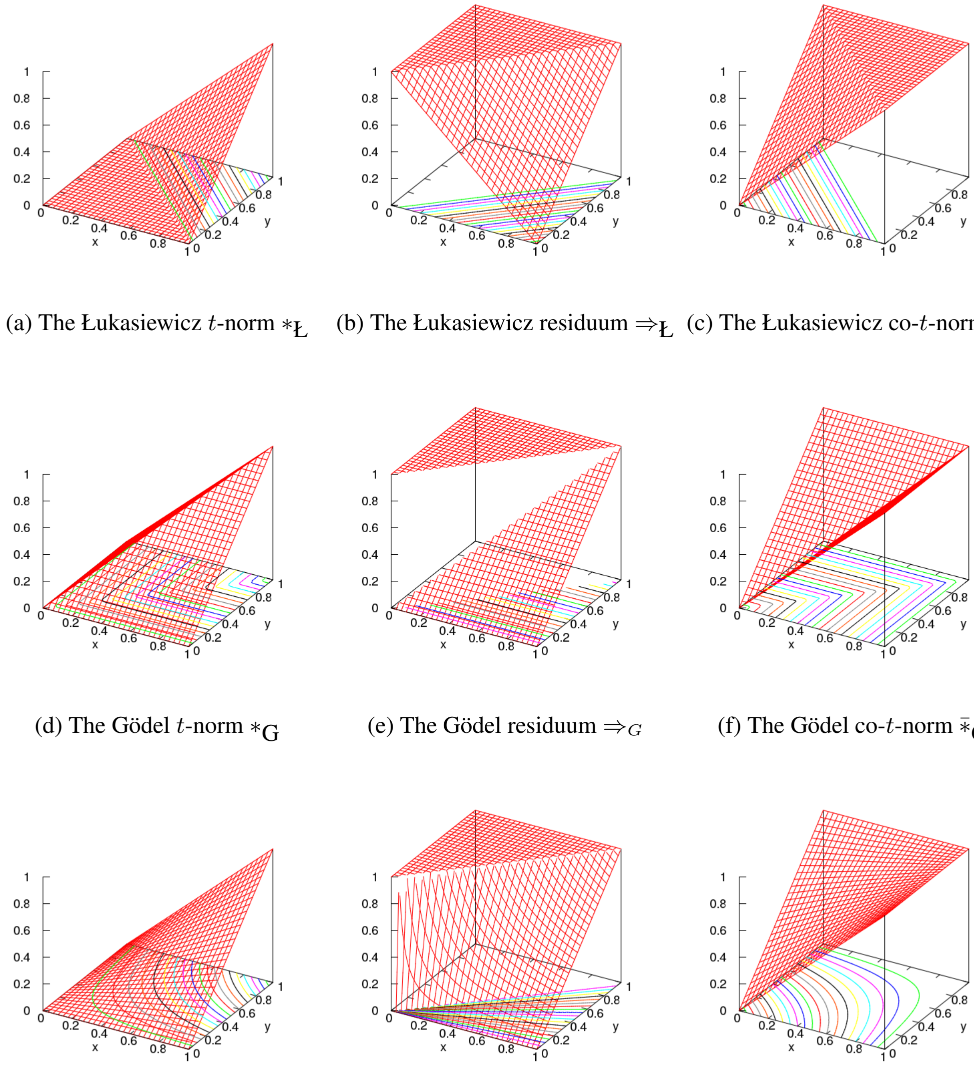
Hájek's *basic logic* (BL) [7] denotes the logic of *all* continuous t-norms. The three *fundamental t-norms* are listed in Table 1.1, namely the Łukasiewicz t-norm $*_{\text{Ł}}$, the Gödel t-norm $*_{\text{G}}$, and the product t-norm $*_{\Pi}$. All other continuous t-norms can be constructed from them (up to isomorphisms) by a so-called *ordinal sum construction* as demonstrated in [37].

	t-norm	residuum	co-t-norm
Łukasiewicz	$x *_{\text{Ł}} y = \max(0, x+y-1)$	$x \Rightarrow_{\text{Ł}} y = \min(1, 1-x+y)$	$x \bar{*}_{\text{Ł}} y = \min(x+y, 1)$
Gödel	$x *_{\text{G}} y = \min(x,y)$	$x \Rightarrow_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$	$x \bar{*}_{\text{G}} y = \max(x,y)$
Product	$x *_{\Pi} y = x \cdot y$	$x \Rightarrow_{\Pi} y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases}$	$x \bar{*}_{\Pi} y = x + y - x \cdot y$

Table 1.1: The three fundamental t-norms

[6] The residuum $\Rightarrow_*$ of a continuous t-norm $*$ is characterized by $x * y \leq z$ iff $y \leq (x \Rightarrow_* z)$. This relation also uniquely defines $\Rightarrow_*$ if $*$ is only a left-continuous t-norm. The logic formed by all left-continuous t-norms is called *monoidal t-norm logic* (MTL) and belongs to the broader class of substructural logics.

[7] There have been very recent proposals within the fuzzy logics community to call this logic the 'Hájek Logic' H honoring Hájek's contributions to mathematical fuzzy logic. However, since this notation has not caught on yet, I here still use Hájek's original denotation BL.



(a) The Łukasiewicz t-norm $*_{Ł}$ (b) The Łukasiewicz residuum $\Rightarrow_{Ł}$ (c) The Łukasiewicz co-t-norm $\bar{*}_{Ł}$

(d) The Gödel t-norm $*_{\mathrm{G}}$ (e) The Gödel residuum $\Rightarrow_{G}$ (f) The Gödel co-t-norm $\bar{*}_{\mathrm{G}}$

(g) The Product t-norm $*_{\Pi}$ (h) The Product residuum $\Rightarrow_{P}$ (i) The Product co-t-norm $\bar{*}_{\Pi}$

Figure 1.2: Graphs of the three fundamental t-norms, their residua, and their co-t-norms

Łukasiewicz logic $\mathbf{Ł}_\infty$[8] will play a central role in the analysis of semantic games in Chapter 2 because of a unique feature: The truth functions for all connectives in $\mathbf{Ł}_\infty$ are continuous. As shown in Table 1.1 the residuum of the Łukasiewicz t-norm is the only continuous one, and also among all other t-norms; for a proof we refer to [19]. As all other connectives can be expressed using strong conjunction and implication, their associated truth functions are continuous as well. According to Hájek's design choices above any assignment (*valuation*) v of values in $[0,1]$ to propositional variables is extended to arbitrary formulas as follows:

$$\begin{array}{ll} v(\bot) = 0 & v(\neg\phi) = 1 - v(\phi) \\ v(\phi \wedge \psi) = \min(v(\phi), v(\psi)) & v(\phi \& \psi) = \max(0, v(\phi) + v(\psi) - 1) \\ v(\phi \vee \psi) = \max(v(\phi), v(\psi)) & v(\phi \to \psi) = \min(1, 1 - v(\phi) + v(\psi)) \end{array}$$

The quantifiers $\forall$ and $\exists$ are introduced as

$$v(\forall x\, \phi(x)) = \inf_{v' \overset{x}{\sim} v} v'(\phi(x)) \qquad \text{and} \qquad v(\exists x\, \phi(x)) = \sup_{v' \overset{x}{\sim} v} v'(\phi(x))$$

where $v' \overset{x}{\sim} v$ denotes a valuation identical to v except for its assignment to x. If one stipulates that there exist constants for all domain elements (as we will do in the context of semantic games), these quantifiers can be defined without changing the evaluation function v, but by quantifying over all domain elements instead.

Often $\mathbf{Ł}_\infty$ is based on the full syntax, as specified above. Alternatively, one may initially only define $v(\bot)$ and $v(\phi \to \psi)$ as above and introduce the other connectives in terms of '$\bot$' and '$\to$':

$$\begin{aligned} \neg A &=_{\mathit{def}} A \to \bot, \\ A \& B &=_{\mathit{def}} \neg(A \to \neg B), \\ A \wedge B &=_{\mathit{def}} A \& (A \to B), \text{ and} \\ A \vee B &=_{\mathit{def}} ((A \to B) \to B) \wedge ((B \to A) \to A). \end{aligned}$$

In the literature on fuzzy logic and on many-valued logics in general frequently only the propositional $\wedge$, $\vee$, and $\neg$ connectives are considered. We call this fragment of $\mathbf{Ł}_\infty$, together with the standard quantifiers ($\forall$, $\exists$), *weak Łukasiewicz logic* $\mathbf{Ł}^w$ here. This fragment is sometimes also referred to as *Kleene-Zadeh logic* KZ (see, e.g., [1]). The restrictions of $\mathbf{Ł}_\infty$ and $\mathbf{Ł}^w$ to the propositional part will be denoted by $\mathbf{Ł}_p$ and $\mathbf{Ł}^w_p$, respectively.

[8] Łukasiewicz logic $\mathbf{Ł}_\infty$ as presented here is sometimes also referred to as *infinite valued* Łukasiewicz logic in the literature, distinguishing it from the *finite valued* Łukasiewicz logics $\mathbf{Ł}_n$. Originally, Łukasiewicz introduced such logics for reasoning about future contingents.

Moreover, all truth functions are piecewise linear and, vice versa, by McNaughton's famous theorem [66] all continuous piecewise linear functions with integer coefficients of the form $[0,1]^n \to [0,1]$ can be expressed within $\mathbf{Ł}_\infty$.

Gödel logic is sometimes also called intuitionistic fuzzy logic as it is an intermediate logic between classical and intuitionistic logic: An axiomatization is readily obtained from an axiom system of intuitionistic logic by adding the prelinearity axiom $(\phi \to \psi) \vee (\psi \to \phi)$. The Gödel t-norm and its residuum are listed in Table 1.1. Observe that no arithmetic functions are applied in the definition of the truth functions, only comparisons. Thus the truth value of a formula can only assume the value 1 or the truth value of one of its atomic subformulas. This also means that the validity of a formula does not depend on the absolute values assigned by an according valuation function, but only on their relative order (except for the special truth value 1). This property makes Gödel logic a robust candidate for many practical applications. Furthermore, for Gödel logic the truth functions for strong and weak conjunction coincide and therefore only one type of conjunction is considered in the literature.

Product logic is the third of the fundamental t-norm based logics based on the product t-norm. As depicted in Figure 1.2h its residuum is non-continuous at $(0,0)$. Product logic has to be kept apart from probabilistic reasoning: Interpreting truth values as probabilities may suggest that the truth value of $p\&q$ coincides with the joint probability that two events p and q obtain, if they are independent. However notice that implication and negation (and of course weak conjunction) do not resemble their probabilistic counterparts, so this motivation cannot be carried over to arbitrary complex formulas.

For all t-norm based fuzzy logics, analogously to the distinction between strong and weak conjunction, sometimes also the notion of strong disjunction is considered in the literature [37], contrasting the above defined (weak) max-disjunction. Strong disjunction is based on the respective co-t-norm $\bar{*}$ which is taken as its truth function. It is defined as the dual of an arbitrary t-norm $*$ as

$$x \mathbin{\bar{*}} y =_{df} 1 - ((1-x) * (1-y)).$$

Thus co-t-norms are always commutative, associative, monotone, and have 0 as their neutral element. The co-t-norms for the three fundamental fuzzy logics are depicted in Table 1.1. Strong disjunction '$\otimes$' can straightforwardly be expressed via

$$A \otimes B =_{\text{def}} \neg(\neg A \& \neg B).$$

The t-norms and their respective co-t-norms will resurface in Chapter 4 as bounds describing how the size of contexts changes when applying logical operations such as conjunction, disjunction, implication, or negation.

For the propositional part, there exist analytic proof systems for all these three t-norm based logics, typically using hypersequents, as explored in great depth by Metcalfe [67]. For first order t-norm based fuzzy logics it turns out that only Gödel logic is recursively axiomatizable.

Fuzzy logic and vagueness

Fuzzy logic is not directly tied to vagueness; it is employed in all kinds of engineering applications to handle imprecise information such as washing machines, electric heatings, or rice cookers. There, the fuzziness is due to the volatility of sensoric data and does not exhibit the characteristics of vagueness explained above. For other purposes such as expert systems or traffic planning fuzzy logic may be employed, because a deeper rigorous analysis using probability theory might just be not feasible.

In the 1970s, Zadeh applied his fuzzy logic approach to natural language semantics for modeling the meaning of vague expressions. In his paper 'The concept of a linguistic variable and its application to approximate reasoning' [95] he introduced the so-called notion of a non-numeric *linguistic variable*, an expression like 'age' whose values themselves can be linguistic such as 'young' or 'old' instead of numeric. In a given model these values are mapped to truth degrees between 0 and 1, i.e., to the real unit interval $[0, 1]$ and may be modified by *linguistic hedges* such as 'very' or 'rather'. Also, Zadeh introduced the notion of a *fuzzy quantifier* [96]—expressions like 'few', 'most', 'about a half'—using tools of fuzzy set theory. Essentially, the size of a fuzzy set is computed by a suitable measure functions, and a truth functions over $[0, 1]$ are associated with a fuzzy quantifier fixing its meaning. Fuzzy quantifiers are investigated in greater depth in Chapter 3 by means of semantic games.

Zadeh and his followers agree that vagueness exhibits context dependent features like context dependent standards for vague predicates. This kind of context dependence is, however, not modeled *within* a fuzzy model, instead the model is only subject to the current context.

Although Zadeh uses the word 'linguistic', he does not place his analysis in the setting of mainstream linguistic research. Lakoff [58] explored the analysis of vagueness by fuzzy logic from a linguistic point of view, however at least since Kamp [47] fuzzy logic was discarded by most linguists for the analysis of natural language meaning. A common counterexample to the naive use of fuzzy logic for vagueness goes as follows: Consider two people, John and Bob, where John is taller than Bob, and a fuzzy model assigning the truth values 0.5 to the proposition 'John is tall' and 0.4 to the proposition 'Bob is tall'. The sentence 'If Bob is tall then John is tall' receives the true value 1 as expected by the definition of implication. However, using the standard involutive negation 'John is not true' also receives the truth value 0.5 and thus the sentence 'If Bob is tall then John is not tall' again comes out true, although it is clearly false. This example already gives a hint that a scalar truth value alone might not suffice to capture the vague content of a proposition; other examples focus on the conjunction of a formula and its negation. Sauerland [79] re-evaluates such arguments in hindsight of more modern fuzzy logics, showing that these cannot be easily rebutted by e.g. changing the truth function for negation. Moreover, Pinkal [75] notes that fuzzy logic in this respect suffers from over-precisification. He asks "How should we decide whether a certain sentence – simple or complex – is 0.72 or 0.73 or even 0.82 "true" '[9]. An answer to this question will be given in the next chapter; Giles's Game shows how to interpret fuzzy truth values as expected win or loss in an evaluation game.

Fuzzy Quantification

Zadeh's notion of 'fuzzy logic in a broad sense' as outlined above also encompasses an approach to natural language semantics by fuzzy logic. In particular his seminal paper 'A computational approach to fuzzy quantifiers in natural language' [96] shows how to model vague quantifiers like *few*, *most*, *about a half*, or *about ten* with tools from fuzzy set theory.

In formal semantics, models for quantification in natural language usually refer to the work of Barwise and Cooper [7] about *generalized quantifiers*. Contrasting the quantifiers '$\forall$' and '$\exists$' familiar from classical logic, a generalized quantifier may (and usually does) take several arguments. For example, the quantifier 'most' as used in 'Most students smoke.' takes two arguments: (i) the so-called *range* (the set of all students) and (ii) the *scope* (the set of all smoking individuals). According to the theory of generalized quantifiers, such quantifiers are modeled as relations between sets of individuals. Following the standard approach, the sentence above is true iff the set of smoking individuals who are also students is at least half the size of the set of all students. Zadeh refers to this principle, however he uses fuzzy sets and relations: The concepts in question need not be crisp, and, moreover, the quantifier itself may be modeled by a fuzzy relation between these concepts as witnessed by the statement 'Few good students smoke often'. Here, 'good students' and individuals, who 'smoke often' both denote fuzzy sets, while the binary quantifier 'few' denotes a fuzzy relation between those sets, i.e., it is associated with a truth function associating a value of the real unit interval $[0, 1]$ with two such fuzzy sets. Such quantifiers are called *fully fuzzy*. On the other hand, *semi-fuzzy* quantifiers can only be applied to crisp sets. Take as an example the statement 'Few students are under eighteen years old' and assume that both properties 'being a student' and 'being under eighteen years old' are crisp. Then, in this case, 'few' may be modeled by a semi-fuzzy quantifier characterized by a *fuzzy relation* over crisp sets.

In the literature, the analysis of fuzzy quantification often starts with unary quantifiers, and then lifts the obtained fuzzy models to binary (and n-ary) quantification in another step—we will

[9] [75], Chapter 5, Page 162

also take on this approach below in Chapter 3. Essentially this is done by narrowing the universe of discourse to the range of the quantifier. Following Liu and Kerre [62], unary quantifiers can be divided into four types with respect to their involvement of membership degrees and degrees of truth:

Type I: the quantifier is precise and its scope is crisp;

Type II: the quantifier is precise, but may have a fuzzy scope;

Type III: the quantifier is fuzzy, but its scope is crisp;

Type IV: the quantifier as well as its scope are fuzzy.

The classical universal and existential quantifiers '$\forall$' and '$\exists$' are Type I quantifiers. They are usually lifted to fuzzy logics, i.e., to Type II quantifiers, by returning as truth value the infimum and supremum, respectively, of the membership degrees of the fuzzy set corresponding to the scope as defined in the previous section. Intuitively this means that the quantifier may be applied to a vague scope, but does itself not introduce any additional vagueness. On the other hand, *many*, *about half*, or *very few*, require models of at least Type III, these are the semi-fuzzy quantifiers described above. Finally, fully fuzzy quantifiers as primarily investigated by Zadeh [96] constitute Type IV quantifiers.

In spite of Zadeh directly modeling fully fuzzy quantifiers, the literature on fuzzy quantification points out severe problems regarding the linguistic adequateness of such models. Particularly Glöckner in a more recent monograph [35] systematically explores where fully fuzzy models defined in the tradition of Zadeh provide counter-intuitive predictions. Take as an example the statement 'About half of Austria is cloudy': We model 'about half' as a unary fully fuzzy quantifier, with the universe of discourse identified with all locations in Austria. According to Zadeh, 'cloudy' is modeled by a fuzzy set which for each location determines its degree of cloudiness. An adequate model for the quantifier 'about half' then takes some cardinality measure of this set and computes a fuzzy truth value which is associated with the statement. If half of Austria is completely cloudy (membership degree 1) and half of Austria is completely clear (membership degree 0), this statement should surely be evaluated to 1. However, consider another scenario where all of Austria is borderline cloudy (membership degree 0.5). Zadeh advocates the so-called Σ-count for computing the cardinality of fuzzy sets evaluates, which delivers the expected result for the first scenario, but also renders the second one perfectly true, which arguably goes against our intuitions about the statement. Glöckner shows that also for other fuzzy cardinality measures presented in the literature on fuzzy quantification the same problem pertains, or others arise. His solution is a framework for modeling only semi-fuzzy quantification in the first place, where models are much less controversial. Then, in a separate step Glöckner shows how semi-fuzzy quantifiers can be lifted to fuzzy sets in a systematic way while retaining a catalogue of desirable properties. The central premise underlying this approach states, that two different linguistically adequate models for fully fuzzy quantifiers already disagree for crisp arguments at some point. This justifies defining models also for fully fuzzy quantifiers by initially restricting the analysis only to crisp arguments, i.e., to semi-fuzzy quantifiers. While the approach to fuzzy quantification presented in this thesis does not directly use Glöckner's framework, Chapter 3 proceeds analogously in focusing on models for semi-fuzzy quantification.

1.6 Structure of the Thesis

The remainder of this thesis is organized as follows. Chapter 2 further explores Giles's Game, the central tool I use for giving coherent interpretations to truth values. In spite of Giles's original motivation in the context of physical theories, I show how his game can be obtained as a straightforward extension of Hintikka's evaluation game for classical logic. By further generalizing Giles's game and abstracting away from Giles concrete choice of game rules we see that the original game re-emerges from a broader class of semantic games.

Chapter 3 uses Giles's game to define classes of (semi-)fuzzy quantifiers focusing on proportionality quantifiers. By this approach the game also serves as a justification for the choice of particular models for fuzzy quantifiers. Giles's game is extended by the principle of random witness selection which leverages modeling fuzzy quantifiers. Depending on the point when the players of Giles's game get to see the randomly chosen witnesses we define the classes of *blind choice quantifiers* and *deliberate choice quantifiers*. Blind choice quantifiers are characterized by piecewise linear functions and provide promising candidates for practical application, deliberate choice quantifiers on the other hand have to be transformed in order to provide linguistically plausible models.

Chapters 4 revisits Chris Barker's dynamic approach to vagueness. We show how some issues with Barker's presentation can be resolved and how a degree-based semantics can be recovered from Barker's notion of context. As it turns out, the resulting semantics is not truth functional, but t-norms and co-t-norms familiar from fuzzy logics emerge as tight boundaries for context updates with compound predicates. Focusing on so-called *saturated contexts*, where features of vague predicates are kept independent, finally allows us to recover truth functions for connectives.

Chapter 5 presents Stewart Shapiro's contextualist approach of 'Vagueness in Context'. We present a Hintikka-style evaluation game where game states explicitly reference to the truth values in question. Moreover, we show how to obtain a semantic game in the tradition of Giles's game, where this explicit reference can be omitted. This game is no longer a two-player zero-sum game, as it may happen that none of the players have a winning strategy—namely if the formula in question is indefinite. Thus a third player, *Nature*, is introduced. Moreover we explore connections between Shapiro's delineation-based and Barker's scale-based approach. As it turns out, if a situation can be modeled within both approaches, they both enable the same inferences from a set of vague statements as long as the new information is non-conflicting.

The thesis is based on the following publications:

- C.G. Fermüller and C. Roschger. Bridges between contextual linguistic models of vagueness and t-norm based fuzzy logic. In F. Montagna, editor, *Petr Hájek on Mathematical Fuzzy Logic* volume 6 of *Outstanding Contributions to Logic*, pages 91–114. Springer, 2015

- C.G. Fermüller and C. Roschger. From games to truth functions: A generalization of Giles's game. *Studia Logica. Special issue on Logic and Games*, 102(2): pages 389–410. Springer, 2014.

- C.G. Fermüller and C. Roschger. Randomized game semantics for semi-fuzzy quantifiers. *Logic Journal of the IGPL. Special issue on Non-classical Modal and Predicate Logics*, 22(3): pages 413–439. Oxford University Press, 2014.

- C. Roschger. Comparing context updates in delineation and scale based models of vagueness. In *Reasoning under Vagueness - Logical, Philosophical, and Linguistic Perspectives*. P. Cintula, C.G. Fermüller, L. Godo, P. Hájek (eds.), College Publications, 2012.

- C. Roschger. Evaluation games for Shapiro's logic of vagueness in context. In M. Peliš, editor, *The Logica Yearbook 2009*, pages 231–246.College Publications, 2010.

CHAPTER 2

Giles's Game

Already in the 1970s Robin Giles, [33] and, in more detail, [32], presented a logic for reasoning about physical theories with dispersive experiments, where repeated trials of the same experiment may yield different results—the most familiar examples arising from quantum mechanics. Giles's aim is to provide *tangible meaning* for propositions about such experiments. Giles motivates this agenda by showing that an axiomatic definition of truth based on (objective) probability values does not satisfactorily attach meaning to such propositions. Instead, Giles refers to the so-called particle rules of Lorenzen's dialogue games for intuitionistic and classical logic [63, 64] for a dialogue game reducing arguments about logically complex assertions to arguments about atomic assertions. To evaluate the latter, Giles assigns a dispersive experiment to each atomic proposition and lets the players bet on the corresponding results. Thus, truth is ultimately defined in terms of winning or losing money and this is what Giles calls *tangible meaning*. Later he applied this game in the context of fuzzy reasoning [34] with the aim of attaching an empirical notion of meaning to the term *fuzzy set*.

As we see, Giles's game was originally not conceived as a game for *vague* expressions. This chapter shows how the formal apparatus of Giles's game can be viewed as a game-based characterization of (certain) many-valued logics, particularly Łukasiewicz logic $\mathbf{Ł}_\infty$following [23]. First, let us recapitulate Hintikka's classical evaluation game [41] and a straight-forward extension to the many valued setting of *weak Łukasiewicz logic* $\mathbf{Ł}^w$. We will see how Giles's game can be characterized within this setting as a further generalization to more complex game states. Finally, we will explore a more general variant of Giles's game abstracting away from concrete payoff functions and dialogue rules, and see how Giles's game re-emerges from a broader family of games.

2.1 Hintikka's Evaluation Game

As shown by Hintikka [40, 41], building on an idea of Henkin, the Tarskian notion of truth can be characterized by a two person constant-sum game played on a first order formula with respect to a given model. Such a characterization provides a semantic framework that goes beyond mere definitions of truth functions. It suggests an analysis of logical truth and validity in game theoretical terms and thus opens a formal pragmatic approach to logic that has proved to be very fruitful and led to the study of well motivated variants of classical logic. Particularly independence friendly logic (IF logic; see [42, 65]) arises when the assumption of perfect information is dropped and a player does not necessarily know or recall past moves. Following [23] I will present the classical evaluation game in a slightly unusual terminology that will make the later transition to Giles's game more transparent.

The $\mathcal{H}$-game. There are two players, say *me* and *you*, who can both act in the roles of either the *attacker* or the *defender* of a formula.[1] The game is played with respect to a given classical first order interpretation M, where all domain elements are witnessed by constants. M can thus be identified with an assignment of 0 (*false*) or 1 (*true*) to the variable free atoms of the language. For the sake of simplicity, let us assume that there are no function symbols in the language, but an unlimited supply of constants, variables, and relation symbols for each finite arity. By $v_M(F)$ we denote the truth value to which F evaluates in M. Observe that in this setting the truth conditions for the (first order) quantifiers $\exists$ and $\forall$ can be stated by referring to a constant and without changing the current valuation.

At every state of the game either I or you act as the defender of some sentence (closed formula) F, the opponent player is the attacker. Accordingly, moves by the defender may be referred to as *defenses* and moves by the attacker as *attacks*. We will say that player **X** *asserts* F, if **X** is the defender of F at the given state where **X** is either me or you.

The game starts with *my* assertion of some formula and proceeds according to the following rules corresponding to the form of the currently considered formula.

($R_\wedge$) If *I* assert $F \wedge G$ then *you* attack by pointing either to the left or to the right subformula. As corresponding defense, *I* then have to assert either F or G, according to your choice.[2]

[1] Naming the players *me* and *you* is due to Giles [32, 33], Hintikka and Sandu call them *Myself* and *Nature*. Moreover, the player's the roles in the game are called *verifier* and *falsifier*, in [42]. This is not only closer to Giles's terminology, but assists in disentangling the players' roles from the reference to the classical truth values. Notice that, in stating the rules of the game, Hintikka and Sandu only refer to the roles, not to the identity of the players. While this allows for a more compact presentation, it hides the fact that, in any formal presentation of the game, one has to keep track of the current assignment of roles to the two players. For Giles's game the distinction between the players and their roles will turn out even more crucial below: there, a player can simultaneously act as a defender of some formulas while attacking a formula to be defended by his opponent. In order to make the later transition to Giles's game more transparent, let us therefore include the reference to the players in stating the rules.

[2] Note the duality of the rules for $\wedge$ and $\vee$. A version of conjunction where both conjuncts have to be asserted

$(R_\vee)$ If *I* assert $F \vee G$ then *I* have to assert either F or G at my own choice.

$(R_\neg)$ If *I* assert $\neg F$ then you have to assert F. In other words, our roles are switched: the game continues with you as defender and me as attacker (of F).

$(R_\forall)$ If *I* assert $\forall x F(x)$ then you attack by picking c and *I* have to defend by asserting $F(c)$.

$(R_\exists)$ If *I* assert $\exists x F(x)$ then *I* have to pick a constant c and assert $F(c)$.

Note that $(R_\vee)$ and $(R_\exists)$ only involve a move by *me*. However, we may speak of an empty attack by *you*, followed by *my* defense, also in these cases. Also the role switch in $(R_\neg)$ may be viewed as triggered by *my* defense to *your* attack of $\neg F$. Note also, that although no rule for attacking formulas of the form $F \to G$ is given explicitly, one can easily be derived as a 'shortcut' for $\neg F \vee G$:

$(R_\to)'$ If *I* assert $F \to G$ then I have to assert G or you have to assert F at my own choice. In other words, if I choose the latter, the game continues with you as defender and me as attacker of F.

In this manner we arrive at a uniform format of rules and corresponding rounds in a run of the game: each round consists of an attack followed by a defense. We have only stated rules for states where *I* am the defender and *you* are the attacker of the currently considered formula. The rules for *you* defending a formula are completely dual.

More formally, each state of the $\mathcal{H}$-game is identified with the currently asserted sentence and a role assignment (either *I* am the defender and you the attacker, or vice versa). The role assignment remains unchanged in all state transitions, except for the one explicitly triggered by $(R_\neg)$. Let us denote the game state where I assert (defend) A by $[\mid A]$, and the game state where you are the defender by $[A \mid]$. This notation is due to Giles [33] and already hints at his notion of compound game states.

Note that in standard game theory, the notion of a game position usually also includes the choices made by the players *so far*. As we are only concerned with games of perfect information here, i.e. both players know their opponent's move, the history *how* a certain sentence and role assignment were obtained does not matter and can be omitted for the sake of simplicity. Let us, moreover, focus on so-called *extensive-form games*, i.e. games where all states and possible moves are explicitly specified by a *game tree*.

Definition 2. (Game tree) An *extensive-form game* G is specified by a rooted tree T where (i) the root of T corresponds to the initial position in G, (ii) each node of T is labeled with one

will be considered for $\mathcal{G}$-games, in Section 2.3.

player, (iii) each outgoing edge correspond to possible choices by that player, and (iv) leaf nodes correspond to final game states.

The game G is called of *finite* depth if T is of finite depth. A *run* of G is a path in T between the root node and a leaf.

Each final game state is assigned a player designating the winner of a run ending at this state.

Hintikka's game is a so-called *zero sum game*: After each run of the game one player is declared the winner of the game, while the other player loses.

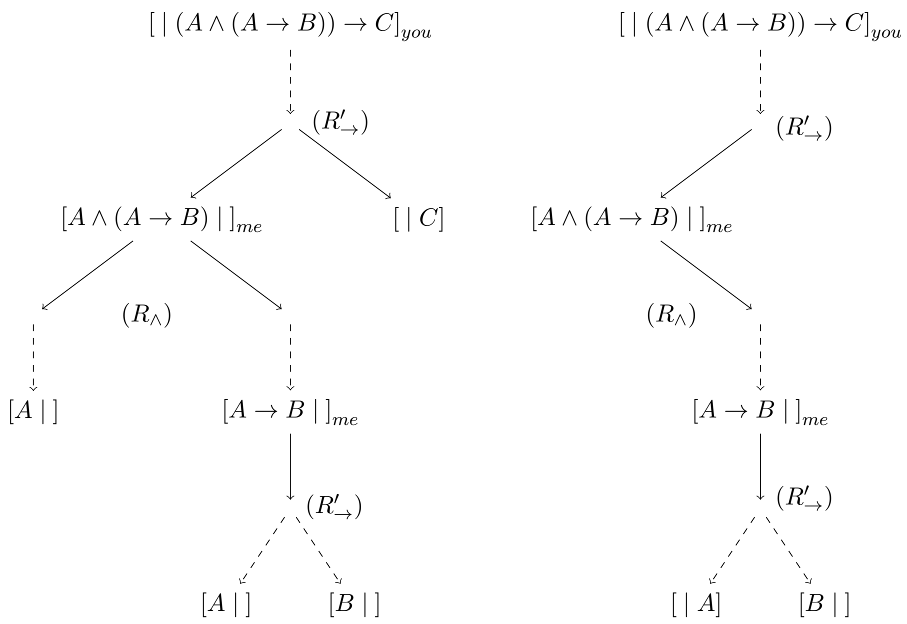
Definition 3. (Zero sum game) A game is zero sum if each leaf node of the associated tree is labeled with one player, the *winner* in the corresponding final game state.

$\mathcal{H}$-games can be straightforwardly viewed as two-player zero sum games with perfect information. A run of an $\mathcal{H}$-game is a sequence of states beginning with a sentence defended by *me*, where each successor state results from the previous one in accordance with the specified rules. Thus attack moves strictly alternate with corresponding defense moves. Each round consists of two state transitions, where only the second one, the defense move, changes the currently asserted formula, i.e., the according game tree contains intermediate nodes. Once we arrive at an atomic formula, the run of the game ends. In such a final state, where *I* assert an atomic formula A, we say that *I* win if $v_M(A) = 1$ and *I* lose if $v_M(A) = 0$. Analogously, in a final state, where *you* assert an atomic formula A, we say that *I* win if $v_M(A) = 0$ and *I* lose if $v_M(A) = 1$. We call the game starting with my assertion of F the $\mathcal{H}$-game for F under M.

Definition 4. (Strategy) Let G be a game and T its game tree. A strategy S for a player X is a subtree of T containing the root node such that (i) for every non-leaf node labeled with X, S contains exactly *one* of its successor nodes, (ii) for every other non-leaf node, S contains *all* of its successor nodes in T.
If X wins at all leaf nodes in S, then S is called a *winning strategy* for X.

Zermelo's theorem [97] states that every zero-sum game with perfect information of finite depth is determined: The according game tree always contains a winning strategy for one of the players. Note that the game tree itself can be infinitely branching: for any model with an infinite domain the dialogue rules $(R_\forall)$ and $(R_\exists)$ allow *infinitely* many different moves when attacking or defending the formula. However, the number of moves in any given run of the game remains finite.

Figure 2.1a shows the game tree of the $\mathcal{H}$-game for $(A \wedge (A \to B)) \to C$. The game starts with my assertion of the formula. Your (empty) attack can be answered in two ways, according to $(R_\to)'$: either I choose to defend C (and the game ends) or the game continues with your assertion of $A \wedge (A \to B)$. In the latter case, as you are now the defender, I can attack according



(a) The game tree (b) A strategy for me

Figure 2.1: The $\mathcal{H}$-game for $(A \wedge (A \to B)) \to C$ including intermediary states. States labeled by $[\cdot \mid \cdot]_{me}$ denote a new round where *I* am to move first (solid arrows), while states labeled by $[\cdot \mid \cdot]_{you}$ denote a new round where *you* are to move first (dashed arrows).

to $(R_\wedge)$ by either pointing to the left or to the right conjunct. If pointing to the left conjunct, the game ends with your assertion of A, otherwise the game continues with your assertion of $A \to B$. After me attacking this formula, the game ends with either me asserting A or you asserting B, depending on your according defense move. Figure 2.1b shows a possible strategy for this game determining for me which move to make at each possible game state. Consider a model M with $v_M(A) = 1$ and $v_M(B) = 0$. Then this strategy is a winning strategy for me and ensures that I will win the game regardless of your decisions.

While this presentation of Hintikka's game looks different from traditional descriptions by referring directly to the players instead of only the current roles, the game itself remains the same. Hintikka shows that it characterizes classical logic:

Theorem 1 (Hintikka)**.** *A sentence F is true in an interpretation M (in symbols: $v_M(F) = 1$) iff I have a winning strategy in the $\mathcal{H}$-game for F under M.*

2.2 Hintikka's Game for Many-Valued Logics

The aim is to provide a similarly elegant characterization of *graded truth* for first order fuzzy logics. As a first step, let us transfer $\mathcal{H}$-games into a many-valued setting by following an idea of Giles [32, 33] and reformulating the winning condition in terms of *dispersive* experiments. This way, intermediate truth values naturally arise from Hintikka's classical game corresponding to expected risks of payments. Without changing the game any further, we will see how this game captures weak Łukasiewicz logic $\mathbf{Ł}^w$ as defined in Section 1.5.

Let us conceive of the evaluation of the atomic formula A at the final state of an $\mathcal{H}$-game as a *(binary) experiment* E_A that either *fails*, meaning $v_M(A) = 0$, or *succeeds*, meaning $v_M(A) = 1$. Moreover, we stipulate that I have to pay 1€ to you if I lose the game; i.e. at a final game state, I have to pay if I am the defender of A and the corresponding experiment fails, or if you are the defender and the experiment succeeds. Otherwise, no payments are done. Hence winning strategies turn into strategies for avoiding payment. Note that even when the roles of attacker and defender are switched, it is *me*, not *you*, who has to pay upon losing the game. This is necessary to ensure that expected payments (inversely) correspond to truth values. Giles's extended game scenario allows one to restore perfect symmetry, as we will see in Section 2.3. So far this just amounts to an alternative way to present the original game. Giles's main innovation is to consider *dispersive* experiments: they may show different results upon repetition, where the individual trials of the experiment are understood as independent events. Of course, $\mathsf{E}_\perp$ remains non-dispersive: it simply always fails. Such experiments frequently occur in the context of quantum mechanics, however also vague language may be modeled by trials of dispersive experiments: while in concrete dialogues competent language users either usually accept or don't accept grammatical utterances, vagueness is reflected in a brittleness or dispersiveness of such highly context dependent decisions. (See, e.g, [5, 82].) In order to arrive at 'degrees of truth' for an atomic statement A in such a model, one assumes that the dialogue partners associate a fixed success probability $\pi(\mathsf{E}_A)$ to the experiment E_A. The result of E_A may be thought of as an answer to the question "Do you accept A (at this instance)?" If A is vague and this question is asked repeatedly to different (random) people independently of each other, the answers will exhibit dispersion: While some people will agree with A, others will not, all without compromising their competence of the English language.

By $\langle A\rangle^r = 1 - \pi(\mathsf{E}_A)$ let us denote the *risk* associated with A, i.e., the *expected* (average) loss of money associated with my assertion of A. The function $\langle\cdot\rangle^r$ that maps each atomic sentence into a failure probability of the corresponding experiment is called *risk value assignment*. Risk value assignments can be straight-forwardly translated to many-valued interpretations by $\langle A\rangle^r_M = 1 - v_M(A)$. Moreover, let us extend the notion of risk to game states: $\langle\ \mid G\rangle^r$ denotes my final risk in a game where I am defending and you are attacking G assuming we both play rationally, i.e. we try to minimize our risk. Conversely, If I am the attacker and you are the defender of G this value is denoted by $\langle G \mid \rangle^r$.

The setting of randomized payoff for $\mathcal{H}$-games leads to a characterization of weak propositional Łukasiewicz logic $\mathbf{Ł}_p^w$, as shown in the following theorem.

Theorem 2. *A $\mathbf{Ł}_p^w$-sentence F is evaluated to $v_M(F) = x$ in interpretation M iff in the $\mathcal{H}$-game for F under the corresponding risk value assignment $\langle\cdot\rangle_M$ I have a strategy that limits my expected risk to $(1-x)$€, while you have a strategy that ensures that my expected risk is not below this value.*

Proof. Note that probability and risk are only involved in the definition of the payoff at final (atomic) states. The game itself does not contain any random moves. Since the game is of finite depth one can compute optimal *pure* strategies by backward induction.

If F is atomic then $\langle F \mid \rangle^r = 1 - \langle F \rangle^r_M$ and $\langle \mid F \rangle^r = \langle F \rangle^r_M$ and thus my risk is $v_M(F)^r$ in the former case and $1 - v_M(F)$ in the latter case, as required. Otherwise, by induction on the complexity of F that $\langle \mid F \rangle^r = 1 - v_M(F)$ we have:

- If I assert $\neg G$, the game continues with your assertion of G and with our roles switched. This means, whenever I will win a game where I initially assert $\neg G$, I will lose the game where you initially assert G. Since winning/losing the game directly corresponds to paying 1€, the risk value for $\langle \mid \neg G \rangle^r$ reduces to $\langle G \mid \rangle^r = 1 - \langle \mid G \rangle^r$, just like in the truth function for $\neg$.

- If I assert $G \vee H$ then I will pick G or H according to where my associated expected risk is smaller and the game will continue with my corresponding assertion. Therefore $\langle \mid G \vee H \rangle^r = \min(\langle \mid G \rangle^r, \langle \mid H \rangle^r)$, and thus $v_M(G \vee H) = \max(v_M(G), v_M(H))$ amounts to $1 - \min(\langle \mid G \rangle^r, \langle \mid H \rangle^r) = 1 - \langle \mid G \vee H \rangle^r$.

- If I assert $G \wedge H$ then you will pick G or H according to where my associated risk, i.e., your expected gain, is higher. Therefore $\langle \mid G \wedge H \rangle^r = \max(\langle \mid G \rangle^r, \langle \mid H \rangle^r)$, and correspondingly $v_M(G \wedge H) = \min(v_M(G), v_M(H)) = 1 - \max(\langle \mid G \rangle^r, \langle \mid H \rangle^r) = 1 - \langle \mid G \wedge H \rangle^r$.

The cases where you defend and I attack F are completely dual. □

Note that $\mathbf{Ł}_p^w$as presented in Section 1.5 does not feature a binary connective $\rightarrow$. In the literature $G \rightarrow H$ is often defined by $\neg G \vee H$, however note that this operation is different from the residuated implication operator in $\mathbf{Ł}_\infty$. Therefore Giles's game below introduces a new dialog rule for disjunction different from $R_\rightarrow$ for the $\mathcal{H}$-game.

Note also that we are only interested in final risk as payoff values of the game, not in actual final payments due to particular results of experiments. Since individual trials of experiments are independent events, truth functionality is preserved. Consider for example a game for $A \vee \neg A$. While I will finally have to pay either 1€ or nothing, depending on a trial of E_A, my risk, i.e. my optimal expected loss under the risk value assignment corresponding to interpretation M is $\min(\langle A \rangle_M, 1 - \langle A \rangle_M)$€(depending on my defense either we end up with me or you asserting A), which indeed amounts to $(1 - v_M(A \vee \neg A))$€.

There is a slight complication in lifting Theorem 2 to the first order level: in a $[0,1]$-valued interpretation M witnessing domain elements for quantified sentences may not exist. More precisely, we may have $v_M(\forall x F(x)) < v_M(F(c))$ and $v_M(\exists x F(x)) > v_M(F(c))$ for all constants c. Consider for example a model M with the natural numbers (without 0) as domain D, and the predicate P interpreted by $v_M(P(c)) = 1 - \frac{1}{c}$ (remembering that we stipulated above to identify domain elements with constants). Then $v_M(P(c)) < 1$ for all constants c, but $v_M(\exists x P(x)) = 1$. For this reason we define the following general notion for games with randomized payoff (as in our new version of the $\mathcal{H}$-game, above, and in $\mathcal{G}$-games, introduced below).

Definition 5. A game with randomized payoff is *r-valued for player* ***X*** if, for every $\epsilon > 0$, **X** has a strategy that guarantees that her expected loss is at most $(r + \epsilon)$€, while her opponent has a strategy that ensures that the loss of **X** is at least $(r - \epsilon)$€. We call r the *risk for* ***X*** in that game.

This notion allows us to state the generalization of Theorem 2 to **Ł**^{w} concisely:

Theorem 3. *A* ***Ł***^{w}*-sentence F is evaluated to $v_M(F) = x$ in interpretation M iff the $\mathcal{H}$-game for F under risk value assignment $\langle \cdot \rangle_M$ is $(1 - x)$-valued for me.*

Proof. Building on the proof of Theorem 2, it only remains to consider the induction steps for quantified sentences:

- If I assert $\exists x F(x)$, then the game continues with my assertion of $F(c)$ for a constant c picked by me in a manner that minimizes my risk. In fact, since there might be no domain element witnessing the infimum $v_M(\exists x F(x)) = \inf_{c \in D}(v_M(F(c)))$, we can only ensure that, for any given $\delta > 0$, $\langle | \; \exists x F(x) \rangle^r = \langle | \; F(c) \rangle^r = 1 - v_M(\exists x F(x)) + \delta$.

- If I assert $\forall x F(x)$, the game continues with my assertion of $F(c)$, where c is chosen by you to maximize my risk. Therefore, analogously, we obtain $\langle | \; \forall x F(x) \rangle^r = \langle | \; F(c) \rangle^r = 1 - v_M(\forall x F(x)) - \delta$ for some $\delta > 0$.

The cases where you are the defender of a quantified formula are dual. □

Note that the value ϵ mentioned in Definition 5 does not directly correspond to δ as used in the above proof, but rather results from the accumulation of appropriate δs. For a formula with k quantifiers, one might for example set all δs uniformly to $\frac{\epsilon}{k+1}$.

2.3 Giles's Game

Robin Giles in the 1970s [32, 33] introduced an evaluation game that was intended to provide 'tangible meaning' to reasoning about statements with dispersive semantic tests as described above. The main difference with respect to Hintikka games is that now both players are allowed to assert a multiset of formulas at the same time. Observe that for the $\mathcal{H}$-game for any formula G the final risk value always amounts to the risk value $\langle P\rangle^r_M$ of an atomic subformula P of G or its invert $1 - \langle P\rangle^r_M$ (or, if G contains quantifiers, the final risk value approaches $\langle P\rangle^r_M$ or $1 - \langle P\rangle^r_M$) since at a final game state only one atomic subformula of G is asserted. Therefore, extending Hintikka's game to a more complex notion of game state just seems a natural choice, when striving to model full Łukasiewicz logic $\mathbf{Ł}_\infty$.

For the logical rules of his game Giles referred not to Henkin or Hintikka, but to Lorenzen's dialogue game semantics for intuitionistic logic [64]. In particular, the following rule for implication was proposed:

$(R_\rightarrow)$ If *I* assert $F \rightarrow G$ then *you* may attack by asserting F, which obliges me to defend by asserting G. (Analogously if *you* assert $F \rightarrow G$.)

In contrast to $\mathcal{H}$-games, such a rule introduces game states, where more than one formula may be asserted by each of us at the same time. Since, in general, it matters whether we assert the same statement just once or more often —particularly whether we place a bet on the outcome of an experiment or several bets on repetitions of that experiment— game states are now denoted as pairs of multisets of formulas.

Definition 6. (Tenet) The *tenet* Γ of a player (me or you) is the finite multiset $[F_1, \ldots, F_n]$ of formulas asserted by that player at a given state of the game. The tenet Γ is *atomic* if all formulas in Γ are atomic.

Let us denote atomic tenets by lower Greek letters $\gamma, \delta, \ldots$ and arbitrary tenets by upper Greek letters $\Gamma, \Delta, \ldots$. Moreover, let us write $[\Gamma, \Delta]$ to denote the union of the tenets Γ and Δ as well as $[\Gamma, F]$ instead of $[\Gamma, [F]]$, etc.

Definition 7. (Game state) A *game state*, $[\Gamma \mid \Delta]$ consists of two tenets, where Γ is *your* tenet and Δ is *my* tenet. A game state is *atomic* if both Γ and Δ are atomic.

Again, assume that a binary experiment E_A is associated with every atomic A with corresponding risk $\langle A \rangle^r = 1 - \pi(\mathsf{E}_A)$. Payments, however, are now fully dual: let us stipulate that I have to pay 1€ to you whenever an instance of an experiment corresponding to one of my atomic assertion fails, while you have to pay me 1€ for each instance of a failing experiment corresponding to one of your atomic assertions. My risk value, i.e. the expected total amount of money (in €) that I have to pay to you at the exhibited final state is readily calculated by:

$$\langle A_1, \ldots, A_n \mid B_1, \ldots, B_m \rangle^r = \sum_{1 \le i \le m} \langle B_i \rangle^r - \sum_{1 \le j \le n} \langle A_j \rangle^r .$$

Note that the risk can be negative in $\mathcal{G}$-games, i.e., the risk values of the relevant propositions may be such that I expect a net payment by you to me.

As an example consider the state $[P, P \mid Q]$, where you assert P twice and I assert Q once. Three trials of experiments are involved in the corresponding evaluation: two trials of E_P, one for each of your assertions, and one trial of E_Q to test my assertion of Q. If $\langle P \rangle^r = 0.2$, i.e., if the probability that the experiment E_P yields a positive result is 0.8, and $\langle Q \rangle^r = 0.5$ then $\langle P, P \mid Q \rangle^r = 0.1$. This means that my expected loss of money according to our betting scheme is 0.1€. If, on the other hand, $\langle P \rangle^r = \langle Q \rangle^r = 0.5$, then $\langle P, P \mid Q \rangle^r = -0.5$, which means that I expect an (average) gain of 0.5€.

The rules $(R_\wedge)$, $(R_\vee)$, $(R_\forall)$, and $(R_\exists)$ as defined above for $\mathcal{H}$-games remain unchanged for $\mathcal{G}$-games. By adding the above implication rule $(R_\to)$ and defining $\neg F = (F \to \bot)$ we arrive at Giles's game for Łukasiewicz logic. The according dialogue rule $R_{(\neg)}$ can be formulated as

- If I assert $\neg F$ then you may attack by asserting F, which obliges me to pay 1€ to you, i.e. to assert $\bot$. (Analogously if you assert $\neg F$.)

Remember that for a $\mathcal{H}$-game the game state consisted of only one formula and a role assignment, therefore it was always determined which player had to attack next and which one had to defend. For $\mathcal{G}$-games this is not determined; Giles however proofs that the order in which formulas are attacked has no influence on the final risk value [33]. Instead, both players are allowed to attack one of their opponent's asserted non-atomic formulas. After the attack and the corresponding defense the attacked formula is removed from the defendant's tenet and the game continues.

Notice that Łukasiewicz logic $\mathbf{Ł}_\infty$ as presented in Section 1.5 also has a second type of conjunction, so-called 'strong conjunction' (&), while this connective is not considered by Giles. From the point of view of contemporary mathematical fuzzy logic [37,67] strong conjunction is the central connective of a t-norm based fuzzy logic and the Łukasiewicz t-norm is one of the three fundamental t-norms. Nevertheless, from Giles's own perspective of providing 'tangible meaning' of logical connectives, it is somewhat odd to consider exclusively a dialogue rule for conjunction, where only one of the conjuncts has to be defended. One might rather be tempted to add the following rule:

- If I assert $F \wedge' G$ then, if you attack, I am obliged to assert both F and G.

However, it is easy to see that this has undesirable effects. E.g., there is no strategy for avoiding positive risk when initially asserting $\bot \to (\bot \wedge' \bot)$. If you attack my initial assertion, I have to assert $\bot \wedge' \bot$ to defend my claim. If you attack this assertion in the next round, we end up in the game state $[\bot \mid \bot, \bot]$, and I will always have to pay you 1€, just as if I had initially asserted $\bot$. This is surely undesirable since $\bot \to (\bot \wedge' \bot)$ is a tautology of $\mathbf{Ł}_\infty$. More profoundly, one cannot any longer limit one's risk associated with asserting a single formula by 1€. Therefore Giles defends his choice of conjunction rule, originating with Lorenzen [63], by referring to what he calls the *principle of limited liability* stating that both players always have a strategy to limit their risk, i.e. their *expected* loss to 1€.

As pointed out in [20], one can formulate a simple rule that is adequate for strong conjunction $\&$ in $\mathbf{Ł}_\infty$:

($R_\&$) If I assert $F\&G$ then, if you attack, I am obliged to assert either both F and G, or else to assert $\bot$. (Analogously if you assert $F\&G$.)

Remember that asserting $\bot$ obliges one to pay the agreed upon maximal 'fine' of 1€ for asserting a statement that cannot be verified by a corresponding trial of a dispersive experiment ($\langle \bot \rangle^r = 1$). In this sense this rule, too, is motivated by the principle of limited liability.

There is yet another form of the principle of limited liability already present in Giles's rule for implication: instead of attacking $F \to G$ by asserting F to force the opponent to assert G, a player may choose not to attack $F \to G$ at all. Since the risk associated with F may be higher than the risk associated with G, the latter choice (no attack) amounts to an option that limits my risk originating with your assertion of $F \to G$—a risk that I would not be able to avoid if the rules of the game required that every asserted implication is to be attacked.

These two forms of limiting risk can be formulated in a slightly more abstract way:

Limited liability for defense (LLD): A player can always choose to just assert $\bot$ in reply to an attack by his opponent.

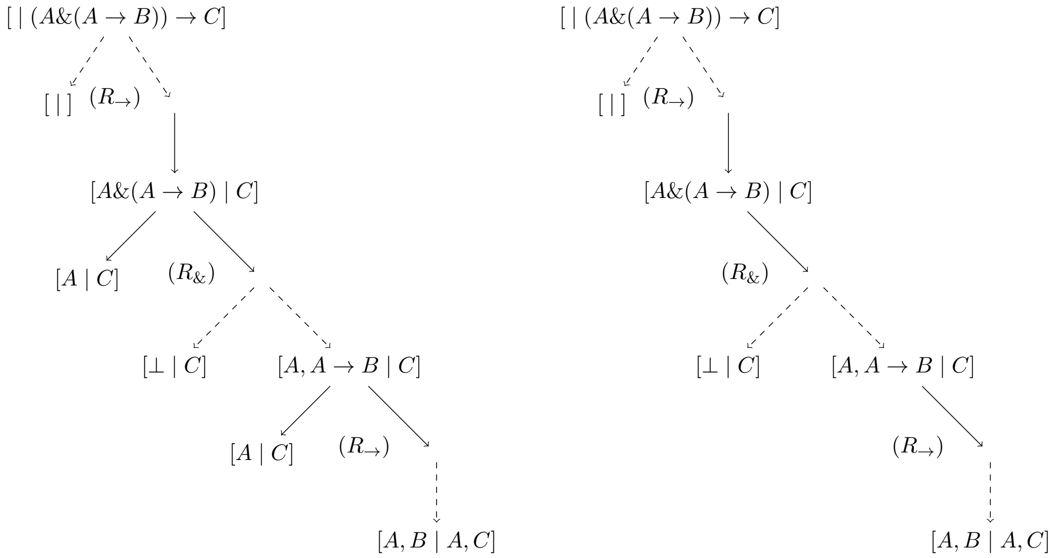
Limited liability for attack (LLA): A player can always declare not to attack an occurrence of a formula that has been asserted by his opponent.

The principle of limited liability can be seen as two general rules for Giles's game, not just for implication and strong conjunction. It is straight-forward to check that the proof of Theorem 4 as presented in [20] remains essentially unchanged if LLD and LLA are uniformly imposed on all specific dialogue rules. This is due to the fact that for the other dialogue rules, a rational player never has to invoke this principle. Take for example the rule $(R_\wedge)$: If you attack my assertion of $A\wedge$B, then invoking LLD and responding with $\perp$ will never bear less risk for me than asserting either A or B. Conversely, you will can only gain from attacking my assertion, thus it is never favorable for you to invoke LLA and grant my assertion of $A \wedge B$.

With the principle of limited liability ensuring that the risk arising from asserting a formula can always be hedged by 1€ we can also formulate a dialogue rule corresponding to strong disjunction $\otimes$ as defined in Section 1.5. Similarly as for conjunction it might to seem odd to have a rule for disjunction where I immediately have to make a decision for one of the conjunctions when asserting $A \vee B$. E.g., if A and B denote atomic formulas, I may be convinced that at least one of the experiments E_A and E_B will fail, but I don't know which one. The following dialogue rule allows me to assert both disjuncts, however you have to pay me 1€ in advance. If you think that the disjunction is false you will attack because you expect to receive 1€ in return for each of the two disjuncts. Otherwise, you will as a rational player refrain from attacking by invoking LLA instead.

$(R_\otimes)$ If I assert $F \otimes G$ then, if you attack by asserting $\perp$, I am obliged to assert both F and G. (Analogously if you assert $F \otimes G$.)

Figure 2.2a shows the game tree of the $\mathcal{G}$-game for $(A\&(A \to B)) \to C$. The game starts with my assertion of the starting formula. If you choose not to attack, according to LLA, the game ends with both players asserting nothing. Otherwise, you attack by asserting $A\&(A \to B)$ and I defend by asserting B. If I attack your assertion you can defend by invoking LLD and assert $\perp$, otherwise you have to add both conjuncts to your tenet. (I also could have invoked LLA and granted your assertion. However, LLA for Rule $(R_\&)$ can safely be ignored, since it never brings any advantages for the attacker.) Next, I can attack or grant your assertion of $A \to B$; if I choose the latter, we end up in $[A \mid C]$, otherwise I attack by asserting A and you defend by asserting B and the game ends. Figure 2.2b shows a strategy which ensures that my risk is non-positive (I don't expect to lose any money) whenever $\langle B\rangle^r > \langle C\rangle^r$: There are three possible final game states: $[\,\mid\,]$, $[\perp \mid C]$, and $[A, B \mid A, C]$. At the first one, clearly no player has to pay anything. At the second one I even expect to gain some money if $\langle C\rangle^r < 1$, but I will never lose money, and at the third one our payments from asserting A will cancel out in the long run, and I expect B to fail more often than C. Therefore this strategy ensures zero risk, and the game is $1-$valued for me (you can always set my risk to 0 by not attacking in the first place). Note that, the strategy I will choose crucially depends on the model. Fermüller and Metcalfe [20] show how the notion of a strategy can be generalized to *disjunctive strategies* which are independent of a particular model.



(a) The game tree

(b) A strategy for me

Figure 2.2: The $\mathcal{G}$-game for $(A\&(A \to B)) \to C$ including intermediary states. Dashed and solid arrows denote moves by you and me, respectively.

The above description might yet be too informal to see in which sense every $\mathcal{G}$-game, just like an $\mathcal{H}$-game, constitutes an ordinary two-person zero-sum extensive game of finite depth with perfect information. For this purpose one has to be a bit more precise than Giles and introduce the notion of a *regulation*, a labeling which determines at each state which player is to move next. A regulation is *consistent* if it selects only a player who can make a valid move by the rules of Giles's game. For example, if you don't assert any non-atomic formulas, then a consistent regulation must not appoint me to attack any of your formulas. Additionally, one has to model the selection of a non-atomic formula to be attacked as an explicit move in the game. Giles's proof that the order, in which the players attack their opponent's formulas, has no influence on the outcome, then amounts to proving that the final risk value of a $\mathcal{G}$-game is the same for all consistent regulations.

Any given consistent regulation determines the tree representing a concrete $\mathcal{G}$-game for a formula F (a game with initial state $[\mid F]$). By identifying the formula F selected for attack with the formula exhibited in the corresponding game rule one obtains for every such state the next two levels of successor states: the first level registers the possible choices (if any) for attacking F (including, in the case of an implication, the option to apply the principle of limited liability LLA and thus just to remove F) and the second level registers the options for defending F according to the rule (also possibly invoking LLD and resorting to $\perp$). Analogously to $\mathcal{H}$-games, these three levels correspond to a *round* in the game. Like for randomized $\mathcal{H}$-games, every run of a $\mathcal{G}$-game is a sequence of states ending in a final state, the risk of which constitutes its payoff. For a detailed formal presentation, including examples, see [20].

We arrive at the following characterization of strong Łukasiewicz logic $\mathbf{Ł}_\infty$ by $\mathcal{G}$-games:

Theorem 4 ([20], based on [33])**.** *A $\mathbf{Ł}_\infty$-sentence F is evaluated to $v_M(F) = x$ in interpretation M iff every $\mathcal{G}$-game for F under risk value assignment $\langle\cdot\rangle_M$ is $(1-x)$-valued for me.*

Remark. There are also variants of Giles's game corresponding to Gödel logic G and Product logic Π (see [12] and [18]). The main idea is to change the evaluation part of the game: For Gödel logic, I select one formula in *your* tenet, and you select one formula in *my* tenet (if our tenets are not empty). Only the experiments associated with these two formulas are done and again, each player has to pay 1€ if the experiment chosen by his opponent fails. For Product logic all experiments are done as suggested by Giles, however I have to pay you 1€ unless *all* of the experiments associated with formulas in my tenet succeed, and vice versa. However, only changing the evaluation scheme does not suffice: one also needs to adjust the dialogue rule $(R_\to)$ either by forcing players to invoke LLA and not to attack an assertion $F \to G$ in certain cases, or by extending the game state by an additional flag governing who will be declared *winner* of the game. Moreover, these adjustments come at a price: An analogy to Theorem 4 no longer holds, instead only truth in a model is characterized: a formula F is true in M, i.e. $v_M(F) = 1$, if and only if I have a winning strategy for the game starting with my assertion of F; the direct correspondence between payoff and truth values, however, is lost. This is due to Łukasiewicz logic being the only fuzzy logic where all connectives are *continuous*. Consider the formula $A \to B$ with $v_M(A) = 0$. If also $v_M(B) = 0$ we have $v_M(A \to B) = 0$ for both G and Π. If, however, $v_M(B)$ approaches 0, the truth value $v_M(A \to B)$ amounts to 1.

2.4 Generalizing Giles's Game

In the following we aim at a more general framework that nevertheless preserves the essential and arguably quite desirable features of Giles's game. We will follow the presentation [22]. In contrast to the game variants for Gödel logic G and product logic Π as pointed out in Section 2.3, let us retain the direct correspondence between payoff and truth values. Also, we retain Giles's notion of a game state given by two mulitsets of formulas (the players' tenets) as well as the. Moreover, we retain the 'frame rules' of Giles's game governing how formulas are attacked and defended.

Instead of talking about specific rules for particular logical connectives, we investigate which dialogue rules rules can in principle be formulated within Giles's framework. We will also look at the evaluation of final (atomic) game states from a more abstract perspective that is independent on philosophical motivations regarding proper forms of reasoning in physics.

General payoff functions. Ignoring the specific details of Giles's story about bets on dispersive experiments, we see that the proposed betting scheme boils down to an ordinary payoff function (in the game theoretic sense), i.e., an assignment of real numbers to all final states of the game. Probabilities, risk, and amounts of money to be paid by either me or you merely provide an *interpretation* of those real numbers. This observation motivates the formulation of general principles for assigning payoff values to atomic states.

Definition 8 (Payoff)**.** A *payoff* function assigns a value $\in \mathbb{R}$ to every atomic game state. The payoff of the game state $G = [\gamma \mid \delta]$ is denoted as $\langle G \rangle$. Instead of $\langle [\gamma \mid \delta] \rangle$ we write $\langle \gamma \mid \delta \rangle$.

Note that *the payoff* of an atomic game state corresponds to *my payoff* in Giles's diction. As the game is zero sum, *your payoff* for the same state is directly inverse to mine. This is also expressed in Payoff Principle 2, below. Of course, a payoff function has to obey certain *payoff principles* in order to ensure that the game characterizes a logic at all, particularly a truth functional one. For example, we must ensure that, as for Giles's game, the order in which the players attack their opponents' assertions has no effect on the outcome of the game. The following notion of *context independence* will turn out to be the first and most basic step towards this aim.

Payoff Principle 1 (Context independence)**.** A payoff function $\langle \cdot \mid \cdot \rangle$ is *context independent* if for all atomic tenets $\gamma, \delta, \gamma', \delta', \gamma''$, and δ'' the following holds: If $\langle \gamma' \mid \delta' \rangle = \langle \gamma'' \mid \delta'' \rangle$ then $\langle \gamma, \gamma' \mid \delta', \delta \rangle = \langle \gamma, \gamma'' \mid \delta'', \delta \rangle$.

Context independence entails that the payoff for a state $[\gamma, \gamma' \mid \delta, \delta']$ is solely determined by the payoffs of its sub-states $[\gamma \mid \delta]$ and $[\gamma' \mid \delta']$. This property will be crucial to achieve a truth functional (compositional) semantics.

Proposition 1. *Let $\langle \cdot \mid \cdot \rangle$ be a context independent payoff function and let $G = [\gamma, \gamma' \mid \delta, \delta']$ be an atomic game state. Then there exists an associative and commutative binary operation $\oplus$ on $\mathbb{R}$ such that $\langle G \rangle = \langle \gamma \mid \delta \rangle \oplus \langle \gamma' \mid \delta' \rangle$.*

Proof. Assume that $\langle \gamma \mid \delta \rangle = \langle \gamma'' \mid \delta'' \rangle = x$ and that $\langle \gamma' \mid \delta' \rangle = \langle \gamma''' \mid \delta''' \rangle = y$. Then $\langle \gamma'', \gamma''' \mid \delta'', \delta''' \rangle = \langle \gamma, \gamma''' \mid \delta, \delta''' \rangle = \langle \gamma, \gamma' \mid \delta, \delta' \rangle$ by applying context independence twice. Thus we may write $\langle \gamma, \gamma' \mid \delta, \delta' \rangle = x \oplus y$ since the overall payoff really depends only on x and y: substituting $\langle \gamma \mid \delta \rangle$ and $\langle \gamma' \mid \delta' \rangle$ with other game states having the same payoff does not change the outcome. Associativity and commutativity of $\oplus$ directly follow from the fact that tenets are multisets. $\square$

Remark. Let us call $\oplus$ as specified in Proposition 1 the *aggregation function corresponding to* $\langle \cdot \mid \cdot \rangle$. In Giles's original game the function $\oplus$ is ordinary addition, which motivates this notation.

The next payoff principle codifies that the game is zero-sum: in a fair game *your* payoff in an atomic state should amount to *my* payoff in another game where we have switched our tenets. In a zero-sum game these to payoffs add up to 0.

Payoff Principle 2 (Symmetry)**.** A payoff function $\langle \cdot \mid \cdot \rangle$ is *symmetric* if $\langle \gamma \mid \delta \rangle = - \langle \delta \mid \gamma \rangle$ for all atomic tenets γ and δ.

If $\langle \cdot \mid \cdot \rangle$ is context independent and symmetric then the payoff of an arbitrary atomic game state can be decomposed as follows:

$$\begin{aligned}
&\langle A_1, \ldots, A_n \mid B_1, \ldots, B_m \rangle \\
&= \langle A_1 \mid \rangle \oplus \ldots \oplus \langle A_n \mid \rangle \oplus \langle \mid B_1 \rangle \oplus \ldots \langle \mid B_m \rangle \\
&= - \langle \mid A_1 \rangle \oplus \ldots \oplus - \langle \mid A_n \rangle \oplus \langle \mid B_1 \rangle \oplus \ldots \oplus \langle \mid B_m \rangle .
\end{aligned}$$

Note that symmetry implies that $\langle \gamma \mid \gamma \rangle = 0$. In other words, the payoff is 0 in an atomic state where your tenet is identical to mine. Moreover, this shows that the payoff of a complex state can be decomposed into the payoffs of states where your tenet is empty and my tenet contains only one formula.

Proposition 2. *Let $\langle \cdot \mid \cdot \rangle$ be a context independent and symmetric payoff function. Then*
(i) $-$ distributes over the corresponding aggregation function $\oplus$, i.e., for all payoff values x and y, $-(x \oplus y) = -x \oplus -y$,
(ii) $-$ is inverse to $\oplus$, i.e., $x \oplus -x = 0$ holds for all values x, and
(iii) 0 is neutral for $\oplus$, i.e., $x \oplus 0 = x$ holds for all values x.

Proof.
(i) Let $[\gamma_1 \mid \delta_1]$ and $[\gamma_2 \mid \delta_2]$ be two atomic states such that $\langle \gamma_1 \mid \delta_1 \rangle = x$ and $\langle \gamma_2 \mid \delta_2 \rangle = y$. Then

$$\begin{aligned}
-(x \oplus y) &= -(\langle \gamma_1 \mid \delta_1 \rangle \oplus \langle \gamma_1 \mid \delta_2 \rangle) && \text{by definition of } x, y \\
&= - \langle \gamma_1, \gamma_2 \mid \delta_1, \delta_2 \rangle && \text{by Proposition 1} \\
&= \langle \delta_1, \delta_2 \mid \gamma_1, \gamma_2 \rangle && \text{by Payoff Principle 2 (symmetry)} \\
&= \langle \delta_1 \mid \gamma_1 \rangle \oplus \langle \delta_2 \mid \gamma_2 \rangle && \text{by Proposition 1} \\
&= - \langle \gamma_1 \mid \delta_1 \rangle \oplus - \langle \gamma_2 \mid \delta_2 \rangle && \text{by Payoff Principle 2 (symmetry)} \\
&= -x \oplus -y && \text{by definition of } x, y.
\end{aligned}$$

(ii) Let $[\gamma \mid \delta]$ be an atomic game state such that $\langle \gamma \mid \delta \rangle = x$. Then

$$\begin{aligned} x \oplus -x &= \langle \gamma \mid \delta \rangle \oplus - \langle \gamma \mid \delta \rangle && \text{by definition of } x \\ &= \langle \gamma \mid \delta \rangle \oplus \langle \delta \mid \gamma \rangle && \text{by Payoff Principle 2 (symmetry)} \\ &= \langle \gamma, \delta \mid \gamma, \delta \rangle && \text{by Proposition 1} \\ &= \langle \gamma \mid \gamma \rangle \oplus \langle \delta \mid \delta \rangle && \text{by Proposition 1} \\ &= 0 \oplus 0 && \text{by Payoff Principle 2 (symmetry)} \\ &= 0 && \text{by Proposition 1.} \end{aligned}$$

(iii) Let $[\gamma \mid \delta]$ be an atomic game state such that $\langle \gamma \mid \delta \rangle = x$. Note that symmetry entails that $\langle | \rangle = 0$. Then

$$\begin{aligned} x \oplus 0 &= \langle \gamma \mid \delta \rangle \oplus \langle | \rangle && \text{by definition of } x \\ &= \langle \gamma \mid \delta \rangle && \text{by Payoff Principle 1 (context independence)} \\ &= x && \text{by definition of } x. \end{aligned}$$

□

Given Proposition 2 we can rewrite the decomposition of the payoff for an atomic state $[A_1, \ldots, A_n \mid B_1, \ldots, B_m]$ as

$$\begin{aligned} &\langle A_1, \ldots, A_n \mid B_1, \ldots, B_m \rangle \\ &= \langle | B_1 \rangle \oplus \ldots \oplus \langle | B_m \rangle \oplus - \langle | A_1 \rangle \oplus \ldots \oplus - \langle | A_n \rangle \\ &= \langle | B_1 \rangle \oplus \ldots \oplus \langle | B_m \rangle \oplus -(\langle | A_1 \rangle \oplus \ldots \oplus \langle | A_n \rangle) \end{aligned}$$

Payoff Principle 3 (Monotonicity)**.** A payoff function $\langle \cdot \mid \cdot \rangle$ is *monotone* if for all tenets γ,δ,γ', δ', γ'', and δ'' the following holds: if $\langle \gamma' \mid \delta' \rangle \leq \langle \gamma'' \mid \delta'' \rangle$ then $\langle \gamma, \gamma' \mid \delta', \delta \rangle \leq \langle \gamma, \gamma'' \mid \delta'', \delta \rangle$.

Proposition 3. *Let $\langle \cdot \mid \cdot \rangle$ be a monotone and context independent payoff function and $\oplus$ the corresponding aggregation function. Then for all payoff values x, y, and z:*
(i) if $y \leq z$ then $x \oplus y \leq x \oplus z$,
(ii) $\min$ *and* $\max$ *distribute over* $\oplus$*, i.e.,* $\min(x \oplus y, x \oplus z) = x \oplus \min(y, z)$ *and* $\max(x \oplus y, x \oplus z) = x \oplus \max(y, z)$.

Proof. (i) Let $G = [\gamma \mid \delta]$, $G' = [\gamma' \mid \delta']$, and $G'' = [\gamma'' \mid \delta'']$ be three atomic states such that $\langle G \rangle = x$, $\langle G' \rangle = y$, and $\langle G'' \rangle = z$. Then the premise $y \leq z$ amounts to $\langle \gamma' \mid \delta' \rangle \leq \langle \gamma'' \mid \delta'' \rangle$ and $x \oplus y \leq x \oplus z$ to $\langle \gamma \mid \delta \rangle \oplus \langle \gamma' \mid \delta' \rangle \leq \langle \gamma \mid \delta \rangle \oplus \langle \gamma'' \mid \delta'' \rangle$ or, equivalently, to $\langle \gamma, \gamma' \mid \delta, \delta' \rangle \leq \langle \gamma, \gamma'' \mid \delta, \delta'' \rangle$, which is just an instance of Payoff Principle 3.

(ii) We only consider the equation for $\min$; the argument for $\max$ is analogous. Assume that $y \leq z$ holds. Then, by *(i)*, $x \oplus y \leq x \oplus z$ holds for all x and thus also $\min(x \oplus y, x \oplus z) = x \oplus y = x \oplus \min(y, z)$. On the other hand, if $z \leq y$ then $x \oplus z \leq x \oplus y$ and thus also $\min(x \oplus y, x \oplus z) = x \oplus z = x \oplus \min(y, z)$. □

Proposition 4. *Every context independent and symmetric payoff function induces via its aggregation function an archimedean totally ordered abelian group $(G, \oplus)$ with (some subset of) the reals as base set and and* 0 *as neutral element.*

Proof. Proposition 1 states that $\oplus$ is associative and commutative. Payoff Principle 2 ensures that each element has an inverse and that 0 is the neutral element, thus $(G, \oplus)$ forms an abelian group.

A group is archimedean if for any two values y and x with $y > x > 0$ there exists $n \in \mathbb{N}$ such that $n * x > y$ (where $n*$ denotes the n-fold application of $\oplus$). Suppose $(G, \oplus)$ is not archimedean. Then there exists y such that $n*x \leq y$ for all n. Hence the set $S = \{n*x : n \in \mathbb{N}\}$ is (upwards) bounded, and therefore S has a least upper bound $y_0 \in \mathbb{R}$. For every n, also $(n+1) * x \in S$, thus $(n+1) * x \leq y_0$. We have

$$\begin{aligned}
(n+1) * x \leq y_0 \text{ iff } & (n * x) \oplus x \leq y_0 & \\
\text{iff } & ((n * x) \oplus x) \oplus -x \leq y_0 \oplus -x & \text{by Proposition 3(i)} \\
\text{iff } & (n * x) \oplus (x \oplus -x) \leq y_0 \oplus -x & \text{by Proposition 1} \\
\text{iff } & n * x \oplus 0 \leq y_0 \oplus -x & \text{by Proposition 2(ii)} \\
\text{iff } & n * x \leq y_0 \oplus -x & \text{by Proposition 2(iii)}\ .
\end{aligned}$$

Note that $-x < 0$, thus $y_0 \oplus -x \leq y_0$ by Proposition 3, and $y_0 \oplus -x \neq y_0$ since this would imply that $x = 0$:

$$\begin{aligned}
y_0 \oplus -x = y_0 \text{ iff } & y_0 \oplus -x \oplus -y_0 = y_0 \oplus -y_0 & \text{by Proposition 3(i)} \\
\text{iff } & (y_0 \oplus -y_0) \oplus -x = y_0 \oplus -y_0 & \text{by Proposition 1} \\
\text{iff } & 0 \oplus -x = 0 & \text{by Proposition 2(ii)} \\
\text{iff } & -x = 0 & \text{by Proposition 2(iii)} \\
\text{iff } & x = 0 & \text{by Proposition 2(ii)}
\end{aligned}$$

contradicting the assumption that $x > 0$. Summarizing, we have $n*x \leq y_0 \oplus -x < y_0$, hence y_0 cannot be a least upper bound of S contradicting the assumption that $n * x$ is upwards bounded by y_0. Therefore, $(G, \oplus)$ is archimedean. □

Definition 9 (Discriminating)**.** We call a payoff function $\langle \cdot \mid \cdot \rangle$ *discriminating* if it is context independent, symmetric, and monotone.

We will see below that under some very general conditions on the form of logical dialogue rules, to be investigated in the next section, discriminating payoff functions can be extended to arbitrary game states.

A general format of logical dialogue rules. Let us now focus on logical connectives and look for dialogue rules that regulate the stepwise reduction of states with logically complex assertions to final atomic states. Since we aim for full generality, we will not consider conjunction, disjunction, implication, etc., separately, but rather specify a generic format of dialogue rules for arbitrary n-ary connectives ($n \geq 1$). It turns out that two simple and general 'dialogue principles', in combination with discriminating payoff functions, suffice to guarantee that a truth-functional semantics can be extracted from the corresponding game. The logical rules considered in Giles's game (in the previous section) are all instances of this more general format of a dialogue rule. Also, a round proceeds analogously to Giles's game: In general, the attacking player may choose first between one of several available forms of attacking a particular formula, as witnessed by the rule for (weak) conjunction in the original game. Likewise, as exemplified in Giles's rule for disjunction, a rule may also involve a choice by the defending player.

Definition 10. (Round) A round in the game is a sequence of three consecutive moves:

1. (Selection) Player X picks an occurrence of a compound formula $\diamond(F_1, \ldots, F_n)$ from Y's current tenet for attack,

2. (Attack) X chooses the form of attack according to the dialogue rule for $\diamond$,

3. (Defense) Y chooses the way in which he wants to reply to the given attack on the indicated occurrence of $\diamond(F_1, \ldots F_n)$

for X and Y being you or me, or vice versa. The game continues according to Y's chosen defense.

Dialogue Principle 1 (Decomposition)**.** A (dialogue) rule for an n-ary connective $\diamond$ is *decomposing* if in any corresponding round of the game exactly one occurrence of a compound formula $\diamond(F_1, \ldots, F_n)$ is removed from the current state and (possibly zero) occurrences of $F_1, \ldots, F_n$ and of propositional constants are added to obtain the successor state.

According to the decomposition principle, a dialogue rule for an n-ary connective $\diamond$ can be identified with a tree as shown in Figure 2.3. Referring to Definition 10, a round consists of three moves: the first one determines which compound formula of the defender's tenet is to be attacked. The corresponding dialogue rule lists possible attack moves and, for each attack move, possible defense moves. Thus, if you choose to attack an assertion of $\diamond(F_1, \ldots, F_n)$ made by me (with the corresponding rule defined as in Figure 2.3), then your attack consists of selecting one of the m subtrees of the dialogue rule. (Note that the number m of valid attack moves solely depends on the dialogue rule and is not related to the arity n.) My valid defense moves are specified by the leaf nodes of that subtree, with each leaf node fixing our new tenets. Note that the attacked formula is always removed from my tenet and also, that the other formulas asserted by us remain unchanged.[3]

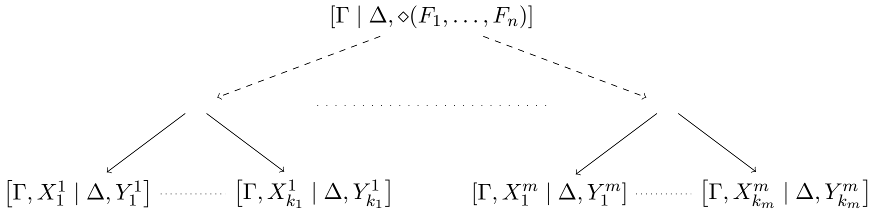


Figure 2.3: Generic dialogue rule for $\diamond(F_1, \ldots, F_n)$, where you have m attack moves and for each attack i, I have k_i defense moves, and X_j^i and Y_j^i, for $1 \leq j \leq k_i$, $1 \leq i \leq m$, are multisets of zero or more occurrences of the formulas $F_1, \ldots, F_n$ and of propositional constants.

To illustrate this dialogue rule format by a concrete example, consider the case of your attack on my assertion of $F \to G$ in a variant of Giles's game where both forms of the principle of limited liability are imposed. The resulting version of the implication rule is depicted in Figure 2.4.

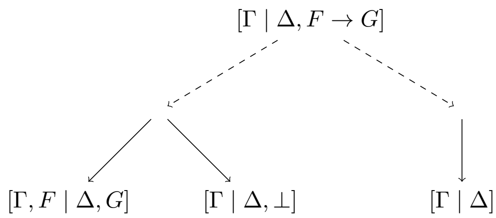


Figure 2.4: Implication rule (your attack) with two-fold principle of limited liability

Using the notation of Figure 2.3, this rule is specified by $n = 2$ (as $\to$ is a binary connective), $m = 2$ (as there exist two different attack forms), $k_1 = 2, k_2 = 1$ (for the first attack there exist two possible defense move, while for the second attack only one defense move is possible), and $X_1^1 = [F], Y_1^1 = [G], X_2^1 = [\,], X_2^1 = [\bot], X_1^2 = Y_1^2 = [\,]$. LLA, i.e., the possibility for the attacker to grant a formula, corresponds to the right subtree where the attacked assertion is simply removed. Otherwise, if you attack my assertion of $F \to G$, I may invoke LLD and assert $\bot$ as a defense. Thus, LLD requires each subtree (except for the one corresponding to LLA, of

[3] There is a subtle difference with respect to Giles's formulation of a dialogue rule. In Giles's original game, an attack itself may involve asserting sub-formulas of the attacked assertion (and propositional constants). For example, attacking $F \to G$ requires the attacker to assert G (see Section 2.3). Here, the notion of attack and defense moves are more abstract and do not need to correspond to asserting (or selecting) a formula. For example, even after your attack, the multiset X of formulas added to your tenet may depend on my defense move. For all the dialogue rules proposed by Giles, this means that the multisets X_j^o in Figure 2.3 coincide for each $o \in \{1, \ldots, m\}$.

course) to contain one branch, where only $\bot$ is added to the defendant's tenet. If we both do not invoke the principle of limited liability, the game continues in with an assertion of G added to my tenet in exchange for your assertion of F.

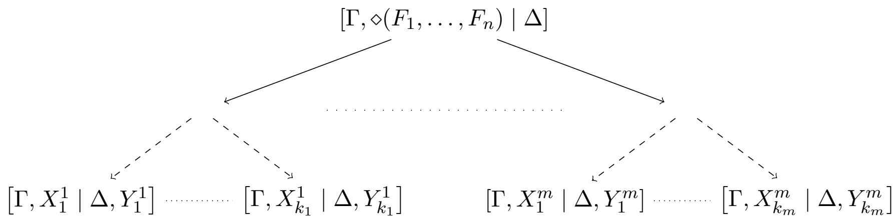
Each occurrence of a formula can be attacked at most once: after the round it is simply removed from the state and, moreover, only subformulas (and constants) may be added to a player's tenet, but not the attacked formulas itself. Therefore the decomposition principle ensures that the game eventually terminates. Note that subformulas may be duplicated; thus the number of connectives of a formula F cannot be taken as an upper bound for the length of the game starting with $[\mid F]$.

The second principle that we want to maintain in generalizing Giles's game is player neutrality, i.e., role duality: you and me have the very same obligations and rights in attacking or defending a particular type of formula.

Dialogue Principle 2 (Duality)**.** A rule $\delta_\diamond$ for my (your) assertion of a formula of the form $\diamond(F_1, \dots, F_n)$ is called *dual* to the rule $\delta'_\diamond$ for your (my) assertion of $\diamond(F_1, \dots, F_n)$, if $\delta_\diamond$ is obtained from $\delta'_\diamond$ by just switching the roles of the players.

A dialogue game *has dual rules* if for every dialogue rule of the game there is dual rule.

Figure 2.5 depicts the generic dialogue that is dual to that in Figure 2.3. Note that now *I* am the one who, in attacking your assertion of $\diamond(F_1, \dots, F_n)$, is free to move first in the tree, whereas the second layer now refers to *your* choices when defending against my attack.



where m, k_i, X^i_j, and Y^i_j are defined in Figure 2.3.

Figure 2.5: Generic dialogue rule dual to that in Figure 2.3

Note that since the format of decomposing rules allows for a choice between different types of attacks as well as corresponding replies, we may speak without loss of generality of *the* dialogue rule for a connective $\diamond$ if the game has dual rules.

Lifting payoffs to valuations of general states. Remember that the payoff function $\langle \cdot \mid \cdot \rangle$ specifies my payoff at an atomic game state. We extend this function to arbitrary game states such that $\langle \cdot \mid \cdot \rangle$ specifies my *maximal enforceable* payoff value, i.e., the maximal (final) payoff value I can be sure to reach regardless of your choices. Observe that, whenever I am attacking or defending, I will choose my move such that my payoff amounts to the *maximum* of my maximal enforceable payoffs of the succeeding game states. Conversely, whenever you are attacking or defending, my maximal enforceable payoff amounts to the *minimum* of my maximal enforceable payoffs of the following game states. Applying this observation recursively is a well-known game theoretic principle called *backward induction*. For the dialogue rules as given above, backward induction yields the following *min-max conditions* for non-atomic game states:

$$\langle \Gamma \mid \diamond(F_1, \ldots F_n), \Delta \rangle = \min_{1 \le i \le m} \max_{1 \le j \le k_i} \left\langle \Gamma, X^i_j \mid \Delta, Y^i_j \right\rangle \tag{2.1}$$

$$\langle \diamond(F_1, \ldots F_n), \Gamma \mid \Delta \rangle = \max_{1 \le i \le m} \min_{1 \le j \le k_i} \left\langle \Gamma, Y^i_j \mid \Delta, X^i_j \right\rangle \tag{2.2}$$

where m, k_i, X^i_j, and Y^i_j are defined as in Figure 2.3. If you are attacking, Condition 2.1 reflects that you move first by selecting an appropriate attack move. Thus my payoff amounts to the minimum of the maximal enforceable payoffs I can ensure by selecting a suitable defense move. Conversely, if I am attacking I will choose an attack such that my payoff is maximal whatever defense move you choose.

Observe that the first move of a round—selection of the formula to attack—does not change my maximal enforceable payoff: As dialogue rules are decomposing and thus preserve the context all compound formulas must be attacked at some point. As tenets are multisets, the order in which formulas are added to tenets does not matter. For the same reason, if both players are asserting a compound formula, it does not matter which player attacks first—in the end the same final game state will be reached no matter in which order the players attack their opponents' formulas. Therefore, the min-max conditions 2.1 and 2.2 indeed define an extension of $\langle \cdot \mid \cdot \rangle$ to arbitrary game states, called the *extended payoff* function. This fact is more formally expressed by the following Proposition:

Proposition 5. *For any discriminating payoff function $\langle \cdot \mid \cdot \rangle$, there exists a unique extension to arbitrary (complex) game states obeying the min-max conditions 2.1 and 2.2.*

Proof. As the game is zero-sum with perfect information, the game is determined and for each round the game value is readily calculated by backward induction. We prove that it is invariant of which player is to move next and which of his opponent's complex formulas he chooses to attack by induction on the complexity of a game state[4]:

For an atomic game state $[\gamma \mid \delta]$ the payoff is given by $\langle \cdot \mid \cdot \rangle$. For a game state where only one complex formula is asserted either Condition 2.1 or Condition 2.2 determines the payoff

[4]Note that induction just on the number of connectives occurring in the players' tenets does not suffice: A decomposing dialogue rule may well increase that number by duplicating formulas. Therefore, let us consider the multiset (instead of the sum) of the numbers of connectives for each assertion at a game state with the usual multiset order.

since there is only one formula to attack. If more than one complex formula is asserted let us distinguish three cases:

(Case 1) Consider a game state $[\Gamma, \diamond_1(F_1,\dots,F_{n_1}) \mid \Delta, \diamond_2(G_1,\dots,G_{n_2})]$ where both players assert a complex formula. Let us denote the corresponding constants and multisets for the dialogue rule of $\diamond_1$ by $n_1, k_{1i}, m_1, {}^1X^i_j$ and ${}^1Y^i_j$ and analogously for $\diamond_2$ by $n_2, k_{2i}, m_2, {}^2X^i_j$ and ${}^2Y^i_j$ as described in Figure 2.3. If I am to move next and I attack your assertion of $\diamond_1(F_1,\dots,F_{n_1})$ condition 2.2 dictates that my enforceable payoff is calculated as

$$\begin{aligned}&\langle \Gamma, \diamond_1(F_1,\dots,F_{n_1}) \mid \Delta, \diamond_2(G_1,\dots,G_{n_2})\rangle\\ &= \max_{1\le i\le m_1} \min_{1\le j\le k_{1i}} \left\langle \Gamma, {}^1Y^i_j \mid \Delta, {}^1X^i_j, \diamond_2(G_1,\dots,G_{n_2})\right\rangle\end{aligned}$$

The resulting game state after this round is less complex than the previous one, thus by the induction hypothesis there is exactly one function $\langle\cdot \mid \cdot\rangle$ calculating my maximal enforceable payoff as

$$\max_{1\le i\le m_1} \min_{1\le j\le k_{1i}} \min_{1\le p\le m_2} \max_{1\le q\le k_{2p}} \left\langle \Gamma, {}^1Y^i_j, {}^2X^p_q \mid \Delta, {}^1X^i_j, {}^2Y^p_q\right\rangle .$$

Analogously, if you are to move first and to attack, my enforceable payoff is calculated by

$$\begin{aligned}&\langle \Gamma, \diamond_1(F_1,\dots,F_{n_1}) \mid \Delta, \diamond_2(G_1,\dots,G_{n_2})\rangle\\ &= \min_{1\le p\le m_2} \max_{1\le q\le k_{2p}} \left\langle \Gamma, {}^2X^p_q, \diamond_1(F_1,\dots,F_{n_1}) \mid \Delta, {}^2Y^p_q\right\rangle\\ &= \min_{1\le p\le m_2} \max_{1\le q\le k_{2p}} \max_{1\le i\le m_1} \min_{1\le j\le k_{1i}} \left\langle \Gamma, {}^1Y^i_j, {}^2X^p_q \mid \Delta, {}^1X^i_j, {}^2Y^p_q\right\rangle .\end{aligned}$$

By exchanging the minima and maxima, we see that my enforceable payoff is the same, regardless of which player is to move next.

(Case 2) Consider a game state $[\Gamma, \diamond_1(F_1,\dots,F_{n_1}), \diamond_2(G_1,\dots,G_{n_2}) \mid \Delta]$ where you assert two complex formula. Again, let us denote the corresponding constants and multisets for the dialogue rules of $\diamond_1$ and $\diamond_2$ as above. If I attack the the first one of these assertion, $\diamond_1(F_1,\dots,F_{n_1})$, by Condition 2.2 and the induction hypothesis my enforceable payoff is calculated as

$$\begin{aligned}&\langle \Gamma, \diamond_1(F_1,\dots,F_{n_1}), \diamond_2(G_1,\dots,G_{n_2}) \mid \Delta\rangle\\ &= \max_{1\le i\le m_1} \min_{1\le j\le k_{1i}} \left\langle \Gamma, {}^1Y^i_j, \diamond_2(G_1,\dots,G_{n_2}) \mid \Delta, {}^1X^i_j\right\rangle\\ &= \max_{1\le i\le m_1} \min_{1\le j\le k_{1i}} \max_{1\le p\le m_2} \min_{1\le q\le k_{2p}} \left\langle \Gamma, {}^1X^i_j, {}^2X^p_q \mid \Delta, {}^1Y^i_j, {}^2Y^p_q\right\rangle .\end{aligned}$$

If I choose to attack the second assertion first my enforceable payoff amounts to

$$\begin{aligned}&\langle \Gamma, \diamond_1(F_1,\dots,F_{n_1}), \diamond_2(G_1,\dots,G_{n_2}) \mid \Delta\rangle\\ &= \max_{1\le p\le m_2} \min_{1\le q\le k_{2p}} \left\langle \Gamma, {}^2Y^p_q, \diamond_1(F_1,\dots,F_{n_1}) \mid \Delta, {}^2X^p_1\right\rangle\\ &= \max_{1\le p\le m_2} \min_{1\le q\le k_{1p}} \max_{1\le i\le m_1} \min_{1\le j\le k_{1i}} \left\langle \Gamma, {}^1X^i_j, {}^2X^p_q \mid \Delta, {}^1Y^i_j, {}^2Y^p_q\right\rangle\end{aligned}$$

and again, both payoffs coincide.

(Case 3) The case where I assert two complex formulas is analogous to Case 2.

Thus, for any complex game state the order in which the players attack their opponents' formulas has no influence on the game value, and my maximal enforceable payoff is calculated using backwards induction by the minmax-conditions 2.1 and 2.2. □

The above notions of context independence, symmetry, and monotonicity for payoff functions by definition refer only to atomic game states. However, by inspecting Definitions 1, 2, and 3 it is obvious that neither these properties, nor those expressed in Propositions 1, 2, and 3 depend on the atomicity of the formulas in a corresponding tenet. Therefore we can speak without ambiguity of context independence, symmetry, and monotonicity for arbitrary functions from general states to real numbers, not just for proper payoff functions.

Theorem 5. *Let $\eth$ be a dialogue game with a discriminating payoff function and decomposing dual rules. Then the extended payoff function denoting my enforceable payoff is context independent, symmetric, and monotone.*

Proof. Given a discriminating payoff function $\langle \cdot \mid \cdot \rangle$ with corresponding aggregation function $\oplus$, we define a function v from (arbitrary) game states to the real numbers inductively as follows:

$$\begin{array}{ll}
(a) & v([\mid G]) = \langle \mid G \rangle \\
(b) & v([\mid \Delta]) = \bigoplus_{G \in \Delta} v([\mid G]) \\
(c) & v([\Gamma \mid \Delta]) = v([\mid \Delta]) \oplus -v([\mid \Gamma]) \\
(d) & v([\mid \diamond(G_1, \ldots G_n)]) = \min\limits_{1 \le i \le m} \max\limits_{1 \le j \le k_i} v(\big[X^i_j \mid Y^i_j\big])
\end{array}$$

where B denotes only atomic and G arbitrary formulas. The constants and sets m, k_i, X^i_j, and Y^i_j are defined as in Figure 2.3.

We prove that v indeed calculates my enforceable payoff, i.e., it coincides with $\langle \cdot \mid \cdot \rangle$ on atomic states and fulfills the min-max conditions. Moreover we show that it is context independent, symmetric, and monotone.

It is straightforward to check that $v([\gamma \mid \delta])$ indeed coincides with $\langle \gamma \mid \delta \rangle$ for all atomic states $[\gamma \mid \delta]$. Taking our clue from this observation we will from now on usually write $\langle \Gamma \mid \Delta \rangle$ instead of $v([\Gamma \mid \Delta])$, even if the tenets Γ and Δ are not atomic.

The symmetry of $v([\cdot \mid \cdot])$ immediately follows from its definition, where (here as well as further on) we freely exploit the commutativity and associativity of $\oplus$.

$$\begin{aligned}
(-v([\Gamma \mid \Delta] =) - \langle \Gamma \mid \Delta \rangle &= -(\langle \mid \Delta \rangle \oplus - \langle \mid \Gamma \rangle) && \text{by definition of } v \text{ (c)} \\
&= - \langle \mid \Delta \rangle \oplus \langle \mid \Gamma \rangle && \text{by Proposition 2(i)} \\
&= \langle \Delta \mid \Gamma \rangle && \text{by definition of } v \text{ (c)}
\end{aligned}$$

Note that the definition of v directly entails that, just like the payoff at atomic states, the enforceable payoff at arbitrary states can also be obtained from the enforceable payoffs for sub-states by applying $\oplus$: we will refer to *merging* of and *partitioning*, respectively. More precisely:

$$\begin{aligned}
\langle \Gamma, \Gamma' \mid \Delta', \Delta \rangle &= \langle \mid \Delta', \Delta \rangle \oplus - \langle \mid \Gamma, \Gamma' \rangle && \text{by definition of } v \text{ (c)} \\
&= (\langle \mid \Delta' \rangle \oplus \langle \mid \Delta \rangle) \oplus -(\langle \mid \Gamma' \rangle \oplus \langle \mid \Gamma \rangle) && \text{by definition of } v \text{ (b)} \\
&= \langle \mid \Delta' \rangle \oplus \langle \mid \Delta \rangle \oplus - \langle \mid \Gamma' \rangle \oplus - \langle \mid \Gamma \rangle && \text{by Proposition 2} \\
&= \langle \Gamma' \mid \Delta' \rangle \oplus \langle \Gamma \mid \Delta \rangle && \text{by definition of } v \text{ (c).}
\end{aligned}$$

Given this fact, it is easy to see that $\langle \cdot \mid \cdot \rangle$ is context independent. Let $[\Gamma' \mid \Delta']$, $[\Gamma'' \mid \Delta'']$ be two game states such that $\langle \Gamma' \mid \Delta' \rangle = \langle \Gamma'' \mid \Delta'' \rangle$. Then for arbitrary tenets Γ and Δ

$$\begin{aligned}
\langle \Gamma, \Gamma' \mid \Delta', \Delta \rangle &= \langle \Gamma' \mid \Delta' \rangle \oplus \langle \Gamma \mid \Delta \rangle && \text{by partitioning} \\
&= \langle \Gamma'' \mid \Delta'' \rangle \oplus \langle \Gamma \mid \Delta \rangle && \text{by assumption} \\
&= \langle \Gamma, \Gamma'' \mid \Delta'', \Delta \rangle && \text{by merging.}
\end{aligned}$$

Monotonicity also straightforwardly carries over from atomic to arbitrary game states. Let $[\Gamma' \mid \Delta']$, $[\Gamma'' \mid \Delta'']$ be two game states such that $\langle \Gamma' \mid \Delta' \rangle \leq \langle \Gamma'' \mid \Delta'' \rangle$. Then for arbitrary tenets Γ and Δ

$$\begin{aligned}
\langle \Gamma, \Gamma' \mid \Delta', \Delta \rangle &= \langle \Gamma' \mid \Delta' \rangle \oplus \langle \Gamma \mid \Delta \rangle && \text{by partitioning} \\
&\leq \langle \Gamma'' \mid \Delta'' \rangle \oplus \langle \Gamma \mid \Delta \rangle && \text{by assumption and Proposition 3(i)} \\
&= \langle \Gamma, \Gamma'' \mid \Delta, \Delta'' \rangle && \text{by merging.}
\end{aligned}$$

It remains to check that the min-max conditions are satisfied. For states of the form $[\Gamma \mid \Delta, \diamond(F_1, \ldots F_n)]$ we obtain min-max condition (2.1) as follows:

$$\begin{aligned}
&\langle \Gamma \mid \Delta, \diamond(F_1, \ldots F_n) \rangle \\
&= \langle \Gamma \mid \Delta \rangle \oplus \langle \mid \diamond(F_1, \ldots F_n) \rangle && \text{by partitioning} \\
&= \langle \Gamma \mid \Delta \rangle \oplus \min\nolimits_{1 \leq i \leq m} \max\nolimits_{1 \leq j \leq k_i} \left(\left\langle X^i_j \mid Y^i_j \right\rangle \right) && \text{by definition of } v \text{ (d)} \\
&= \min\nolimits_{1 \leq i \leq m} \max\nolimits_{1 \leq j \leq k_i} \left(\langle \Gamma \mid \Delta \rangle \oplus \left\langle X^i_j \mid Y^i_j \right\rangle \right) && \text{by Proposition 3(ii)} \\
&= \min\nolimits_{1 \leq i \leq m} \max\nolimits_{1 \leq j \leq k_i} \left(\left\langle \Gamma, X^i_j \mid Y^i_j, \Delta \right\rangle \right) && \text{by merging.}
\end{aligned}$$

The dual min-max condition (2.2) exploits the symmetry of $\langle \cdot \mid \cdot \rangle$:

$$\begin{aligned}
&\langle \Gamma, \diamond(F_1, \ldots F_n) \mid \Delta \rangle && \\
&= -\langle \Delta \mid \Gamma, \diamond(F_1, \ldots F_n) \rangle && \text{by symmetry} \\
&= -\min_{1\le i\le m} \max_{1\le j\le k_i} \left(\left\langle \Delta, X^i_j \mid Y^i_j, \Gamma \right\rangle \right) && \text{by min-max condition (2.1)} \\
&= \max_{1\le i\le m} \min_{1\le j\le k_i} \left(- \left\langle \Delta, X^i_j \mid Y^i_j, \Gamma \right\rangle \right) && \text{by Proposition 3(ii)} \\
&= \max_{1\le i\le} \min_{1\le j\le k_i} \left(\left\langle Y^i_j, \Gamma \mid \Delta, X^i_j \right\rangle \right) && \text{by symmetry,}
\end{aligned}$$

where m, k_i, X^i_j, and Y^i_j are defined as in Figure 2.3. □

Remark. The duality of dialogue rules is used only indirectly in the above proof: it is reflected in the corresponding duality of the two min-max conditions and in the symmetry of the extended payoff function.

Corollary 1. *Let* $\eth$ *be a game with discriminating payoff function and decomposing dual rules. Then for each connective* $\diamond$ *there is a function* $f_\diamond$ *such that* $\langle \mid \diamond(F_1, \ldots F_n) \rangle = f_\diamond \left(\langle\mid F_1 \rangle, \ldots, \langle\mid F_n \rangle \right)$ *for all formulas* $F_1, \ldots, F_n$*, where* $\langle \cdot \mid \cdot \rangle$ *denotes the extended payoff function of Theorem 5.*

Proof. Applying min-max condition (2.1) as well as context independence and symmetry, we obtain

$$\begin{aligned}
&\langle \mid \diamond(F_1, \ldots, F_n) \rangle \\
&= \min_{1\le i\le m} \max_{1\le j\le k_i} \left\langle X^i_j \mid Y^i_j \right\rangle \\
&= \min_{1\le i\le m} \max_{1\le j\le k_i} \left(\left\langle \mid Y^i_j \right\rangle \oplus \left\langle X^i_j \mid \right\rangle \right) \\
&= \min_{1\le i\le m} \max_{1\le j\le k_i} \left(\left\langle \mid Y^i_j \right\rangle \oplus - \left\langle \mid X^i_j \right\rangle \right) \\
&= \min_{1\le i\le m} \max_{1\le j\le k_i} \left(\bigoplus_{Y \in Y^i_j} \langle \mid Y \rangle \oplus - \bigoplus_{X \in X^i_j} \langle \mid X \rangle \right),
\end{aligned}$$

where $\oplus$ is the aggregation function corresponding to $\langle \cdot \mid \cdot \rangle$; m, k_i, Y^i_j, and X^i_j obviously again refer to the dialogue rule for $\diamond(F_1, \ldots F_n)$ as exhibited in Figure 2.3. Note that the X^i_js and Y^i_js are multisets containing only the formulas $F_1, \ldots, F_n$ and propositional constants, which of course are evaluated to constant real numbers. Therefore that last expression defines the required function $f_\diamond$. □

Remark. To emphasize that $f_\diamond$ is of type $\mathbb{R}^n \mapsto \mathbb{R}$ it can be rewritten as

$$f_\diamond(x_1, \ldots, x_m) = \min_{1\le i\le n} \max_{1\le j\le k_i} \left(\bigoplus_{y \in \overline{Y^i_j}} y \oplus - \bigoplus_{x \in \overline{X^i_j}} x \right),$$

where $\overline{Y^i_j}$ is a multiset of real numbers defined with respect to the multiset of formulas Y^i_j as follows: $\overline{Y^i_j} = \{\overline{F} \mid F \in Y^i_j\}$, where $\overline{F} = x_i$ when $F = F_i$ for $1 \leq i \leq n$ and $\overline{F} = \langle \mid F \rangle$ if F is a propositional constant and analogously for $\overline{X^i_j}$. As an example, consider the dialogue rule depicted in Figure 2.4. The according function $f_{\rightarrow}$ amounts to

$$f_{\rightarrow}(x_1, x_2) = \min(\max(x_2 \oplus -x_1, -1), 0)$$

assuming the payoff $\langle \mid \bot \rangle = -1$.

Duality of the rules entails that also the extended payoff of a game state where you initially assert a complex formula $\diamond(F_1, \ldots F_n)$ can be calculated by $f_\diamond$ using $\langle \diamond(F_1, \ldots F_n) \mid \rangle = -\langle \mid \diamond(F_1, \ldots F_n) \rangle = -f_\diamond(\langle \mid F_1 \rangle, \ldots, \langle \mid F_n \rangle)$. By identifying payoff values with truth values we may thus claim to have extracted a unique truth function for $\diamond$ from a given payoff function and any decomposing dialogue rule for $\diamond$ (as defined in Dialogue Principle 1). However, as we will see in the next section, standard truth functions for affected logics usually are based on different sets of truth values. To obtain those truth functions from an appropriate game we have to use certain bijections between payoff values $\mathcal{V}_{\text{payoff}}$ and truth values $\mathcal{V}_{\text{truth}}$. The relation between these two functions is visualized by

$$\begin{tikzcd}
\mathcal{V}_{\text{payoff}} \arrow[r, "f_\diamond"] & \mathcal{V}_{\text{payoff}} \arrow[d, "\sigma^{-1}"] \\
\mathcal{V}_{\text{truth}} \arrow[u, "\sigma"] \arrow[r, "\tilde{\diamond}"] & \mathcal{V}_{\text{truth}}
\end{tikzcd}$$

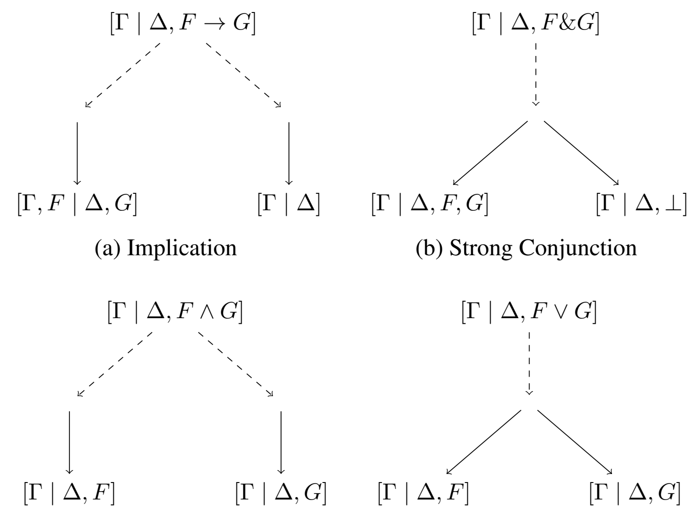
where σ is a bijection between truth and payoff values, and $\tilde{\diamond}$ is the truth function for the connective $\diamond$. In the following we survey which logics are captured by this general framework of semantic games.

Łukasiewicz logic. Giles's original game for Łukasiewicz logic can be recovered as an instance of this more abstract framework for games. For this reason we identify Giles's original notion of *risk*—the expected loss of money—with *inverted payoff*.

The assignment of risk $\langle \cdot \mid \cdot \rangle^r$ to atomic states, as defined for Giles's original game, amounts to a discriminating payoff function (according to Definition 9), i.e. Giles's notion of a risk value associated with an atomic game state is context independent, symmetric, and monotone. However, while rational players strive to *minimize* their risk, payoff is *maximized*. Therefore, let us identify the payoff of an atomic game state with its *inverted* risk and set

$$\begin{aligned}\langle F_1,\ldots,F_n \mid G_1,\ldots,G_m\rangle &= -\langle F_1,\ldots,F_n \mid G_1,\ldots,G_m\rangle^r \\ &= -\big(\sum_{1\le i\le m}\langle G_i\rangle^r - \sum_{1\le j\le n}\langle F_j\rangle^r\big) \\ &= -\sum_{1\le i\le m} -\langle\mid G_i\rangle + \sum_{1\le j\le n} -\langle\mid F_j\rangle \\ &= \sum_{1\le i\le m}\langle\mid G_i\rangle - \sum_{1\le j\le n}\langle\mid F_j\rangle\end{aligned}$$

for arbitrary (complex) game states. As we see, the aggregation function corresponding to $\langle\cdot\mid\cdot\rangle$ is ordinary addition $+$. Figure 2.6 presents the dialogue rules in the format of Figure 2.3. Because of duality we only have to consider your attacks on my assertions explicitly.



(a) Implication (b) Strong Conjunction (c) Conjunction (d) Disjunction

Figure 2.6: Giles's game with strong conjunction (your attack/my defense)

Note that discriminating payoff functions have 0 as neutral element because of symmetry (Payoff Principle 2). Moreover, the payoff[5] of a game starting with my assertion of a single formula ranges between 0, and -1: If it was greater than 0, you would invoke the attacking principle of limited liability LLA in the first step limiting my payoff to 0, and if it was smaller than -1, I would invoke the defending principle of limited liability LLD in the first round by asserting $\perp$ and thus fixing it to -1. Observe moreover, that asserting $\perp$ adds -1 to my payoff

[5]Remember that the *extended payoff* as defined above denotes the maximal final payoff which I can ensure at an arbitrary game state, while (plain) *payoff* is only defined for atomic game states. For the remainder of this chapter, we however abandon this distinction and subsume both these functions under the notion of *payoff*.

(and therefore subtracts 1). Likewise for a *true* formula is 0 my payoff remains unchanged. Therefore, if we want to match the functions $f_{\to}$, $f_{\&}$, $f_{\wedge}$, and $f_{\vee}$ extracted from these dialogue rules according to Corollary 1 with standard truth functions over $[0,1]$ we have to transform payoff into truth values by adding 1.

Indeed, using this transformation the functions extracted from the rules in Figure 2.6 coincide with the standard truth functions for $\mathbf{Ł}_\infty$, reviewed in Section 2.3. The relation between risk values $\mathcal{V}_{\text{risk}}$, payoff values $\mathcal{V}_{\text{payoff}}$, and truth values $\mathcal{V}_{\text{truth}}$ can be visualized as follows:

$$\begin{tikzcd}
 & \mathcal{V}_{\text{payoff}} \arrow[r, "f_\diamond"] & \mathcal{V}_{\text{payoff}} \arrow[d, "x \mapsto x-1"] \arrow[dr, <->, "x \mapsto -x"] & \\
\mathcal{V}_{\text{risk}} \arrow[ur, <->, "x \mapsto -x"] \arrow[dr, <->, "x \mapsto 1-x"'] & & & \mathcal{V}_{\text{risk}} \\
 & \mathcal{V}_{\text{truth}} \arrow[uu, "x \mapsto x+1"] \arrow[r, "\tilde{\diamond}"] & \mathcal{V}_{\text{truth}} \arrow[ur, <->, "x \mapsto 1-x"'] &
\end{tikzcd}$$

where we have $\mathcal{V}_{\text{truth}} = [0,1]$, $\mathcal{V}_{\text{payoff}} = [-1,0]$, and $\mathcal{V}_{\text{risk}} = [0,1]$.

Let us illustrate the case for implication. The rule for my assertion of $F \to G$ gives you a choice between asserting F to force me to assert G or else to declare that you will not attack this assertion at all. We obtain the following instance of min-max condition 2.1:

$$\langle | \ F \to G \rangle = \min(\langle F \mid G \rangle, \langle | \rangle) = \min(0, \langle | \ G \rangle - \langle | \ F \rangle).$$

Here we omit the maximum since $k_1 = 1$ and $k_2 = 1$ (refering to Figure 2.3, i.e. there is only one possible defense move. Adding 1 yields the truth function $v(F \to G) = 1 + \langle | \ F \to G \rangle = \min(1, 1 + \langle | \ G \rangle + 1 - (\langle | \ F \rangle + 1)) = \min(1, 1 - v(F) + v(G))$. The truth functions for the other connectives of $\mathbf{Ł}_\infty$ are obtained in the same manner.

Finite valued Łukasiewicz logics. Instead of considering arbitrary risk (and therefore also arbitrary truth values) from $[0,1]$, one may restrict the set of permissible risk values (equivalently: truth values) to $V_n = \{\frac{i}{n-1} \mid 1 \le i < n\}$, for some $n \ge 2$. Since V_n is closed with respect to addition, subtraction, as well as min and max, truth functions for all *finite valued* Łukasiewicz logics $\mathbf{Ł}_n$ are obtained just like those for $\mathbf{Ł}_\infty$.

Note that classical logic coincides with $\mathbf{Ł}_2$. Hence, classical logic can be modeled by a version of Giles's game where the experiments that determine the payoffs are not dispersive: Every atomic proposition A is simply true or false, entailing a determinate payment of 1€ for every assertion of A in case it is false. For every assignment of risk values 0 or 1 to atomic formulas I have a strategy for avoiding (net) payment in a game starting with my assertion of a formula F, if F is true under that assignment; on the other hand, if F is false, my best strategy limits my payment to you to 1€.

Cancellative hoop logic. A more interesting case is cancellative hoop logic **CHL** [17]. The truth value set of **CHL** is $(0, 1]$; correspondingly we consider a language without the propositional constant $\bot$; as negation $(\neg)$ is defined using $\bot$ it is removed as well. The truth functions for implication and strong conjunction are given as

$$v(F\&G) = v(F) \cdot v(G) \qquad v(F \to G) = \left\{ \begin{array}{cl} \frac{v(G)}{v(F)} & \text{if } v(F) \geq v(G) \\ 1 & \text{else.} \end{array} \right\}$$

just as for product logic Π. Note that a naive adaptation of Giles's Game for Π fails as pointed out in Section 2.3—where all counterexamples involve the truth value 0 (as the corresponding residuum is non-continuous at 0). Indeed, starting from Giles's game but using $\log$ as transformation function from truth to payoff values (and, conversely, $\exp$ as transformation function from payoff to truth values), we arrive at a game-based characterization of **CHL**:

$$\begin{tikzcd} \mathcal{V}_{\text{payoff}} \arrow[r, "f_\diamond"] & \mathcal{V}_{\text{payoff}} \arrow[d, "x \mapsto \exp x"] \\ \mathcal{V}_{\text{truth}} \arrow[u, "x \mapsto \log x"] \arrow[r, "\tilde{\diamond}"] & \mathcal{V}_{\text{truth}} \end{tikzcd}$$

where $\mathcal{V}_{\text{truth}} = (0, 1]$ and $\mathcal{V}_{\text{payoff}} = (-\infty, 0]$. Observe that for implication $\to$ this yields the correct truth function: Adopting Giles's dialogue rules (see Figure 2.6) and payoff scheme we have $\langle| \; F \to G \rangle = \min(0, \langle| \; G\rangle - \langle| \; F\rangle)$ (as shown above), and thus for a **CHL**-interpretation M we have

$$\begin{aligned} \tilde{\to}(x, y) &= \exp(\min(0, \log y - \log x)) = \\ &= \min(1, y/x) \end{aligned}$$

matching the definition of the connective $\to$ above. In the same manner addition $(+)$ over $(-\infty, 0]$ maps into multiplication $(\cdot)$ over $(0, 1]$. However, the payoff function extracted from the dialogue rule for (strong) conjunction $\&$ of Giles's game (with risk inverted into payoff) is $\langle| \; F\&G\rangle = \max(-1, \langle| \; G\rangle + \langle| \; F\rangle)$ rather than the required $+$. Moreover, the defending principle of limited liability LLD allows one to assert $\bot$ as a valid defense to an attack on $F\&G$. Since for **CHL** we have removed $\bot$ from the language, we also drop the defending principle of limited liability LLD. Hence we obtain

$$\begin{aligned} \tilde{\&}(x, y) &= \exp(\log x + \log y) = \\ &= x \cdot y \end{aligned}$$

matching the definition of the connective above.

Abelian logic. So far we have only considered logics where the set of truth values is a proper subset of $\mathbb{R}$ and where we had to explicitly transform payoff values into truth values and vice versa. But there is an interesting and well studied logic, namely Slaney and Meyer's Abelian logic **A** [30, 68, 70], which coincides with one of Casari's logics for modeling comparative reasoning in natural language [10, 11], where arbitrary real valued payoffs in a Giles-style game can be directly interpreted as truth values. The truth value set of **A** indeed is $\mathbb{R}$. The truth functions for implication ($\rightarrow$) is subtraction and the truth function for strong conjunction ($\&$) is addition over $\mathbb{R}$. In addition, $\max$ and $\min$ serve as truth functions for disjunction ($\vee$) and weak conjunction ($\wedge$), respectively.

The game based characterization of **A** is particularly simple: just drop both forms of the principle of limited liability, LLA and LLD, from Giles's game. In other words, every assertion made by the opposing player, including those of the form $F \rightarrow G$, has to be attacked; moreover the only permissible reply to an attack on $F\&G$ is to assert both A and B. (The latter rule has already been used for **CHL**, above.) The functions that can be extracted from the resulting dialogue rules according to Corollary 1 are precisely those mentioned above: $f_{\rightarrow} = -$, $f_{\&} = +$, $f_{\wedge} = \min$, and $f_{\vee} = \max$.

Alternative aggregation functions. In all the above examples, the aggregation function $\oplus$ corresponding to the respective payoff function has been addition ($+$). This raises the question, whether in fact $\oplus$ always has to be $+$. This question is of some interest, since every truth function that can be extracted from a Giles-style game is built up from $\oplus$, $-$, $\min$, $\max$, and some bijective transformation function between payoff and truth values (like $+1$ for $\mathbf{Ł}_\infty$, and $\exp$ for **CHL**).

To settle this question in the negative it suffices to check that for any assignment v of reals to atomic propositions

$$\langle \gamma \mid \delta \rangle = \sqrt[3]{\sum_{q \in \delta} v(q)^3} - \sqrt[3]{\sum_{p \in \gamma} v(p)^3}$$

is a discriminating payoff function with $\oplus(x, y) = \sqrt[3]{x^3 + y^3}$ as corresponding aggregation function. However, this payoff function can be reduced to $+$ by adjusting the mapping σ from payoff values into truth values: Consider for example Giles's dialogue rule for (strong) conjunction (see Figure 2.6). We obtain

$$\langle | \, F\&G \rangle = \max(-1, \langle | \, F \rangle \oplus \langle | \, G \rangle) = \max\Big(-1, \sqrt[3]{\langle | \, F \rangle^3 + \langle | \, G \rangle^3}\Big)$$

and thus transforming payoffs into truth values we obtain

$$\&(x, y) = \sigma^{-1}(\max(\sigma(-1), \sqrt[3]{\sigma(x)^3 + \sigma(y)^3})).$$

Now assume another game with the same dialogue rules, standard addition + as aggregation function $\oplus$, and $\sigma'(x) = \sigma(x)^3$ as transformation between truth and payoff values. Its inverse amounts to $\sigma'^{-1}(x) = \sigma^{-1}(\sqrt[3]{x})$. The payoff $\langle| F\&G\rangle$ is calculated as $\max(-1, \langle| F\rangle + \langle| G\rangle)$ just as for Giles's game. Transforming this payoff to a truth value again yields

$$\begin{aligned}
\&(x,y) &= \sigma'^{-1}(\max(\sigma'(-1), \sigma'(x) + \sigma'(y))) \\
&= \sigma^{-1}\big(\sqrt[3]{(\max(\sigma(-1)^3, \sigma(x)^3 + \sigma(y)^3)}\big) \\
&= \sigma^{-1}\big(\max(\sqrt[3]{\sigma(-1)^3}, \sqrt[3]{\sigma(x)^3 + \sigma(y)^3})\big) \\
&= \sigma^{-1}\big(\max(\sigma(-1), \sqrt[3]{\sigma(x)^3 + \sigma(y)^3})\big)
\end{aligned}$$

matching the truth function in the previous game, where we used a non-standard aggregation function. As can easily be seen, this also holds for arbitrary dialogue rules. More generally, we observe that for *any* aggregation function the ordered group $G = (\mathbb{R}; \leq, \oplus, 0, -)$ is isomorphic to $(\mathbb{R}; \leq, +, 0, -)$. This is provided by Hölder's Theorem [44]. Since G is archimedean (Proposition 4) the theorem states that G is isomorphic to a *sub*group of $(\mathbb{R}; \leq, +, 0, -)$. Therefore we may safely assume that the aggregation function amounts to standard addition $(+)$ without loss of generality.

CHAPTER 3

Games for Vague Quantifiers

This chapter provides a characterization of (classes of) fuzzy quantifiers in terms of Giles's game as presented above. We focus on *proportional quantifiers* such as *about a half*, *at most roughly a third*, etc., which solely depend on the ratio of elements in the scope and those in the universe (set to the quantifier's range). Zadeh calls these 'fuzzy quantifiers of the second kind' [96], to be distinguished from *absolute quantifiers*, like *about ten* or *at least (roughly) a thousand*, which Zadeh calls 'fuzzy quantifiers of the first kind'. How the quantifier 'many' fits into this classification has independently been subject to investigation by various linguists. In particular, Partee [73] argues that 'many' is ambiguous and has absolute and proportional readings depending on the context. Here, we here however focus solely on proportional quantifiers and thus also only on proportional readings of 'many'.

Figure 3.1 illustrates possible models for the fuzzy (proportional) quantifiers 'few', 'most', and 'about a half' as given by Zadeh.

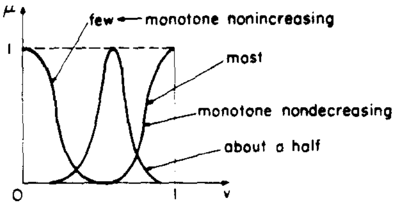


Figure 3.1: Proportional quantifiers — reproduced from [96] (Figure 3).

The value v on the x-axis refers to the proportion of domain elements which also belong to the quantifier's scope, whereas μ on the y-axis refers to the degree of truth of the resulting quantified statement. Zadeh's approach to fuzzy quantification however has several independent drawbacks. First, it does in no way attach *meaning* to truth values computed by such a quantifier. Therefore one can not justify why exactly the truth function for, e.g., 'about a half' depicted in Figure 3.1 is an adequate model and not some other function. There are proposals like voting semantics [59], acceptability semantics [71], re-randomising semantics [43], approximation semantics [9], and of course Robin Giles's game based semantics as presented in the previous chapter, which address this challenge for fuzzy logics in general. These have, however, not been extended to cover also fuzzy quantification beyond the Type II quantifiers '$\forall$' and '$\exists$'. Zadeh's figure surely is meant as a deliberately vague suggestion regarding the general form of plausible models, rather than as fixing the meaning of these quantifiers by particular truth functions. However the space of possible candidate functions remains uncountably large: even if we impose restrictions like continuity, monotonicity, or symmetry, it is left unclear what parameters can be set in order to justify the choice of particular functions. Moreover, extending this picture to fully fuzzy (Type IV) quantification, the value v on the x-axis cannot refer to a proportion of elements, but rather to some cardinality measure on the fuzzy set in question, adding yet another dimension of variability. Glöckner [35] surveys different fuzzy cardinality measures used in the literature for defining (fully) fuzzy quantifiers. He then illustrates problematic situations where quantifiers defined on top of all of these measures yield properties which arguably go against our intuitions for their use, see e.g. Glöckner's criticism of the Σ-count proposed by Zadeh in Section 1.5.

This chapter shows how meaning can be attached to (semi-)fuzzy quantifiers by a generalisation of Giles's game. This way, we retain Giles's characterization of Łukasiewicz logic enriched with semi fuzzy proportional quantifiers in terms of an evaluation game. Moreover, different candidate functions emerging from the game can be characterized by a small set of parameters which are reflected in the corresponding dialogue rule. As a 'side effect' the resulting quantifiers are compatible with Łukasiewicz logic, one of the three fundamental t-norm based fuzzy logics used for many applications and the only one, where all connectives are continuous (see Section 1.5). Other popular approaches to fuzzy quantification like Glöckner's do not share this property since they do not model implication by residuation (as do t-norm based fuzzy logics) but by the material implication $\phi \to \psi$ defined by $\neg\phi \vee \psi$. Thus, they can not be analyzed as deductive (mathematical) fuzzy logics as introduced by Hájek [37].

3.1 Giles Game with Random Witnesses

We first show how to extend Giles's game for $\mathbf{Ł}_\infty$as described in Section 2.3 in a simple way by introducing a new semi-fuzzy quantifier Π. The following sections then build on this extension to introduce more complex dialogue rules which aid to single out more interesting classes of semi-fuzzy proportional quantifiers, all while maintaining Giles's idea of providing 'tangible meaning' to logical connectives in terms of bets on the results of dispersive experiments.

Observe that the only difference between the rules $(R_\forall)$ and (R_E) as presented in Section 2.3 for defending assertions $\forall x F(x)$ and $\exists x F(x)$, respectively, is that either the defender or the attacker has to pick the constant c that determines the new formula $F(c)$ which is added to the defender's tenet. Considering the randomized setting of $\mathcal{G}$-games, the following rule for a new type of (monadic) quantifier Π seems quite natural:

(R_Π) If I assert $\Pi x F(x)$ then I have to assert $F(c)$ for a randomly picked c (and analogously for your assertion of $\Pi x F(x)$).

The random choice refers to a uniform distribution of the domain. One might want to consider also other ways of randomly picking domain elements, and thus other probability distributions. However note that there is an agreement in the literature on fuzzy quantification (see, e.g., [74, 87]) that a necessary condition for a quantifier to be called *logical* is the *domain invariance* of its semantics: it ought to be invariant with respect to isomorphisms between domains, essentially restricting the witness selection process to uniform distributions in our case. Additionally to Π, also for the other quantifiers considered in the following Sections 3.2 and 3.3 logicality is maintained by insisting on random choices with respect to a uniform distribution. In the following we will call the formula $F(c)$ where c has been chosen randomly a *random instance* of F.

While (R_Π), in principle, can be applied to arbitrary $\mathbf{Ł}_\infty$-formulas F in the scope of Π, we will view Π as a *semi-fuzzy quantifier* and hence insist on classical formulas in its scope for reasons. In this we follow Glöckner [35] as explained in Section 1.5 to first focus on the semantics of semi-fuzzy quantifiers and to extend these models to fully fuzzy quantifiers in a separate step. Glöckner calls such a step a *quantifier fuzzyfication mechanism* and gives axioms for plausible models of fully fuzzy quantifiers. Particularly Díaz-Hermida et al. [16] provide such a quantifier fuzzyfication mechanism suitable for finite domains.

More formally, we specify the language for the logic $\mathbf{Ł}_\infty(\Pi)$ as follows:

$$\begin{array}{lcl} \gamma & ::= & \oplus \mid \hat{P}(\vec{t}) \mid \neg\gamma \mid (\gamma \vee \gamma) \mid (\gamma \wedge \gamma) \mid \forall v\gamma \mid \exists v\gamma \\ \varphi & ::= & \gamma \mid \tilde{P}(\vec{t}) \mid \neg\varphi \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \to \varphi) \mid (\varphi \& \varphi) \mid \forall v\varphi \mid \exists v\varphi \mid \Pi v\gamma \end{array}$$

where $\hat{P}$ and $\tilde{P}$ are meta-variables for classical and for general (i.e., possibly fuzzy) predicate symbols, respectively; v denotes variables, and $\vec{t}$ denotes a sequence of terms, i.e. either object variable or constant symbol, matching the arity of the preceding predicate symbol. Note the language is two-sorted: the scope of the quantifier Π always consists of a classical formula, while on the outer level the syntax is as for $\mathbf{Ł}_\infty$ itself. In the following two sections we will also extend $\mathbf{Ł}_\infty$ with other (classes of) semi-fuzzy quantifiers in an analogous way. We denote these extensions with $\mathbf{Ł}_\infty(Qs)$ where Qs is a list of semi-fuzzy quantifiers.

Moreover, observe that randomness in Giles's original game is exhibited only for the betting part, i.e., at final (atomic) game states. The risk value of a game coincides with the risk value of the final game state reached, provided both players play rationally. Here we introduce randomness at the dialogue stage when attacking according to rule (R_Π). Since we focus on semi-fuzzy quantifiers, these two sources randomness are kept separate from each other: After attacking $\Pi x \hat{F}(x)$, the resulting assertion $\hat{F}(c)$ is crisp, so only trials of non-dispersive experiments will be conducted once a final game state is reached. The players' risk values for the quantified statement however may range between 0 and 1. From a game-theoretic point of view random witness selection can be formalised by means of introducing a third player called *Nature* who is indifferent of the final game state, thus, from my or your points of view, acts randomly. A strategy for me (or you) then fixes a move for all of *Nature*'s and your (or my, respectively) choices. In order to stay within Giles's framework we here refrain from formally introducing *Nature* as a third player which simplifies the game's presentation and analysis.

The following operator $\mathit{Prop}_x\, \hat{G}(x)$ denotes the semantic counter-part to the quantifier Π as specified by the dialogue rule (R_Π). For a finite domain D it computes the proportion of elements in D satisfying some classical formula $\hat{G}(x)$. As motivated above we focus on a uniform distribution over D. Thus this proportion coincides with the probability that a randomly chosen element satisfies $\hat{G}$.

Definition 11. Let $\hat{G}(x)$ be a classical formula and $v_M(\cdot)$ a corresponding evaluation function over the finite domain D. Then

$$\mathit{Prop}_x\, \hat{G}(x) = \frac{\sum_{c \in D} v_M(\hat{G}(c))}{|D|}$$

The following theorem states that rule (R_Π) matches the extension of the valuation function for $\mathbf{Ł}_\infty$ to $\mathbf{Ł}_\infty(\Pi)$ by $v_M(\Pi x F(x)) = \mathit{Prop}_x\, F(x)$.

Theorem 6. *A $\mathbf{Ł}_\infty(\Pi)$-sentence F is evaluated to $v_M(F) = x$ in an interpretation M iff every $\mathcal{G}$-game for F augmented by rule (R_Π) is $(1-x)$-valued for me under the risk value assignment $\langle\cdot\rangle_M$.*

In the next section we will define the more general class of *blind choice quantifiers* containing the quantifier Π defined here. Thus, Theorem 6 will turn out to be an instance of a more general result to be proved there.

As pointed out in Section 1.5, linguists employ a more general notion of quantification than for (classical) logics. In fact, natural language quantifiers are typically binary, as in *About half* *of all Austrians are male*, rather than unary as in *About half (of all individuals) are male* where the universe of discourse is set to all Austrians. Observe that for the classical quantifiers $\forall$ and $\exists$ their binary counterparts can straightforwardly expressed using unary quantification. The (binary quantified) statements 'All Austrians are male' and 'There exists a male Austrian' are readily encoded by $\forall x\ (Austrian(x) \to male(x))$ and $\exists x\ (Austrian(x) \wedge male(x))$, respectively. Proportional binary quantifiers like *about half*, *many*, *at least a third*, however, cannot in general be reduced in this manner. We therefore define analogously as above:

Definition 12. Let $\hat{G}(x)$ and $\hat{R}(x)$ be classical formulas and $v_M(\cdot)$ a corresponding evaluation function over the finite domain D. Then

$$^{\hat{R}}\,\mathit{Prop}_x\,\hat{G}(x) = \frac{\sum_{c \in D} v_M(\hat{G}(c) \wedge \hat{R}(c))}{|D \cap \hat{R}|}.$$

Note that by $\hat{R}$ we here denote the set of elements x that satisfy the crisp predicate $\hat{R}(x)$. The term $^{\hat{R}}\,\mathit{Prop}_x\,\hat{G}(x)$ thus denotes the proportion of elements x in $\hat{R}$ which satisfy $\hat{G}(x)$. It is undefined if $\hat{R}$ is empty. This allows us to formulate a dialogue rule for the a binary version version of the Π quantifier as defined above:

(R_{Π^2}) If I assert $^{\hat{R}}\Pi x \hat{B}(x)$ then I have to assert $\hat{B}(c)$ where c is a randomly picked element of $\hat{R}$ (and analogously for your assertion of $^{\hat{R}}\Pi x B(x)$).

If the classical formula $\hat{R}(x)$ is atomic then it is clear what it means to randomly pick an element of $\hat{R}$ (if one exists at all—otherwise the quantifier is left undefined here). If however $\hat{R}(x)$ is of arbitrary logical complexity, then we may remain within our game semantical setup by employing $\mathcal{H}$-games to find an appropriate random witness as follows:

1. Pick a random domain element c.

2. Initiate an $\mathcal{H}$-game where a Proponent **P** defends $\hat{R}(c)$ against an Opponent **O**.

3. If **P** wins the $\mathcal{H}$-game, then I and you continue the main $\mathcal{G}$-game with the constant c. Otherwise, continue at step 1.

Observe that the $\mathcal{H}$-game is not played by me and you, but by two new players **P** and **O**. This entails that the players' objectives in these subgame are independent from my and your objectives in the $\mathcal{G}$-game. By Theorem 1 **P** wins the $\mathcal{H}$-game against the rational Opponent **O** if and only if $\hat{A}(c)$ is true, i.e., if $c \in \hat{A}$. If the $\mathcal{H}$-game was played by same players me and you, one of us could deliberately lose the $\mathcal{H}$-game, just to end up with an advantageous witness c in the $\mathcal{G}$-game. Note that the indicated procedure and therefore the main $\mathcal{G}$-game will fail to terminate if the range $\hat{A}$ is empty. This is in accordance with the above definition that leaves $v_M(^{\hat{A}}\mathsf{Q}xF(x))$

undefined if the range is empty. According to Barwise and Cooper [7] this matches intuitions about natural language quantifiers applied to an empty range.

Note that, although fuzzy binary quantifiers are extensional, they cannot directly be reduced to their unary counterparts as follows: For a unary semi-fuzzy quantifier Q define its restriction to $\hat{R}$, denoted ${}^{\hat{R}}\mathsf{Q}x\hat{G}(x)$, by setting $v_M({}^{\hat{R}}\mathsf{Q}xF(x)) = v_{M'}(\mathsf{Q}x\hat{G}(x))$, where M' denotes the interpretation that results from M by restricting the domain of M to $\hat{R}$. This, however, evaluates $\hat{G}(c)$ with respect to the restricted interpretation M'. Consider the statement 'Most Austrians know someone from abroad'. If we restrict the domain to only Austrians in the first place, in the restricted model there will be no people from abroad and thus nobody will know someone from abroad. This is the reason for introducing the operator ${}^{\hat{R}}$ *Prop* above, which uses the restricted domain only for counting the proportion, but not for evaluating $v_M(\hat{G}(c))$.

3.2 Blind Choice Quantifiers

In the dialogue rule (R_Π) above, I—the defendant—have to assert a random instance of the quantified formula. Dialogue rules for other (semi-)fuzzy quantifiers can be introduced by continuing the game with several bets *for* and *against* such random instances: Betting for a formula $F(c)$ just means to assert $F(c)$, while betting against $F(c)$ is equivalent to betting for $\neg F(c)$ and thus amounts to an assertion of $\perp$ while the opponent asserts $F(c)$. Such a bet against $F(c)$ can be interpreted as paying 1€ for a betting ticket regarding $F(c)$ that entitles one to receive whatever payment resulting from one's opponent assertion of $F(c)$ according to the results of associated dispersive experiments made at the end of the game. Depending on different attack moves and according defense moves for each attack, the number of bets placed by the players may vary.

As explained in Section 2.3, a round of a game consists of a player's *attack* of an assertion made by the other player, followed by a *defense* of that latter player, where the principle of limited liability for defense (LLD) states that asserting $\perp$ is always a valid defense (see Section 2.3). By the other form of the principle of limited liability (LLA), the attacker, instead of attacking an assertion in some specific way, may grant the assertion which will consequently be deleted from the current state of the game. In general, when an assertion of $\mathsf{Q}x\hat{F}(x)$ is attacked, the round results in a state where both players are placing certain numbers of bets for or against various random instances of $\hat{F}(c)$. Thus we arrive at a rich set of possible quantifier rules. This setting can even be generalized by allowing not only random instances, but letting the players choose some instances (as for Giles's original rules $(R_\exists)$ and $(R_\forall)$); in the following we will however only be interested in random instances as it is them, which provide additional complexity over the classical quantifiers.

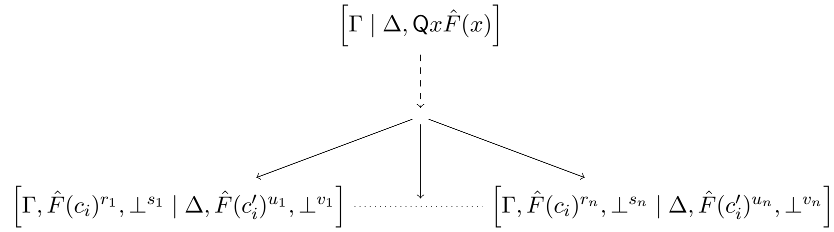


Figure 3.2: Schematic blind choice quantifier rule — my possible defenses to a particular attack by you. The dashed line denotes your attacking move, while solid lines denote possible defense moves by me.

In this section we will investigate the family of *blind choice quantifiers* defined as follows.

Definition 13. Q is a *(semi-fuzzy) blind choice quantifier* if it can be specified by a game rule satisfying the following two conditions:

(i) An attack on $\mathsf{Q}x\hat{F}(x)$ followed by a defense move results in a state where both players have placed a certain number (possibly zero) of additional bets *for* and *against* random instances of $\hat{F}(x)$ (or choose to invoke the principle of limited liability).

(ii) The identity of the random constants is revealed to the players only at the end of the round; i.e., after an attack has been chosen by the one player and a corresponding defense move has been chosen by the other player.

We call these *blind choice quantifiers* because of clause (ii): The players may invoke the principle of limited liability (both LLA and LLD), but without knowing which constants have been chosen. The next section will introduce the class of so-called *deliberate choice quantifiers*, where the constants are immediately revealed to the players—before the defending player decides, for and against which instances he wants to place a bet. As it turns out, this subtle modification has a significant influence on the resulting risk (and thus truth) values.

Figure 3.2 depicts possible state transitions involved in the application of a blind choice quantifier rule. Γ and Δ denote arbitrary multisets of formulas; $\hat{F}(x)$ is a classical formula that forms the scope of the sentence $\mathsf{Q}x\hat{F}(x)$ asserted by me and attacked by you; $\perp^k$ denotes k occurrences of $\perp$; and $\hat{F}(c_i)^k$ is used as an abbreviation for the k assertions of random instances $\hat{F}(c_1), \ldots, \hat{F}(c_k)$. Note that in general there is more than one way in which you may attack my assertion of $\mathsf{Q}x\hat{F}(x)$—Figure 3.2 only shows the scheme for one particular attack. A presentation of a full rule consists of a finite number of instances of this scheme. The intermediate state between these two moves is not labeled which means that the effect of the different attacks is shown only at the end of a full round, i.e., only after also a corresponding defense has been chosen. The attacking principle of limited liability (LLA) implies that you (the attacker) may choose to simply remove the exhibited occurrence of $\mathsf{Q}x\hat{F}(x)$ from the state. In other words, every rule includes as a valid attack move an instance of Figure 3.2 that consists of only one branch ($n = 1$), where $r_1 = s_1 = u_1 = v_1 = 0$, i.e., where the game continues with $[\Gamma \mid \Delta]$. For me, as defender, the principle of limited liability implies that in any other instance of the schematic tree there is a branch i with $r_i = s_i = u_i = 0$ and $v_i = 1$, i.e., where I reply to your attack by asserting $\bot$ and the game continues with $[\Gamma \mid \Delta, \bot]$.

As above for the dialogue rule (R_Π) (and also for the other, original rules of Giles's game) we assume that for every rule for my assertion of a formula, there is a corresponding rule for your assertion of the same formula, with our roles switched. We therefore explicitly state and investigate rules for my assertions of quantified formulas only.

We define the blind choice quantifiers L^k_m and G^k_m as follows and call them *fundamental* blind choice quantifiers. As shown below, all other blind choice quantifiers can be constructed from these fundamental ones.

($R_{\mathsf{L}^k_m}$) If I assert $\mathsf{L}^k_m x\hat{F}(x)$ then you may attack by betting for k random instances of $\hat{F}(x)$, while I bet against m random instances of $\hat{F}(x)$.

($R_{\mathsf{G}^k_m}$) If I assert $\mathsf{G}^k_m x\hat{F}(x)$ then you may attack by betting against m random instances of $\hat{F}(x)$, while I bet for k random instances of $\hat{F}(x)$.

Although not explicitly mentioned, the principle of limited liability remains in force. Therefore, the defender may also respond to an attack by asserting $\bot$ (LLD). However, as stated by condition *(ii)* of Definition 13 the random constants used to obtain the mentioned instances of $\hat{F}(x)$ are only revealed to the players after they have placed their bets. Thus, if the defender wants to invoke LLD, he must do so, before the constants are revealed. Conversely, the attacking player may invoke LLA before constants are chosen. If none of the players invokes the principle of limited liability the following successor game states are reached:

$$\begin{aligned} \text{for } \mathsf{L}^k_m x\hat{F}(x): &\quad \left[\Gamma, \hat{F}(c_i)^{k+m} \mid \Delta, \bot^m\right] \\ \text{for } \mathsf{G}^k_m x\hat{F}(x): &\quad \left[\Gamma, \bot^m \mid \Delta, \hat{F}(c_i)^{k+m}\right] \end{aligned}$$

Thus, for my assertions of $\mathsf{L}^k_m x\hat{F}(x)$ and $\mathsf{G}^k_m x\hat{F}(x)$ the according dialogue rules can be depicted as shown in Figure 3.3 and analogously for your assertions.

We claim that these rules match the extension of Ł_∞ to $\text{Ł}_\infty(\mathsf{L}^k_m, \mathsf{G}^k_m)$ by

$$v_M(\mathsf{L}^k_m x\hat{F}(x)) = \min(1, \max(0, 1 + k - (m+k)\, \mathit{Prop}_x\, \hat{F}(x))) \text{ and} \tag{3.1}$$

$$v_M(\mathsf{G}^k_m x\hat{F}(x)) = \min(1, \max(0, 1 - k + (m+k)\, \mathit{Prop}_x\, \hat{F}(x))). \tag{3.2}$$

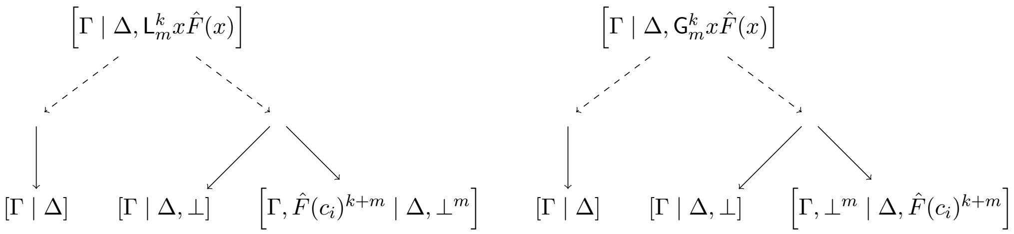


Figure 3.3: The dialogue rules $R_{\mathsf{L}^k_m}$ and $R_{\mathsf{G}^k_m}$. The dashed lines denote different possible attack moves for you, while solid lines denote possible defense moves for me.

Theorem 7. *A $Ł_\infty(\mathsf{L}^k_m, \mathsf{G}^k_m)$-sentence F is evaluated to $v_M(F) = x$ in an interpretation M iff every $\mathcal{G}$-game for F augmented by the rules $(R_{\mathsf{L}^k_m})$ and $(R_{\mathsf{G}^k_m})$ is $(1-x)$-valued for me under risk value assignment $\langle\cdot\rangle_M$.*

Proof. Relative to the proof of Theorem 4 (see [20,32,33]) we only have to consider states of the form $\left[\Gamma \mid \Delta, \mathsf{L}^k_m x \hat{F}(x)\right]$ and $\left[\Gamma \mid \Delta, \mathsf{G}^k_m x \hat{F}(x)\right]$. For your assertions of L^k_m- or G^k_m-quantified formulas, the cases are completely dual. In fact, since G^k_m is treated analogously to L^k_m, we may focus on states of the form $\left[\Gamma \mid \Delta, \mathsf{L}^k_m x \hat{F}(x)\right]$ without loss of generality. Like for the other connectives, we obtain the total risk at such a state by separating the risk for the exhibited assertion and the risk for the rest of the state and adding these values

$$\left\langle \Gamma \mid \Delta, \mathsf{L}^k_m x \hat{F}(x) \right\rangle = \langle \Gamma \mid \Delta \rangle + \left\langle \mid \mathsf{L}^k_m x \hat{F}(x) \right\rangle .$$

It remains to show that the reduction of the quantified formula to instances according to rule $(R_{\mathsf{L}^k_m})$ results in a risk that corresponds to the specified truth function 3.1 if both players play rationally. According to Figure 3.3 the three possible successor states are $\left[\hat{F}(c_i)^{k+m} \mid \bot^m\right]$, $[\mid]$, and $[\mid \bot]$. In the first case, revealing the constants to the players already fixes the amount of money I have to pay, since only classical formulas are involved—again, assuming both players act rationally: I have to pay m€ to you for my m assertions of $\bot$, while for each of your $k+m$ assertions you have to pay me either 0€ or 1€. In total I thus lose between $-k$€ and m€, depending on the random constants c_i. The risk value of the game state *before* the identities of the constants are revealed to the players is therefore calculated as the *expected* value for this amount. As the selection processes of the random witnesses are independent of each other[1] my

[1]One might want to insist that $k+m$ *different* witnesses are selected. This interpretation arguably might be more intuitive in motivating such quantifiers for a particular natural language expression. As a result, the random witnesses are no longer independent of each other. However, my loss now is hypergeometrically distributed and thus my *expected* value remains the same leaving the rest of our analysis intact.

loss is binomially distributed and its expect value is readily computed as:

$$m - \sum_{i=0}^{k+m} i \cdot (\mathit{Prop}_x\, \hat{F}(x))^{k+m-i}(1 - \mathit{Prop}_x\, \hat{F}(x))^i \binom{k+m}{i} =$$
$$m - (k+m)(1 - \mathit{Prop}_x\, \hat{F}(x)) = -k + (k+m)\, \mathit{Prop}_x\, \hat{F}(x)).$$

The second case (state $[\,|\,]$, carrying risk 0) arises if you choose to grant my assertion of the formula, which you, playing rationally, will do if the above expression is below 0. The third case (state $[\,|\perp]$, carrying risk 1) arises if I invoke the principle of limited liability to hedge my expected loss. Thus we obtain

$$\left\langle \mid \mathsf{L}_m^k x \hat{F}(x) \right\rangle = \min(1, \max(0, -k + (k+m)\, \mathit{Prop}_x\, \hat{F}(x))) = 1 - v_M(\mathsf{L}_m^k x \hat{F}(x))$$

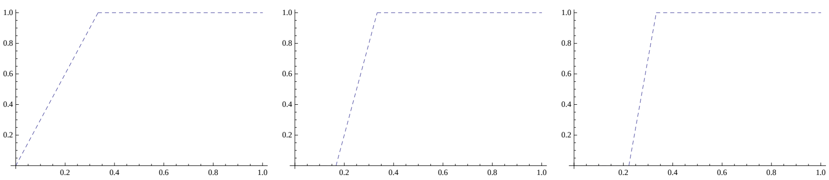
which means that the claimed correspondence between the truth function and the risk resulting from playing rationally holds. □

At least a third. As a motivation to model natural language quantification by blind choice quantifiers, consider the following quote by Barbara Partee [73]:

> [...] when the restrictor does involve a very open-ended set, "proportion" [...] becomes ill-defined, and some extended sense of "frequency" may be needed, including an atemporal sense conceptualized in terms of an imagined survey of the given domain.

For modeling 'at least a third' we take a closer look at quantifiers of the form G_{2s}^s. We argue that these quantifiers can be used to model the natural language expression 'at least a third'. Note that the attacker of $\mathsf{G}_{2s}^s x\hat{F}(x)$ is supposed to believe that $\hat{F}(x)$ holds for less than a third of all domain elements (otherwise he would invoke the attacking principle of limited liability (LLA) and grant the assertion). Consequently he will agree to place $2s$ bets *against* random instances of $\hat{F}(x)$ if the defender places s bets *for* such random instances. This kind of witness selection can also be seen as a survey of the domain as suggested by Partee. Observe that the blind choice here is irrelevant—showing the constants to the players *before* they place their bets for and against the corresponding instances might yield a dialogue rule which more intuitively corresponds to a survey. However, no matter which player picks which constants, the resulting game state will always be the same.

Figure 3.4 shows the resulting truth functions for sample sizes of $3, 6$, and of 9 elements, where the horizontal axis corresponds to $\mathit{Prop}_x\, \hat{F}(x)$ and the vertical axis to $v_M(\mathsf{G}_{2s}^s x\hat{F}(x))$. Functions like these are routinely suggested to represent natural language quantifiers like



(a) $\mathsf{G}_2^1 x \hat{F}(x)$ (b) $\mathsf{G}_4^2 x \hat{F}(x)$ (c) $\mathsf{G}_6^3 x \hat{F}(x)$

Figure 3.4: Truth functions for $\mathsf{G}_{2s}^s x \hat{F}(x)$

'at least about a third' in the fuzzy logic literature.[2] However no justification beyond intuitive plausibility is usually given. In contrast, our model allows one to extract such truth function from an underlying semantic principle: namely the willingness to bet on randomly chosen witnesses that support or refute the statement in question.

In formal semantics, linguists often distinguish between semantic and pragmatic phenomena like *imprecision*, while in fuzzy logics, these are more frequently intermingled with each other. For example the intended *meaning* of one of these quantifiers surely is that the proportion of elements in the domain satisfying $\hat{F}$ is at least $1/3$. On the other hand, *pragmatic* aspects might govern, how 'loosely' we are ready to interpret the quantifier: Strictly speaking, 'at least a third' is not fuzzy at all. However, in many cases we are to judge the quantifier as 'almost true' if the proportion is just slightly less to $1/3$. Our model separates both concerns: The proportion targeted by the quantifier is determined by the ratio of k and $k+m$ (in this case s and $3s$) and the level of precision is determined by the sample size: The larger the sample size is, the stricter we are in interpreting 'at least a third'.

As noted above, the quantifiers L_m^k and G_m^k are only (very restricted) examples of blind choice quantifiers. Nevertheless, they turn out to be expressive enough to define *all* blind choice quantifiers in the context of weak Łukasiewicz logic $\mathbf{Ł}^w$:

Theorem 8. *All blind choice quantifiers can be expressed using quantifiers of the form L_m^k and G_m^k, conjunction $\wedge$, disjunction $\vee$, and $\perp$.*

Proof. As illustrated in Figure 3.2 above, the game state resulting from an attack and a corresponding defense of my assertion of a blind choice quantifiers (also possibly invoking the principle of limited liability) is always of the form $\left[\Gamma, \hat{F}(c_i)^r, \perp^s \mid \Delta, \hat{F}(c_i')^u, \perp^v\right]$. Analogously to the proof of Theorem 7, the associated risk before the identities of the constants are revealed is computed as

$$\langle \Gamma \mid \Delta \rangle + v - s + (u-r)(1 - \mathit{Prop}_x\, \hat{F}(x)).$$

[2] For example in [35] trapezoidal functions like the ones in Figure 3.4 are explicitly suggested for natural language quantifiers of this kind.

Remember that $\hat{F}(c_i)^k$ is short hand notation for k (in general) different random instances of $F(x)$. As a first step towards a simplified uniform presentation of arbitrary blind choice quantifiers, note the following. Instead of picking $u+r$ random constants we can rather investigate the game state $\left[\Gamma, \hat{F}(c)^r, \bot^s \mid \Delta, \hat{F}(c)^u, \bot^v\right]$ where only one random constant c is picked, since this modification does not change the *expected* risk. As a further step, note that game states where assertions of $\hat{F}(c)$ are made by *both* players show redundancies in the sense that there are equivalent game states where $\hat{F}(c)$ occurs only in one of the two multisets of assertions that represent a state. Likewise for game states with assertions of $\bot$ made by both players. Depending on v, s, u, and r, an equivalent game state is given by:

$$
\begin{array}{lll}
(1) & \left[\Gamma, \hat{F}(c)^{r-u} \mid \Delta, \bot^{v-s}\right] & \text{if } v > s \text{ and } r > u, \\
(2) & \left[\Gamma, \hat{F}(c)^{r-u}, \bot^{s-v} \mid \Delta\right] & \text{if } v \leq s \text{ and } r > u \\
(3) & \left[\Gamma \mid \Delta, \hat{F}(c)^{u-r}, \bot^{v-s}\right] & \text{if } v > s \text{ and } r \leq u, \\
(4) & \left[\Gamma, \bot^{s-v} \mid \Delta, \hat{F}(c)^{u-r}\right] & \text{if } v \leq s \text{ and } r \leq u.
\end{array}
$$

Note that states of type (2) are redundant, since you would rather invoke the principle of limited liability, resulting in $[\Gamma \mid \Delta]$, than to make an assertion without being compensated by any assertions made by me. On the other hand, states of type (3) reduce to state $[\Gamma \mid \Delta, \bot]$, since I may invoke the principle of liability. For states of type (1) I will invoke the principle of limited liability if $v - s > r - u$. Similarly, you will invoke the principle of limited liability to ensure that only those states of type (4) have to be considered where $s - v \leq u - r$. But, for appropriate choices of k and m, this leaves us with states that result from the rules for either $\mathsf{L}_m^k x \hat{F}(x)$ or for $\mathsf{G}_m^k x \hat{F}(x)$.

Finally observe that all of my defenses to your attack on $\mathsf{Q}x\hat{F}(x)$ lead to successor states which are reached also by suitable instances of $\mathsf{G}_m^k x \hat{F}(x)$, of $\mathsf{L}_m^k x \hat{F}(x)$, (or $\bot$). Hence my risk for that attack amounts to the minimum of the risk values for these successor states, which in turn equals the risk value for asserting the disjunction of these instances. Similarly, since you can choose between several attacks on $\mathsf{Q}x\hat{F}(x)$ in the first place, my risk for $\mathsf{Q}x\hat{F}(x)$ amounts to the maximum of the risks for these attacks. Hence it is equal to the risk of the conjunction of these disjunctions. $\square$

About half. As an example consider the family of quantifiers H_t^s, defined by the game rule depicted in Figure 3.5. We suggest that H_t^s induces plausible fuzzy models for the natural language quantifier ‘about half’. Figure 3.6 shows the truth functions for three different quantifiers of this family, where the horizontal axis corresponds to $\mathit{Prop}_x\, \hat{F}(x)$ and the vertical axis to $v_M(\mathsf{H}_t^s x \hat{F}(x))$.

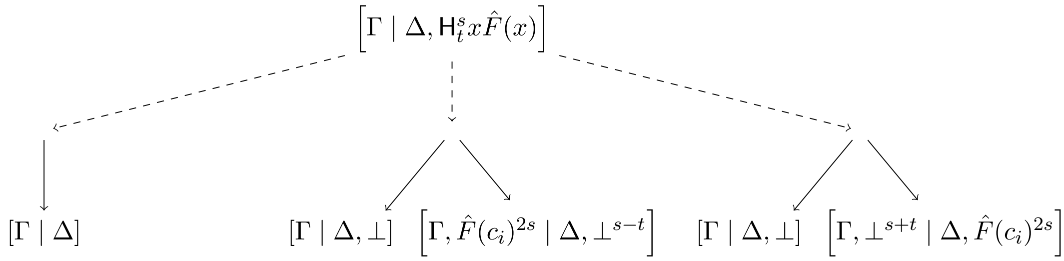
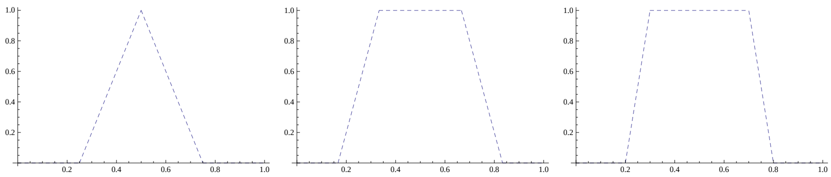


Figure 3.5: The dialogue rule $R_{\mathsf{H}^s_t}$. Dashed lines denote different attacks by you, while solid lines denote possible defense moves by me for each of your attacks.



(a) $\mathsf{H}^2_1 x\hat{F}(x)$ (b) $\mathsf{H}^3_1 x\hat{F}(x)$ (c) $\mathsf{H}^4_2 x\hat{F}(x)$

Figure 3.6: Truth functions for $\mathsf{H}^s_t x\hat{F}(x)$

Similar as for G^s_{2s} above, the two parameters of H^s_t can be interpreted as follows: s determines the sample size (i.e. the number of random instances involved in reducing the quantified formula), while t may be called the *tolerance*, since the smaller t gets, the closer *Prop*$_x$ $\hat{F}(x)$ has to be to $1/2$ if $\mathsf{H}^s_t x\hat{F}(x)$ is to be evaluated as perfectly true. If $t = 0$ (zero tolerance) then $v_M(\mathsf{H}^s_0 x\hat{F}(x)) = 1$ if only if *Prop*$_x$ $\hat{F}(x) = 1/2$ in M. By increasing t (while maintaining the same sample size s) the range of values for *Prop*$_x$ $\hat{F}(x)$ that guarantee $v_M(\mathsf{H}^s_0 x\hat{F}(x)) = 1$ grows symmetrically around $1/2$. On the other hand, maintaining the same value t and increasing s, yields truth functions which are less and less ready to yield ‘almost true’ if the proportion exceeds that tolerance. Hence, t might be interpreted as determining the quantifier’s meaning while s determines its level of precision.

As an instance of Theorem 8 we obtain that $\mathsf{H}^s_t x\hat{F}(x)$ is equivalent to $\mathsf{G}^{s+t}_{s-t} x\hat{F}(x) \wedge \mathsf{L}^{s+t}_{s-t} x\hat{F}(x)$. Your second option to attack at the center of Figure 3.5 corresponds to the rule for G^{s+t}_{s-t} and the one at the right hand side corresponds to the rule for L^{s+t}_{s-t}. The left sub tree corresponds to the fact that the attacker may choose to grant the formula.

Next we show how arbitrary blind choice quantifiers can be reduced to the quantifier Π introduced in Section 3.1 if the connectives of strong Łukasiewicz logic $\mathbf{Ł}_\infty$ are available.

Theorem 9. *The blind choice quantifiers* G^k_m *and* L^k_m *can be expressed in* $\mathbf{Ł}_\infty(\Pi)$ *via the fol-*

lowing reductions:

$$v_M(\mathsf{G}^k_m x\hat{F}(x)) = v_M(\neg((\neg\Pi x\hat{F}(x))^{m+1})\&(\Pi x\hat{F}(x))^{k-1})$$
$$v_M(\mathsf{L}^k_m x\hat{F}(x)) = v_M(\neg((\Pi x\hat{F}(x))^{k+1})\&(\Pi x\neg\hat{F}(x))^{m-1})$$

for all natural numbers m and k and ϕ^n denoting $\phi\&\ldots\&\phi$, n times.

Proof. Note that the truth functions of $\mathsf{G}^k_m x\hat{F}(x)$ and $\mathsf{L}^k_m x\hat{F}(x)$ depend only on *Prop*$_x\,\hat{F}(x)$, while the random choice quantifier Π is directly represented by the truth function *Prop*$_x\,\hat{F}(x)$. Hence the equivalences can easily be checked by computing the truth value of the respective right hand side formula and comparing it to the truth function for the corresponding quantifier. □

Corollary 2. *All blind choice quantifiers can be expressed in **Ł**$_\infty(\Pi)$.*

The corollary follows directly from Theorems 8 and 9. By a less direct route, one could also employ (a constructive proof of) McNaughton's theorem [66] to obtain such reductions.

We finally point out a related fact: any linear function $f(x) = m_1x + m_0$ with integer coefficients m_1 and m_0 capped to the unit interval $[0,1]$ can be expressed via instances of G^k_m and L^k_m and $\bot$. We distinguish the two cases (1) $m_1 \geq 0$ and (2) $m_1 < 0$. Case (1): If $m_0 > 1$ we take the truth function of L^0_0, constantly yielding 1; if $m_0 + m_1 < 1$ we use $\bot$, constantly yielding 0; otherwise we use $\mathsf{G}^{1-m_0}_{m_0+m_1-1}$. Case (2): If $m_0 < 1$ we use $\bot$; if $m_0+m_1 > 1$ we use L^0_0; otherwise we use $\mathsf{L}^{m_0-1}_{1-m_0-m_1}$. This can readily be checked by inserting the respective values for k and m into Formulas 3.1 and 3.2.

3.3 Deliberate Choice Quantifiers

In the previous section we surveyed the family of blind choice quantifiers and concluded that these quantifiers all amount to piecewise linear truth functions. A much more general class of quantifiers arises by dropping condition *(ii)* of Definition 13, i.e., the randomly chosen constants are immediately visible to the players. As an example of this class we investigate the family of so-called *deliberate choice quantifiers*, specified by the following schematic game rule, where $\hat{F}$ is a classical formula:

($R_{\Pi^k_m}$) If I assert $\Pi^k_m x\hat{F}(x)$ then, if you attack, $k+m$ constants are chosen randomly and I have to pick k of those constants, say $c_1,\ldots,c_k$, and bet for $\hat{F}(c_1),\ldots,\hat{F}(c_k)$, while simultaneously betting against $\hat{F}(c'_1),\ldots,\hat{F}(c'_m)$, where $c'_1,\ldots,c'_m$ are the remaining m random constants. (Analogously for your assertion of $\Pi^k_m x\hat{F}(x)$.)

Recall that both forms of the principle of limited liability remains in place: after the constants are chosen, I may invoke LLD and assert $\bot$ (i.e., agree to pay 1€) instead of betting as indicated above. Therefore I have $1 + \binom{k+m}{k}$ possible defenses to your attack on my assertion of $\Pi^k_m x\hat{F}(x)$: either I choose to hedge my loss by asserting $\bot$ or I pick k out of the $k+m$ random constants to proceed as indicated.

We claim that this rule matches the extension of **Ł**$_\infty$ to **Ł**$_\infty(\Pi^k_m)$ by

$$v_M(\Pi^k_m \hat{F}(x)) = \binom{k+m}{k}(\mathit{Prop}_x\, \hat{F}(x))^k(1 - \mathit{Prop}_x\, \hat{F}(x))^m.$$

Theorem 10. *A* **Ł**$_\infty(\Pi^k_m)$*-sentence F is evaluated to $v_M(F) = x$ in interpretation M iff every $\mathcal{G}$-game for F augmented by rule $(R_{\Pi^k_m})$ is $(1-x)$-valued for me under risk value assignment $\langle\cdot\rangle_M$.*

Proof. Analogously as in the proof of Theorem 7, we only have to consider states of the form $\left[\Gamma \mid \Delta, \Pi^k_m x\hat{F}(x)\right]$. Again, we can separate the risk for the exhibited assertion from the risk for the remaining assertions:

$$\left\langle \Gamma \mid \Delta, \Pi^k_m x\hat{F}(x)\right\rangle = \langle \Gamma \mid \Delta\rangle + \left\langle \mid \Pi^k_m x\hat{F}(x)\right\rangle.$$

It remains to show that my optimal way to reduce the exhibited quantified formula to instances as required by rule $(R_{\Pi^k_m})$ results in a risk that corresponds to the specified truth function. Remember that the principle of limited liability is in place and that $\hat{F}(x)$ is classical. This means that I either finally have to pay 1€ for my assertion of $\Pi^k_m x\hat{F}(x)$ (by LLD) or do not have to pay anything at all. The latter is only the case if all my bets for $\hat{F}(c_1), \ldots, \hat{F}(c_k)$, as well as all my bets against $\hat{F}(c'_1), \ldots, \hat{F}(c'_m)$, for $c_1, \ldots, c_k$, $c'_1, \ldots, c'_m$ as specified in rule $(R_{\Pi^k_m})$, succeed. Let the random variable K denote the number of chosen elements c on which my bet is successful; i.e., where $\left\langle \hat{F}(c)\right\rangle^r = 0$. Then K is binomially distributed and the probability that this event obtains (the inverse of my associated risk) is readily calculated to be

$$\binom{k+m}{k} \mathit{Prop}_x\, \hat{F}(x)^k(1 - \mathit{Prop}_x\, \hat{F}(x))^m.$$

This matches the truth function as given above. □

At a first glance, the deliberate choice quantifier Π^k_m might seem suitable for modeling the natural language quantifier *about k out of $m+k$*. However, a look at the corresponding graph for $\left\langle \Pi^1_1 x\hat{F}(x)\right\rangle$ reveals that the risk for asserting $\Pi^1_1 x\hat{F}(x)$ is always larger than 0.5. In other words the statement is at most true to the degree 0.5, which is clearly not desirable for a semi-fuzzy quantifier.

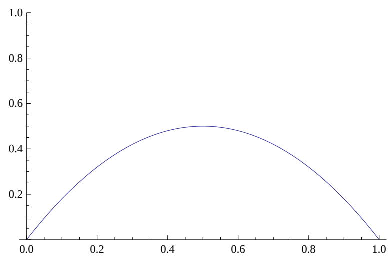


Figure 3.7: Truth value for $\Pi^1_1 x\hat{F}(x)$ (depending on p)

An additional mechanism is needed to obtain more appropriate models of natural quantifier expressions like *about half*. While there are many ways to achieve the desired effect, we confine ourselves here to a particularly simple operator that nicely fits our semantic framework, since it arises by simply multiplying involved bets. Given a number $n \geq 2$ and a semi-fuzzy quantifier Q we specify the quantifier $\mathsf{W}_n(\mathsf{Q})$ by the following rule.

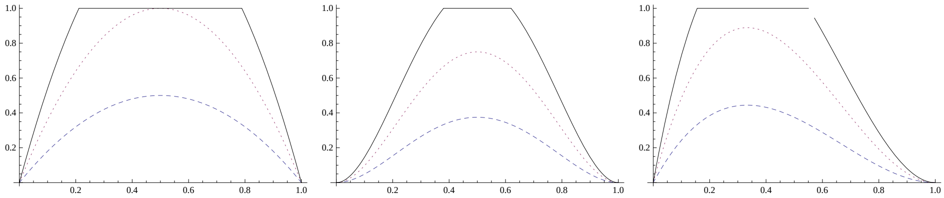
($\mathsf{W}_n(\mathsf{Q})x\hat{F}(x)$) If I assert $\mathsf{W}_n(\mathsf{Q})x\hat{F}(x)$ then you have to place n bets *against* $\mathsf{Q}x\hat{F}(x)$ while I have to bet *for* $\mathsf{Q}x\hat{F}(x)$ just once. (Analogously for your assertion of $\mathsf{W}_n(\mathsf{Q})x\hat{F}(x)$.)

Note that W_n is acting here as a *quantifier modifier*; for any semi-fuzzy quantifier Q, $\mathsf{W}_n(\mathsf{Q})$ still denotes a semi-fuzzy quantifier. The principle of limited liability remains in place, hence the game state $\left\langle \Gamma \mid \Delta, \mathsf{W}_n(\mathsf{Q})x\hat{F}(x) \right\rangle$ is reduced to $\left\langle \Gamma, \bot^n \mid \Delta, \mathsf{Q}x\hat{F}(x)^{n+1} \right\rangle$, or to $\langle \Gamma \mid \Delta \rangle$, depending on whether it is is preferable from the attacker's point of view to attack or to grant the assertion of $\mathsf{Q}x\hat{F}(x)^{n+1}$. (The defender never has to invoke the principal of limited liability in optimal strategies.) Moreover, similarly as in Theorem 9, W_n can be expressed using negation and strong conjunction by

$$v_M(\mathsf{W}_n(\mathsf{Q})x\hat{F}(x)) = v_M(\neg(\neg \mathsf{Q}x\hat{F}(x))^{n+1}).$$

The truth functions for some quantifiers of type $\mathsf{W}_n(\Pi^k_m)$ are presented in the following figure:
The quantifier $\mathsf{W}_3(\Pi^2_2)$ may be considered as formal fuzzy counterpart of the informal expression *about half*. Likewise, $\mathsf{W}_3(\Pi^1_2)$ may be understood as model of *about a third*. Moreover, $\mathsf{W}_3(\Pi^1_1)$ might serve as a model of *very roughly half*, whereas $\mathsf{W}_2(\Pi^1_1)$ might be appropriate as fuzzy model of the (unhedged) determiner *half*.

Of course, one might claim that combining suitable instances of the W_n operator with suitable instances of Π^k_m comes in an ad-hoc manner. Nevertheless the resulting quantifiers still only depend on three discrete parameters and also provide a characterization in terms of Giles's



(a) $\mathsf{W}_i(\Pi^1_1)x\hat{F}(x)$ (b) $\mathsf{W}_i(\Pi^2_2)x\hat{F}(x)$ (c) $\mathsf{W}_i(\Pi^1_2)x\hat{F}(x)$

Figure 3.8: W_i-modified proportional quantifiers — the graphs correspond to the cases $i = 1$, $i = 2$, and $i = 3$ from bottom to top in each diagram.

game. Thus, unlike many other semi-fuzzy quantifiers proposed in the literature, they easily integrate with full Łukasiewicz logic **Ł**$_\infty$.

In a similar manner, deliberate choice quantifiers can be used to generate plausible candidate models for the proportional reading of *many*. In particular, consider a model where asserting (the formal counterpart of) 'Many [domain elements] are $\hat{F}$'. If someone willingly asserts such a statement, this means that he is ready to place a certain number of bets for random instances of $\hat{F}(x)$. This clearly amounts to considering the family of quantifiers Π^i_0. The more bets the player is ready to place on $\hat{F}(x)$, the stronger is his accepted interpretation of 'many'. The corresponding truth functions are depicted in Figure 3.9.

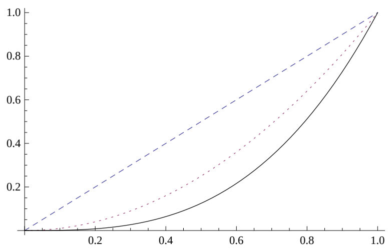


Figure 3.9: Truth functions for $\Pi^i_0 x\hat{F}(x)$ (depending on p) for $i = 1, 2, 3$ from top to bottom.

Like for 'about half' etc, above, one may want to evaluate 'Many [domain elements] are $\hat{F}$' as perfectly true (truth value 1) even if *Prop*$_x$ $\hat{F}(x)$ is somewhat smaller than 1. Again, this can be achieved by employing the W_n-operator, which requires the attacker to place several bets against the contended assertion.

CHAPTER 4

Relating Barker's Dynamics of Vagueness to Fuzzy Logics

As motivated in Section 1.4 we take on Chris Barker's dynamic approach to vagueness [5] as an example of a scale-based linguistic account of vagueness. Section 4.1 explores Barker's approach in greater depth. By focusing on linearly ordered arithmetic scales—instead of arbitrary partially ordered ones as considered by Barker—we are able to simplify Barker's formalism. Also, this simplification allows us to resolve some problematic issues with Barker's presentation, particularly his definitions of so-called *predicate modifiers*.

Following our presentation in [24] and [21], we show in Section 4.2 how t-norm based fuzzy logics can be recovered within this approach by measuring the size of contexts. We introduce logical operators on contexts and observe that the three fundamental t-norms as defined in Section 1.5 emerge as marginal cases. Finally, restricting our attention further to so-called *saturated contexts* in Section 4.3 yields a model where the size of the context after applying such a logical operator can be directly computed: we observe that the Gödel t-norm and co-t-norm arise as truth functions for conjunction and disjunction, respectively.

4.1 The 'Dynamics of Vagueness' revisited

Linguists commonly agree that vagueness strongly exhibits context dependence. Partly, the notion of context can be modeled statically, e.g., to determine an intended comparison class for the use of a vague predicate: The predicate 'tall' may refer to different standards of tallness when talking about people or about trees. But, more than that, Kamp [47], Klein [52], and particularly

Lewis [61] observe that uttering a vague statement itself affects the subsequent context following that statement. Barker's approach, which he called the 'Dynamics of Vagueness' models these context changes by means of so-called *dynamic semantics* [36,39]. This tool has been successfully employed in formal semantics to handle, e.g., anaphoric relations or the projection of presuppositions. Typically, the *meaning* of a statement is identified with its *context change potential*—a function operating on contexts. Depending on the application of dynamic semantics, contexts may be modeled differently. For example, when trying to determine anaphoric relations, a context usually consists of possible (valid) assignments of variables to objects, whereas for vagueness the context typically keeps record of current standards of acceptance and of current knowledge about relevant objects.

Barker models contexts as sets of possible worlds, where worlds provide complete precisifications of the vague predicates in question. For all vague predicates under consideration a world provides a threshold value and for all individuals (objects) under consideration a world determines to which degree the individual has this property. Thus in each world all predicates are precise, i.e., all (relevant) atomic propositions are either true or false: If, at a world, the threshold value of some graded property is below the actual degree to which an individual has this property, the property applies, and otherwise it does not. Note that Barker's approach is usually subsumed under 'degree-based' approaches to vagueness, see, e.g., [89]. Here we rather use the term 'scale based' to avoid confusion with fuzzy logics: 'degrees' here are degrees of applicability of vague predicates to individuals, but not degrees of truth. This view is common to most linguistic approaches to vague (gradable) predicates. While predicates may apply to individuals *to some degree*, a statement itself is either true or false in a given context.

Strictly speaking, Barker models the interaction of two distinct phenomena with this approach: *(i)* Different local threshold values for a vague predicate cause uncertainty due to the vagueness of that predicate while *(ii)* different local degrees of applicability of these predicates to individuals reflect uncertainty due to epistemic ignorance. Barker therefore distinguishes between *(i)* the descriptive and *(ii)* the meta-linguistic use of language. The statement 'Bill is tall' may be meant to communicate information about Bill's height to someone who does not know him, but may also be meant to establish a standard of tallness by making this utterance. Following his example, Bill might be standing right next to the speaker and hearer, and someone asks "What are people like where Bill comes from?". If the answer is "In that country Bill is tall" the speaker wants to convey the standard of tallness in 'that country'. Note that both speaker and hearer have (nearly) perfect knowledge of Bill's height anyway. Barker notes that, in general both descriptive and meta-linguistic use of language are intertwined and therefore cannot be treated separately.

More formally, we consider a context C and a possible world $w \in C$ in this context. Following Barker we will take ‘tall’ as a prototypical vague predicate for the remainder of this section and denote by $[\![tall]\!]$ the *meaning* of tall. A *delineation* function $\delta(w)(\alpha)$ maps atomic vague predicates to threshold values. These values represent local standards of acceptance. For example $d = \delta(w)({}^{\uparrow}tall)$ yields the standard of absolute tallness in the world w, say, in cm; i.e., every individual which is at least d cm tall in w will be accepted as ‘tall’ in w. Note that Barker uses ‘$[\![tall]\!]$’ instead of ‘${}^{\uparrow}tall$’ here, which amounts to a circularity: $[\![tall]\!]$ is used inside of the definition of $[\![tall]\!]$. However, in order to indicate a purely referential use we introduce the notation ${}^{\uparrow}tall$.

At this point we slightly depart from Barker’s presentation. He assumes that degrees for gradable predicates refer to some partially ordered scale and notes that abstract predicates like ‘stupid’ may require a non-linearly ordered scale. However, we choose to follow Kennedy [51] and Klein [53] and assume that degrees refer to a *linearly ordered* arithmetic scale. Such scales are arguably sufficient for predicates like ‘tall’. Barker formally models the meaning of predicates like $[\![tall]\!]$ and predicate modifiers like $[\![very]\!]$, $[\![definitely]\!]$, and $[\![clearly]\!]$. He claims that these modifiers can be iterated to obtain the meaning of $[\![very\ very\ tall]\!]$ or $[\![definitely\ tall]\!]$, but as noted below, it is not obvious how his formalism can accomplish this. Restricting to linear scales, where operations such as addition and subtraction are defined, allows us to simplify Barker’s definitions and also enables us to resolve these technical problems.

We assume that for each vague predicate under consideration, there exists a function like tall, where $\mathsf{tall}(w)(x)$ denotes the degree to which the individual x is tall in the world w. Note that this definition differs from Barker’s: he defines $\mathsf{tall}(d, x)$ to return the set of worlds where x is tall to at least d. Such a definition is necessary for a non-total scale if an individual can be stupid to some degree d_1 and also to some degree d_2, but neither $d_1 \leq d_2$ nor $d_2 \leq d_1$. However, since we restricted ourselves to linear scales, our definition (which is also used by Kennedy [51]) will be useful in modeling the meaning of predicates and predicate modifiers. We moreover stipulate that the vague predicates under consideration as well as the relevant individuals are the same in each world in a context C.

Consider the statement $\phi =$ ‘Bill is tall’. The meaning $[\![\phi]\!]$ of this statement is a function on contexts transforming the context *before* uttering ϕ to a new context which arises from taking into account the information conveyed by the utterance. In fact, only a simple form of such update functions is needed, namely *filters* where $[\![\phi]\!](C) \subseteq C$ for all contexts C—the result $[\![\phi]\!](C)$ being the set of worlds in C that survive the update of C with the assertion that ϕ. (A *filter* here is just a function f which maps any set to one of its subsets, i.e. $f : \mathcal{P}(S) \to \mathcal{P}(S)$ with $f(X) \subseteq X$ for all $X \in \mathcal{P}(S)$). Observe that the notion of ‘truth at a world’ can straightforwardly be defined in terms of such an update function: ϕ is true (accepted) at c if $c \in [\![\phi]\!](C)$ and ϕ is false (rejected) at c if $c \notin [\![\phi]\!](C)$.

Using the delineation function δ as defined above, we define the dynamic meaning of the atomic predicate 'tall' as follows:

$$[\![tall]\!] \quad =_{df} \quad \lambda x \lambda C.\{w \in C : \delta(w)({}^{\uparrow}tall) \leq \mathsf{tall}(w)(x)\}.$$

This means, $[\![tall]\!]$ is a function which takes as its first argument an individual and returns a function on contexts. Hence, the meaning of 'Bill is tall' amounts to $[\![\textit{Bill is tall}]\!] = [\![tall]\!](\mathsf{b})$ where the constant b refers to Bill. Barker also defines a context-dependent assignment function $g(w)$ analogously to $\delta(w)$, which keeps track of the individuals being referred to at a world. However, since this is irrelevant for our aims here, i.e., modeling vagueness, we tacitly assume that b always refers to Bill.

Among other features, this semantic setup allows Barker to capture the intuitive difference of the meaning of the adjectives 'very', 'definitely', and 'clearly'. They are implemented as *predicate modifiers*, i.e., the first argument of $[\![very]\!]$, $[\![definitely]\!]$, or $[\![clearly]\!]$ is the predicate (e.g. $[\![tall]\!]$) that is to be modified. Barker's presentation of $[\![very]\!]$, and also of other predicate modifiers, however, has two subtle problems regarding the type of $\delta(w)({}^{\uparrow}very)$ and regarding the composition of predicate modifiers to obtain predicates like 'very very tall' or 'definitely very tall'. In the following we show how to enhance his definitions in order to circumvent these issues while maintaining his original intentions regarding the semantics of 'very', 'definitely', and 'clearly'.

The predicate modifier $[\![very]\!]$, according to Barker, checks whether an individual has a property *by some margin*. This margin, in turn, is vague, i.e., world dependent. It is denoted by $\delta(w)({}^{\uparrow}very)$. Thus the vagueness of $[\![\textit{very tall}]\!]$ is modeled by a twofold context dependence: (1) the standard for tallness may obviously vary from world to world, but (2) even for a fixed threshold value of tallness, there may be different different margins which must be exceeded in order to count as 'very tall'. Therefore, $[\![\textit{Bill is very tall}]\!] = ([\![very]\!]([\![tall]\!]))(\mathsf{b})$ is an update that filters out exactly those worlds w of a given context where Bill does not exceed the local standard of tallness $\delta(w)({}^{\uparrow}tall)$ by the local margin $\delta(w)({}^{\uparrow}very)$.

An interesting feature of Barker's representation of $[\![very]\!]$ is that the value $\delta(w)({}^{\uparrow}very)$ does not depend on the vague predicate in question. This implies that by stating 'Bill is very tall' one communicates also the intended use of the word 'very', thus possibly affecting how the sentence 'Bill is very stupid' will be evaluated in a subsequent context. However, this assumes that 'tall', 'stupid', and 'very' all refer to the same scale. Even on the same scale it is hard to see why the (absolute) margin involved in uttering 'very huge' should be the same as for uttering 'very tiny'. Therefore, we will stipulate that this margin for each world may differ for different predicates, and therefore is denoted by $\delta(w)({}^{\uparrow}very, {}^{\uparrow}tall)$. Note that $\delta(w)$ is polymorphic: for simple predicates such as 'tall' it has only one argument. However, if the first argument is a reference to a modifier like $[\![\textit{very}]\!]$ (or $[\![\textit{clearly}]\!]$ as seen below), then a reference to a predicate is expected as second argument. Moreover, if the second argument is itself already modified by a predicate modifier, we simply discard that modifier. Technically this is accomplished by setting $\delta(w)(^{\uparrow}\textit{very}, ^{\uparrow}\alpha(\textit{tall})) = \delta(w)(^{\uparrow}\textit{very}, ^{\uparrow}\textit{tall})$.

Secondly, in Barker's original setup it is not possible to iterate predicate modifiers, although this is claimed and intended by Barker. For example 'very very tall' cannot be represented as $[\![\textit{very}]\!]([\![\textit{very}]\!]([\![\textit{tall}]\!]))$. The reason for this is that Barker's definition of $[\![\textit{very}]\!]$ uses the local threshold value at the world w of the modified predicate α and adjusts it to a new degree d (as expressed by his notation $w[d/^{\uparrow}\alpha]$). This does not work if α itself is a composite predicate such as 'very tall' in our example, because there simply is no local threshold value for 'very tall' registered by the delineation δ. (The situation gets even more involved when turning to other complex predicates such as 'very clearly tall'.) Instead, for each world we only have both threshold values $\delta(w)(^{\uparrow}\textit{very}, ^{\uparrow}\textit{tall})$ and $\delta(w)(^{\uparrow}\textit{tall})$ at our disposal when defining the meaning of $[\![\textit{very very tall}]\!]$. We solve this problem by introducing the function $\Delta(w, C)(\alpha, x)$ denoting the difference between the threshold value for α and the actual degree to which α applies to the individual x in the world w and the context C. For a simple (atomic) predicate such as 'tall' we have

$$\Delta(w, C)(^{\uparrow}\textit{tall}, x) =_{\textit{df}} \mathsf{tall}(w)(x) - \delta(w)(^{\uparrow}\textit{tall}).$$

Observe that the argument C is not used when calculating the difference Δ for atomic predicates. However, for other more complex predicates like 'clearly tall' (see below) Δ is dependent on whole the context instead only on w. Based on this function we can define the predicate modifier $[\![\textit{very}]\!]$ as

$$[\![\textit{very}]\!] =_{\textit{df}} \lambda\alpha\lambda x\lambda C\ \{w \in C : \Delta(w, C)(^{\uparrow}\alpha, x) \geq \delta(w)(^{\uparrow}\textit{very}, ^{\uparrow}\alpha)\}.$$

Thus, Bill is very tall at a world w, if the margin $\Delta(w)(^{\uparrow}\alpha, \mathsf{b})$ by which he is tall at w exceeds the local standard of 'very tallness' $\delta(w)(^{\uparrow}\textit{very}, ^{\uparrow}\alpha)$. (Since this standard is positive, only such worlds come under consideration where Bill is tall initially). By defining the behavior of Δ on predicates modified by 'very' as follows, $[\![\textit{very}]\!]$ becomes fully iterable:

$$\Delta(w, C)(^{\uparrow}\textit{very}(\alpha), x) =_{\textit{df}} \Delta(w)(^{\uparrow}\alpha, x) - \delta(w)(^{\uparrow}\textit{very}, ^{\uparrow}\alpha).$$

This implies that now the *meaning* of a predicate α is not only determined by its associated filter $[\![\alpha]\!]$ but also by the behaviour of Δ on α: If we modify the predicate 'very tall' by some other modifier like, e.g., 'clearly very tall' (see below), then also $\Delta(w, C)(^{\uparrow}\textit{very tall}, x)$ is relevant in computing $[\![\textit{clearly}]\!]([\![\textit{very tall}]\!])$, it thus surely contributes to its meaning. In fact, we can even express $[\![\textit{tall}]\!]$ and $[\![\textit{very tall}]\!]$ by their respective Δ-functions: $[\![\textit{tall}]\!] = \lambda x \lambda C\ \{w \in C : \Delta(w, C)(^{\uparrow}\textit{tall}, x) \geq 0\}$ and likewise for $[\![\textit{very tall}]\!]$.

Consider the statement ‘Bill is very very tall’. The associated filter on contexts is computed as follows:

$$\begin{aligned}
&([\![\textit{very}]\!]([\![\textit{very}]\!]([\![\textit{tall}]\!])))(\mathsf{b}) = \\
&\lambda C\ \{w \in C : \Delta(w, C)(^{\uparrow}\textit{very(tall)}, \mathsf{b}) \geq \delta(w)(^{\uparrow}\textit{very}, [\![\textit{very}]\!]([\![\textit{tall}]\!]))\} = \\
&\lambda C\ \{w \in C : \Delta(w, C)(^{\uparrow}\textit{tall}, \mathsf{b}) - \delta(w)(^{\uparrow}\textit{very}, [\![\textit{tall}]\!]) \geq \delta(w)(^{\uparrow}\textit{very}, [\![\textit{tall}]\!])\} = \\
&\lambda C\ \{w \in C : \mathsf{tall}(w)(\mathsf{b}) - \delta(w)(^{\uparrow}\textit{tall}) - \delta(w)(^{\uparrow}\textit{very}, [\![\textit{tall}]\!]) \geq \delta(w)(^{\uparrow}\textit{very}, [\![\textit{tall}]\!])\} = \\
&\lambda C\ \{w \in C : \mathsf{tall}(w)(\mathsf{b}) - \delta(w)(^{\uparrow}\textit{tall}) \geq 2 * \delta(w)(^{\uparrow}\textit{very}, [\![\textit{tall}]\!])\}.
\end{aligned}$$

Thus, Bill is considered very very tall in all worlds where he exceeds the standard for tallness by twice the margin for being counted as ‘very tall’.

As seen above, for each individual world w, the update function for $[\![\textit{very}]\!]$ refers only to information present in w. In contrast, Barker suggests to model ‘definitely’ as a kind of modal operator, that takes into account also global information:

$$[\![\textit{definitely}]\!] =_{\textit{df}} \lambda \alpha \lambda x \lambda C\ \{w \in \alpha(x)(C) : \forall d(w[d/^{\uparrow}\alpha] \in C \rightarrow w[d/^{\uparrow}\alpha] \in \alpha(x)(C))\}.$$

This means that a world $w \in C$ survives the update with $[\![\textit{Bill is definitely tall}]\!]$ if and only if all worlds in C in which Bill has the same height as in w judge Bill as tall according to their local standard. Note that the hearer may be uncertain about Bill’s actual height. This uncertainty is reflected in the model if Bill has different heights (degrees of tallness) in different worlds of the context. Consequently, in general, ‘definitely tall’ is not just equivalent to ‘tall’ in all worlds of the context’. However, if there is no uncertainty about Bill’s height, i.e., if Bill has the same height in all worlds, then $[\![\textit{definitely tall}]\!]$ does not filter out any world ($[\![\textit{definitely tall}]\!](\mathsf{b})(C) = C$) in case Bill’s height is above all the local standard for tallness and filters out all worlds ($[\![\textit{definitely tall}]\!](\mathsf{b})(C) = \emptyset$) otherwise.

Again this definition, as given by Barker, poses an obstacle when iterating predicate modifiers such as in ‘definitely very tall’: the use of $w[d/^{\uparrow}\alpha]$ does not (yet) scale up to composite predicates. However defining $w[d/^{\uparrow}\alpha(\textit{tall})] =_{\textit{df}} w[d/^{\uparrow}\textit{tall}]$ for composite predicates yields a robust notion of substitution in a world, i.e., we discard all predicate modifiers and only change the threshold value of the underlying atomic predicate in the respective world. Evaluated at a particular world $w \in C$ the sentence ‘Bill is definitely very tall’ can then be understood as intended, namely as *Bill is very tall* in all worlds in C in which he has the same height as in w. Note that there is no direct analogon to $\Delta(w)(^{\uparrow}\textit{very}, x)$ for defining $\Delta(w, C)(^{\uparrow}\textit{definitely}, x)$, since, unlike for ‘very’, there is no world dependent margin $\delta(w)(^{\uparrow}\textit{definitely}, ^{\uparrow}\alpha)$ for ‘definitely’. This perfectly matches the intuition that applying the modifier 'very' to the predicate 'definitely tall' arguably sounds rather odd, while applying 'definitely' to 'very tall' seems quite appropriate. The presented model captures this intuition by insisting that 'definitely', in contrast to 'very', is not gradable. Nevertheless both, 'very' and 'definitely', are understood as vague adjectives, in the sense of being systematically context dependent. Observe that the definition of $[\![\textit{definitely}]\!]$ does not directly refer to the function Δ. Therefore, 'definitely' can be iterated to obtain predicates like $[\![\textit{definitely}]\!]([\![\textit{definitely}]\!]([\![\textit{tall}]\!]))$. However, note that this predicate coincides with $[\![\textit{definitely}]\!]([\![\textit{tall}]\!])$.

Barker further introduces $[\![\textit{clearly}]\!]$ as a gradable alternative to $[\![\textit{definitely}]\!]$. In fact, the following presentation of the meaning of 'clearly' combines essential elements of $[\![\textit{very}]\!]$ as well as of $[\![\textit{definitely}]\!]$. Note that this version is significantly simpler than in Barker's original presentation due to our focus on linear arithmetic scales:

$$[\![\textit{clearly}]\!] =_{\textit{df}} \lambda\alpha\lambda x\lambda(C)\ \{w \in C : \forall d(w[d/{}^{\uparrow}\alpha] \in C \\ \rightarrow \Delta(w[d/{}^{\uparrow}\alpha], C)(\alpha, x) \geq \delta(w)({}^{\uparrow}\textit{clearly}, {}^{\uparrow}\alpha))\}.$$

The reference ${}^{\uparrow}\textit{clearly}$ as an argument of $\delta(w)$ indicates that, like 'very', 'clearly' itself is vague (and gradable): $\delta(w)({}^{\uparrow}\textit{clearly}, {}^{\uparrow}\alpha)$ returns a world dependent margin for α analogously to $\delta(w)({}^{\uparrow}\textit{very}, {}^{\uparrow}\alpha)$. However there is an essential difference between $[\![\textit{very}]\!]$ and $[\![\textit{clearly}]\!]$: while for 'very tall' one compares the local standard of tallness with the local value for an individual x's height in each world w, 'clearly tall' checks whether for all worlds where x has the same height as in w the individual x is 'tall' even by the margin $\delta(w)({}^{\uparrow}\textit{clearly})$. This comparison of all worlds in the context that share the same height is analogous to the definition of $[\![\textit{definitely}]\!]$. Therefore, in order to be clearly tall, an individual must be definitely tall by some world dependent margin.

In contrast to 'definitely', composing 'clearly' and 'very' to, e.g., 'clearly very tall' but also to 'very clearly tall' intuitively seems just fine. Defining $\Delta(w, C)({}^{\uparrow}\textit{clearly}(\alpha), x)$ accordingly as below enables iterating 'clearly'. Calculating the margin Δ to which an individual clearly is 'tall' takes into account all worlds in the context C where the individual has the same height as in the current world:

$$\Delta(w, C)({}^{\uparrow}\textit{clearly}(\alpha), x) =_{\textit{df}} \min_{\{d: w[d/{}^{\uparrow}\alpha] \in C\}} \{\Delta(w[d/{}^{\uparrow}\alpha]\})(\alpha, x) - \delta(w)({}^{\uparrow}\textit{clearly}, {}^{\uparrow}\alpha)\}.$$

Figure 4.1 shows an example context illustrating different threshold values for $[\![\textit{very}]\!]$ and $[\![\textit{clearly}]\!]$. Observe that Bill is tall, whenever $\mathsf{tall}(w)(\mathsf{b}) \geq \delta(w)({}^{\uparrow}\textit{tall})$ Bill is definitely tall in worlds w_4 to w_7 since these are all worlds where he measures $190cm$ and he is tall in all of them. Finally, Bill is clearly tall in w_6 and w_7 since in both cases he is tall by at least $\delta(w)(^{\uparrow}clearly, {}^{\uparrow}tall)$ centimeters.

w	$\mathsf{tall}(w, \mathsf{b})$	$\delta(w)(^{\uparrow}tall)$	$\delta(w)(^{\uparrow}very, {}^{\uparrow}tall)$	$\delta(w)(^{\uparrow}clearly, {}^{\uparrow}tall)$	ϕ_{tall}	ϕ_{very}	$\phi_{definitely}$	$\phi_{clearly}$
w_1	180	185	5	10				
w_2	185	190	10	5				
w_3	185	180	5	5	✓	✓		
w_4	190	185	10	10	✓		✓	
w_5	190	185	5	10	✓	✓	✓	
w_6	190	185	5	5	✓	✓	✓	✓
w_7	190	185	10	5	✓		✓	✓

Table 4.1: Example of a context C with b denoting 'Bill', evaluating the sentences 'Bill is tall' (ϕ_{tall}), 'Bill is very tall' (ϕ_{very}), 'Bill is definitely tall' ($\phi_{definitely}$), and 'Bill is clearly tall' ($\phi_{clearly}$).

4.2 Measuring Contexts

In the following we demonstrate how Barker's dynamic, context based approach to the meaning of gradable predicates like 'tall' can be connected to fuzzy logic. The key idea is to identify such predicates in a given context with fuzzy sets. Consider for example the statement 'Bill is tall'. By measuring the relative size of the context C *before* and *after* processing this statement, i.e., $[\![tall]\!](\mathsf{b})(C)$, we compute the membership degree of b, denoting Bill, in the respective fuzzy set.

Remember that update functions associated with a statement always are filters, i.e., only worlds of the current context are filtered out, but no new worlds are introduced into the context. Correspondingly, we introduce the notion of an *element filter*. These are filters parameterized by an element of the universe. Element filters that we have already encountered in the previous section are, e.g., $[\![tall]\!]$ but also $[\![very]\!]([\![tall]\!])$, where for a given element a both $[\![tall]\!](\mathsf{a})$ and $([\![very]\!]([\![tall]\!]))(\mathsf{a})$ are filters on contexts. As already implicitly assumed above (following Barker), we stipulate that the relevant element is in the universe of the context to which the filter is applied—otherwise the result simply remains undefined.

Given a context C we extract a fuzzy set from the meaning $\alpha = [\![P]\!]$ of a predicate P by applying for each element a of the universe the filter $\alpha(\mathsf{a})$ to C and measuring the amount of worlds surviving this update. In the following we consider only finite sets of worlds as contexts and moreover stipulate (as above) that all considered contexts share the same universe U. As argued above, adjectives like 'tall' or 'heavy' at the first glance refer to continuous scales. Here we however argue that the scales of *perceived* heights or weights are discrete by imposing some level of granularity that is due to our perception and to cognitive limitations. This allows one to straightforwardly determine the membership degree of an individual a in the fuzzy set $[\alpha]_C$ by counting the worlds in C before and after applying the filter α. Of course, this approach can be generalized to infinite contexts by imposing suitable probability measures on possible worlds.

We identify fuzzy sets with their membership functions and obtain:

Definition 14. Let C be a context over a universe U and α an element filter. Then the fuzzy set $[\alpha]_C$ is given by

$$[\alpha]_C : U \to [0,1] : \quad x \mapsto \frac{|\alpha(x)(C)|}{|C|}$$

For simplicity we treat only unary predicates here, for n-ary predicates a fuzzy set of n-tuples of domain elements can be defined analogously. The only differences are that a corresponding element filter is a filter parameterized by an n-tuple, thus the domain of the corresponding fuzzy set is D_C^n.

Note that the collection of fuzzy sets $[\alpha]_C$ for all relevant element filters α carries less information than C itself: different contexts of different size (and, of course, referring to different threshold values) may well yield the same fuzzy set for a given predicate. Also within a fixed context, different predicates may yield the same fuzzy set. It is therefore not possible to 'revert' this mapping, i.e., to reconstruct the context from a given fuzzy set.

Barker does not mention logical connectives in his paper 'The Dynamics of Vagueness' [5] and, indeed, logical connectives typically are not in the scope of linguistic treatments of vagueness. We therefore introduce the logical operators *and*, *or*, and *not* not on the propositional level, but rather to act on predicates in a straightforward manner and explore how they relate to the corresponding operations on the extracted fuzzy sets. Again we focus on monadic predicates, but the concepts can straight-forwardly be extended to relations of higher arity.

Extending the framework of Barker, we model compound predicates (like 'tall and clever'), built up from logically simpler predicates ('tall', 'clever'), as follows.

Definition 15.

- $[\![\mathit{and}]\!] =_{\mathit{df}} \lambda\alpha\lambda\beta\lambda x\lambda C\ \alpha(x)(C) \cap \beta(x)(C)$
- $[\![\mathit{or}]\!] =_{\mathit{df}} \lambda\alpha\lambda\beta\lambda x\lambda C\ \alpha(x)(C) \cup \beta(x)(C)$
- $[\![\mathit{not}]\!] =_{\mathit{df}} \lambda\alpha\lambda x\lambda C\ C\backslash(\alpha(x)(C))$

Note that in the above definition $\alpha = [\![A]\!]$ and $\beta = [\![B]\!]$ are element filters representing the meaning of the predicates A and B, respectively. The compound predicate 'A and B' then is analyzed as $[\![A \textit{ and } B]\!] = ([\![\mathit{and}]\!](A))(B)$ which again is an element filter.

In general, applying $[\![A \textit{ and } B]\!]$ is not equivalent to applying the element filters $[\![A]\!]$ and $[\![B]\!]$ consecutively. We may additionally define

- $[\![\textit{and}^{>}]\!] =_{df} \lambda\alpha\lambda\beta\lambda x\lambda C.\beta(x)(\alpha(x)(C))$

Then $[\![A\,\textit{and}^{>}\,B]\!]$ is not only different from $[\![A\,\textit{and}\,B]\!]$, but also from $[\![B\,\textit{and}^{>}\,A]\!]$, thus commutativity is lost. This form of conjunction might be argued to correspond to the expressions 'and moreover' or 'and even'. Note that in this model the non-commutativity of $[\![A\,\textit{and}^{>}\,B]\!]$ arises only if one of the vague predicates 'A' and 'B' is built up using modalities like 'definitely' or 'clearly'. Otherwise, if no modalities are involved, all worlds are tested individually and independently of the context in which they are appearing. Consequently exactly those worlds survive the update where both 'A' and 'B' hold. Thus both forms of conjunction coincide. However, consider the predicates 'tall and definitely tall' and 'tall *and*$^{>}$definitely tall'. A sentence like 'Bill is tall and even definitely tall' could have as intended meaning that 'Bill is tall' sets the focus to only those worlds where Bill actually is tall and the second conjunct 'and even definitely tall' is evaluated with respect to that smaller set of possible worlds. Consider Table 4.2. In this context we have $[\![\textit{tall}]\!](\mathsf{b})(C) = \{w_1, w_2, w_3\}$ and $[\![\textit{definitely tall}]\!](\mathsf{b})(\{w_1, w_2, w_3\}) = \{w_1, w_2, w_3\}$, thus $[\![\textit{tall and}^{>}\textit{ definitely tall}]\!](\mathsf{b})(C) = \{w_1, w_2, w_3\}$. If we however evaluate $[\![\textit{definitely tall}]\!](\mathsf{b})$ directly with respect to C, then w_1 does not survive this update because of the possible world w_4. Therefore $[\![\textit{tall and}^{>}\textit{ definitely tall}]\!]$ is different from $[\![\textit{definitely tall and}^{>}\textit{ tall}]\!]$ and from $[\![\textit{tall and definitely tall}]\!]$.

In natural language one can also find *exclusive* disjunction, e.g. 'Bill is either tall or clever' (but not both). Such an exclusive disjunction can be modeled as well in the obvious way:

$$[\![\textit{or}^{\triangle}]\!] =_{df} \lambda\alpha\lambda\beta\lambda x\lambda C\;\; (\alpha(x)(C) \cup \beta(x)(C)) \setminus (\alpha(x)(C) \cap \beta(x)(C))\,.$$

Material implication is be expressed by composing $[\![\textit{not}]\!]$ and $[\![\textit{or}]\!]$, as usual:

$$[\![\textit{if}]\!] =_{df} \lambda\alpha\lambda\beta\lambda C.(C\backslash\alpha(x)(C)) \cup \beta(x)(C).$$

This operator may be used to model predicates of the form 'B if A'. In other words, we take 'Bill is heavy if tall' as synonymous to 'Bill is heavy or not tall'. Of course, one may be sceptical about the use of material implication in natural language—consider, e.g., 'Bill is not tall if tall'. The corresponding context update is survived by all worlds where Bill is not tall; in the case of context C (Table 4.2) we have $[\![\textit{tall if not tall}]\!](b)(C) = \{w_4, w_5\}$. However we define a corresponding operator here mainly for comparing it with familiar connectives from fuzzy logic below.

Consider the operator *and*. The membership degree of an individual x in the fuzzy set $[A\,\textit{and}\,B]_C$ is determined analogously to atomic predicates as in Definition 14: We apply the filter $[\![A\,\textit{and}\,B]\!](x)$ to the context C and calculate the fraction of worlds in C that survive this update. For the sake of readability we write $[X]_C$ instead of $[[\![X]\!]]_C$. Naturally, as the aim is to extract *truth-functional* fuzzy logics from Barker's contextual approach, the question arises whether we can determine $[A\, and\, B]_C(x)$ from the membership degrees $[A]_C(x)$ and $[B]_C(x)$ alone. This, of course, would give us a fully truth functional semantics for *and*, *or*, and *not*. However, as already indicated, fuzzy sets abstract away from the internal structure of contexts that may show various possible dependencies of worlds.

Consider again the example context C consisting of the five possible worlds w_1 to w_5 as shown in Table 4.2. Furthermore, let $[\![Bill]\!] = \mathsf{b}$ be in the universe of discourse and let tall, clever, and heavy be the denotations of the unary predicates 'tall', 'clever', and 'heavy', respectively, just as demonstrated for 'tall' above. Then $[\![heavy]\!]$ is an element filter where

w	$\delta(w)(^{\uparrow}tall)$	$\mathsf{tall}(w)(\mathsf{b})$	$\delta(w)(^{\uparrow}clever)$	$\mathsf{clever}(w)(\mathsf{b})$	$\delta(w)(^{\uparrow}heavy)$	$\mathsf{heavy}(w)(\mathsf{b})$
w_1	170	175	100	105	80	75
w_2	160	170	120	125	75	70
w_3	170	180	100	95	90	100
w_4	180	175	105	100	85	75
w_5	170	165	110	115	70	65

Table 4.2: Example context C

$[\![heavy]\!](\mathsf{b})(C) = \{w_3\}$. Accordingly, $[heavy]_C(\mathsf{b}) = 1/5$. Likewise we have $[clever]_C(\mathsf{b}) = [tall]_C(\mathsf{b}) = 3/5$. Since these latter are equal, also the membership degrees of b in the fuzzy sets $[tall\ and\ heavy]_C$ and $[clever\ and\ heavy]_C$, respectively, had to be equal if the (context update) meaning of 'and' were truth functional. But we obtain $[\![tall\ and\ heavy]\!](\mathsf{b})(C) = \{w_3\}$, thus $[tall\ and\ heavy]_C(\mathsf{b}) = 1/5$, while, on the other hand, $[clever\ and\ heavy]_C(\mathsf{b}) = 0$. Note that by extracting the three fuzzy sets from the corresponding element filters we lose the information about the specific overlap of the corresponding updates in the given context.

The following bounds describe optimal knowledge about membership degrees for fuzzy sets extracted from logically compound predicates with respect to membership degrees referring to the corresponding components.

Proposition 6. *Let C be a context, $d \in U$, and let $\alpha = [\![A]\!]$ and $\beta = [\![B]\!]$ be two element filters. Then the following bounds are tight:*

- $\max\{0, [\alpha]_C(d) + [\beta]_C(d) - 1\} \leq [A\, and\, B]_C(d) \leq \min\{[\alpha]_C(d), [\beta]_C(d)\}$,
- $\max\{[\alpha]_C(d), [\beta]_C(d)\} \leq [A\, or\, B]_C(d) \leq \min\{1, [\alpha]_C(d) + [\beta]_C(d)\}$,
- $[not\ A]_C(d) = 1 - [\alpha]_C(d)$.

Proof. The value $1 - [\alpha]_C(d)$ for negation follows directly from the relevant definitions.

For conjunction and disjunction note that the membership degree $[\alpha]_C(u)$ can—according to Definition 14—be identified with the probability that a randomly chosen possible world w survives the corresponding update $[\![\alpha]\!](u)$. The operators 'and' and 'or' then calculate the conjunction and disjunction, respectively of these events. The given bounds arise in the extremal cases where the two sets $\alpha(d)(C)$ and $\beta(d)(C)$ are maximally disjoint or maximally overlapping and thus directly follow from the Fréchet inequalities [28]. $\square$

Note that $*_{\mathrm{G}} = \min$ and $\bar{*}_{\mathrm{G}} = \max$ are the Gödel t-norm and co-t-norm, respectively, and $*_{\text{Ł}} = \lambda x, y. \max\{0, x + y - 1\}$ and $\bar{*}_{\text{Ł}} = \lambda x, y. \min\{1, x + y\}$ are the Łukasiewicz t-norm and co-t-norm, respectively, as introduced in Section 1.5. In other words, Proposition 6 shows that the truth functions of (strong) conjunction and (strong) disjunction in Gödel and Łukasiewicz logic correspond to opposite extremal cases of context based evaluations of conjunction and disjunction.

The bounds for the material implication $[\![\textit{if}]\!]$, as defined above, can be derived easily as well:

$$\max\{1 - [\alpha]_C(d), [\beta]_C(d)\} \leq [\text{`if '}A\text{` then '}B]_C(d) \leq \min\{1, 1 - [\alpha]_C(d) + [\beta]_C(d)\}.$$

Note the emergence of the residual Łukasiewicz implication as defined in Section 1.5 as upper bound. This operation is also called the R-implication induced by the Łukasiewicz t-norm [55]. The lower bound amounts to the so-called S-implication induced by the Gödel t-norm. This operation is obtained by defining $\phi \to \psi =_{\mathit{df}} \neg\phi \vee \psi$ and taking as a truth function for disjunction the co-t-norm associated with the Gödel t-norm (while retaining the involutive negation).

Consider two fuzzy sets $[\alpha]_C$, $[\beta]_C$ and some $\mathsf{a} \in D_C$. Then the tightness of these bounds implies that we can, for any value v in the given interval, find two element filters $\alpha' = [\![A]\!]$ and $\beta' = [\![B]\!]$ such that $[\alpha]_C(\mathsf{a}) = [\alpha']_C(\mathsf{a})$, $[\beta]_C(\mathsf{a}) = [\beta']_C(\mathsf{a})$ and $[A\,\mathit{and}\,B]_C(\mathsf{a}) = v$. (Analogously for $[A\,\mathit{or}\,B]_C(\mathsf{a})$). Therefore the above bounds indicate to which extend the behavior of the operators on predicates can be approximated by truth functional semantics.

Remark. Although not motivated linguistically, Jeff Paris [72] obtains essentially the same bounds for truth functions approximating probabilities. In order to determine a single truth value for a compound statement he suggests to take the arithmetic mean value of these calculated lower and upper bounds. However, such a truth function is not associative and thus hardly relates to a corresponding connective of fuzzy logics.

The above analysis of logical predicate operators can be easily lifted to the propositional level. For a sentence like 'Bill is tall' its meaning $[\![\textit{Bill is tall}]\!]$ is a filter, rather than an element filter. Logical connectives on propositions can be defined in analogy to Definition 15:

Definition 16.

- $[\![\phi \wedge \psi]\!] =_{\mathit{df}} \lambda C\ [\![\phi]\!](C) \cap [\![\psi]\!](C)$

- $[\![\phi \vee \psi]\!] =_{df} \lambda C\, [\![\phi]\!](C) \cup [\![\psi]\!](C)$
- $[\![\neg\phi]\!] =_{df} \lambda C\, C \backslash [\![\phi]\!](C)$

Likewise we may augment:

- $[\![\phi \to \psi]\!] =_{df} \lambda C\, (C \backslash [\![\phi]\!])(C) \cup [\![\psi]\!](C)$,
- $[\![\phi \wedge^{>} \psi]\!] =_{df} \lambda C\, ([\![\psi]\!]([\![\phi]\!]))(C)$, and
- $[\![\phi \vee^{\triangle} \psi]\!] =_{df} \lambda C\, ([\![\phi]\!](C) \cup [\![\psi]\!](C)) \setminus ([\![\phi]\!](C) \cap [\![\psi]\!](C))$.

Similarly to the predicate level we can associate a 'degree of truth' $\|\phi\|_C$ for every proposition ϕ formed this way by applying the filter $[\![\phi]\!]$ to the context C:

$$\|\phi\|_C =_{df} \frac{|[\![\phi]\!](C)|}{|C|}.$$

In other words, we identify the degree of truth of ϕ in a context C with the proportion of worlds in C that survive the update with the filter $[\![\phi]\!]$. Referring to the example context C specified in Table 4.2, 'Bill is tall' is true to degree $3/5$ in C since three out of five worlds in C classify Bill's height as above the relevant local standard of tallness. Note that by this way of determining truth values the compound statement 'Bill is tall and Bill is heavy' receives exactly the same truth value as the atomic statement 'Bill is tall and heavy' (but with a compound predicate).

As above, since contexts allow us to model specific constraints on the worlds, i.e., possible precisifications, one cannot a priori assume that threshold values and the degrees, to which individuals have vague properties are independent of each other. Therefore, in general, there are no truth functions that determine $\|\phi \wedge \psi\|_C$ and $\|\phi \vee \psi\|_C$ in terms of $\|\phi\|_C$ and $\|\psi\|_C$ alone. However the optimal bounds of Proposition 6 also apply at the level of sentences. In particular:

- $*_{Ł}(\|\phi\|_C, \|\psi\|_C) \leq \|\phi \wedge \psi\|_C \leq *_{G}(\|\phi\|_C, \|\psi\|_C)$,
- $\bar{*}_{G}(\|\phi\|_C, \|\psi\|_C) \leq \|\phi \vee \psi\|_C \leq \bar{*}_{Ł}(\|\phi\|_C, \|\psi\|_C)$, and
- $I_G(\|\phi\|_C, \|\psi\|_C) \leq \|\phi \to \psi\|_C \leq \|\phi\|_C \Rightarrow_{Ł} \|\psi\|_C$.

where $*_{G}(\bar{*}_{G})$ and $*_{Ł}(\bar{*}_{Ł})$ are the Gödel and Łukasiewicz t-norms (co-t-norms), $\Rightarrow_{Ł}$ is the residuum of the Łukasiewicz t-norm, and I_G denotes the S-implication induced by the Gödel t-norm (see [55]), respectively.

4.3 Saturated Contexts

Having determined bounds for truth functions applied to arbitrary contexts above, we now investigate a special class of contexts, called *saturated contexts*. There, contexts can be completely determined by a few parameters, which enables us to directly calculate membership degrees in the respective fuzzy sets. In a saturated context the degrees to which predicates apply to individuals as well as all relevant thresholds values are defined by intervals. All values (up to a certain level of granularity) in the given interval are assumed to occur in that context uniformly distributed. Moreover, we assume that the intervals for different attributes and corresponding threshold values are independent of each other. This means that, e.g., an adequate saturated context for uttering 'Bill is tall' is completely defined by giving lower and upper bounds for Bill's height (denoted by h^l_{b} and h^u_{b}) and for possible threshold values for tallness (denoted by tall^l and tall^u). There are, however, no dependencies between these values: for each threshold value between tall^l and tall^u there are possible worlds in the context for all degrees of Bill's height from h^l_{b} to h^u_{b}.

Saturated contexts thus are natural when the hearer of a statement only assumes those bounds, but no further information, e.g., about dependencies between the values or about varying likelihood for the individual possible values, is available.

In the previous subsection we stipulated contexts to be *finite* sets of possible worlds and argued why this is a natural assumption in linguistics, due to the granularity imposed by limits of distinguishabilty. From now on, however, we will treat the sets of degrees available in a context as intervals of real numbers. Those intervals can be seen as the limit case for ever higher (more fine-grained) levels of granularity. To motivate this move, consider (again) the statement 'Bill is tall' and a hearer, who only knows about Bill's height that it is between $h^l_{\mathsf{b}} = 179cm$ and $h^u_{\mathsf{b}} = 181cm$. Moreover, for the sake of simplicity, let the hearer be certain that it is adequate (in the given context C) to call a person 'tall' if and only it its height is at least $\mathsf{tall}^l = \mathsf{tall}^u = 180cm$. If the granularity is too coarse and the interval only includes the three values $179cm$, $180cm$, and $181cm$ as possible values for Bill's height, then in two out of these three possible worlds Bill is judged to be tall, hence $[\mathsf{tall}]_C(\mathsf{b}) = 2/3$. This value however is just an artifact imposed by the very low (coarse-grained) level of granularity: If the degrees $179cm$, $179.5cm$, $180cm$, $180.5cm$ and $181cm$ were available in a context C', Bill would be considered tall in 3 out of these 5 worlds, thus $[\mathsf{tall}]_{C'}(\mathsf{b}) = 3/5$. For higher levels of granularity the value $[\mathsf{tall}]_C(\mathsf{b})$ approaches $1/2$. In other words, the proportion of worlds in the given context where 'Bill''s height is above the threshold of tallness, is arbitrarily close to $1/2$ for sufficient high levels of granularities. Therefore, from now on we will only be interested in the limit case where we can interpret $[h^l_{\mathsf{b}}, h^u_{\mathsf{b}}]$ as a real interval and calculate $[\mathsf{tall}]_C(\mathsf{j}) = 1/2$, corresponding to the intuition that exactly half of this interval of possible values for *Bill*'s height is cut off by the given threshold value for tallness. Definition 14 only applies to finite contexts, as it requires counting the possible worlds within. However, the value $[\mathsf{tall}]_C(\mathsf{b})$ can also be interpreted as the probability that a randomly chosen possible world w survives the corresponding update $[\![tall]\!](\mathsf{b})$, where we assume a uniform distribution over all worlds of the original context. This enables us to analyze the relevant limit cases directly.

In saturated contexts there are no dependencies between possible values or varying likelihoods of them. Hence we can compute the fuzzy membership value $[\mathsf{tall}]_C(\mathsf{b})$ given the values h^l_{b}, h^u_{b}, tall^l, and tall^u alone. For the actual computation, we have to distinguish between six cases, depending on the relative position of the two intervals $[h^l_{\mathsf{b}}, h^u_{\mathsf{b}}]$ and $[\mathsf{tall}^l, \mathsf{tall}^u]$ as follows:

case (a): $h^l_{\mathsf{b}} > \mathsf{tall}^u$	case (b): $h^u_{\mathsf{b}} < \mathsf{tall}^l$
case (c): $h^u_{\mathsf{b}} \geq \mathsf{tall}^u, h^l_{\mathsf{b}} \leq \mathsf{tall}^l$	case (d): $h^u_{\mathsf{b}} \leq \mathsf{tall}^u, h^l_{\mathsf{b}} \geq \mathsf{tall}^l$
case (e): $h^l_{\mathsf{b}} > \mathsf{tall}^l, h^u_{\mathsf{b}} \geq \mathsf{tall}^u \geq h^l_{\mathsf{b}}$	case (f): $h^u_{\mathsf{b}} < \mathsf{tall}^u, h^u_{\mathsf{b}} \geq \mathsf{tall}^l \geq h^l_{\mathsf{b}}$.

Either both intervals are completely disjoint (cases (a) and (b)), one of them is contained in the other (cases (c) and (d)), or they are properly overlapping (cases (e) and (f)).

Figures 4.1 and 4.2 depict all these six cases. The hatched area on the left hand side represents the possible worlds in a saturated contexts determined by the boundary values of the two intervals before applying the update. The hatched area on the right hand side represents the worlds after applying the update that corresponds to accepting 'Bill is tall': all worlds under the diagonal are eliminated by the element filter $[\![tall]\!](\mathsf{b})$. For cases (a) and (b) either all or no worlds survive the associated update, thus the fuzzy membership degree in question is 1 (or 0, respectively). We consider case (e) in more depth, the other cases are analogous: The hatched area on the left hand side (before applying the filter) has the size $A = (h^u_{\mathsf{b}} - h^l_{\mathsf{b}})(\mathsf{tall}^u - \mathsf{tall}^l)$. The size of the are triangle which is cut off by applying the update (representing the worlds which are filtered out) has the $(\mathsf{tall}^u - h^l_{\mathsf{b}})^2/2$. Hence the size of the hatched are on the right hand side amounts to $A - (\mathsf{tall}^u - h^l_{\mathsf{b}})^2/2$. Putting these observations together, the probability that a randomly chosen world of the context survives the update—and thus the membership degree for b in $[\mathsf{tall}]_C$—is readily computed as

$$[\mathsf{tall}]_C(\mathsf{b}) = \frac{A - \frac{1}{2}(\mathsf{tall}^u - h^l_{\mathsf{b}})^2}{A} \text{ with } A = (h^u_{\mathsf{b}} - h^l_{\mathsf{b}})(\mathsf{tall}^u - \mathsf{tall}^l).$$

For the remaining three cases the fuzzy membership degree can be computed analogously leading to

$$[\mathsf{tall}]_C(\mathsf{b}) = \begin{cases} 1 & \text{in case (a)} \\ 0 & \text{in case (b)} \\ \frac{1}{A}\left((\mathsf{tall}^u - \mathsf{tall}^l)(h^u_{\mathsf{b}} - \mathsf{tall}^u) + \frac{1}{2}(\mathsf{tall}^u - \mathsf{tall}^l)^2\right) & \text{in case (c)} \\ \frac{1}{A}\left((h^l_{\mathsf{b}} - \mathsf{tall}^l)(h^u_{\mathsf{b}} - h^l_{\mathsf{b}}) + \frac{1}{2}(h^u_{\mathsf{b}} - h^l_{\mathsf{b}})^2\right) & \text{in case (d)} \\ \frac{1}{A}\left(A - \frac{1}{2}(\mathsf{tall}^u - h^l_{\mathsf{b}})^2\right) & \text{in case (e)} \\ \frac{1}{2A}\left(h^u_{\mathsf{b}} - \mathsf{tall}^l\right)^2 & \text{in case (f),} \end{cases}$$

again with $A = (h^u_{\mathsf{b}} - h^l_{\mathsf{b}})(\mathsf{tall}^u - \mathsf{tall}^l)$.

Next, we consider two *independent* predicates, say ‘tall’ and ‘heavy’. Saturated contexts induce a fully compositional semantics for logical connectives such as ‘and’ and ‘or’. In other words, $[\text{‘tall and heavy’}]_C$ is determined only by the values $[\text{‘tall’}]_C$ and $[\text{‘heavy’}]_C$. The independence of the two predicates is crucial here: every possible combination of a degree of height and a degree of weight for Bill and corresponding threshold values for ‘tallness’ and ‘heaviness’ is assumed to occur with equal probability in C.

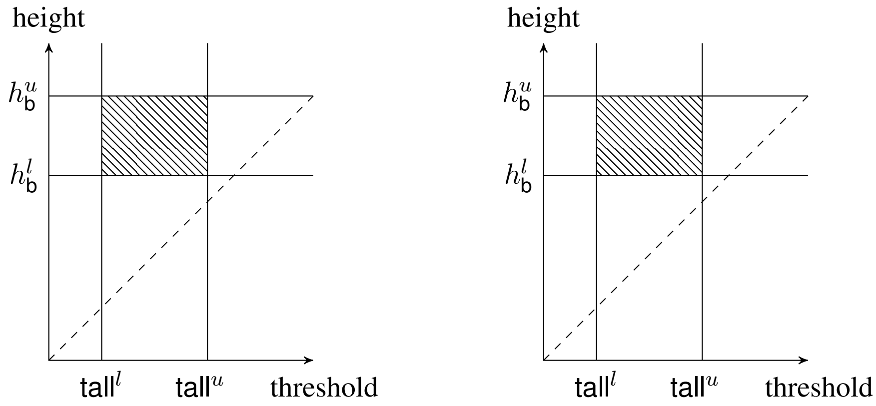
The probability that ‘Bill’ is tall in a randomly selected world $w \in C$ is $[\text{‘tall’}(\mathsf{b})]_C$, while the probability that he is heavy in w is $[\text{‘heavy’}(\mathsf{b})]_C$. The independence implies that the probability of ‘Bill’ being ‘tall and heavy’ at w is modeled as the joint probability, i.e., by the product t-norm $*_\Pi$, which amounts to $x *_\Pi y = x \cdot y$ (see Section 1.5). Analogously, the probability of ‘Bill’ being ‘tall or heavy’ at w is modeled as the probabilistic sum, i.e., by the product co-t-norm $\bar{*}_\Pi$, which amounts to $x \bar{*}_\Pi y = x + y - x \cdot y$:

$$[\text{‘tall and heavy’}]_C(\mathsf{b}) = [\text{‘tall’}]_C(\mathsf{b}) *_\Pi [\text{‘heavy’}]_C(\mathsf{b}),$$
$$[\text{‘tall or heavy’}]_C(\mathsf{b}) = [\text{‘tall’}]_C(\mathsf{b}) \bar{*}_\Pi [\text{‘heavy’}]_C(\mathsf{b}).$$

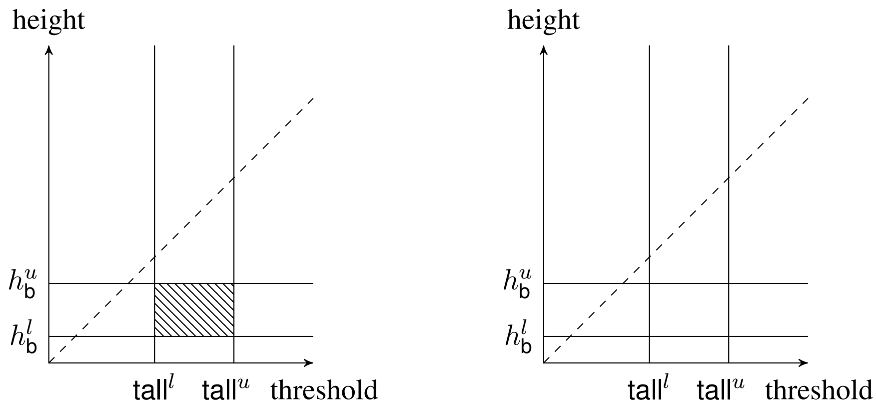
As above, this analysis can be lifted with an analogous argument to the sentential level in order to model, e.g., ‘Bill is tall and Jane is heavy’ and ‘Bill is tall or Jane is heavy’ as

$$\|\text{‘Bill is tall’} \wedge \text{‘Jane is heavy’}\|_C = \|\text{‘Bill is tall’}\|_C *_\Pi \|\text{‘Jane is heavy’}\|_C,$$
$$\|\text{‘Bill is tall’} \vee \text{‘Jane is heavy’}\|_C = \|\text{‘Bill is tall’}\|_C \bar{*}_\Pi \|\text{‘Jane is heavy’}\|_C.$$

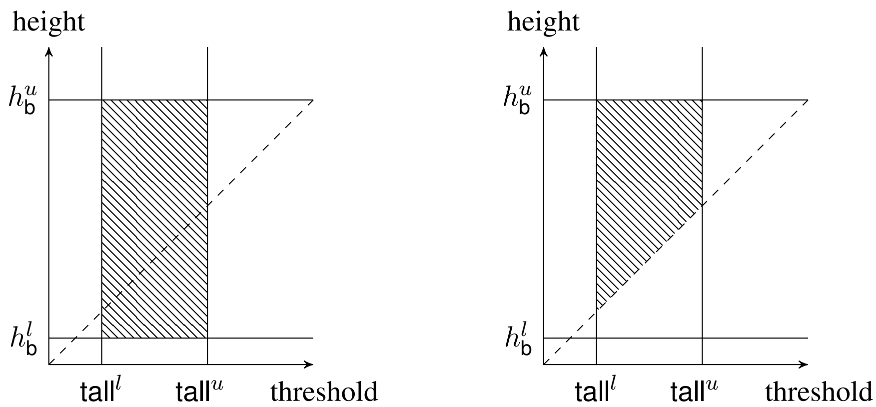
However, observe that it is crucial here that the two statements consist of two different predicates. Consider ‘Bill is tall and Jane is tall’: This conjunction can not be modeled in a truth-functional way by saturated contexts, in general. While, in a saturated context, Bill’s and Jane’s heights are independents, the respective *judgements of tallness* are not, as they both refer to the same threshold value $\delta(w)(^\uparrow\textit{tall})$ in each world w of the context. In other words, the probability that ‘Bill is tall and Jane is tall’ holds in a randomly selected world is not just



(a) $h_{\mathsf{b}}^l > \mathsf{tall}^u$

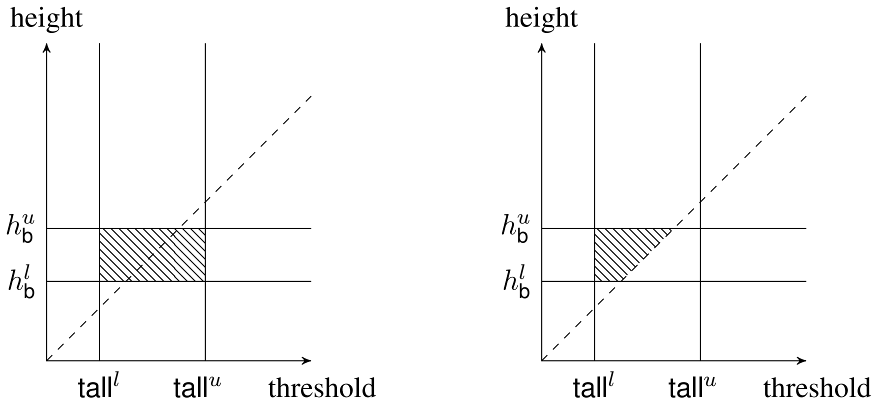


(b) $h_{\mathsf{b}}^u < \mathsf{tall}^l$

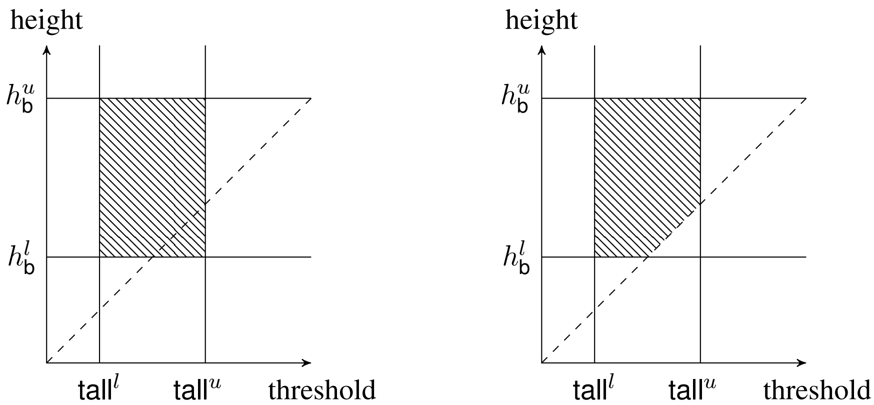


(c) $h_{\mathsf{b}}^u \geq \mathsf{tall}^u, h_{\mathsf{b}}^l \leq \mathsf{tall}^l$

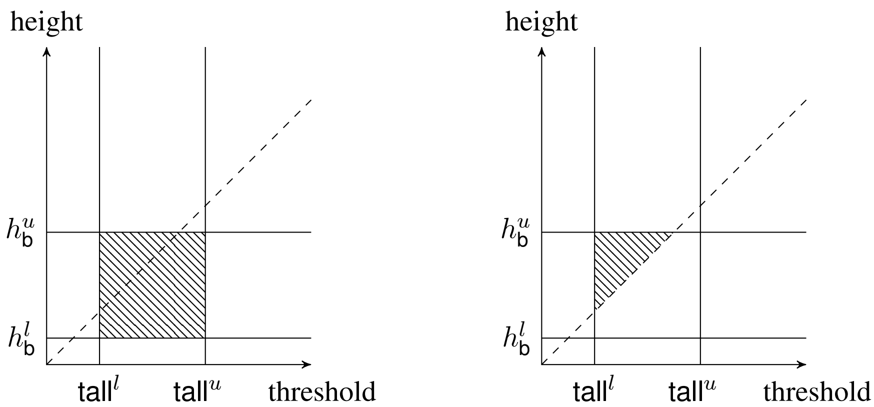
Figure 4.1: Illustration of a saturated context before and after an update with the statement 'Bill is tall' (Cases a-c)



(d) $h_{\mathsf{b}}^u \leq \mathsf{tall}^u, h_{\mathsf{b}}^l \geq \mathsf{tall}^l$



(e) $h_{\mathsf{b}}^l > \mathsf{tall}^l, h_{\mathsf{b}}^u \geq \mathsf{tall}^u \geq h_{\mathsf{b}}^l$



(f) $h_{\mathsf{b}}^u < \mathsf{tall}^u, h_{\mathsf{b}}^u \geq \mathsf{tall}^l \geq h_{\mathsf{b}}^l$

Figure 4.2: Illustration of a saturated context before and after an update with the statement 'Bill is tall' (Cases d-f)

a function of the probability that 'Bill is tall' and the probability that *Jane is tall*, respectively. Consider for example a context, where Bill's heights range between $180cm$ and $185cm$, while Jane is strictly smaller; her heights range between $170cm$ and $175cm$. Then for *any* threshold value for tallness, Jane being tall entails that Bill is also tall at the same world. Thus, judgements of Jane's and Bill's status of tallness are not independent, while their actual heights are. Rather, for arbitrary saturated contexts, one has to take into account the particular intervals of heights for both 'Bill' and 'Jane' in relation to the possible thresholds to obtain the value for the compound statement.

Finally, we consider a special case of saturated contexts where truth functionality can be recovered. Assume, there is *perfect knowledge* about the height of the individuals under consideration, but nevertheless there is still vagueness in the meaning of 'tall'. Thus all possible worlds agree on their values for the heights of Bill, $\mathsf{tall}(w)(\mathsf{b})$, and of Jane, $\mathsf{tall}(w)(\mathsf{j})$, while differing in their threshold value $\delta(w)({}^{\uparrow}\textit{tall})$ for tallness. As above, if Bill is taller than Jane, each world which judges Jane as tall, also judges Bill as tall, and also vice versa. Therefore the membership degree of a conjunction (or disjunction) amounts to the minimum (or maximum, respectively) of the components' fuzzy membership degrees:

$$\|\text{'Bill is tall'} \wedge \text{'Jane is tall'}\|_C = \min(\|\text{'Bill is tall'}\|_C, \|\text{'Jane is tall'}\|_C)$$
$$\|\text{'Bill is tall'} \vee \text{'Jane is tall'}\|_C = \max(\|\text{'Bill is tall'}\|_C, \|\text{'Jane is tall'}\|_C).$$

In other words, the Gödel t-norm and co-t-norm appear as truth-functions for conjunction and disjunction in saturated contexts with perfect knowledge.

CHAPTER 5

On Contextual Vagueness

As already explained in Section 1.3 we focus on Stewart Shapiro's account of "Vagueness in Context" [82, 83] as an example of a contextual approach to vagueness. First we give a more detailed overview of Shapiro's formal machinery. Then we show how the resulting logic can be characterized by evaluation games in Hintikka's and Giles's frameworks (as defined in Chapter 2). Moreover, we will see how Shapiro's framework can be set into relation with Chris Barker's scale-based approach presented in the previous chapter. Although Shapiro analyzes vagueness by means of partial valuations and does not refer to scales and degrees, we will see, following [77], under which conditions both frameworks can be regarded as equivalent, and in which situations one framework is superior to the other. Finally, Kyburg and Morreau's framework for modeling vague predicates is presented. Kyburg and Morreau do not refer to Shapiro's approach, but instead combine supervaluation with the theory of belief revision [2]. Nevertheless we will see how this framework, from a technical point of view, can be seen as a further specialization of Shapiro's approach.

5.1 Vagueness in Context

Stewart Shapiro [82, 83] presents a model for reasoning with vague propositions with a special focus on Sorites situations [45] as explained in Section 1.2. He maintains that the extensions and anti-extensions of vague predicates such as *bald* crucially depend on the context which evolves during a conversation. At the beginning this context contains facts which are externally determined—and thus are not subject to vagueness here.[1] The extensions and anti-extensions

[1] 'Context' as used by Shapiro has to be distinguished from the semantic context of vague predicates. Context here means the context of a conversation containing common facts and views shared by conversationalists as well as

of vague predicates may undefined for many objects, the so-called *borderline cases*. During a conversation these partial interpretations are precisified, such that borderline cases, which have been undecided so far, get assigned to the (anti-)extension of a vague predicate.

We recapitulate Shapiro's version of a so-called *forced march Sorites* argument: Assume, 1000 men are lined up ordered according to their amount and arrangement of hair, where the first man has no hair at all and the last one has full hair. A group of conversationalists is asked to decide whether man $\#1$ is bald, then man $\#2$, and so on, where all conversationalists must establish a common judgement for each man $\#i$. The principle of *tolerance* dictates that they cannot judge man $\#i$ to be not bald, when at the same time judging man $\#i - 1$ to be bald, considering that the amount and arrangement of hair of these two men differ only marginally. Its precise formulation as used in [83] is as follows:

> Suppose that two objects a, b in the field of P differ only marginally in the relevant respect (on which P is tolerant). Then if one competently judges a to have P, then she cannot competently judge b in any other manner.

This implies that there exists no complete precisification of the predicate 'bald' consistent with the facts that the first and last men are bald and not bald, respectively, with tolerance in force.

Relatedly, *open texture* means that if the status of baldness of a man is still undecided at a point, i.e., that man belongs neither to the extension nor to the anti-extension of 'bald', a speaker can freely decide to call him tall or not tall without compromising his competency (of the English language). Note that the notion of 'competent speaker' is also vague; this is where the model can be extended to higher order vagueness (see Section 1.3). Furthermore, *judgment dependence* entails that the extensions and anti-extensions for borderline cases of vague predicates are solely determined by such decisions.

In a Sorites series, the speakers initially will unequivocally judge 'bald', but at some point they will begin to discuss among each other and finally switch to 'not bald'. How can such a switch be modeled with tolerance in force? Shapiro maintains, that at the same time as man $\#i + 1$ is judged as not bald, former judgements about the previous men are implicitly retracted from the conversational record. In this manner, tolerance is not necessarily violated by such a switch.

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assumptions made during the conversation. The semantic context of a vague predicate on the other hand determines a suitable comparison class. For example, uttering 'tall' may denote different predicates when used in a conversation about children or in a conversation about basketball players. As layed out in Section 1.3, contextual approaches to vagueness often discard this kind of contextual variability as not being crucial for vagueness, since even within a fixed comparison class one can easily form a Sorites series.

For modeling the conversational record more formally, Shapiro uses a Kripke-style directed acyclic structure, called *frame*. Each frame, denoted $\langle W, M \rangle$, consists of a set of worlds W with one designated world $M \in W$, called the *base* of the frame. A world is a valuation of atoms assigning either *true*, *false*, or *indefinite* to all predicates in question[2]; all worlds in a frame are over the same domain. The world N' is called a sharpening of the world N, denoted as $N' \succeq N$, if and only if each atom which is *true* or *false* at N is also *true* or *false*, respectively, at N'. At the base M propositions are fixed which are determined outside the current conversation. It is required that for all $N \in W$ that N is a sharpening of M. As $\succeq$ is a partial order, a frame can be 'unraveled' to a tree of precisifications starting with the base M as root and duplicating shared worlds. As explained above, in contrast to supervaluationist approaches [27], the completability requirement is not enforced in the presence of tolerance. This means that we do not require that at the leaves of the tree structure a vague predicate P is decided for all objects, i.e., we may leave P undecided for some objects.

In a forced march Sorites situation initially only the externally determined facts are available on the conversational record; these are reflected in M. As mentioned above, in the beginning the conversationalists will repeatedly vote for 'bald'. Making these judgments corresponds to moving alongside a branch, away from the root M and thus precisifying the asserted statements. At some point, however, they will switch to 'not bald'. With the principle of tolerance in force they have to withdraw some statements from the conversational record; this amounts to jumping to another branch in the frame. (In his treatment of higher order vagueness, Shapiro models between which worlds such jumps are allowed.)

Shapiro argues that the notion of *determinate truth* in a frame is best characterized by *forcing*. A formula ϕ is forced at a sharpening N, if for each sharpening $N' \succeq N$ there is a further sharpening $N'' \succeq N'$ such that ϕ is *true* at N''. Intuitively, ϕ being forced at N means that ϕ will eventually get *true*: a formula ϕ is *determinately true* if ϕ is forced at the base of F. Forcing here is the counterpart to the notion of *supertruth* in supervaluationist theories as layed out in Section 1.3, which is simply defined as 'true in all complete precisifications' . Indeed, if a frame contains *complete sharpenings* for N, i.e., sharpenings where all predicates are completely precisified, these two notions coincide. Moreover, the notions of validity and, more generally, consequence are defined in terms of forcing: $\Gamma \models \phi$ if and only if ϕ is forced at every sharpening in every frame in which all formulas of Γ are forced. Shapiro also introduces the following notion: A formula ϕ is weakly forced at N if there is no sharpening $N' \succeq N$ such that ϕ is *false* at N'. Thus instead of ϕ being 'eventually true', weak forcing means that ϕ will never get false. Again, if all sharpenings can be made precise within a frame, this notion coincides

[2] Shapiro defends the notion that there are conceptually only two truth values, *true* and *false*; *indefinite* is to be interpreted as the absence of a classical truth value.

with forcing and supertruth.

Of course not all possible frames are adequate for a given (Sorites) situation. For example, we don't consider frames which contain partial interpretations where man $\#i$ is declared to be bald, but another man $\#j$ with $j > i$, who has more hair, is judged not to be bald. Such constraints on adequate frames are called *penumbral connections*. Typically, the penumbral connection above is required to hold locally at each world in a frame. Sometimes it is however more fruitful to model penumbral connections globally for the whole frame, e.g., by requiring that some proposition is forced at the base. The principle of tolerance can be formulated as such a 'global' penumbral connection as seen below.

Local operators. Shapiro defines local logical connectives for negation '$\neg$', conjunction '$\wedge$', disjunction '$\vee$', and implication '$\rightarrow$'. These all adhere to the strong Kleene truth tables as given by Figure 1.1a in Section 1.3. Also the standard first order quantifiers $\forall$ and $\exists$ are present with the expected semantics: A formula $\forall x\ \phi(x)$ is *true* if $\phi(c)$ is true for all constants c, it is false if $\phi(c)$ is false for some constant c, and it is indefinite otherwise. Analogously, $\exists x\ \phi(x)$ is *true* if $\phi(c)$ is true for some constant c, it is false if $\phi(c)$ is false for all constants c, and it is indefinite otherwise (assuming a language where we identify each domain element with a constant).

Note that all these connectives and quantifiers obey to a twofold monotonicity principle: If a (compound) formula ϕ is *true* (or *false*) at a world N then ϕ is *true* (or *false*, respectively) also at all sharpenings N' of N. We will call these two notions positive and negative monotonicity. By definition of the relation $\succeq$, also atomic propositions adhere to this principle.

Non-local operators. Additionally to the standard logical connectives and quantifiers, Shapiro introduces new non-local ones operating on whole subtrees instead of a single sharpening. One of them is the new non-local implication '$\Rightarrow$' with the following semantics:

> The formula $\phi \Rightarrow \psi$ is *true* at a world N if and only if for every sharpening $N' \succeq N$ it holds that ϕ is true at N' implies that ψ is true at N'.

This connective is used extensively by Shapiro to define penumbral connections as seen by the following example: assume a Sorites situation as explained above. Then we can stipulate as a penumbral connection, that for all x and y the formulas $(B(y) \wedge M(x,y)) \Rightarrow B(x)$ and $(\neg B(x) \wedge M(x,y)) \Rightarrow \neg B(y)$, with $B(x)$ if x is bald and $M(x,y)$ iff x has more hair than y, hold at the base (and thus at all sharpenings). This ensures that at a sharpening where a man is judged 'bald', all men with less hair are judged 'bald' as well. Vice versa, at a sharpening where a man is judged 'not bald', all men with more hair as are judged 'not bald' as well.

In order to preserve monotonicity we also have to give a falsehood condition for each new connective. Just stating that $\phi \Rightarrow \psi$ is *false* if it is not *true* would violate monotonicity; this can be seen in the following example frame:

$$\begin{array}{ccc} & M & \\ \swarrow & & \searrow \\ N : P(a), \neg P(b) & & N' : P(a), P(b) \end{array}$$

At the base M the formula $P(a) \Rightarrow P(b)$ is not *true* because the condition is violated at N, but it is *true* at N'. Thus, positive and negative monotonicity in force, we have to leave $P(a) \Rightarrow P(b)$ undecided at M. Therefore, Shapiro makes use of the so-called *stable failure*:

> The formula $\phi \Rightarrow \psi$ is *false* at the sharpening N if and only if there is no sharpening N' of N such that $\phi \Rightarrow \psi$ is *true* at N.

This ensures that also falsehood is preserved in the tree structure: If a formula $\phi \Rightarrow \Psi$ is *false* at a world N, it is also *false* at each sharpening of N.

Another new connective is the intuitionistic-style negation '$-$':

> The formula $-\phi$ is *true* at a world N if and only if there is no sharpening $N' \succeq N$ where ϕ is true.

Stable failure dictates the following falseness condition for the new negation operator:

> The formula $-\phi$ is *false* at a world N if and only if there is no sharpening $N' \succeq N$ where $-\phi$ is true.

Note that monotonicity prevents us from introducing forcing as an operator $\Box$ in the object language such that $\Box\phi$ is true at a world N if ϕ is forced at N and false if ϕ is not forced at N: It is easy to construct a frame where ϕ is not forced at a world N, but where ϕ is forced at a sharpening N' of N violating the (negative) monotonicity principle for $\Box\phi$. However, using the new non-local negation operator, forcing can be expressed as follows: a formula ϕ is forced at a world N if and only if $\neg - \phi$ is *true* at N. Thus, if ϕ is not forced at N, the formula $\neg - \phi$ is either false or indefinite at N.

Shapiro observes that, as in supervaluationist theories, a formula $\exists x\ \phi(x)$ can be forced at a sharpening N without $\phi(a)$ being forced at N for any particular witness a. In order to make the existence of such witnesses expressible in the object language, he introduces a new global existential quantifier E with the following semantics:

The formula $Ex\ \phi(x)$ is *true* at N if and only if there exists a such that $\phi(a)$ is forced at N. The formula $Ex\ \phi(x)$ is *false* at N if and only if there exists no sharpening $N' \succeq N$ such that $Ex\ \phi(x)$ is true at N'.

Similarly it is possible to define the new global universal quantifier A:

The formula $Ax\ \phi(x)$ is *true* at N if and only if for all x it holds that $\phi(x)$ is forced at N. The formula $Ax\ \phi(x)$ is *false* at N if and only if there exists no sharpening $N' \succeq N$ such that $Ax\ \phi(x)$ is true at N'.

Using these new non-local operators it is also possible to express the tolerance principle within the object language (as argued by Shapiro): Consider two predicates, $B(x)$ expressing that individual x is 'bald' and $R(x, y)$ expressing that x and y's amount and arrangement of hair differs only marginally. Then the tolerance principle dictates that the formula $\forall x\, \forall y\ (R(x, y) \Rightarrow -B(x) \wedge \neg B(y))$ holds at the base of the frame under consideration (and, thus, at all worlds).

We obtain the following lemma which will be useful in the next section:

Lemma 1. *Let ϕ be a formula of the form $-\psi$ or $Ax\ \psi(x)$. Then ϕ is* indefinite *at a world N in a frame $\mathcal{F}$ if and only if there exist sharpenings $N' \succeq N$ and $N'' \succeq N$ such that ϕ is* true *at N' and* false *at N''.*

Proof. If ϕ is *true* at N' and *false* at N'' then, due to positive and negative monotonicity, it must be *indefinite* at N.

For proving the converse, consider $\phi = Ax\ \psi(x)$, and assume that ϕ is *indefinite* at N. Stable failure dictates that ϕ would be false at N if there was no $N' \succeq N$ such that ϕ is true at N'. Thus, by the falseness condition for the 'A' operator there exists at least one sharpening $N' \succeq N$ where ϕ is true. Next, assume that there doesn't exist a sharpening where ϕ is *false*. Then ϕ is either *true* or *indefinite* at every sharpening $N'' \succeq N$ of N. As just argued, if ϕ is *indefinite* at N'' there exists a further sharpening N'' where ϕ is *true*. This means that ϕ is forced at N. Shapiro [82] (Section 4.3, page 100) shows that a formula $Ax\ \psi(x)$ is forced at a sharpening if and only if it is *true* at that sharpening, and consequently we conclude that ϕ is *true* at N'' contradicting the above assumption. Thus there exists a sharpening $N'' \succeq N$ such that ϕ is *false* at N''. For the global negation '$-$' we can reason analogously. $\square$

Remark. Lemma 1 does not hold for formulas of the form $\phi \Rightarrow \psi$ or $Ex.\phi(x)$. Such formulas can be *indefinite* at a sharpening N and *true* at all further sharpenings of N. We consider the following frame

$$\begin{array}{c} M : P(a) \\ | \\ N : P(a), P(b) \end{array}$$

and observe that the formula $P(a) \Rightarrow P(b)$ is true at N but indefinite at M.

5.2 A Hintikka-style Semantic Game

Shapiro rests his approach to vagueness on conversational situations. Extensions and anti-extensions of vague predicates always depend on the context at some point during the conversation. For example, forced march Sorites situations, as defined in the previous section, amount to a conversation between a group of 'judges' with one person repeatedly asking whether certain people of the Sorites series are to be considered as bald or as not. When it comes to asserting more complex propositions than just atomic ones like 'bald', dialogue games seem to be a natural consolidation of this idea. There, the semantic status of a complex proposition is determined without leaving Shapiro's setting of a dialogue between conversationalists. The game provides an explicit mechanism for the evaluation of compound formulas. Observe that Shapiro's falseness conditions for non-local connective all refer to *stable failure*–thus falseness is characterized only indirectly by referring to the truth conditions of the connective. Moreover, indefiniteness of propositions is defined only implicitly: A formula is indefinite at a sharpening N if it is neither true nor false at N. Using the dialogue game these implicit references are replaced by an explicit mechanism checking whether a formula is true, false, or indefinite at a given sharpening in a frame.

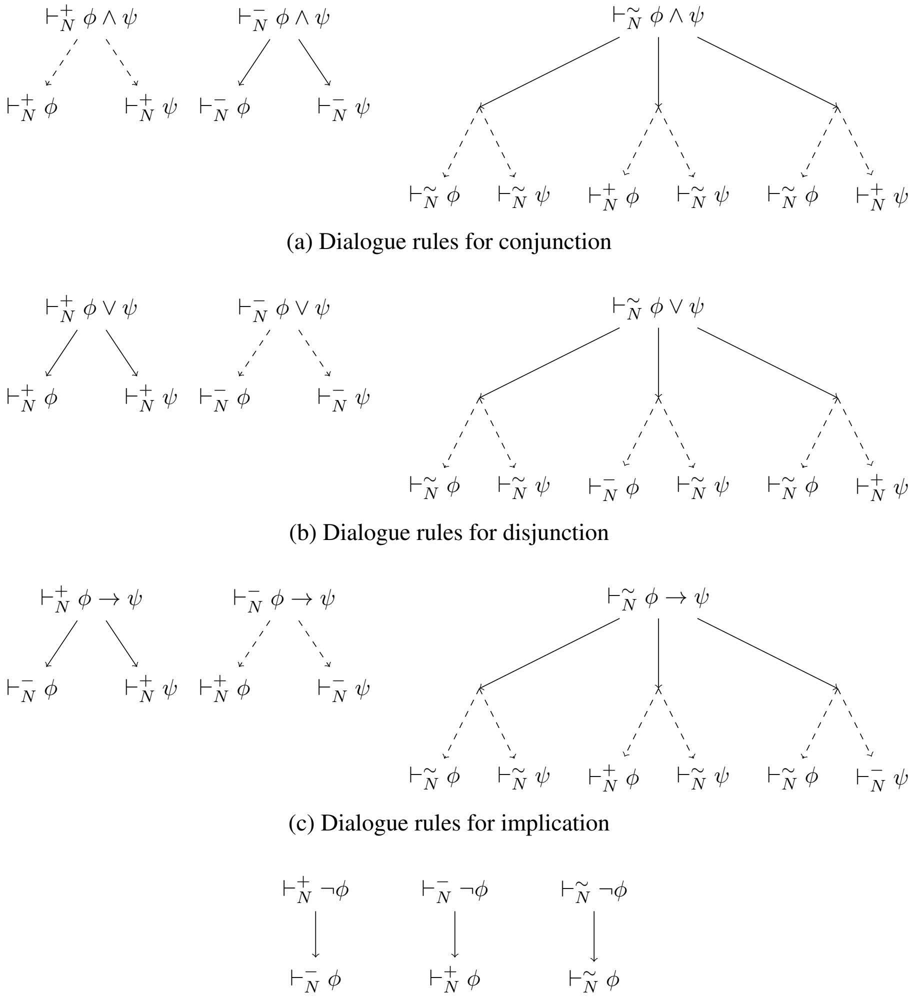
This section provides such a dialogue game in the tradition of Hintikka's evaluation game as defined in Section 2.1, essentially following [76]. Another semantic game related to Giles's game setting is presented in the next section.

There are two players. As motivated in Chapter 2 we call them *me* and *you*, deviating from Hintikka's original game. Given a fixed frame $\langle W, M \rangle$, I, the proponent in Hintikka's diction, initially assert that a formula ϕ is is either *true*, *false*, or *indefinite* at some world $N \in W$. This is denoted by me asserting either $\vdash^{+}_{N} \phi$, $\vdash^{-}_{N} \phi$, or $\vdash^{\sim}_{N} \phi$, respectively. Thus, a game state is determined by three components: (i) the formula in question, (ii) the current world, and (iii) the formula's asserted semantic status at that world. As we will see below, non-local operators require dialogue rules where the current world is changed from N to a sharpening $N' \succeq N$. Note that we do not need a role assignment (as for Hintikka's original game) stating which player is making that assertion—it is always me who asserts the semantic status. This change allows us to directly account for the third truth value, indefiniteness. During the game the initial formula ϕ is decomposed step by step according to the dialogue rules until, in the end, I asserts the semantic status of only an atomic formula $P(a)$. Assume that the game ends at the sharpening N'. If at this point I assert $\vdash^{+}_{N'} P(a)$ and if a is in the extension of P at N' then I am the winner of the game, otherwise I lose and you win. Analogously, I win if I assert $\vdash^{-}_{N'} P(a)$ and a is in the anti-extension of P at N', or if I assert $\vdash^{\sim}_{N'} P(a)$ and a is neither in the extension nor in the anti-extension of P at N'. Moreover, we will consider only models where we can identify each domain element with a constant symbol. This stipulation will turn out to be crucial for the formulation of dialogue rules for quantifiers. Note that the game is a finite two-player zero-sum game with perfect information. By Zermelo's Theorem [97] we conclude that the game is determined, i.e., either I or you always have a winning strategy.

Dialogue Rules. As described above, at each point in the game exactly one formula is asserted by me to be *true*, *false*, or *indefinite* at a certain world. The dialogue rules specify how this formula is to be further reduced to a subformula and which player has to make which choices. For instance, consider the dialogue rule for conjunction as given in Figure 5.1a: Nodes with outgoing solid edges denote game states where I have to move next, while nodes with outgoing dashed edges denote game states where you have to move next. If I assert $\vdash^{+}_{N} \phi \wedge \psi$ than you can choose whether I have to further assert $\vdash^{+}_{N} \phi$ or $\vdash^{+}_{N} \psi$ at the same world. For $\vdash^{-}_{N} \phi \wedge \psi$, on the other hand, I myself may choose. If I assert $\vdash^{\sim}_{N} \phi \wedge \psi$, I first choose whether to commit myself to asserting that both ϕ and ψ are *indefinite* at N, or that only ϕ is *indefinite* and ψ is *true*, or vice versa. In response you choose one of the two corresponding assertions.

Observe that these dialogue rules rules can be obtained directly from the strong Kleene truth tables as defined in Figure 1.1a in Section 1.3. For local quantifiers we proceed in the same way: Figure 5.2a shows the dialogue rules for the universal quantifier. Note that there are moves labeled either 'I choose a' or 'You choose a'. For $\vdash^{+}_{N} \forall x.\phi(x)$ you have to choose a constant a and the game proceeds with my assertion $\vdash^{+}_{N} \phi(a)$, whereas for $\vdash^{-}_{N} \forall x.\phi(x)$ the choice is mine. Remember that we are only interested in models where we can identify each domain element with a constant. In the third case, $\vdash^{\sim}_{N} \forall x.\phi(x)$, first you choose whether you want me to select an element a and assert $\vdash^{\sim}_{N} \phi(a)$ or if you rather opt to select a by yourself, but let me choose whether to assert $\vdash^{+}_{N} \phi(a)$ or $\vdash^{\sim}_{N} \phi(a)$. This rule can be informally motivated by observing that $\forall x.\phi(x)$ is *indefinite* if and only if for all instances a of x it holds that $\phi(a)$ is either *true* or *indefinite* and, moreover, there is at least one instance a' such that $\phi(a')$ is *indefinite*. Again, the dialogue rule for the existential quantifier is obtained analogously. Note that these dialogue rules in a three-valued setting are instances of the more general dialogue rules for many-valued logics presented in [25].

The dialogue rules above are all *local* in the sense that the current world always remains the



(a) Dialogue rules for conjunction

(b) Dialogue rules for disjunction

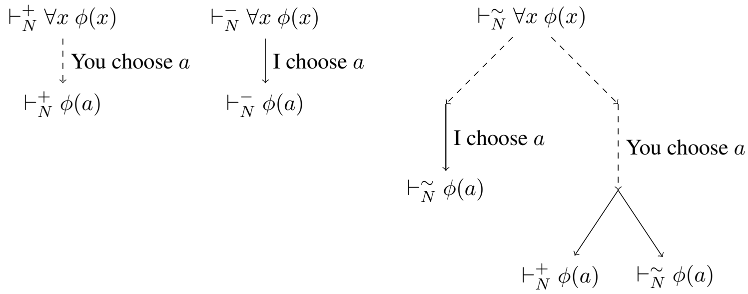
(c) Dialogue rules for implication

(d) Dialogue rules for negation

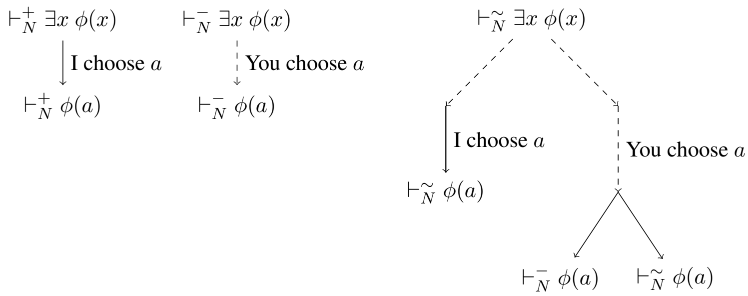
Figure 5.1: Dialogue rules for local connectives. Dashed and solid arrows denote moves by you and me, respectively.

same. Rules for the other, non-local, connectives involve choosing sharpenings of the current world by the players.

The rule for the non-local universal quantifier 'A' is given in Figure 5.3a. In the rule for $\vdash^{+}_{N} Ax.\phi(x)$ first you select a domain element a and then choose a sharpening N' of N. Next, I choose yet a further sharpening N'' of N' and assert $\vdash^{+}_{N''} \phi(a)$. According to Shapiro's definition, in order for $Ax\ \phi(x)$ to be *true* at N, after you have chosen a, the formula $Ax\ \phi(x)$ must be forced at N. By alternatively selecting further sharpenings this way, we obtain a literal translation of Shapiro's forcing condition to dialogue rules. The rule $\vdash^{-}_{N}\ Ax\ \phi(x)$ involves



(a) Dialogue rules for the local universal quantifier

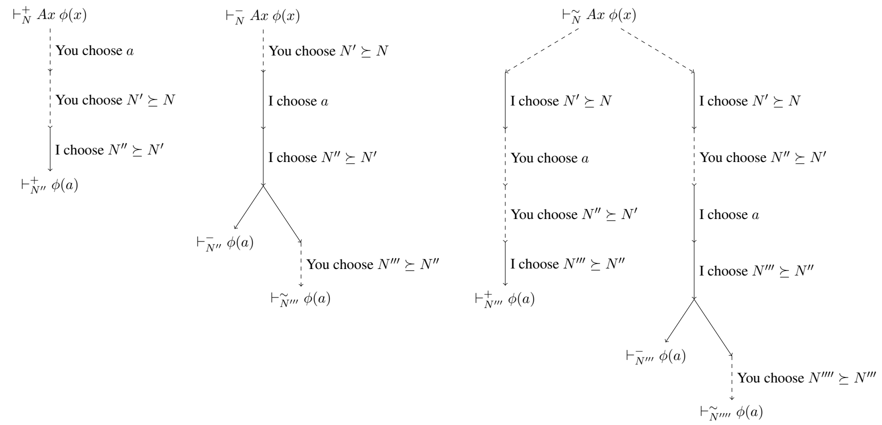


(b) Dialogue rules for the local existential quantifier

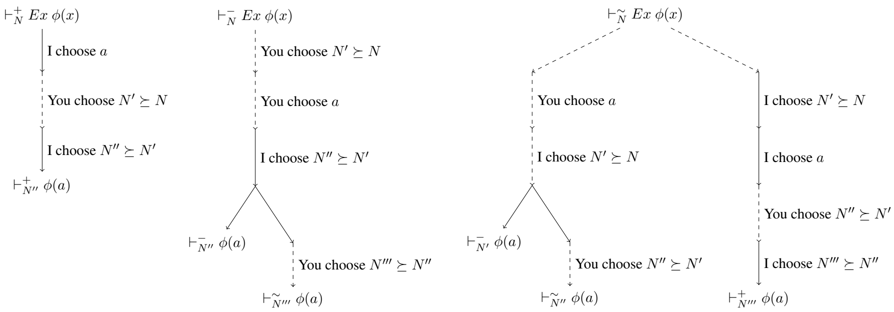
Figure 5.2: Dialogue rules for local quantifiers. Dashed and solid arrows denote moves by you and me, respectively.

Shapiro's definition of the stable failure of 'A'. According to this principle I have to show that there is no sharpening of N where $Ax\ \phi(x)$ is true. Thus, after you choose $N' \succeq N$, I select a domain element a and then show that $\phi(a)$ is not forced at N'. This means that I must find a further sharpening such that either $\phi(a)$ is false there, or is indefinite and remains so in all further sharpenings. According to the third rule, if I assert $\vdash_N^\sim\ Ax\ \phi(x)$, I must be prepared both to find a further sharpening where $Ax\ \phi(x)$ is true and to find one where it is false, respectively. Note that, after the second, move both subtrees directly refer to the dialogue rules for $\vdash_N^\sim\ Ax\ \phi(x)$ $\vdash_N^-\ Ax\ \phi(x)$.

Figure 5.3b shows the dialogue rules for the non-local existential quantifier 'E'. The difference between the rules for $\vdash_N^+\ Ex.\phi(x)$ and $\vdash_N^-\ Ex.\phi(x)$ and their counterparts for the global universal quantifier is only that here I have to pick one domain element instead of you. However, we have to give a fundamentally different rule for $\vdash_N^\sim\ Ex.\phi(x)$: if I assert that $Ex\ \phi(x)$ is indefinite at N, I have to be able to show that it is not true and to show that it is not false at N. The former case amounts to the left branch: for any given element a, I must assert that $\phi(a)$ is not forced by providing a sharpening N' of N where $\phi(a)$ is either false, or indefinite and remains



(a) Dialogue rules for the non-local universal quantifier



(b) Dialogue rules for the non-local existential quantifier

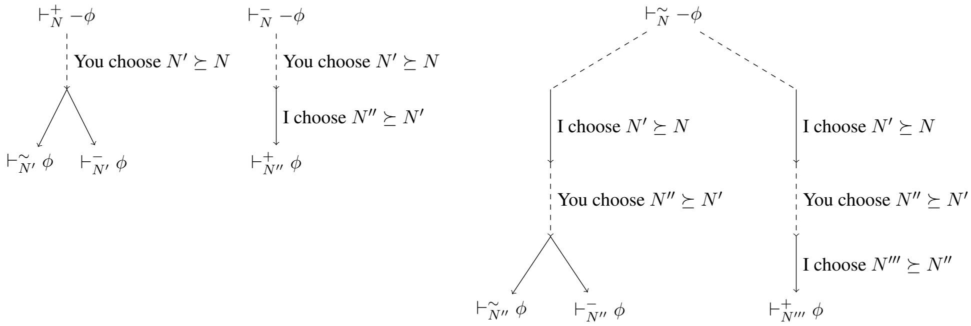
Figure 5.3: Dialogue rules for non-local quantifiers. Dashed and solid arrows denote moves by you and me, respectively.

indefinite at each further sharpening. In the latter case I show that $Ex\ \phi(x)$ is not false at N by providing a sharpening of N where the formula is true. Thus the remaining branch refers to the rule for $\vdash^{+}_{N} Ex\ \phi(x)$.

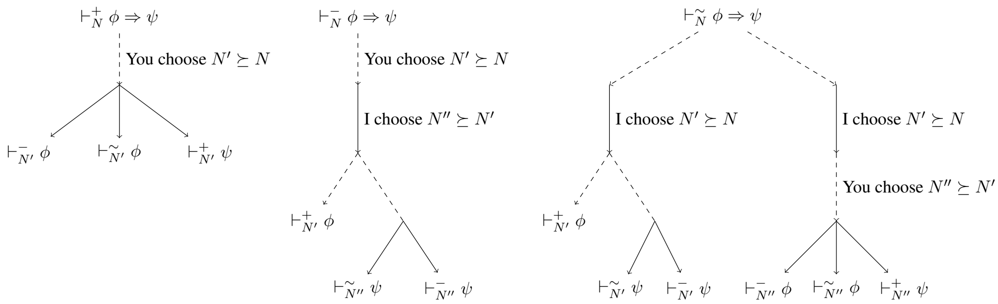
The dialogue rules for the non-local negation '$-$' and non-local implication '$\Rightarrow$' as shown in Figure 5.4 are defined in the same manner: Rules for $\vdash^{+}_{N} -\phi$ and $\vdash^{-}_{N} -\phi$ are obtained directly from Shapiro's truth and falsehood conditions; as for the non-local universal quantifier, the rule for $\vdash^{\sim}_{N} -\phi$ again forces me to find further sharpenings where the the formula is true and where it is false.

Since, as noted above, forcing can be expressed in terms of this operator, we can read the dialogue rule for $\vdash^{-}_{N} -\phi$ directly as a rule for forcing: if I want to state that a formula ϕ is forced at a sharpening N, I just assert $\vdash^{-}_{N} -\phi$.

The dialogue rule for the non-local implication '$\Rightarrow$' is obtained similarly: If I assert $\vdash^{+}_{N}$



(a) Dialogue rule for the non-local negation



(b) Dialogue rule for the non-local implication

Figure 5.4: Dialogue rule for non-local connectives. Dashed and solid arrows denote moves by you and me, respectively.

$\phi \Rightarrow \psi$, then you may select a sharpening $N' \succeq N$. Then I must either assert that the premiss ϕ is not true at N' (i.e., it is either false or indefinite) or that the conclusion ψ is true. If I assert $\vdash^{-}_{N} \phi \Rightarrow \psi$, then you select a sharpening $N' \succeq N$, and I select a sharpening $N'' \succeq N'$. There, I must be prepared to assert both $\vdash^{+}_{N''} \phi$ and one of $\vdash^{\sim}_{N''} \psi$ and $\vdash^{-}_{N''} \psi$. Finally, if I assert $\vdash^{\sim}_{N} \phi \Rightarrow \Psi$, the dialogue rule is specified in an analogous fashion as for the non-local existential quantifier. Essentially, I must be prepared to show that the formula is neither true nor false.

Adequacy of the game. We claim that the dialogue rules are adequate for Shapiro's logic in the following sense:

Theorem 11. *Given a frame $\mathcal{F}$ and a sharpening N in $\mathcal{F}$, a formula ϕ is*

- true *at N iff I have a winning strategy for the game where I initially assert $\vdash^{+}_{N} \phi$,*
- false *at N iff I have a winning strategy for the game where I initially assert $\vdash^{-}_{N} \phi$, and*
- indefinite *at N iff I have a winning strategy for the game where I initially assert $\vdash^{\sim}_{N} \phi$.*

Proof. We proof the claim that the game rules are adequate for Shapiro's logic by induction on the complexity of ϕ. If ϕ is atomic, this is obvious. Otherwise, applying one of the dialogue rules reduces ϕ to a less complex formula. For the local operators we distinguish the following cases:

$\psi \wedge \chi$: Assume that ϕ is of the form $\psi \wedge \chi$ and I assert that ϕ is true at N. If $\psi \wedge \chi$ is true at N, then both ψ and χ are true at N. By induction, I have winning strategies for both $\vdash_N^+ \psi$ and $\vdash_N^+ \chi$, and, thus, I have a winning strategy no matter which branch you choose. On the other hand, if $\psi \wedge \chi$ is not true at N, then at least either ψ or χ is not true at N. By induction, I have no winning strategy for $\vdash_N^+ \psi$ or for $\vdash_N^+ \chi$ (or both). As the following game state depends on your choice, I have no winning strategy for $\vdash_N^+ \psi \wedge \chi$.

If I assert that $\psi \wedge \chi$ is false at N we reason analogously: If $\psi \wedge \chi$ is false at N, then either ψ or χ (or both) are false at N. By induction, I have an according winning strategy for $\vdash_N^- \psi$ or for $\vdash_N^- \chi$. Since it is my choice whether the game continues with ψ or with χ, I have a winning strategy for $\vdash_N^- \psi \wedge \chi$. On the other hand, if $\psi \wedge \chi$ is not false at N, neither ψ nor χ are false at N, and, by induction, I have no winning strategy for $\vdash_N^- \psi$ nor for $\vdash_N^- \chi$, and. Thus I have no winning strategy for $\vdash_N^- \psi \wedge \chi$.

Assume that I assert that $\psi \wedge \chi$ is indefinite at N. If this claim holds then the strong Kleene truth tables (see Figure 1.1a) show that either both ψ and χ are indefinite at N or one of them is true and the other one is indefinite at N. In the first case, selecting the leftmost branch yields a winning strategy for me, since then, depending on your choice, I have to assert either $\vdash_N^\sim \phi$ or $\vdash_N^\sim \chi$ and, by induction, I have winning strategies for both these game states. Analogously, if one of ψ and χ is true and the other one indefinite at N, by selecting the middle or the right branch, by induction, I can ensure that I have a winning strategy for both of your choices.

$\neg\psi, \psi \vee \chi, \psi \to \chi$: We observe that also for the other local connectives '$\neg$', '$\vee$', and '$\to$' the dialogue rules directly correspond to the according Kleene truth tables in an analogous fashion as for '$\wedge$'.

$\forall x\ \psi(x)$: Assume that ϕ is of the form $\forall x\ \psi(x)$ and I assert that ϕ is true at N. If $\forall x\ \psi(x)$ is true at N, then for all domain elements a it holds that $\psi(a)$ is true at N. By the induction hypothesis, I have a winning strategy for $\vdash_N^+ \psi(a)$. Thus, no matter which domain element a you choose, I also have a winning strategy for $\vdash_N^+ \forall x\ \psi(x)$. On the other hand, if $\forall x\ \psi(x)$ is not true at N, then there exists an element a such that $\psi(a)$ is not true at N. If you select a, then I have to assert that $\psi(a)$ is true at N, and again, by induction, I have no winning strategy for $\vdash_N^+ \psi(a)$, thus I also have no winning strategy for for $\vdash_N^+ \forall x\ \psi(x)$.

If I assert that $\forall x\ \psi(x)$ is false at N, I have a winning strategy if and only if I can find an element a such that I have a winning strategy for $\vdash_N^- \psi(a)$. By induction, this is the case if $\psi(a)$ is false at N. If $\forall x\ \psi(a)$ is false at N then there exists such an element, yielding a winning strategy for me. If, however, $\forall x\ \psi(a)$ is not false at N, then $\psi(a)$ is true or indefinite at N for all elements a. Thus I have no winning strategy for $\vdash_N^- \psi(a)$.

Finally, assume that I assert that $\forall x\ \psi(x)$ is indefinite at N. If the formula is indeed indefinite at N, then $\psi(a)$ is indefinite at N for some element a_p and true or indefinite for all others. If you select the left branch then I choose a_p. By induction, I have a winning strategy for $\vdash_N^\sim \psi(a_p)$. If you choose the right branch, then, since $\psi(a)$ is true or indefinite at N, I have, by induction, a winning strategy for $\vdash_N^+ \psi(a)$ or for $\vdash_N^\sim \psi(a)$, respectively. Thus, I have a winning strategy for all of your choices of a. Since I have winning strategies for both branches, I also have a winning strategy for $\vdash_N^\sim \forall x\ \psi(x)$. If $\forall x\ \psi(x)$ is not indefinite at N then either $\psi(a)$ is true for all elements a or false for some element a rendering the formula true or false, respectively. In the first case, I, by induction, have no winning strategy for $\vdash_N^\sim \psi(a)$, thus, I have also no winning strategy for $\vdash_N^\sim \forall x\ \psi(x)$. In the second case, if you select a such that $\psi(a)$ is false at N then I, by induction have no winning strategies for $\vdash_N^+ \psi(a)$ or for $\vdash_N^\sim \psi(a)$, and thus I have no winning strategy for $\vdash_N^\sim \forall x\ \psi(x)$.

$\exists x\ \psi(x)$**:** For the local existential quantifier '$\exists$' we reason analogously as for '$\forall$'.

For the non-local operators we distinguish the following cases:

$Ax\ \psi(x)$**:** Assume that ϕ is of the form $Ax\ \psi(x)$ and I assert that ϕ is true at N. I have a winning strategy if and only if for all your choices of a and all sharpenings $N' \succeq N$, I can find a further sharpening N'', such that I have a winning strategy for $\vdash_{N''}^+ \psi(a)$. By induction, this is the case if and only if $\psi(a)$ is true at N'', i.e., if $\psi(a)$ is forced at N.

Assume that I assert that $Ax\ \psi(x)$ is false at N. Stable failure dictates that this is the case if and only if $Ax\ \psi(x)$ is not true at any sharpening $N' \succeq N$. Thus, if $Ax\ \psi(x)$ is false at N, then for all sharpenings $N' \succeq N$ there exist elements a such that $\psi(a)$ is not forced at N', which means that $\psi(a)$ does not eventually get true, i.e., there exists a sharpening $N'' \succeq N'$ where $\psi(a)$ is either false, or for all further sharpenings $N''' \succeq N''$ the proposition $\psi(a)$ remains indefinite. In the first case, I have, by induction, a winning strategy for $\vdash_{N''}^- \psi(a)$, while in the second case I have a winning strategy for $\vdash_{N'''}^\sim \psi(a)$ for all of your choices of $N'' \succeq N'$. Choosing a and N'' accordingly yields a winning strategy for me. On the other hand, if $Ax\ \psi(x)$ is not false at N, then there exists a sharpening $N' \succeq N$ where the formula is true, i.e., where for all elements a, $\psi(a)$ is forced at N'. However then there exists no sharpening $N'' \succeq N'$ where $\psi(a)$ is either false or remains indefinite at all further sharpenings $N''' \succeq N''$. Thus, by induction I have no winning strategy for $\vdash^{-}_{N''}\ \psi(a)$ and there exists a sharpening $N''' \succeq N''$ such that I have no winning strategy for $\vdash^{\sim}_{N'''}\ \psi(a)$. Thus, I have no winning strategy for $\vdash^{-}_{N}\ Ax\ \psi(x)$.

Assume that I assert that $Ax\ \psi(x)$ is indefinite at N. By Lemma 1 this claim holds if and only if there exist sharpenings where the formula is true and where it is false. Observe that left branch of the dialogue rule amounts to asserting that $Ax\ \psi(x)$ is true at N' while the right branch amounts to asserting that $Ax\ \psi(x)$ is false at N'. I thus have a winning strategy if and only if I can find two sharpenings where the formula is true and false, respectively.

$Ex\ \psi(x)$: Assume that ϕ is of the form $Ex\ \psi(x)$. If I assert that ϕ is either true or false at some world N, we reason analogously as for $Ax\ \psi(x)$ (only changing the player to select the element a).

Remember that Lemma 1 does not hold for the operator 'E'. Assume that I assert that ϕ is indefinite at N. This claim holds, if and only if ϕ is neither true nor false at N which means that ϕ is not true at N, but there exists a sharpening $N' \succeq N$ where ϕ is true. Observe that the left branch corresponds to the dialogue rule for $\vdash^{-}_{N}\ Ax\ \psi(x)$, i.e., if you choose that branch, I have a winning strategy if and only if $Ex\ \psi(x)$ is not true at N. Moreover, if you choose the right branch, I have a winning strategy if and only if I can find a sharpening $N' \succeq N$ where $Ex\ \psi(x)$ is true. Observe that the right branch directly amounts to the dialogue rule for $\vdash^{+}_{N}\ Ex\ \psi(x)$. I thus have a winning strategy for $\vdash^{\sim}_{N}\ Ex\ \psi(x)$ if and only if $Ex\ \psi(x)$ is indeed indefinite at N.

$-\psi$: Assume that ϕ is of the form $-\psi$ and I assert that ϕ is true at N. If this claim holds, then there exists no sharpening $N' \succeq N$ where ϕ is true. Thus, regardless of your choice of N', the formula is either false or indefinite at N' and, by induction, I have a winning strategy for $\vdash^{-}_{N'}\ \psi$ or for $\vdash^{\sim}_{N'}\ \psi$. I therefore also have a winning strategy for $\vdash^{+}_{N}\ -\psi$. On the other hand, if the claim does not hold, then there exists a sharpening $N' \succeq N$ where ψ is true. If you select such a sharpening, then I have, by induction no, winning strategy for $\vdash^{-}_{N'}\ \psi$ nor for $\vdash^{\sim}_{N'}\ \psi$, and thus I have no winning strategy for $\vdash^{+}_{N}\ -\psi$.

Assume that I assert that $-\psi$ is false at N. Shapiro argues that this is the case if and only if ψ is forced at N ([82], Section 4.2, Theorem 7 on Page 96). Thus I have a winning strategy if and only if for each of your choices of $N' \succeq N$ I can find another sharpening $N'' \succeq N'$ such that ψ holds at N''. By induction, this is the case if and only if, I have a winning strategy for $\vdash^{+}_{N''} \psi$.

Assume I assert that $-\psi$ is indefinite at N. Again, Lemma 1 states that this is the case if and only if there exist sharpenings where the formula is true and where it is false. As above, observe that both branches of contain copies of the respective dialogue rules for $\vdash^{+}_{N'} -\psi$ and for $\vdash^{-}_{N'} -\psi$.

$\psi \Rightarrow \chi$**:** Assume that ϕ is of the form $\psi \Rightarrow \chi$ and I assert that ϕ is true at N. This claim holds if and only if for all sharpenings $N' \succeq N$, whenever ψ is true at N', so is also χ, i.e., if and only if for all sharpenings $N' \succeq N$, either ψ is false or indefinite or χ is true at N. By induction, I then (and only then) have a winning strategy for $\vdash^{-}_{N'} \psi$, for $\vdash^{\sim}_{N'} \psi$, or for $\vdash^{+}_{N'} \chi$. Thus I also have a winning strategy for $\vdash^{+}_{N} \psi \Rightarrow \chi$ if and only if the formula is true at N.

Assume that I assert that ϕ is false at N. If this claim holds, there is no sharpening $N' \succeq N$ where the formula is true. This means that for all sharpenings $N' \succeq N$ there exists a sharpening $N'' \succeq N'$ which violates the condition that χ is true whenever ψ is true. This condition is violated if ψ is true at N'' while χ is not, i.e., χ is either indefinite or false at N''. By induction I have a winning strategy both for $\vdash^{+}_{N''} \psi$ and either for $\vdash^{\sim}_{N''} \chi$ or for $\vdash^{-}_{N''} \chi$. Thus I also have a winning strategy for $\vdash^{-}_{N} \psi \Rightarrow \chi$. On the other hand, if ϕ is not false at N, there exists a sharpening $N' \succeq N$ where ϕ is true. If you choose such a sharpening, no matter which sharpening $N'' \succeq N'$ I choose, the condition, that either ψ is indefinite or false, or χ is true at N'', holds. In the first case, by induction, I do not have a winning strategy for $\vdash^{+}_{N''} \psi$, in the second case I do not have winning strategies neither for $\vdash^{\sim}_{N''} \chi$ nor for $\vdash^{-}_{N''} \chi$. Thus I have no winning strategy for $\vdash^{-}_{N} \psi \Rightarrow \chi$.

Assume that I assert that ϕ is indefinite at N. This claim holds, if and only if ϕ is neither true nor false at N which means that ϕ is not true at N, but there exists a sharpening $N' \succeq N$ where ϕ is true. Observe that the left branch corresponds to the dialogue rule for $\vdash^{-}_{N} \psi \Rightarrow \chi$, i.e., if you choose that branch, I have a winning strategy if and only if $\psi \Rightarrow \chi$ is not true at N. Moreover, if you choose the right branch, I have a winning strategy if and only if I can find a sharpening $N' \succeq N$ where $\psi \Rightarrow \chi$ is true. Observe that the right branch directly amounts to the dialogue rule for $\vdash^{+}_{N} \psi \Rightarrow \chi$. I thus have a winning strategy for $\vdash^{\sim}_{N} \psi \Rightarrow \chi$ if and only if $\psi \Rightarrow \chi$ is indeed indefinite at N.

□

As noted above, forcing can be expressed in terms of the global negation '$-$'. The following corollary immediately follows from Theorem 11.

Corollary 3. *Given a frame F and a sharpening N in F, a formula ϕ is forced at N in F if and only if I have a winning strategy for the game starting in $\vdash_N^- -\phi$.*

5.3 A Giles-style Semantic Game

The Hintikka-style game presented in the previous section provides a game-based characterization of Shapiro's logic. Using a Hintikka-style game setup, local and non-local connectives and operators are characterized by dialogue rules, which provide an explicit mechanism for checking formulas in Shapiro's logic for truth, falseness, or indefiniteness. In Hintikka's original game for classical logic (as presented in Section 2.1), the players do not need to explicitly refer to the expected valuation of the formula in question; instead a role assignment marks one player the proponent and the other player the opponent. As Shapiro's logic is three-valued, however, such a role assignment is not sufficient: rather, the player *I* has to assert explicitly whether a formula is true, false, or indefinite at some world of evaluation. As a result, the dialogue rules themselves in some cases can hardly be motivated from first principles about how we would naturally use the logical connective. For example, the local connectives are merely direct translations of the strong Kleene truth tables (see Figure 1.1a). This section aims to remedy this situation by presenting a Giles-style evaluation game which provides much more natural dialogue rules. In fact, for the local connectives and operators we retain Giles's own dialogue rules. In using these rules Giles himself referred to the work of Lorenzen [63, 64] who in turn tried to derive logical rules from first principles about correct reasoning. We show, how truth and falseness of a formula can be identified with the existence of a winning strategy for *me* (for the game where *I* initially assert that formula) and *you*, respectively. However, for some games neither *you* nor *me* have a winning strategy—namely, if the initial formula is indefinite at the current world.

We proceed in two stages: first we define a Giles-style semantic game for Shapiro's logic where the current world of evaluation is fixed, amounting to an evaluation in strong Kleene logic (see Section 1.3). Then, in a second step, we introduce additional notations to allow for moves where the players may change the current world of evaluation within a given frame.

A semantic game for strong Kleene logic. As in Giles's original game presented in Section 2.3, we consider game states, where both players—*me* and *you*—may assert multisets of formulas, their tenets. These are denoted $[\Gamma \mid \Delta]$ where Γ is *your* tenet while Δ is *my* tenet. If we want to stress that a tenet is atomic, i.e., that it contains only atomic formulas, we write γ and δ, respectively. Initially, *I* am the proponent asserting the formula F in question, thus the game starts in $[\ \mid F]$. As for Giles's original game, in each round a player picks one of the other player's non-atomic assertions. Valid attacks and defenses are specified by the following dialogue rules (see Figure 2.6 for a graphical representation):

($R_\to$) If *I* assert $F \to G$ then *you* may attack by asserting F, which obliges *me* to defend by asserting G.

($R_\wedge$) If *I* assert $F \wedge G$ then *you* attack by pointing either to the left or to the right subformula. As corresponding defense, *I* then have to assert either F or G, according to your choice.

($R_\vee$) If *I* assert $F \vee G$ then *I* have to assert either F or G at *my* own choice.

($R_\neg$ If *I* assert $\neg F$ then *you* may attack by asserting F, which obliges *me* to assert $\bot$.

($R_\forall$) If *I* assert $\forall x F(x)$ then *you* attack by picking c and *I* have to defend by asserting $F(c)$.

($R_\exists$) If *I* assert $\exists x F(x)$ then *I* have to pick a constant c and assert $F(c)$.

For *my* attacks on *your* assertions the dialogue rules are completely dual.

After an attack and its associated defense, the formula is removed from the defending player's tenet and the game continues with the next round until an atomic game state is reached. Remember that for Giles's game it is possible to define also a second rule for conjunction which prompts *me* to assert both conjuncts. Such a rule can also be formulated for this game; as seen below, here both rules are equivalent, unlike as in Giles's game.

($R'_\wedge$) If *I* assert $F \wedge' G$ then, if *you* attack, I am obliged to assert both F and G or, alternatively to assert $\bot$ (and analogously if *you* assert $F \wedge' G$).

As we see, we completely retain the structure of Giles's game as well as the dialogue part of the game, where the assertion of a complex formula is decomposed into assertions of atomic formulas. We must, however, change the betting part for evaluating atomic game states. We no longer associate atomic assertions with (trials of) dispersive experiments, but with possibly vague (borderline) propositions. Still, if *I* assert a false proposition at a atomic game state, I have to pay 1€ to *you* while *you* pay 1€ to *me* for each of *your* false propositions. If the status of an atomic proposition is unsettled at the current world of evaluation, we both agree to consult a randomly selected neutral third party. (If one wants to uphold Giles's experiment metaphor, this act of consulting may be interpreted as an experiment. However, since the players cannot attach any probabilities to its outcome, it has to be modeled differently.) If this third party judges the assertion to be false, the asserting player is obliged to pay 1€ to his opponent. Note that the third party is selected for evaluating each assertion individually. Thus, if an atomic game state contains several assertions of the same formula, these are evaluated independently. From a game-theoretic point of view this is modeled by a three player game: we introduce an additional player called *Nature* (following Hintikka's diction). At an atomic game state *Nature* selects for each assertion of a proposition, which is indefinite at the current world of evaluation, whether it has to be regarded as true or as false. Thus, atomic game states are not necessarily *final* any more, since *Nature* may still make a fixed number of moves. In the end, *my* payoff amounts to the amount of money I receive from *you*, while *your* payoff is inverse to *mine*. *Nature* is always assigned the payoff zero. Finally, either *I* or *you* are declared the winner of the game depending on our payoffs just as in Giles's game.

Definition 17 (Winning conditions)**.** At a final game state, *I* am declared the winner of the game, iff *I* do not lose any money, i.e., iff *my* payoff is at least zero. Otherwise, *your* payoff is greater than zero and *you* are declared the winner of the game.

A winning strategy for *me* is a strategy which ensures that the game reaches a final state where *I* am declared the winner of the game, i.e., *my* payoff is non-negative regardless of *your* or *Nature*'s moves. A winning strategy for *you* is defined analogously. Note that for some game states neither of us may have a winning strategy. For example, if we both assert a formula which is indefinite, none of us has, because the winner of the game is determined solely by *Nature*'s choices. However, if the game reaches an atomic game state where *I* win the game, even if *Nature* judges all of *my* assertions of indefinite atomic propositions as false, while all of *yours* as true, *I* have a winning strategy and vice versa for *you*. This is reflected by the following two kinds of valuation and the notions of optimistic versus pessimistic payoff assignment:

Definition 18 (Valuation)**.** We define the *optimistic* valuation function $v^{\uparrow}$ and the *pessimistic* valuation function $v^{\downarrow}$ for an atomic formula $P(a)$ at the current world of evaluation as follows:

$$v^{\uparrow}(P(a)) = \begin{cases} 0 & \text{if } P(a) \text{ is false} \\ 1 & \text{otherwise} \end{cases} \qquad v^{\downarrow}(P(a)) = \begin{cases} 1 & \text{if } P(a) \text{ is true} \\ 0 & \text{otherwise} \end{cases} .$$

Definition 19 (Optimistic vs. pessimistic payoff)**.** For an atomic game state $[\gamma \mid \delta]$ we define *my optimistic* and *my pessimistic* payoff, denoted $\langle \gamma \mid \delta \rangle^{\uparrow}$ and $\langle \gamma \mid \delta \rangle^{\downarrow}$, respectively, as

$$\langle \gamma \mid \delta \rangle^{\uparrow} =_{df} \sum_{G \in \gamma} (1 - v^{\uparrow}(G)) - \sum_{D \in \delta} (1 - v^{\downarrow}(D))$$
$$\langle \gamma \mid \delta \rangle^{\downarrow} =_{df} \sum_{G \in \gamma} (1 - v^{\downarrow}(G)) - \sum_{D \in \delta} (1 - v^{\uparrow}(D)).$$

We only define *my* optimistic and pessimistic payoff and note that these are inverse to *your* pessimistic and optimistic payoffs, respectively.

Lemma 2. *Consider an atomic game state* $[\gamma \mid \delta]$*.* I *have a winning strategy if and only if* $\langle\gamma \mid \delta\rangle^{\downarrow} \geq 0$ *while* you *have a winning strategy if and only if* $\langle\gamma \mid \delta\rangle^{\uparrow} < 0$*.*

Proof. Observe that $\langle\gamma \mid \delta\rangle^{\downarrow}$ denotes *my* payoff for the worst case (from *my* point of view) for *Nature*'s choices. Thus, if $\langle\gamma \mid \delta\rangle^{\downarrow} \geq 0$, *my* actual payoff will never fall below zero. Vice versa, observe that $-\langle\gamma \mid \delta\rangle^{\uparrow}$ denotes *your* payoff for the worst case (from *your* point of view). Thus if, $-\langle\gamma \mid \delta\rangle^{\uparrow} > 0$, *your* actual payoff will always exceed zero. □

We observe that the valuation functions $v^{\uparrow}$ and $v^{\downarrow}$ can be extended to arbitrary formulas in an obvious way: we just check whether a *compound* formula locally evaluates to true, false, or indefinite at the current world according to the Kleene truth tables (see Figure 1.1a). Based on these more general valuation function, also the notions of *my* optimistic and pessimistic payoff $\langle\gamma \mid \delta\rangle^{\uparrow}$ and $\langle\gamma \mid \delta\rangle^{\downarrow}$ can be extended from atomic game states in a straight-forward way:

Definition 20 (Optimistic vs. pessimistic payoff)**.** For an arbitrary game state $[\Gamma \mid \Delta]$ we define *my optimistic* and *my pessimistic* payoff, denoted $\langle\Gamma \mid \Delta\rangle^{\uparrow}$ and $\langle\Gamma \mid \Delta\rangle^{\downarrow}$, respectively, as

$$\langle\Gamma \mid \Delta\rangle^{\uparrow} =_{df} \sum_{G\in\Gamma}(1 - v^{\uparrow}(G)) - \sum_{D\in\Delta}(1 - v^{\downarrow}(D))$$
$$\langle\Gamma \mid \Delta\rangle^{\downarrow} =_{df} \sum_{G\in\Gamma}(1 - v^{\downarrow}(G)) - \sum_{D\in\Delta}(1 - v^{\uparrow}(D)).$$

Remark. Note that these extended payoff functions for non-atomic game states do not a priori correspond to payoff values in the game, they are at this point only to be seen as extensions assigning some value to game states.

Lemma 3. *Consider an arbitrary game state* $\mathcal{G} = [\Gamma \mid \Delta]$*. Then*

(i) I *have a winning strategy for the game starting in* $\mathcal{G}$ *iff* $\langle\Gamma \mid \Delta\rangle^{\downarrow} \geq 0$ *and*

(ii) you *have a winning strategy for the game starting in* $\mathcal{G}$ *iff* $\langle\Gamma \mid \Delta\rangle^{\uparrow} < 0$*.*

Proof. We first prove claim (i) by induction on the complexity of $\mathcal{G}$ (i.e., the number of connectives in $\mathcal{G}$). If $[\Gamma \mid \Delta]$ is atomic, the claim holds since we have reached an atomic state and, by Lemma 2, I have a winning strategy iff $\langle\Gamma \mid \Delta\rangle^{\downarrow} \geq 0$. If $[\Gamma \mid \Delta]$ is not atomic, we distinguish the following cases:

$[\Gamma \mid \Delta', F \to G]$: If *you* attack *my* assertion of $F \to G$, the succeeding game state, according to rule $(R_{\to})$, will be either $[\Gamma \mid \Delta']$ or $[F, \Gamma \mid \Delta', G]$ at *your* choice. Thus, we need to show that $\langle\Gamma \mid \Delta', F \to G\rangle^{\downarrow} \geq 0$ iff both $\langle\Gamma \mid \Delta'\rangle^{\downarrow} \geq 0$ and $\langle F, \Gamma \mid \Delta', G\rangle^{\downarrow} \geq 0$. By induction we then conclude that $\langle\Gamma \mid \Delta', F \to G\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for both states; therefore also $\langle\Gamma \mid \Delta', F \to G\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[\Gamma \mid \Delta', F \to G]$.

If $F \to G$ is true at the current world of evaluation, then $\langle \Gamma \mid \Delta', F \to G\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow}$ by Definition 20 and thus $\langle \Gamma \mid \Delta', F \to G\rangle^{\downarrow} \geq 0$ iff $\langle \Gamma \mid \Delta'\rangle^{\downarrow} \geq 0$. Moreover, we have $\langle F, \Gamma \mid \Delta', G\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow}$ if both F and G are true or false, or F is false and G is indefinite. Otherwise, if F is false and G is true or indefinite we have $\langle F, \Gamma \mid \Delta', G\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow}+1$. In any case, according to the strong Kleene truth tables, $\langle F, \Gamma \mid \Delta', G\rangle^{\downarrow} \geq 0$ if $\langle \Gamma \mid \Delta', F \to G\rangle^{\downarrow} \geq 0$ whenever $F \to G$ is true at the current world of evaluation.

If $F \to G$ is false or indefinite at the current world of evaluation, then $\langle \Gamma \mid \Delta', F \to G\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow} - 1$, thus if $\langle \Gamma \mid \Delta', F \to G\rangle^{\downarrow} \geq 0$ then also $\langle \Gamma \mid \Delta'\rangle^{\downarrow} \geq 0$. Moreover, $\langle F, \Gamma \mid \Delta', G\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow} - 1$ if F is true or indefinite and G is false or indefinite. Thus, according to the strong Kleene truth tables, we have $\langle F, \Gamma \mid \Delta', G\rangle^{\downarrow} = \langle \Gamma \mid \Delta', F \to G\rangle^{\downarrow}$, and therefore $\langle F, \Gamma \mid \Delta', G\rangle^{\downarrow} \geq 0$ iff $\langle \Gamma \mid \Delta', F \to G\rangle^{\downarrow} \geq 0$, whenever $F \to G$ is false or indefinite at the current world of evaluation.

$[F \to G, \Gamma' \mid \Delta]$: If *I* attack *your* assertion of $F \to G$, the succeeding game state, according to rule ($R_{\to}$), will be either $[\Gamma' \mid \Delta]$ or $[G, \Gamma' \mid \Delta, F]$ at *my* choice. Thus, we need to show that $\langle F \to G, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff either $\langle \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ or $\langle G, \Gamma' \mid \Delta, F\rangle^{\downarrow} \geq 0$ (or both). By induction we then conclude that $\langle \Gamma', F \to G \mid \Delta\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for one of these two states; therefore also $\langle \Gamma', F \to G \mid \Delta\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[\Gamma', F \to G \mid \Delta]$.

If $F \to G$ is true or indefinite at the current world of evaluation, then $\langle F \to G, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow}$ and thus $\langle F \to G, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff $\langle \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$. Moreover, $\langle G, \Gamma' \mid \Delta, F\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow}$ if F is true and G is either true or indefinite and if G is false and F is either false or indefinite. Otherwise, if F is false or indefinite and G is true or indefinite, we have $\langle G, \Gamma' \mid \Delta, F\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow} - 1$. In any case, according to the strong Kleene truth tables, we have $\langle F \to G, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ if $\langle G, \Gamma' \mid \Delta, F\rangle^{\downarrow} \geq 0$ whenever $F \to G$ is true or indefinite at the current world of evaluation.

If $F \to G$ is false at the current world of evaluation, then $\langle F \to G, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow} + 1$ and thus $\langle F \to G, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ if $\langle \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$. Moreover, we have $\langle G, \Gamma' \mid \Delta, F\rangle^{\downarrow} = \langle F \to G, \Gamma' \mid \Delta\rangle^{\downarrow}$ if F is true and G is false. According to the strong Kleene truth tables, we have $\langle F \to G, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff $\langle G, \Gamma' \mid \Delta, F\rangle^{\downarrow} \geq 0$ whenever $F \to G$ is false at the current world of evaluation.

$[\Gamma \mid \Delta', F \wedge G]$: If *you* attack *my* assertion of $F \wedge G$, the succeeding game state, according to rule ($R_{\wedge}$), will be either $[\Gamma \mid \Delta', F]$ or $[\Gamma \mid \Delta', G]$ at *your* choice. Thus, we need to show that $\langle \Gamma \mid \Delta', F \wedge G\rangle^{\downarrow} \geq 0$ iff both $\langle \Gamma \mid \Delta', F\rangle^{\downarrow} \geq 0$ and $\langle \Gamma \mid \Delta', G\rangle^{\downarrow} \geq 0$. By induction we then conclude that $\langle \Gamma \mid \Delta', F \wedge G\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for both of these two states; therefore also$\langle \Gamma \mid \Delta', F \wedge G\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[\Gamma \mid \Delta', F \wedge G]$.

If $F \wedge G$ is true, then both F and G are true at the current world of evaluation. Thus $\langle \Gamma \mid \Delta', F \wedge G\rangle^{\downarrow} = \langle \Gamma \mid \Delta', F\rangle^{\downarrow} = \langle \Gamma \mid \Delta', G\rangle^{\downarrow}$ and the claim obviously holds.

Assume that $F \wedge G$ is false or indefinite at the current world of evaluation. If F is false or indefinite then $\langle \Gamma \mid \Delta', F \wedge G\rangle^{\downarrow} = \langle \Gamma \mid \Delta', F\rangle^{\downarrow}$, thus $\langle \Gamma \mid \Delta', F \wedge G\rangle^{\downarrow} \geq 0$ iff $\langle \Gamma \mid \Delta', F\rangle^{\downarrow} \geq 0$. Moreover, $\langle \Gamma \mid \Delta', F \wedge G\rangle^{\downarrow} = \langle \Gamma \mid \Delta', G\rangle^{\downarrow}$ if also G is false or indefinite. Otherwise, if G is true, we have $\langle \Gamma \mid \Delta', F \wedge G\rangle^{\downarrow} = \langle \Gamma \mid \Delta', G\rangle^{\downarrow} - 1$. In any case, we have $\langle \Gamma \mid \Delta', G\rangle^{\downarrow} \geq 0$ if $\langle \Gamma \mid \Delta', F \wedge G\rangle^{\downarrow} \geq 0$. If F is true, then according to the strong Kleene truth tables, G is either false or indefinite, and we reason analogously as above.

$[F \wedge G, \Gamma' \mid \Delta]$: If *I* attack *your* assertion of $F \wedge G$, the succeeding game state, according to rule $(R_\wedge)$, will be either $[F, \Gamma' \mid \Delta]$ or $[G, \Gamma' \mid \Delta]$ at *my* choice. Thus, we need to show that $\langle F \wedge G, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff either $\langle F, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ or $\langle G, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ (or both). By induction we then conclude that $\langle F \wedge G, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for one of these two states; therefore also $\langle F \wedge G, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[F \wedge G, \Gamma' \mid \Delta]$.

If $F \wedge G$ is true or indefinite at the current world of evaluation, then according to the strong Kleene truth tables, both F and G are not false. Thus $\langle F \wedge G, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle F, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle G, \Gamma' \mid \Delta\rangle^{\downarrow}$ and the claim obviously holds.

Assume that $F \wedge G$ is false at the current world of evaluation. If F is false then we have $\langle F \wedge G, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle F, \Gamma' \mid \Delta\rangle^{\downarrow}$, thus $\langle F \wedge G, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff $\langle F, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$. Moreover, $\langle F \wedge G, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle G, \Gamma' \mid \Delta\rangle^{\downarrow}$ if also G is false. Otherwise, if G is true or indefinite, we have $\langle F \wedge G, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle G, \Gamma' \mid \Delta\rangle^{\downarrow} + 1$. In any case, we have $\langle F \wedge G, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ if $\langle G, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$. If F is true or indefinite, then according to the strong Kleene truth tables, G is false, and we reason analogously as above.

$[\Gamma \mid \Delta', F \vee G]$,$[F \vee G, \Gamma' \mid \Delta]$: For assertions of $F \vee G$ we reason analogously as above for $F \wedge G$.

$[\Gamma \mid \Delta', \neg F]$: If *you* attack *my* assertion of $\neg F$, the succeeding game state, according to rule $(R_\neg)$, will be $[F, \Gamma \mid \Delta', \bot]$. If $\neg F$ is false or indefinite at the current world of evaluation, then according to the strong Kleene truth tables, F is true or indefinite. Otherwise, if $\neg F$ is true, then F is false. In both cases we have $\langle \Gamma \mid \Delta', \neg F\rangle^{\downarrow} = \langle F, \Gamma \mid \Delta', \bot\rangle^{\downarrow}$ and thus $\langle \Gamma \mid \Delta', \neg F\rangle^{\downarrow} \geq 0$ iff $\langle F, \Gamma \mid \Delta', \bot\rangle^{\downarrow} \geq 0$. By induction we conclude that

$\langle \Gamma \mid \Delta', \neg F\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for the succeeding state; therefore also $\langle \Gamma \mid \Delta', \neg F\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[\Gamma \mid \Delta', \neg F]$.

$[\neg F, \Gamma' \mid \Delta]$: If *I* attack *your* assertion of $\neg F$ we reason analogously as for *your* attack on *my* assertion of $\neg F$: The succeeding game state is $[\bot, \Gamma' \mid \Delta, F]$ and for all valuations of F at the current world we have $\langle \neg F, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle \bot, \Gamma' \mid \Delta, F\rangle^{\downarrow}$. By induction we conclude that $\langle \neg F, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for the succeeding state; therefore also $\langle \neg F, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[\neg F, \Gamma' \mid \Delta]$.

$[\Gamma \mid \Delta', \forall x\, F(x)]$: If *you* attack *my* assertion of $\forall x\, F(x)$, the succeeding game state, according to rule $(R_\forall)$, will be $[\Gamma \mid \Delta', F(a)]$ where the constant a is chosen by *you*. If $\forall x\, F(x)$ is true at the current world of evaluation, then for all of your choices of a also $F(a)$ is true, thus $\langle \Gamma \mid \Delta', \forall x\, F(x)\rangle^{\downarrow} = \langle \Gamma \mid \Delta', F(a)\rangle^{\downarrow}$ and thus $\langle \Gamma \mid \Delta', \forall x\, F(x)\rangle^{\downarrow} \geq 0$ iff $\langle \Gamma \mid \Delta', F(a)\rangle^{\downarrow} \geq 0$. Otherwise, if $\forall x\, F(x)$ is false or indefinite, then for some choice of a also $F(a)$ is false or indefinite, thus the equality also holds. For all of *your* other choices where $F(a)$ is true we have $\langle \Gamma \mid \Delta', \forall x\, F(x)\rangle^{\downarrow} = \langle \Gamma \mid \Delta', F(a)\rangle^{\downarrow} - 1$ and thus $\langle \Gamma \mid \Delta', F(a)\rangle^{\downarrow} \geq 0$ if $\langle \Gamma \mid \Delta', \forall x\, F(x)\rangle^{\downarrow} \geq 0$.

By induction we conclude that $\langle \Gamma \mid \Delta', \forall x\, F(x)\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for all possible successor states; therefore also $\langle \Gamma \mid \Delta', \forall x\, F(x)\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[\Gamma \mid \Delta', \forall x F(x)]$.

$[\forall x\, F(x), \Gamma' \mid \Delta]$: If *I* attack *your* assertion of $\forall x\, F(x)$, the succeeding game state, according to rule $(R_\forall)$, will be $[F(a), \Gamma' \mid \Delta]$ where the constant a is chosen by *me*. If $\forall x\, F(x)$ is true or indefinite at the current world of evaluation, then for all of *my* choices of a also $F(a)$ is true or indefinite, thus $\langle \forall x\, F(x), \Gamma' \mid \Delta\rangle^{\downarrow} = \langle F(a), \Gamma' \mid \Delta\rangle^{\downarrow}$ and thus $\langle \forall x\, F(x), \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff $\langle F(a), \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$. Otherwise, if $\forall x\, F(x)$ is false, then for some choice of a also $F(a)$ is false, thus the equality also holds. For all of *my* other choices where $F(a)$ is true or indefinite we have $\langle \forall x\, F(x), \Gamma' \mid \Delta\rangle^{\downarrow} = \langle F(a), \Gamma' \mid \Delta\rangle^{\downarrow} + 1$ and thus $\langle \forall x\, F(x), \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ if $\langle F(a), \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$.

By induction we conclude that $\langle \forall x\, F(x), \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for a successor state depending on *my* choice; therefore also $\langle \forall x\, F(x), \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[\forall x\, F(x), \Gamma' \mid \Delta]$.

$[\Gamma \mid \Delta', \exists x\, F(x)]$**,** $[\exists x\, F(x), \Gamma' \mid \Delta]$: For assertions of $, \exists x\ F(x)$ we reason analogously as above for $, \forall x\, F(x)$.

Summarizing, we have shown, that *I* have a winning strategy for the game starting in $[\Gamma \mid \Delta]$ if $\langle \Gamma \mid \Delta\rangle^{\downarrow} \leq 0$ and therefore *I* have a winning strategy for the game starting in $[\,\mid F]$ iff F is true at the current world. We still need to prove that *you* have a winning strategy for that game iff F is false. Analogously as above we prove claim (ii) stating that *you* have a winning strategy for the game starting in $[\Gamma \mid \Delta]$ if $\langle \Gamma \mid \Delta \rangle^{\uparrow} > 0$ by induction on the complexity of $[\Gamma \mid \Delta]$. Again, if $[\Gamma \mid \Delta]$ is atomic, this immediately follows from Lemma 2. If the game state is non-atomic we proceed as above for *my* winning strategies. We therefore focus only on assertions of $F \to G$ and observe that other assertions of compound formulas are treated in a completely analogous way.

$[\Gamma \mid \Delta', F \to G]$: If *you* attack *my* assertion of $F \to G$, the succeeding game state, according to rule $(R_{\to})$, will be either $[\Gamma \mid \Delta']$ or $[F, \Gamma \mid \Delta', G]$ at *your* choice. Thus, we need to show that $\langle \Gamma \mid \Delta', F \to G \rangle^{\uparrow} < 0$ iff either $\langle \Gamma \mid \Delta' \rangle^{\uparrow} < 0$ or $\langle F, \Gamma \mid \Delta', G \rangle^{\uparrow} < 0$ (or both). By induction we then conclude that $\langle \Gamma \mid \Delta', F \to G \rangle^{\uparrow} < 0$ iff *you* have a winning strategy for one of these two states; therefore also $\langle \Gamma \mid \Delta', F \to G \rangle^{\uparrow} < 0$ iff *you* have a winning strategy for $[\Gamma \mid \Delta', F \to G]$.

If $F \to G$ is true or indefinite at the current world of evaluation, then $\langle \Gamma \mid \Delta', F \to G \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow}$ by Definition 20 and thus $\langle \Gamma \mid \Delta', F \to G \rangle^{\uparrow} < 0$ iff $\langle \Gamma \mid \Delta' \rangle^{\uparrow} < 0$. Moreover, we have $\langle F, \Gamma \mid \Delta', G \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow}$ if F is true and G is true or indefinite and also if G is false and F is false or indefinite. Otherwise, if F is false or indefinite and G is true or indefinite, we have $\langle F, \Gamma \mid \Delta', G \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} + 1$. In any case, we have $\langle \Gamma \mid \Delta', F \to G \rangle^{\uparrow} < 0$ if $\langle F, \Gamma \mid \Delta', G \rangle^{\uparrow} < 0$.

If $F \to G$ is false at the current world of evaluation, then $\langle \Gamma \mid \Delta', F \to G \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} - 1$ and thus $\langle \Gamma \mid \Delta', F \to G \rangle^{\uparrow} < 0$ if $\langle \Gamma \mid \Delta' \rangle^{\uparrow} < 0$. Moreover, according to the strong Kleene truth tables, F is true and G is false, thus $\langle F, \Gamma \mid \Delta', G \rangle^{\uparrow} = \langle \Gamma \mid \Delta', F \to G \rangle^{\uparrow}$ and thus $\langle F, \Gamma \mid \Delta', G \rangle^{\uparrow} < 0$ iff $\langle \Gamma \mid \Delta', F \to G \rangle^{\uparrow} < 0$.

$[\Gamma', F \to G \mid \Delta]$: If *I* attack *your* assertion of $F \to G$, the succeeding game state, according to rule $(R_{\to})$, will be either $[\Gamma' \mid \Delta]$ or $[G, \Gamma' \mid \Delta, F]$ at *my* choice. Thus, we need to show that $\langle F \to G, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$ iff both $\langle \Gamma' \mid \Delta \rangle^{\uparrow} < 0$ and $\langle G, \Gamma' \mid \Delta, F \rangle^{\uparrow} < 0$. By induction we then conclude that $\langle \Gamma', F \to G \mid \Delta \rangle^{\uparrow} < 0$ iff *you* have a winning strategy for both of these two states; therefore also $\langle \Gamma', F \to G \mid \Delta \rangle^{\uparrow} < 0$ iff *you* have a winning strategy for $[\Gamma', F \to G \mid \Delta]$.

If $F \to G$ is true at the current world of evaluation, then $\langle F \to G, \Gamma' \mid \Delta \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow}$ and thus $\langle F \to G, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$ iff $\langle \Gamma' \mid \Delta \rangle^{\uparrow} < 0$. Moreover, $\langle G, \Gamma' \mid \Delta, F \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow}$ if F is false and G is false or indefinite or G is true and F is true or indefinite. Otherwise, if F is false and G is true, we have $\langle G, \Gamma' \mid \Delta, F \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow} - 1$. In any case, according to the strong Kleene truth tables, we have $\langle G, \Gamma' \mid \Delta, F \rangle^{\uparrow} < 0$ if $\langle F \to G, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$ whenever $F \to G$ is true at the current world of evaluation.

If $F \rightarrow G$ is either false or indefinite at the current world of evaluation, then $\langle F \to G, \Gamma' \mid \Delta\rangle^{\uparrow} = \langle \Gamma' \mid \Delta\rangle^{\uparrow} + 1$ and thus $\langle \Gamma' \mid \Delta\rangle^{\uparrow} < 0$ if $\langle F \to G, \Gamma' \mid \Delta\rangle^{\uparrow} < 0$. Moreover, we have $\langle G, \Gamma' \mid \Delta, F\rangle^{\uparrow} = \langle \Gamma' \mid \Delta\rangle^{\uparrow} + 1$ if F is true or indefinite and G is false or indefinite. According to the strong Kleene truth tables, we have $\langle F \to G, \Gamma' \mid \Delta\rangle^{\uparrow} < 0$ iff $\langle G, \Gamma' \mid \Delta, F\rangle^{\downarrow} < 0$ whenever $F \rightarrow G$ is false or indefinite at the current world of evaluation.

This shows that *you* indeed have a winning strategy for the game starting in $[\Gamma \mid \Delta]$ if $\langle \Gamma \mid \Delta\rangle^{\uparrow} < 0$ and therefore *you* have a winning strategy for the game starting in $[\ \mid F]$ iff F is false at the current world. □

Theorem 12. *Consider the game $\mathcal{G} = [\ \mid F]$ starting with* my *assertion of F.*

(i) I *have a winning strategy for $\mathcal{G}$ iff F is* true*,*

(ii) you *have a winning strategy for $\mathcal{G}$ iff F is* false*, and*

(iii) *neither of us has a winning strategy for $\mathcal{G}$ iff F is* indefinite

at the current world of evaluation.

Proof. Observe that the extended optimistic and pessimistic payoffs $\langle\ \mid F\rangle^{\uparrow}$ and $\langle\ \mid F\rangle^{\downarrow}$ either amount to 0 or to -1 depending on the valuation of F at the current world of evaluation. We immediately see that $\langle\ \mid F\rangle^{\downarrow} = 0$ iff F is true and $\langle\ \mid F\rangle^{\uparrow} = -1$ iff F is false. Thus, $\langle\ \mid F\rangle^{\downarrow} \geq 0$ iff F is true and $\langle\ \mid F\rangle^{\uparrow} < 0$ iff F is false. By Lemma 3 *I* thus have a winning strategy for $[\ \mid F]$ iff F is true at the current world of evaluation and *you* have a winning strategy for $[\ \mid F]$ iff F is false. The third clause of Theorem 12 follows immediately. □

Remark. We can show in analogously to the proof of Lemma 3 that the alternative dialogue rule for conjunction $(R'_{\wedge})$ is indeed adequate as well: If *you* attack *my* assertion of $A \wedge B$ in the game state $[\Gamma \mid \Delta', A \wedge B]$ then the successor state is either $[\Gamma \mid \Delta', A, B]$ or $[\Gamma \mid \Delta', \bot]$.

If $A \wedge B$ is true at the current world of evaluation, then according to the strong Kleene truth tables, A and B are both true, thus $\langle \Gamma \mid \Delta', A \wedge B\rangle^{\downarrow} = \langle \Gamma \mid \Delta', A, B\rangle^{\downarrow}$ and we have $\langle \Gamma \mid \Delta', A \wedge B\rangle^{\downarrow} \geq 0$ iff $\langle \Gamma \mid \Delta', A, B\rangle^{\downarrow} \geq 0$. Moreover, $\langle \Gamma \mid \Delta', \bot\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow} - 1$ and thus $\langle \Gamma \mid \Delta', A \wedge B\rangle^{\downarrow} \geq 0$ if $\langle \Gamma \mid \Delta', \bot\rangle^{\downarrow} \geq 0$.

If $A \wedge B$ is false or indefinite at the current world of evaluation, then either A or B is false or indefinite (or both are). Thus $\langle \Gamma \mid \Delta', A, B\rangle^{\downarrow} \leq \langle \Gamma \mid \Delta', A \wedge B\rangle^{\downarrow}$ and we have $\langle \Gamma \mid \Delta', A \wedge B\rangle^{\downarrow} \geq 0$ if $\langle \Gamma \mid \Delta', A, B\rangle^{\downarrow} \geq 0$. Moreover, we have $\langle \Gamma \mid \Delta', \bot\rangle^{\downarrow} = \langle \Gamma \mid \Delta', A \wedge B\rangle^{\downarrow}$ and thus $\langle \Gamma \mid \Delta', A \wedge B\rangle^{\downarrow} \geq 0$ iff $\langle \Gamma \mid \Delta', \bot\rangle^{\downarrow} \geq 0$.

Therefore, $\langle \Gamma \mid \Delta', A \wedge B\rangle^{\downarrow} \geq 0$ iff either $\langle \Gamma \mid \Delta', A, B\rangle^{\downarrow} \geq 0$ or $\langle \Gamma \mid \Delta', \bot\rangle^{\downarrow} \geq 0$ (or both). By induction we reason that $\langle \Gamma \mid \Delta', A \wedge B\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[\Gamma \mid \Delta', A, B]$ or for $[\Gamma \mid \Delta', \bot]$ and thus we have $\langle \Gamma \mid \Delta', A \wedge B\rangle^{\downarrow} \geq 0$ iff if *I* have a winning strategy for $[\Gamma \mid \Delta', A \wedge B]$ using Rule $(R'_{\wedge})$. For *my* attack on *your* assertion of $A \wedge B$ using Rule $(R'_{\wedge})$ as well as for *your* winning strategies we reason analogously.

A semantic game for Shapiro's logic. We first introduce two flags which allow the players to modify their assertions. Using the payoff scheme (betting scheme) introduced above, when asserting a formula which is indefinite at the world of evaluation, a neutral third party is asked to judge the proposition. From now on, we allow assertions of *maybe* F and *surely* F for an arbitrary formula F, denoted $F\uparrow$ and $F\downarrow$, respectively. As seen below, these 'hedges' will turn out to be crucial for defining dialogue rules for the non-local operators '$-$' and '$\Rightarrow$'.

In the final betting game, an atomic assertion $A\downarrow$ evaluates to true iff A is true at the current world of evaluation, and the asserting player has to pay 1€ to his opponent otherwise. Similarly, an atomic assertion $A\uparrow$ always evaluates to true iff A is true or indefinite at the current world of evaluation, and the asserting player has to pay 1€ to his opponent otherwise. Therefore assertions marked with $\uparrow$ or $\downarrow$ never involve an external judge.

Definition 21 (Optimistic vs. pessimistic payoff with assertion modifiers)**.** For an atomic game state $\left[\gamma, \gamma^{\uparrow}, \gamma^{\downarrow} \mid \delta, \delta^{\uparrow}, \delta^{\downarrow}\right]$, where $\gamma^{\uparrow}$ and $\delta^{\uparrow}$ denote assertions marked by $\uparrow$, $\gamma^{\downarrow}$ and $\delta^{\downarrow}$ denote assertions marked by $\downarrow$, and γ and δ denote unmarked assertions, we extend the notions of *my optimistic* and *my pessimistic* payoff as follows:

$$\begin{aligned}
\left\langle \gamma, \gamma^{\uparrow}, \gamma^{\downarrow} \mid \delta, \delta^{\uparrow}, \delta^{\downarrow}\right\rangle^{\uparrow} =_{df} & \sum_{G \in \gamma \cup \gamma^{\uparrow}} (1 - v^{\uparrow}(G)) + \sum_{G \in \gamma^{\downarrow}} (1 - v^{\downarrow}(G)) \\
& - \sum_{D \in \delta \cup \delta^{\downarrow}} (1 - v^{\downarrow}(D)) - \sum_{D \in \delta^{\uparrow}} (1 - v^{\uparrow}(D)) \\
\left\langle \gamma, \gamma^{\uparrow}, \gamma^{\downarrow} \mid \delta, \delta^{\uparrow}, \delta^{\downarrow}\right\rangle^{\downarrow} =_{df} & \sum_{G \in \gamma \cup \gamma^{\downarrow}} (1 - v^{\downarrow}(G)) + \sum_{G \in \gamma^{\uparrow}} (1 - v^{\uparrow}(G)) \\
& - \sum_{D \in \delta \cup \delta^{\uparrow}} (1 - v^{\uparrow}(D)) - \sum_{D \in \delta^{\downarrow}} (1 - v^{\downarrow}(D)).
\end{aligned}$$

We observe that Lemma 3 holds also for game states with assertions of the form $F\uparrow$ and $F\downarrow$. Still, $\left\langle \gamma, \gamma^{\uparrow}, \gamma^{\downarrow} \mid \delta, \delta^{\uparrow}, \delta^{\downarrow}\right\rangle^{\uparrow}$ denotes *my* payoff in the worst case (from *my* point of view) for *Nature*'s choices. Thus, *I* have a winning strategy iff $\left\langle \gamma, \gamma^{\uparrow}, \gamma^{\downarrow} \mid \delta, \delta^{\uparrow}, \delta^{\downarrow}\right\rangle^{\uparrow} \geq 0$ and analogously for *your* winning strategies.

For decomposing assertions marked *maybe* or *surely* we extend Giles's dialogue rules given above as follows. For the connectives '$\wedge$' and '$\vee$' and for the quantifiers '$\forall$' '$\exists$' the modifier just carries over to the new assertion. Thus, for example, if *I* assert *maybe* $F \wedge G$ then *I* am entitled to assert either *maybe* F or *maybe* G at *your* choice. Attacking an assertion of *maybe* $\neg F$, however, consists in asserting *surely* F and also attacking *maybe* $F \to G$ consists in asserting *surely* F (and vice versa for *surely* $\neg F$ and *surely* $F \to G$, respectively):

$(R_\to)$ If *I* assert $(F \to G)\downarrow$ then *you* may attack by asserting $F\uparrow$, which obliges *me* to defend by asserting $G\downarrow$. If *I* assert $(F \to G)\uparrow$ then *your* attack consists in asserting $F\downarrow$, and *my* defense consists in asserting $G\uparrow$.

$(R_\wedge)$ If *I* assert $(F \wedge G)\downarrow$ then *you* attack by pointing either to the left or to the right subformula. As corresponding defense, *I* then have to assert either $F\downarrow$ or $G\downarrow$, according to your choice. If *I* assert $(F \wedge G)\uparrow$ then *my* defense consists in asserting either $F\uparrow$ or $G\uparrow$.

$(R_\vee)$ If *I* assert $(F \vee G)\downarrow$ then *I* have to assert either $F\downarrow$ or $G\downarrow$ at *my* own choice. If *I* assert $(F \vee G)\uparrow$ then *my* defense consists in asserting either $F\uparrow$ or $G\uparrow$.

$(R_\neg$ If *I* assert $(\neg F)\downarrow$ then *you* may attack by asserting $F\uparrow$, which obliges *me* to assert $\bot$. If *I* assert $(\neg F)\uparrow$ then *your* attack consists in asserting $F\downarrow$.

$(R_\forall)$ If *I* assert $(\forall x F(x))\downarrow$ then *you* attack by picking c and *I* have to defend by asserting $F(c)\downarrow$. If *I* assert $(\forall x F(x))\uparrow$ then *my* defense consists in asserting $F(c)\uparrow$.

$(R_\exists)$ If *I* assert $(\exists x F(x))\downarrow$ then *I* have to pick a constant c and assert $F(c)\downarrow$. If *I* assert $(\exists x F(x))\uparrow$ then *my* defense consists in asserting $F(c)\uparrow$.

For *my* attacks on *your* assertions the dialogue rules are completely dual.

The correspondence between extended payoff values and winning strategies stated by Lemma 3 is still in force:

Lemma 4. *Consider an arbitrary game state $\mathcal{G} = [\Gamma \mid \Delta]$ where both tenets may include assertions of the form $F\uparrow$. and $F\downarrow$. Then*

(i) I *have a winning strategy for the game starting in $\mathcal{G}$ iff $\langle \Gamma \mid \Delta \rangle^\downarrow \geq 0$ and*

(ii) you *have a winning strategy for the game starting in $\mathcal{G}$ iff $\langle \Gamma \mid \Delta \rangle^\uparrow < 0$.*

Proof. Relative to the proof of Lemma 3 we only need to check attacks on assertions of $F\downarrow$ and $F\uparrow$. We focus on assertions of $F \to G$ at this point and note that the other connectives and quantifiers are treated analogously. For *my* winning strategy we have:

$[\Gamma \mid \Delta', (F \to G)\uparrow]$: If *you* attack *my* assertion $(F \to G)\uparrow$ then the succeeding game state, according to the extended rule $(R_\to)$, will be either $[\Gamma \mid \Delta']$ or $[\Gamma, F\downarrow \mid \Delta', G\uparrow]$ at *your* choice. Thus, we need to show that $\langle \Gamma \mid \Delta', (F \to G){\uparrow}\rangle^{\downarrow} \geq 0$ iff both $\langle \Gamma \mid \Delta'\rangle^{\downarrow} \geq 0$ and $\langle F{\downarrow}, \Gamma \mid \Delta', G{\uparrow}\rangle^{\downarrow} \geq 0$. By induction we then conclude that $\langle \Gamma \mid \Delta', (F \to G){\uparrow}\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for both states; therefore also $\langle \Gamma \mid \Delta', (F \to G){\uparrow}\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[\Gamma \mid \Delta', (F \to G){\uparrow}]$.

If $F \to G$ is true or indefinite at the current world of evaluation, then, according to the strong Kleene truth tables, it is not the case that F is true and G is false. By Definition 21, $\langle \Gamma \mid \Delta', F \to G{\uparrow}\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow}$ and therefore $\langle \Gamma \mid \Delta', F \to G{\uparrow}\rangle^{\downarrow} \geq 0$ iff $\langle \Gamma \mid \Delta'\rangle^{\downarrow} \geq 0$. Moreover, we have $\langle \Gamma, F{\downarrow}\mid \Delta', G{\uparrow}\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow}$ if F is true and G is either true or indefinite and also if G is false and F is either false or indefinite. Otherwise, if F is false or indefinite and G is true or indefinite we have $\langle \Gamma, F{\downarrow}\mid \Delta', G{\uparrow}\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow} + 1$. In any case, according to the strong Kleene truth tables, $\langle \Gamma, F{\downarrow}\mid \Delta', G{\uparrow}\rangle^{\downarrow} \geq 0$ if $\langle \Gamma \mid \Delta', (F \to G){\uparrow}\rangle^{\downarrow} \geq 0$ whenever $F \to G$ is true or indefinite at the current world of evaluation.

If $F \to G$ is false at the current world of evaluation, then, according to the strong Kleene truth tables, F is true and G is false. We have $\langle \Gamma \mid \Delta', (F \to G){\uparrow}\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow} - 1$, thus if $\langle \Gamma \mid \Delta', (F \to G){\uparrow}\rangle^{\downarrow} \geq 0$ then also $\langle \Gamma \mid \Delta'\rangle^{\downarrow} \geq 0$. Moreover, $\langle F{\downarrow}, \Gamma \mid \Delta', G{\uparrow}\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow} - 1 = \langle \Gamma \mid \Delta', (F \to G){\uparrow}\rangle^{\downarrow}$, and therefore we have $\langle F{\downarrow}, \Gamma \mid \Delta', G{\uparrow}\rangle^{\downarrow} \geq 0$ iff $\langle \Gamma \mid \Delta', (F \to G){\uparrow}\rangle^{\downarrow} \geq 0$, whenever $F \to G$ is false at the current world of evaluation.

$[\Gamma \mid \Delta', (F \to G){\downarrow}]$: Observe that this case is equal to *your* attack on *my* (unmodified) assertion of $F \to G$. Since *I* am asserting $F \to G$, the formula is evaluated pessimistically when calculating $\langle \Gamma \mid \Delta', F \to G\rangle^{\downarrow}$ anyways, even without being forced by the $\downarrow$ modifier. Moreover, observe that *my* extended payoff of the (possible) successor state $\langle F{\uparrow}, \Gamma \mid \Delta', G{\downarrow}\rangle^{\downarrow}$ also equals $\langle F, \Gamma \mid \Delta', G\rangle^{\downarrow}$ for the same reason.

$[(F \to G){\downarrow}, \Gamma' \mid \Delta]$: If *I* attack *your* assertion $(F \to G){\downarrow}$ then the succeeding game state, according to the extended rule $(R_{\to})$, will be either $[\Gamma' \mid \Delta]$ or $[\Gamma', G{\downarrow}\mid \Delta, F{\uparrow}]$ at *my* choice. Therefore, we need to show that $\langle (F \to G){\downarrow}, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff either $\langle \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ or $\langle G{\uparrow}, \Gamma' \mid \Delta, F{\downarrow}\rangle^{\downarrow} \geq 0$ (or both). By induction we then conclude that $\langle (F \to G){\downarrow}, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for one of these two states; thus also $\langle (F \to G){\downarrow}, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[(F \to G){\downarrow}, \Gamma' \mid \Delta]$.

If $F \to G$ is true at the current world of evaluation, then, according to the strong Kleene truth tables, F is false or G is true. We have $\langle (F \to G){\downarrow}, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow}$ and therefore $\langle (F \to G){\downarrow}, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff $\langle \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$. Moreover, we have $\langle G{\downarrow}, \Gamma' \mid \Delta, F{\uparrow}\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow} - 1$ if F is false and G is true and otherwise we have $\langle G{\downarrow}, \Gamma' \mid \Delta, F{\uparrow}\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow}$. In any case, according to the strong Kleene truth tables,

$\langle (F \to G){\downarrow}\, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ if $\langle G{\downarrow}, \Gamma' \mid \Delta, F{\uparrow}\rangle^{\downarrow} \geq 0$ whenever $F \to G$ is true at the current world of evaluation.

If $F \to G$ is false or indefinite at the current world of evaluation, then by Definition 21 $\langle (F \to G){\downarrow}, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow} + 1$. Therefore, if $\langle \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$, then also $\langle (F \to G){\downarrow}, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$. Moreover, we have $\langle G{\downarrow}, \Gamma' \mid \Delta, F{\uparrow}\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow} + 1 = \langle (F \to G){\downarrow}, \Gamma' \mid \Delta\rangle^{\downarrow}$ if F is true or indefinite and G is false or indefinite. Thus, according to the strong Kleene truth tables, we have $\langle G{\downarrow}, \Gamma' \mid \Delta, F{\uparrow}\rangle^{\downarrow} \geq 0$ iff $\langle (F \to G){\downarrow}, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$, whenever $F \to G$ is false or indefinite at the current world of evaluation.

$[(F \to G){\uparrow}, \Gamma' \mid \Delta]$: As above, observe that this case is equal to *my* attack on *your* (unmodified) assertion of $F \to G$. Since *you* are asserting $F \to G$, the formula is evaluated optimistically when calculating $\langle F \to G, \Gamma' \mid \Delta\rangle^{\downarrow}$ anyways, even without being forced by the $\uparrow$ modifier. Moreover, observe that *my* extended payoff of the (possible) successor state $\langle G{\uparrow}, \Gamma' \mid \Delta, F{\downarrow}\rangle^{\downarrow}$ also equals $\langle G, \Gamma' \mid \Delta, F\rangle^{\downarrow}$ for the same reason.

For *your* winning strategies we reason analogously. Note that assertions marked by *surely* or *maybe* are always evaluated pessimistically or optimistically, respectively, for both my optimistic and my pessimistic payoff. $\square$

Theorem 13. *Consider two games starting with my assertion of* surely F *and* maybe F*, respectively.*

(i) I *have a winning strategy for* $[\ \mid F{\downarrow}]$ *iff* F *is* true *at the current world of evaluation, and* you *have a winning strategy otherwise.*

(ii) I *have a winning strategy for* $[\ \mid F{\uparrow}]$ *iff* F *is* true *or* indefinite *at the current world of evaluation, and* you *have a winning strategy otherwise.*

Proof. Observe that the extended optimistic and pessimistic payoffs $\langle\ \mid F{\downarrow}\rangle^{\uparrow}$ and $\langle\ \mid F{\uparrow}\rangle^{\downarrow}$ either amount to 0 or to -1 depending on the valuation of F at the current world of evaluation. By Definition 21 we have $\langle\ \mid F{\downarrow}\rangle^{\downarrow} = \langle\ \mid F{\downarrow}\rangle^{\uparrow} = 0$ iff F is true and $\langle\ \mid F{\downarrow}\rangle^{\downarrow} = \langle\ \mid F{\downarrow}\rangle^{\uparrow} = -1$. Similarly, $\langle\ \mid F{\uparrow}\rangle^{\downarrow} = \langle\ \mid F{\uparrow}\rangle^{\uparrow} = 0$ iff F is true or indefinite and $\langle\ \mid F{\uparrow}\rangle^{\downarrow} = \langle\ \mid F{\uparrow}\rangle^{\uparrow} = -1$ otherwise. The theorem thus follows directly from Lemma 4. $\square$

In addition to *local* connectives and quantifiers, Shapiro also defines *non-local ones*, the binary connectives '$-$' and '$\Rightarrow$' as well as the quantifiers 'A' and 'E', as introduced in Section 5.1. Dialogue rules for these operators require the players to change the current world of evaluation. Thus from now on, we stipulate that assertions refer to a particular world inside the current frame. We denote the assertion that an arbitrary formula F holds at the world N as F_N, and similarly we denote the assertion that F surely or maybe holds at N by $F{\downarrow_N}$ and $F{\uparrow_N}$, respectively.[3] The dialogue rules given below require the players to select new sharpenings of N. Correspondingly, we update Definitions 18 and 20 to take into account the world to which the assertion refers.

Definition 22 (Valuation)**.** We define the *optimistic* valuation function $v_N^{\uparrow}$ and the *pessimistic* valuation function $v_N^{\downarrow}$ for an arbitrary formula F as follows:

$$v^{\uparrow}(F)_N = \begin{cases} 0 & \text{if } F \text{ is false at } N \\ 1 & \text{otherwise} \end{cases} \qquad v^{\downarrow}(F)_N = \begin{cases} 1 & \text{if } F \text{ is true at } N \\ 0 & \text{otherwise} \end{cases} .$$

Definition 23 (Optimistic vs. pessimistic payoff with assertion modifiers)**.** For an atomic game state $\left[\gamma, \gamma^{\uparrow}, \gamma^{\downarrow} \mid \delta, \delta^{\uparrow}, \delta^{\downarrow}\right]$, where $\gamma^{\uparrow}$ and $\delta^{\uparrow}$ denote assertions marked by $\uparrow$, $\gamma^{\downarrow}$ and $\delta^{\downarrow}$ denote assertions marked by $\downarrow$, and γ and δ denote unmarked assertions, we define the notions of *my optimistic* and *my pessimistic* payoff as follows:

$$\begin{aligned}
\left\langle \gamma, \gamma^{\uparrow}, \gamma^{\downarrow} \mid \delta, \delta^{\uparrow}, \delta^{\downarrow} \right\rangle^{\uparrow} =_{df} & \sum_{G_N \in \gamma \cup \gamma^{\uparrow}} (1 - v^{\uparrow}(G)_N) + \sum_{G_N \in \gamma^{\downarrow}} (1 - v^{\downarrow}(G)_N) \\
& - \sum_{D_N \in \delta \cup \delta^{\downarrow}} (1 - v^{\downarrow}(D)_N) - \sum_{D_N \in \delta^{\uparrow}} (1 - v^{\uparrow}(D)_N) \\
\left\langle \gamma, \gamma^{\uparrow}, \gamma^{\downarrow} \mid \delta, \delta^{\uparrow}, \delta^{\downarrow} \right\rangle^{\downarrow} =_{df} & \sum_{G_N \in \gamma \cup \gamma^{\downarrow}} (1 - v^{\downarrow}(G)_N) + \sum_{G_N \in \gamma^{\uparrow}} (1 - v^{\uparrow}(G)_N) \\
& - \sum_{D_N \in \delta \cup \delta^{\uparrow}} (1 - v^{\uparrow}(D)_N) - \sum_{D_N \in \delta^{\downarrow}} (1 - v^{\downarrow}(D)_N).
\end{aligned}$$

For the dialogue rules for local connectives and quantifiers presented above, the current world of evaluation carries over from the attacked assertion to the players' new assertions. For example, if *you* choose to attack *my* assertion of $(F \to G)_N$, in the succeeding game state *you* assert F_N while *I* assert G_N. Dialogue rules for non-local operators, however, change the current world: If *you* attack *my* (unmarked) assertion of a non-local proposition, we first agree to consult a neutral third party (as above for evaluating indefinite propositions). This external judge may arbitrarily choose some propositions which are still indefinite at the current world of evaluation. We model this by the third player *Nature* choosing a sharpening $N' \succeq N$. If *you* attack *my* assertion of *surely* F, then *you* may choose this sharpening, but if *you* attack *my* assertion of *maybe* F, then *I* may choose this sharpening (and analogously for *me* attacking your marked assertions).

[3]For assertions of $\bot$ we will omit the corresponding world, since $\bot$ evaluates to false in all possible worlds.

Introducing the intuitionistic-style negation '$-$', as defined by Shapiro, an assertion of $-F$ is true at a world N, iff there is no sharpening of N where F is true. Observe that the following dialogue rule amounts to the defendant being ready to assert *maybe* $\neg F$ at N'' (for *your* attack on *my* assertion of $-F$ the rule is completely dual):

(R_-) If *you* attack *my* assertion of $(-F)_N$ then *Nature* chooses a sharpening $N' \succeq N$. Afterwards, *you* start the attack by choosing a sharpening $N'' \succeq N'$ and asserting $F{\downarrow}_{N''}$, which obliges *me* to assert $\perp$.

If *you* attack *my* assertion of $(-F){\uparrow}_N$ then *I* choose a sharpening $N' \succeq N$. Afterwards, *you* start the attack by choosing a sharpening $N'' \succeq N'$ and asserting $F{\downarrow}_{N''}$, which obliges *me* to assert $\perp$.

If *you* attack *my* assertion of $(-F){\downarrow}_N$ then *you you* start the attack by choosing a sharpening $N' \succeq N$ and asserting $F{\downarrow}_{N'}$, which obliges *me* to assert $\perp$.

Remark. One may be tempted to consider a similar dialogue rule *without* the first move by *Nature*. Naively translating the description of the connective '$-$'—namely that a formula $-F$ is true at at world N iff F is not true at any sharpening of N—into a dialogue rule, would yield a rule where the attacker chooses a sharpening N' of N and asserts that F is not true there. As above, this would be modeled by asserting $\neg F{\uparrow}_{N'}$ which boils down to the attacker asserting $F{\downarrow}_{N'}$ while the defendant asserts $\perp$. Indeed such a dialogue rule is adequate with respect to *my* winning strategy: *I* have a winning strategy for the game starting with my assertion of $(-F)_N$ with this modified dialogue rule, iff $-F$ is true at N. However, the converse does not hold for *your* winning strategy. Consider the following frame with base M:

$$\begin{tikzcd}[column sep=small] & M \arrow[dl, no head] \arrow[dr, no head] & \\ N : P(a) & & N' : \neg P(a). \end{tikzcd}$$

Then $-P(a)$ is true at N' and false at N. Thus, by Shapiro's falseness condition by stable failure $-P(a)$ is indefinite at M. Correspondingly, *I* have no winning strategy, since *you* can initially select N and the game ends in $[P(a){\downarrow}_N \mid \perp]$ where *I* lose. However, this constitutes a winning strategy for *you*, although *you* should only have one if $-P(a)$ is false at M. Using the dialogue rule (R_-) as introduced above, no player has a winning strategy, since the winner depends on *Nature*'s initial choice. This initial selection by *Nature* reflects Shapiro's notion of positive and negative monotonicity, namely, that a formula is true or false at a world, iff it is true or false, respectively, at all sharpenings. Moreover, observe that asserting *maybe* F amounts to stating that F is true at some sharpening, while asserting *surely* F amounts to stating that F is true at all sharpenings.

Similarly, the connective '$\Rightarrow$' introduces an alternative notion of implication. $F \Rightarrow G$ is true at a world N, if for all sharpenings of N, whenever F is true, also G is true.

($R_{\Rightarrow}$) If *you* attack *my* assertion of $(F \Rightarrow G)_N$ then *Nature* chooses a sharpening $N' \succeq N$. Afterwards, *you* may start the attack by choosing a sharpening $N'' \succeq N'$ and asserting $F{\downarrow}_{N''}$, which obliges *me* to assert $G{\downarrow}_{N''}$. Alternatively, *you* may choose not to attack, and the assertion is removed from *my* tenet.

If *you* attack *my* assertion of $(F \Rightarrow G){\uparrow}_N$ then *I* choose a sharpening $N' \succeq N$. Afterwards, *you* may start the attack by choosing a sharpening $N'' \succeq N'$ and asserting $F{\downarrow}_{N''}$, which obliges *me* to assert $G{\downarrow}_{N''}$. Alternatively, *you* may choose not to attack, and the assertion is removed from *my* tenet.

If *you* attack *my* assertion of $(F \Rightarrow G){\downarrow}_N$ then *you* may start the attack by choosing a sharpening $N' \succeq N$ and asserting $F{\downarrow}_{N'}$, which obliges *me* to assert $G{\downarrow}_{N'}$. Alternatively, *you* may choose not to attack, and the assertion is removed from *my* tenet.

Also dialogue rules for the new non-local quantifiers 'A' and 'E' can be introduced. Shapiro defines these rules using the notion of forcing. This is reflected in the dialogue rules by the players alternatively choosing sharpenings.

(R_A) If *you* attack *my* assertion of $(Ax\ F(x))_N$ then *Nature* chooses a sharpening $N' \succeq N$. Afterwards, *you* start the attack by choosing a constant c and a sharpening $N'' \succeq N'$. *I* defend by choosing a further sharpening $N''' \succeq N''$ and asserting $F(c){\downarrow}_{N'''}$.

If *you* attack *my* assertion of $(Ax\ F(x)){\uparrow}_N$ then *I* choose a sharpening $N' \succeq N$. Afterwards, *you* start the attack by choosing a constant c and a sharpening $N'' \succeq N'$. *I* defend by choosing a further sharpening $N''' \succeq N''$ and asserting $F(c){\downarrow}_{N'''}$.

If *you* attack *my* assertion of $(Ax\ F(x)){\downarrow}_N$ then *you* choose a sharpening $N' \succeq N$. Afterwards, *you* start the attack by choosing a constant c. *I* defend by choosing a further sharpening $N'' \succeq N'$ and asserting $F(c){\downarrow}_{N''}$.

(R_E) If *you* attack *my* assertion of $(Ex\ F(x))_N$ then *Nature* chooses a sharpening $N' \succeq N$. Afterwards, *I* choose a constant c and *you* attack by choosing a sharpening $N'' \succeq N'$. *I* defend by choosing a further sharpening $N''' \succeq N''$ and asserting $F(c){\downarrow}_{N'''}$.

If *you* attack *my* assertion of $(Ex\ F(x)){\uparrow}_N$ then *I* choose a sharpening $N' \succeq N$ and a constant c. Afterwards, *you* start the attack by choosing a sharpening $N'' \succeq N'$ and *I* defend by choosing a further sharpening $N''' \succeq N''$ and asserting $F(c)_{N'''}$.

If *you* attack *my* assertion of $(Ex\ F(x))\downarrow_N$ then *you* choose a sharpening $N' \succeq N$. Afterwards, *I* choose a constant c and you choose a sharpening $N' \succeq N$ *I* defend by choosing a further sharpening $N'' \succeq N'$ and asserting $F(c)\downarrow_{N''}$ and

Again, for *my* attack on *your* assertion, the rules are completely dual.

Remark. Observe that for *your* attack on an assertion about these non-local operators marked by $\downarrow$, one sharpening less is chosen than for the other attacks. Strictly adhering to the above scheme, you would first have to select a sharpening $N' \succeq N$ and then start your attack by selecting a further sharpening. These two choices can obviously be subsumed as one move. If *I* attack *your* assertion of $F\downarrow$, we simplify the rule analogously.

Lemma 5. *Consider an arbitrary game state $\mathcal{G} = [\Gamma \mid \Delta]$ where both tenets may include assertions of the form $F\uparrow_N$. and $F\downarrow_N$. Then*

(i) I *have a winning strategy for the game starting in $\mathcal{G}$ iff $\langle \Gamma \mid \Delta \rangle^{\downarrow} \geq 0$ and*

(ii) you *have a winning strategy for the game starting in $\mathcal{G}$ iff $\langle \Gamma \mid \Delta \rangle^{\uparrow} < 0$.*

Proof. Relative to the proof of Lemma 4 we only need to check attacks on (marked and unmarked) assertions of the form $(-F)_N$, $(F \Rightarrow G)_N$, $(Ax\ F(x))_N$, and $(Ex\ F(x))_N$. For *my* winning strategy we distinguish the following cases:

$[\Gamma \mid \Delta', (-F)_N]$: If *you* attack *my* assertion of $(-F)_N$ then the succeeding game state will be $[\Gamma, F\downarrow_{N''} \mid \Delta', \bot]$ where $N'' \succeq N'$ is chosen by *you* after $N' \succeq N$ is chosen by *Nature*. Thus we need to show that $\langle \Gamma \mid \Delta', (-F)_N \rangle^{\downarrow} \geq 0$ iff $\langle \Gamma, F\downarrow_{N''} \mid \Delta', \bot \rangle^{\downarrow} \geq 0$ for all sharpenings $N'' \succeq N$. By induction we then conclude that $\langle \Gamma \mid \Delta', (-F)_N \rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[\Gamma \mid \Delta', (-F)_N]$.

If $-F$ is true at N, then $\langle \Gamma \mid \Delta', (-F)_N \rangle^{\downarrow} = \langle \Gamma \mid \Delta' \rangle^{\downarrow}$ and F is true at no sharpening of N. Thus for all of *Nature*'s and *your* choices of N'', F will never be true at N''. We have $\langle \Gamma, F\downarrow_{N''} \mid \Delta', \bot \rangle^{\downarrow} = \langle \Gamma \mid \Delta' \rangle^{\downarrow} = \langle \Gamma \mid \Delta', (-F)_N \rangle^{\downarrow}$. Therefore, $\langle \Gamma \mid \Delta', (-F)_N \rangle^{\downarrow} \geq 0$ iff $\langle \Gamma, F\downarrow_{N''} \mid \Delta', \bot \rangle^{\downarrow} \geq 0$ for all $N'' \succeq N$'.

If $-F$ is false or indefinite at N, then $\langle \Gamma \mid \Delta', (-F)_N \rangle^{\downarrow} = \langle \Gamma \mid \Delta' \rangle^{\downarrow} - 1$ and there exists a sharpening of N where F is true. Depending on the choice of N'', F will either be true or not true at N'', thus either $\langle \Gamma, F\downarrow_{N''} \mid \Delta', \bot \rangle^{\downarrow} = \langle \Gamma \mid \Delta' \rangle^{\downarrow} - 1$ or $\langle \Gamma, F\downarrow_{N''} \mid \Delta', \bot \rangle^{\downarrow} = \langle \Gamma \mid \Delta' \rangle^{\downarrow}$, respectively. In any case, if $\langle \Gamma \mid \Delta', (-F)_N \rangle^{\downarrow} \geq 0$ then also $\langle \Gamma, F\downarrow_{N''} \mid \Delta', \bot \rangle^{\downarrow} \geq 0$ for all $N'' \succeq N$. As F is true for some $N'' \succeq N$, we have $\langle \Gamma, F\downarrow_{N''} \mid \Delta', \bot \rangle^{\downarrow} = \langle \Gamma \mid \Delta' \rangle^{\downarrow} = \langle \Gamma \mid \Delta', (-F)_N \rangle^{\downarrow}$ for that particular world N'' and thus $\langle \Gamma \mid \Delta', (-F)_N \rangle^{\downarrow} \geq 0$ if for all $N'' \succeq N$ the inequality $\langle \Gamma, F\downarrow_{N''} \mid \Delta', \bot \rangle^{\downarrow} \geq 0$ holds.

$[\Gamma \mid \Delta', (-F)\downarrow_N]$: Observe that this case is equal to *your* attack on my unmodified assertion of $-F$ as above. Since *I* am asserting $-F$, the formula is evaluated pessimistically when calculating $\langle \Gamma \mid \Delta', (-F)\downarrow_N \rangle^{\downarrow}$, even without being forced by the $\downarrow$ modifier. Moreover, observe, that *my* extended payoff of the successor state amounts to $\langle \Gamma, F\downarrow_{N''} \mid \Delta', \bot \rangle^{\downarrow}$ in both cases.

$[\Gamma \mid \Delta', (-F)\uparrow_N]$: By Shapiro's definition of the '$-$' operator, a formula $-F$ is true or indefinite at N iff it is true for some sharpening of N. Observe that *I* have a winning strategy for $[\Gamma \mid \Delta', (-F)\uparrow_N]$ iff there exists a sharpening $N' \succeq N$ such that *I* have a winning strategy for $[\Gamma \mid \Delta', (-F)_{N'}]$: After *I* choose $N' \succeq N$, the dialogue rule from *my* point of view proceeds exactly as for *my* unmarked assertion of $(-F)_{N'}$—for *my* winning strategy it is irrelevant whether a world is chosen by *Nature* and *you*, or just by *you*, since *I* must be prepared for all of *Nature*'s and *your* choices.

$[(-F)_N, \Gamma' \mid \Delta]$: If *I* attack *your* assertion of $(-F)_N$ then the succeeding game state will be $[\bot, \Gamma' \mid \Delta, F\downarrow_{N''}]$ where $N'' \succeq N'$ is chosen by *me* after $N' \succeq N$ is chosen by *Nature*. Thus we need to show that $\langle (-F)_N, \Gamma' \mid \Delta \rangle^{\downarrow} \geq 0$ iff for all sharpenings $N' \succeq N$ there exists a sharpening $N'' \succeq N'$ such that $\langle \bot, \Gamma' \mid \Delta, F\downarrow_{N''} \rangle^{\downarrow} \geq 0$. By induction we then conclude that $\langle (-F)_N, \Gamma' \mid \Delta \rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[(-F)_N, \Gamma' \mid \Delta]$.

If $-F$ is true or indefinite at N then $\langle (-F)_N, \Gamma' \mid \Delta \rangle^{\downarrow} = \langle \Gamma' \mid \Delta \rangle^{\downarrow}$. If $-F$ is true at the sharpening $N' \succeq N$ selected by *Nature*, then no matter which $N'' \succeq N'$ *I* choose, F will be false or indefinite at N''. Thus $\langle \bot, \Gamma' \mid \Delta, F\downarrow_{N''} \rangle^{\downarrow} = \langle \Gamma' \mid \Delta \rangle^{\downarrow}$. Otherwise, if $-F$ is indefinite at N' then F is also indefinite at N'. Thus, if I choose $N'' = N'$, we obtain again $\langle \bot, \Gamma' \mid \Delta, F\downarrow_{N''} \rangle^{\downarrow} = \langle \Gamma' \mid \Delta \rangle^{\downarrow}$. On the other hand, if for all of *Nature*'s choices *I* could find a sharpening where F is true (and thus $\langle \bot, \Gamma' \mid \Delta, F\downarrow_{N''} \rangle^{\downarrow} = \langle \Gamma' \mid \Delta \rangle^{\downarrow} + 1$), then $-F$ would be false at N.

If $-F$ is false at N then $\langle (-F)_N, \Gamma' \mid \Delta \rangle^{\downarrow} = \langle \Gamma' \mid \Delta \rangle^{\downarrow} + 1$. Moreover, by Shapiro's falseness condition by stable failure, this is equivalent to $-F$ being true at no sharpening. Thus, whatever sharpening $N' \succeq N'$ *Nature* chooses, *I* can find find a further sharpening $N'' \succeq N$ where F is true and thus $\langle \bot, \Gamma' \mid \Delta, F\downarrow_{N''} \rangle^{\downarrow} = \langle \Gamma' \mid \Delta \rangle^{\downarrow} + 1 = \langle (-F)_N, \Gamma' \mid \Delta \rangle^{\downarrow}$ and, obviously, there is no choice of N'' which would increase *my* payoff even more.

Summarizing, in both cases we obtain that for all of *Nature*'s choices, I can find a further sharpening N'' such that $\langle \bot, \Gamma' \mid \Delta, F\downarrow_{N''} \rangle^{\downarrow} = \langle (-F)_N, \Gamma' \mid \Delta \rangle^{\downarrow}$ and thus $\langle (-F)_N, \Gamma' \mid \Delta \rangle^{\downarrow} \geq 0$ iff for all $N' \succeq N$ there exists $N'' \succeq N'$ such that $\langle \bot, \Gamma' \mid \Delta, F\downarrow_{N''} \rangle^{\downarrow} \geq 0$.

$[(-F)\!\uparrow_N, \Gamma' \mid \Delta]$: Observe that this case is equal to *my* attack on *your* unmodified assertion of $-F$ as above. Since *you* are asserting $-F$, the formula is evaluated optimistically when calculating $\langle(-F)\!\uparrow_N, \Gamma' \mid \Delta\rangle^{\downarrow}$, even without being forced by the $\uparrow$ modifier. Moreover, observe, that *my* extended payoff of the successor state amounts to $\langle \Gamma', \bot \mid \Delta, F\!\uparrow_{N''}\rangle^{\downarrow}$ in both cases.

$[(-F)\!\downarrow_N, \Gamma' \mid \Delta]$: If *I* attack *your* assertion of $(-F)\!\downarrow_N$ then the succeeding game state will be $[\bot, \Gamma' \mid \Delta, F\!\downarrow_{N'}]$ where $N' \succeq N$ is chosen by *me*. Thus we need to show that $\langle(-F)_N, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff for some sharpening $N' \succeq N$ we have $\langle \bot, \Gamma' \mid \Delta, F\!\downarrow_{N'}\rangle^{\downarrow} \geq 0$. By induction we then conclude that $\langle(-F)\!\downarrow_N, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[(-F)\!\downarrow_N, \Gamma' \mid \Delta]$.

If $-F$ is true at N then $\langle(-F)_N, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow}$. Moreover, no matter which $N' \succeq N$ *I* choose, F will be false or indefinite at N' and thus $\langle \bot, \Gamma' \mid \Delta, F\!\downarrow_{N'}\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow}$. Thus, $\langle(-F)\!\downarrow_N, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff I can find a sharpening N' such that $\langle \bot, \Gamma' \mid \Delta, F\!\downarrow_{N'}\rangle^{\downarrow} \geq 0$.

If $-F$ is false or indefinite at N then $\langle(-F)\!\downarrow_N, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow} + 1$. Moreover, F is true at some sharpening $N' \succeq N$, thus if *I* choose such an N', we have $\langle \bot, \Gamma' \mid \Delta, F\!\downarrow_{N'}\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow} + 1 = \langle(-F)\!\downarrow_N, \Gamma' \mid \Delta\rangle^{\downarrow}$ and, obviously, there is no choice of N' which would increase *my* payoff even more. As above, we obtain that $\langle(-F)\!\downarrow_N, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff I can find a sharpening N' such that $\langle \bot, \Gamma' \mid \Delta, F\!\downarrow_{N'}\rangle^{\downarrow} \geq 0$.

$[\Gamma \mid \Delta', (F \Rightarrow G)_N]$: If *you* attack *my* assertion of $(F \Rightarrow G)_N$ then the succeeding game state will be $[\Gamma, F\!\downarrow_{N''} \mid \Delta', G\!\downarrow_{N''}]$ where $N'' \succeq N'$ is chosen by *you* after $N' \succeq N$ is chosen by *Nature* or it will be $[\Gamma \mid \Delta']$ at *your* choice. Thus we need to show that $\langle \Gamma \mid \Delta', (F \Rightarrow G)_N\rangle^{\downarrow} \geq 0$ iff $\langle \Gamma, F\!\downarrow_{N''} \mid \Delta', G\!\downarrow_{N''}\rangle^{\downarrow} \geq 0$ for all sharpenings $N'' \succeq N$ and also $\langle \Gamma \mid \Delta'\rangle^{\downarrow} \geq 0$. By induction we then conclude that $\langle \Gamma \mid \Delta', (F \Rightarrow G)_N\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[\Gamma \mid \Delta', (F \Rightarrow G)_N]$.

If $F \Rightarrow G$ is true at N, then $\langle \Gamma \mid \Delta', (F \Rightarrow G)_N\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow}$ and G is true at each sharpening of N where also F is true. Thus for all of *Nature*'s and *your* choices of $N'' \succeq N$, if G is true at N'' we have $\langle \Gamma, F\!\downarrow_{N''} \mid \Delta', G\!\downarrow_{N''}\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow}$. Otherwise, if G is not true at N'', we either have $\langle \Gamma, F\!\downarrow_{N''} \mid \Delta', G\!\downarrow_{N''}\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow}$ (if also F is not true at N'') or $\langle \Gamma, F\!\downarrow_{N''} \mid \Delta', G\!\downarrow_{N''}\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow} + 1$ (if F is true at N''). In any case, we have $\langle \Gamma, F\!\downarrow_{N''} \mid \Delta', G\!\downarrow_{N''}\rangle^{\downarrow} \geq 0$ for all $N'' \succeq N$ and also $\langle \Gamma \mid \Delta'\rangle^{\downarrow} \geq 0$ iff $\langle \Gamma \mid \Delta', (F \Rightarrow G)_N\rangle^{\downarrow}$.

If $F \Rightarrow G$ is false or indefinite at N, then $\langle \Gamma \mid \Delta', F \Rightarrow G)_N\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow} - 1$ and there exists a sharpening $N'' \succeq N$ where F is true, but G is not true. If *you* select this sharpening, we have $\langle\Gamma, F{\downarrow}_{N''} | \Delta', G{\downarrow}_{N''}\rangle^{\downarrow} = \langle\Gamma \mid \Delta'\rangle^{\downarrow} - 1 = \langle\Gamma \mid \Delta', (F \Rightarrow G)_N\rangle^{\downarrow}$. For all other sharpenings, we have $\langle\Gamma, F{\downarrow}_{N''} | \Delta', G{\downarrow}_{N''}\rangle^{\downarrow} = \langle\Gamma \mid \Delta'\rangle^{\downarrow}$ (if neither F nor G is true, or both are true) or $\langle\Gamma, F{\downarrow}_{N''} | \Delta', G{\downarrow}_{N''}\rangle^{\downarrow} = \langle\Gamma \mid \Delta'\rangle^{\downarrow} + 1$ (if F is not true, but G is true). In any case, we have $\langle\Gamma, F{\downarrow}_{N''} | \Delta', G{\downarrow}_{N''}\rangle^{\downarrow} \geq 0$ for all N'' and also $\langle\Gamma \mid \Delta'\rangle^{\downarrow} \geq 0$ iff $\langle\Gamma \mid \Delta', F \Rightarrow G)_N\rangle^{\downarrow} \geq 0$.

$[\Gamma \mid \Delta', (F \Rightarrow G){\downarrow}_N]$: Observe that this case is equal to *your* attack on my unmodified assertion of $F \Rightarrow G$ as above. Since *I* am asserting $F \Rightarrow G$, the formula is evaluated pessimistically when calculating $\langle\Gamma \mid \Delta', (F \Rightarrow G){\downarrow}_N\rangle^{\downarrow}$ anyways, even without being forced by the $\downarrow$ modifier. Moreover, observe, that *my* extended payoff of the successor state amounts to $\langle\Gamma, F{\downarrow}_{N''} | \Delta', G{\downarrow}_{N''}\rangle^{\downarrow}$ or to $\langle\Gamma \mid \Delta\rangle^{\downarrow}$ in both cases.

$[\Gamma \mid \Delta', (F \Rightarrow G){\uparrow}_N]$: By Shapiro's definition of the '$\Rightarrow$' operator, a formula $F \Rightarrow G$ is true or indefinite at N iff it is true for some sharpening of N. Observe that *I* have a winning strategy for $[\Gamma \mid \Delta', (F \Rightarrow G){\uparrow}_N]$ iff there exists a sharpening $N' \succeq N$ such that *I* have a winning strategy for $[\Gamma \mid \Delta', (F \Rightarrow G)_{N'}]$: After *I* choose $N' \succeq N$, the dialogue rule from *my* point of view proceeds exactly as for *my* unmarked assertion of $(F \Rightarrow G)_{N'}$—for *my* winning strategy it is irrelevant whether a world is chosen by *Nature* and *you*, or just by *you*, since *I* must be prepared for all of *Nature* and *your* choices.

$[(F \Rightarrow G)_N, \Gamma' \mid \Delta]$: If *I* attack *your* assertion of $(F \Rightarrow G)_N$ then the succeeding game state will be $[G{\downarrow}, \Gamma' \mid \Delta, F{\downarrow}_{N''}]$ where $N'' \succeq N'$ is chosen by *me* after $N' \succeq N$ is chosen by *Nature* or $[\Gamma' \mid \Delta]$ at *my* choice. Thus we need to show that $\langle(F \Rightarrow G)_N, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff for all sharpenings $N' \succeq N$ there exists a sharpening $N'' \succeq N'$ such that $\langle G{\downarrow}_{N''}, \Gamma' \mid \Delta, F{\downarrow}_{N''}\rangle^{\downarrow} \geq 0$ or $\langle\Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$. By induction we then conclude that $\langle(F \Rightarrow G)_N, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[(F \Rightarrow G)_N, \Gamma' \mid \Delta]$.

If $F \Rightarrow G$ is true or indefinite at N then $\langle(F \Rightarrow G)_N, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle\Gamma' \mid \Delta\rangle^{\downarrow}$ and therefore $\langle\Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff $\langle(F \Rightarrow G)_N, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$. Moreover, *I* have no strategy which ensures that $\langle G{\downarrow}_{N''}, \Gamma' \mid \Delta, F{\downarrow}_{N''}\rangle^{\downarrow} > \langle\Gamma' \mid \Delta\rangle^{\downarrow}$: If for all of *Nature*'s choices of $N' \succeq N$ *I* could find such a sharpening $N'' \succeq N'$ then F must be true at N'', while G is not true. However, then then $F \Rightarrow G$ would be false at N.

If $F \Rightarrow G$ is false at N then $\langle(F \Rightarrow G)_N, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle\Gamma' \mid \Delta\rangle^{\downarrow} + 1$. Moreover, by Shapiro's falseness condition by stable failure, this is equivalent to $F \Rightarrow G$ being true at no sharpening of N. Thus, whatever sharpening $N' \succeq N'$ *Nature* chooses, *I* can find find a further sharpening $N'' \succeq N$ where F is true, but G is not and thus $\langle G{\downarrow}_{N''}, \Gamma' \mid \Delta, F{\downarrow}_{N''}\rangle^{\downarrow} = \langle\Gamma' \mid \Delta\rangle^{\downarrow} + 1 = \langle(F \Rightarrow G)_N, \Gamma' \mid \Delta\rangle^{\downarrow}$ and, obviously, there is no other choice of N'' which would increase *my* payoff even more.

Summarizing, in both cases we obtain that for all of *Nature*'s choices, *I* can find a further sharpening N'' such that $\langle G{\downarrow_{N''}}, \Gamma' \mid \Delta, F{\downarrow_{N''}}\rangle^{\downarrow} = \langle (F \Rightarrow G), \Gamma' \mid \Delta\rangle^{\downarrow}$ and thus $\langle (F \Rightarrow G), \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff for all sharpenings $N' \succeq N$ there exists $N'' \succeq N'$ such that $\langle G{\downarrow}, \Gamma' \mid \Delta, F{\downarrow_{N''}}\rangle^{\downarrow} \geq 0$ or $\langle \Gamma \mid \Delta\rangle^{\downarrow} \geq 0$.

$[(F \Rightarrow G){\uparrow_N}, \Gamma' \mid \Delta]$: Observe that this case is equal to *my* attack on *your* unmodified assertion of $F \Rightarrow G$ as above. Since *you* are asserting $F \Rightarrow G$, the formula is evaluated optimistically when calculating $\langle (F \Rightarrow G){\uparrow_N}, \Gamma' \mid \Delta\rangle^{\downarrow}$ anyways, even without being forced by the $\uparrow$ modifier. Moreover, observe, that *my* extended payoff of the successor state amounts to $\langle \Gamma', G{\downarrow_{N''}} \mid \Delta, F{\uparrow_{N''}}\rangle^{\downarrow}$ or to $\langle \Gamma \mid \Delta\rangle^{\downarrow}$ in both cases.

$[(F \Rightarrow G){\downarrow_N}, \Gamma' \mid \Delta]$: If *I* attack *your* assertion of $(F \Rightarrow G){\downarrow_N}$ then the succeeding game state will be $[G{\downarrow_{N'}}, \Gamma' \mid \Delta, F{\downarrow_{N'}}]$ where $N' \succeq N$ is chosen by *me* or $[\Gamma' \mid \Delta]$. Thus we need to show that $\langle (F \Rightarrow G)_N, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff for some sharpening $N' \succeq N$ we have $\langle G{\downarrow_{N'}}, \Gamma' \mid \Delta, F{\downarrow_{N'}}\rangle^{\downarrow} \geq 0$ or $\langle \Gamma \mid \Delta\rangle^{\downarrow} \geq 0$. By induction we then conclude that $\langle (F \Rightarrow G)_N, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[(F \Rightarrow G){\downarrow_N}, \Gamma' \mid \Delta]$.

If $F \Rightarrow G$ is true at N then $\langle (F \Rightarrow G)_N, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow}$. Moreover, for all $N' \succeq N$ we have $\langle G{\downarrow_{N'}}, \Gamma' \mid \Delta, F{\downarrow_{N'}}\rangle^{\downarrow} \leq \langle \Gamma' \mid \Delta\rangle^{\downarrow}$, as otherwise F must be true at N' while G is not. But then $F \Rightarrow G$ would not be true at N.

If $F \Rightarrow G$ is false or indefinite at N then $\langle (F \Rightarrow G){\downarrow_N}, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow} + 1$. Moreover, F is true at some sharpening $N' \succeq N$, while G is not true there. Thus if *I* choose such an $N' \succeq N$, we have $\langle G{\downarrow_{N'}}, \Gamma' \mid \Delta, F{\downarrow_{N'}}\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow} + 1 = \langle (F \Rightarrow G){\downarrow_N}, \Gamma' \mid \Delta\rangle^{\downarrow}$ and, obviously, there is no choice of N' which would increase *my* payoff even more. As above, we obtain that $\langle (F \Rightarrow G){\downarrow_N}, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff I can find a sharpening N' such that $\langle G{\downarrow_{N'}}, \Gamma' \mid \Delta, F{\downarrow_{N'}}\rangle^{\downarrow} \geq 0$ or $\langle \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$.

$[\Gamma \mid \Delta', (Ax\ F(x))_N]$: If *you* attack *my* assertion of $(Ax\ F(x))_N$ then the succeeding game state will be $[\Gamma \mid \Delta', F(c){\downarrow_{N'''}}]$ where first *Nature* chooses $N' \succeq N$, then *you* choose c and $N'' \succeq N'$, and *I* choose $N''' \succeq N''$. Thus we need to show that $\langle \Gamma \mid \Delta', (Ax\ F(x))_N\rangle^{\downarrow} \geq 0$ iff for all constants c and sharpenings $N'' \succeq N$ there exists $N''' \succeq N''$ such that $\langle \Gamma \mid \Delta', F(c){\downarrow_{N'''}}\rangle^{\downarrow} \geq 0$. By induction we then conclude that $\langle \Gamma \mid \Delta', (Ax\ F(x))_N\rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[\Gamma \mid \Delta', (Ax\ F(x))_N]$.

If $Ax\ F(x)$ is true at N, then $\langle \Gamma \mid \Delta', (Ax\ F(x))_N\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow}$ and the formula is also true at all sharpenings $N' \succeq N$. This means, for all constants c, the formula $F(c)$ is forced at N'. Therefore, no matter which sharpening $N' \succeq N$ *Nature* chooses and which constant c *you* choose, for all sharpenings $N'' \succeq N'$, *I* can find a further sharpening $N''' \succeq N''$ where $F(c)$ holds and thus $\langle \Gamma \mid \Delta', F(c){\downarrow}_{N'''}\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow}$ and thus $\langle \Gamma \mid \Delta', F(c){\downarrow}_{N'''}\rangle^{\downarrow} \geq 0$ iff $\langle \Gamma \mid \Delta', (Ax\ F(x))_N\rangle^{\downarrow} \geq 0$.

If $Ax\ F(x)$ is false or indefinite at N, then $\langle \Gamma \mid \Delta', (Ax\ F(x))_N\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow} - 1$ and there exists a sharpening $N' \succeq N$ where F is not true. Thus, there exists a constant c, such that c is not forced at N'. If *you* select this constant c, then *I* cannot find a sharpening $N''' \succeq N''$ where $F(c)$ is true for all of your possible choices of $N'' \succeq N'$. Thus $\langle \Gamma \mid \Delta', F(c){\downarrow}_{N'''}\rangle^{\downarrow} = \langle \Gamma \mid \Delta'\rangle^{\downarrow} - 1$ and thus $\langle \Gamma \mid \Delta', F(c){\downarrow}_{N'''}\rangle^{\downarrow} \geq 0$ iff $\langle \Gamma \mid \Delta', (Ax\ F(x))_N\rangle^{\downarrow} \geq 0$. On the other hand, if *I* could always find such a world where $F(c)$ is true, then $F(c)$ would be forced at N' and if $F(c)$ was forced at all $N' \succeq N$ and for all c, then $Ax\ F(x)$ would be true at N.

$[\Gamma \mid \Delta', (Ax\ F(x)){\downarrow}_N]$: Observe that this case is equal to *your* attack on my unmodified assertion of $Ax\ F(x)$ as above. Since *I* am asserting $Ax\ F(x)$, the formula is evaluated pessimistically when calculating $\langle \Gamma \mid \Delta', (Ax\ F(x)){\downarrow}_N\rangle^{\downarrow}$ anyways, even without being forced by the $\downarrow$ modifier. Moreover, observe, that *my* extended payoff of the successor state amounts to $\langle \Gamma \mid \Delta', F(c){\downarrow}_{N'''}\rangle^{\downarrow}$ in both cases.

$[\Gamma \mid \Delta', (Ax\ F(x)){\uparrow}_N]$: By Shapiro's definition of the 'A' operator, a formula $(Ax\ F(x))$ is true or indefinite at N iff it is true for some sharpening of N. Observe that *I* have a winning strategy for $[\Gamma \mid \Delta', (Ax\ F(x)){\uparrow}_N]$ iff there exists a sharpening $N' \succeq N$ such that *I* have a winning strategy for $[\Gamma \mid \Delta', (Ax\ F(x))_{N'}]$: After *I* choose $N' \succeq N$, the dialogue rule from *my* point of view proceeds exactly as for *my* unmarked assertion of $(Ax\ F(x))_{N'}$— for *my* winning strategy it is irrelevant whether a world is chosen by *Nature* and *you*, or just by *you*, since *I* must be prepared for all of *Nature* and *your* choices.

$[(Ax\ F(x))_N, \Gamma' \mid \Delta]$: If *I* attack *your* assertion of $(Ax\ F(x))_N$ then the succeeding game state will be $[F(c){\downarrow}_{N'''}, \Gamma' \mid \Delta]$ where first *Nature* chooses $N' \succeq N$, then *I* choose c and $N'' \succeq N'$, and *you* choose $N''' \succeq N''$. Thus we need to show that $\langle (Ax\ F(x))_N, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$ iff for all sharpenings $N' \succeq N$ there exists a constant c and a sharpening $N'' \succeq N'$ such that for all sharpenings $N''' \succeq N''$ we have $\langle F(c){\downarrow}_{N'''}, \Gamma' \mid \Delta\rangle^{\downarrow} \geq 0$.

If $Ax\ F(x)$ is true or indefinite at N then $\langle (Ax\ F(x))_N, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow}$ and there exists a sharpening $N' \succeq N$ such that $Ax\ F(x)$ is true at N'. If this world is selected by *Nature*, no matter which constant c *I* choose, as $F(c)$ is forced at N', *you* can find $N''' \succeq N''$ such that $F(c)$ is true at N''' for all of *my* choices of N''. Thus $\langle F(c){\downarrow}_{N'''}, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow}$. For another choice of $N' \succeq N$ by *Nature* where $Ax\ F(x)$ is not true at N' my payoff may be even higher—namely $\langle F(c){\downarrow}_{N'''}, \Gamma' \mid \Delta\rangle^{\downarrow} = \langle \Gamma' \mid \Delta\rangle^{\downarrow} + 1$ if we end up at a world N''' where $F(c)$ is false. However, the highest payoff *I* can enforce remains

$\langle \Gamma' \mid \Delta \rangle^{\downarrow}$. Therefore $\langle (Ax\ F(x))_N, \Gamma' \mid \Delta \rangle^{\downarrow} \geq 0$ iff for all of *Nature*'s choices of $N' \succeq N$, *I* can find a constant c and sharpening $N'' \succeq N'$ to ensure that for the succeeding game state $\langle F(c){\downarrow}_{N'''}, \Gamma' \mid \Delta \rangle^{\downarrow} \geq 0$ holds.

If $Ax\ F(x)$ is false at N then $\langle (Ax\ F(x))_N, \Gamma' \mid \Delta \rangle^{\downarrow} = \langle \Gamma' \mid \Delta \rangle^{\downarrow} + 1$. Moreover, by Shapiro's falseness condition by stable failure, this is equivalent to $Ax\ F(x)$ being true at no sharpening. Thus, whatever sharpening $N' \succeq N'$ *Nature* chooses, *I* can find a constant c such that $F(c)$ is not forced at N'. Thus, there exists a sharpening $N'' \succeq N'$ such that for no further sharpening $N''' \succeq N''$ the formula $F(c)$ is true and we arrive again at at $\langle F(c){\downarrow}_{N'''}, \Gamma' \mid \Delta \rangle^{\downarrow} = \langle \Gamma' \mid \Delta \rangle^{\downarrow} + 1 = \langle (Ax\ F(x))_N, \Gamma' \mid \Delta \rangle^{\downarrow}$.

$[(Ax\ F(x)){\uparrow}_N, \Gamma' \mid \Delta]$: Observe that this case is equal to *my* attack on *your* unmodified assertion of $Ax\ F(x)$ as above. Since *you* are asserting $Ax\ F(x)$, the formula is evaluated optimistically when calculating $\langle (Ax\ F(x)){\uparrow}_N, \Gamma' \mid \Delta \rangle^{\downarrow}$ anyways, even without being forced by the $\uparrow$ modifier. Moreover, observe, that *my* extended payoff of the successor state amounts to $\langle F(c){\downarrow}_{N'''}, \Gamma' \mid \Delta \rangle^{\downarrow}$ in both cases.

$[(Ax\ F(x)){\downarrow}_N, \Gamma' \mid \Delta]$: If *I* attack *your* assertion of $(Ax\ F(x)){\downarrow}_N$ then the succeeding game state will be $[F(c){\downarrow}_{N''}, \Gamma' \mid \Delta]$ where $N'' \succeq N'$ is chosen by *you*, after *I* have chosen a sharpening $N' \succeq N$ and a constant c. Thus we need to show that $\langle (Ax\ F(x)){\downarrow}_N, \Gamma' \mid \Delta \rangle^{\downarrow} \geq 0$ iff there exists a sharpening $N' \succeq N$ and a constant c such that for all sharpenings $N'' \succeq N'$ we have $\langle F(c){\downarrow}_{N''}, \Gamma' \mid \Delta \rangle^{\downarrow} \geq 0$. By induction we then conclude that $\langle (Ax\ F(x)){\downarrow}_N, \Gamma' \mid \Delta \rangle^{\downarrow} \geq 0$ iff *I* have a winning strategy for $[(Ax\ F(x)){\downarrow}_N, \Gamma' \mid \Delta]$.

If $Ax\ F(x)$ is true at N then $\langle (Ax\ F(x)){\downarrow}_N, \Gamma' \mid \Delta \rangle^{\downarrow} = \langle \Gamma' \mid \Delta \rangle^{\downarrow}$. Moreover, no matter which $N' \succeq N$ *I* choose, the formula will be true at N'. This means, that $F(c)$ is forced at N' for all constants c. Thus, *you* can always find a sharpening $N'' \succeq N'$ such that $F(c)$ is true at N'' and thus $\langle F(c){\downarrow}_{N''}, \Gamma' \mid \Delta \rangle^{\downarrow} = \langle \Gamma' \mid \Delta \rangle^{\downarrow} = \langle (Ax\ F(x))_N, \Gamma' \mid \Delta \rangle^{\downarrow}$, and if *you* choose another sharpening $N'' \succeq N'$ *my* payoff is even higher.

If $Ax\ F(x)$ is false or indefinite at N then $\langle (Ax\ F(x)){\downarrow}_N, \Gamma' \mid \Delta \rangle^{\downarrow} = \langle \Gamma' \mid \Delta \rangle^{\downarrow} + 1$. Moreover, there exists a sharpening $N' \succeq N$ where $F(c)$ is not forced for some constant c. If *I* choose this sharpening N' and constant c, then *you* cannot find a sharpening $N'' \succeq N'$ such that $F(c)$ is true at N'' and thus $\langle F(c){\downarrow}_{N''}, \Gamma' \mid \Delta \rangle^{\downarrow} = \langle \Gamma' \mid \Delta \rangle^{\downarrow} + 1 = \langle (Ax\ F(x)){\downarrow}_N, \Gamma' \mid \Delta \rangle^{\downarrow}$.

Summarizing, in both cases we obtain that $\langle (Ax\ F(x))_N, \Gamma' \mid \Delta \rangle^{\downarrow} \geq 0$ iff *I* can find a sharpening $N' \succeq N$ and a constant c such that for all of *your* choices of $N'' \succeq N'$ we have $\langle F(c){\downarrow}_{N''}, \Gamma' \mid \Delta \rangle^{\downarrow} \geq 0$.

$[\Gamma \mid \Delta', (Ex\ F(x))_N]$, $[\Gamma \mid \Delta', (Ex\ F(x))\!\downarrow_N]$, $[\Gamma \mid \Delta', (Ex\ F(x))\!\uparrow_N]$, $[(Ex\ F(x))_N, \Gamma' \mid \Delta]$, $[(Ex\ F(x))\!\downarrow_N, \Gamma' \mid \Delta]$, $[(Ex\ F(x))\!\uparrow_N, \Gamma' \mid \Delta]$: These cases, attacks on assertions of $Ex\ F(x)$, are analogous to the respective attacks on $Ax\ F(x)$. Note that the only difference in the dialogue rules is the player selecting the constant c.

Analogously, we reason for *your* winning strategy:

$[\Gamma \mid \Delta', (-F)_N]$: If *you* attack *my* assertion of $(-F)_N$ then the succeeding game state will be $[\Gamma, F\!\downarrow_{N''} \mid \Delta', \bot]$ where $N'' \succeq N'$ is chosen by *you* after $N' \succeq N$ is chosen by *Nature*. Thus we need to show that $\langle \Gamma \mid \Delta', (-F)_N \rangle^{\downarrow} < 0$ iff for all sharpenings $N' \succeq N$ there exists a sharpening $N'' \succeq N'$ such that $\langle \Gamma, F\!\downarrow_{N''} \mid \Delta', \bot \rangle^{\downarrow} < 0$. By induction we then conclude that $\langle \Gamma \mid \Delta', (-F)_N \rangle^{\downarrow} < 0$ iff *you* have a winning strategy for $[\Gamma \mid \Delta', (-F)_N]$.

If $-F$ is true or indefinite at N, then $\langle \Gamma \mid \Delta', (-F)_N \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow}$. If $-F$ is true at the sharpening $N' \succeq N$ selected by *Nature*, then no matter which $N'' \succeq N'$ *you* choose, F will be false or indefinite at N'' and thus $\langle \Gamma, F\!\downarrow_{N''} \mid \Delta', \bot \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow}$. Otherwise, if $-F$ is indefinite at N', then F is also indefinite at N'. If *you* choose $N'' = N'$, we obtain again $\langle \Gamma, F\!\downarrow_{N''} \mid \Delta', \bot \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow}$. On the other hand, if for all of *Nature*'s choices *you* could find a sharpening where F is true (and thus $\langle \Gamma, F\!\downarrow_{N''} \mid \Delta', \bot \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} - 1$), then $-F$ would be false at N.

If $-F$ is false N, then $\langle \Gamma \mid \Delta', (-F)_N \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} - 1$ and $-F$ is also false for all sharpenings $N' \succeq N$. Thus, there always exists a sharpening $N'' \succeq N'$ where F is true and thus $\langle \Gamma, F\!\downarrow_{N''} \mid \Delta', \bot \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} - 1 = \langle \Gamma \mid \Delta', (-F)_N \rangle^{\uparrow}$.

Summarizing, in both cases we obtain that for all of *Nature*'s choices, *you* can find a further sharpening N'' such that $\langle \Gamma, F\!\downarrow_{N''} \mid \Delta', \bot \rangle^{\uparrow} = \langle \Gamma \mid \Delta', (-F)_N \rangle^{\uparrow}$ and thus $\langle \Gamma \mid \Delta', (-F)_N \rangle^{\uparrow} < 0$ iff for all $N' \succeq N$ there exists $N'' \succeq N'$ such that $\langle \Gamma, F\!\downarrow_{N''} \mid \Delta', \bot \rangle^{\uparrow} < 0$.

$[\Gamma \mid \Delta', (-F)\!\uparrow_N]$: Observe that this case is equal to *your* attack on my unmodified assertion of $-F$ as above. Since *I* am asserting $-F$, the formula is evaluated optimistically when calculating $\langle \Gamma \mid \Delta', (-F)\!\uparrow_N \rangle^{\uparrow}$ anyways, even without being forced by the $\uparrow$ modifier. Moreover, observe, that *your* extended payoff of the successor state amounts to $\langle \Gamma, F\!\downarrow_{N''} \mid \Delta', \bot \rangle^{\uparrow}$ in both cases.

$[\Gamma \mid \Delta', (-F)\!\downarrow_N]$: If *you* attack *my* assertion of $(-F)\!\downarrow_N$ then the succeeding game state will be $[\Gamma, F\!\downarrow_{N'} \mid \Delta', \bot]$ where $N' \succeq N$ is chosen by *you*. Thus we need to show that $\langle \Gamma \mid \Delta', (-F)\!\downarrow_N \rangle^{\uparrow} < 0$ iff for some sharpening $N' \succeq N$ we have $\langle \Gamma, F\!\downarrow_{N'} \mid \Delta', \bot \rangle^{\uparrow} < 0$. By induction we then conclude that $\langle \Gamma \mid \Delta', (-F)\!\downarrow_N \rangle^{\uparrow} < 0$ iff *you* have a winning strategy for $[\Gamma \mid \Delta', (-F)\!\downarrow_N]$.

If $-F$ is true at N then $\langle \Gamma \mid \Delta', (-F)_N \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow}$. Moreover, no matter which $N' \succeq N$ *you* choose, F will be false or indefinite at N' and thus $\langle \Gamma, F{\downarrow}_{N'} \mid \Delta', \bot \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow}$. Thus, $\langle \Gamma \mid \Delta', (-F){\downarrow}_N \rangle^{\uparrow} < 0$ iff *you* can find a sharpening N' such that $\langle \Gamma, F{\downarrow}_{N'} \mid \Delta', \bot \rangle^{\uparrow} < 0$.

If $-F$ is false or indefinite at N then $\langle \Gamma \mid \Delta', (-F){\downarrow}_N \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} - 1$. Moreover, F is true at some sharpening $N' \succeq N$, thus if *you* choose such an N', we have $\langle \Gamma, F{\downarrow}_{N'} \mid \Delta', \bot \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} - 1 = \langle \Gamma \mid \Delta', (-F){\downarrow}_N \rangle^{\uparrow}$ and, obviously, there is no choice of N' which would decrease *my* (and thus increase *your*) payoff even more. As above, we obtain that $\langle \Gamma \mid \Delta', (-F){\downarrow}_N \rangle^{\uparrow} < 0$ iff *you* can find a sharpening N' such that $\langle \Gamma, F{\downarrow}_{N'} \mid \Delta', \bot \rangle^{\uparrow} < 0$.

$[(-F)_N, \Gamma' \mid \Delta]$: If *I* attack *your* assertion of $(-F)_N$ then the succeeding game state will be $[\bot, \Gamma' \mid \Delta, F{\downarrow}_{N''}]$ where $N'' \succeq N'$ is chosen by *me* after $N' \succeq N$ is chosen by *Nature*. Thus we need to show that $\langle (-F)_N, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$ iff $\langle \bot, \Gamma' \mid \Delta, F{\downarrow}_{N''} \rangle^{\uparrow} < 0$ for all sharpenings $N'' \succeq N$. By induction we then conclude that $\langle (-F)_N, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$ iff *you* have a winning strategy for $[(-F)_N, \Gamma' \mid \Delta]$.

If $-F$ is true at N then $\langle (-F)_N, \Gamma' \mid \Delta \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow}$ and F is true at no sharpening $N' \succeq N$. Thus, for all of *Nature* and *my* choices of N'', F will never be true at N'' and $\langle \bot, \Gamma' \mid \Delta, F{\downarrow}_{N''} \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow} = \langle (-F)_N, \Gamma' \mid \Delta \rangle^{\uparrow}$. Therefore, $\langle (-F)_N, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$ iff $\langle \bot, \Gamma' \mid \Delta, F{\downarrow}_{N''} \rangle^{\uparrow} < 0$ for all $N'' \succeq N$.

If $-F$ is either false or indefinite at N then $\langle (-F)_N, \Gamma' \mid \Delta \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow} + 1$. If F is true at the sharpening N'' as chosen by *Nature* and *me*, then $\langle \bot, \Gamma' \mid \Delta, F{\downarrow}_{N''} \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow} + 1 = \langle (-F)_N, \Gamma' \mid \Delta \rangle^{\uparrow}$, and otherwise we have $\langle \bot, \Gamma' \mid \Delta, F{\downarrow}_{N''} \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow} = \langle (-F)_N, \Gamma' \mid \Delta \rangle^{\uparrow} - 1$. In any case, if $\langle (-F)_N, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$ then also $\langle \bot, \Gamma' \mid \Delta, F{\downarrow}_{N''} \rangle^{\uparrow} < 0$. On the other hand, there exists some sharpening $N'' \succeq N$ where F is true and thus $\langle (-F)_N, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$ if $\langle \bot, \Gamma' \mid \Delta, F{\downarrow}_{N''} \rangle^{\uparrow} < 0$ for all $N'' \succeq N$.

$[(-F){\downarrow}_N, \Gamma' \mid \Delta]$: Observe that this case is equal to *my* attack on *your* unmodified assertion of $-F$ as above. Since *you* are asserting $-F$, the formula is evaluated pessimistically when calculating $\langle (-F){\downarrow}_N, \Gamma' \mid \Delta \rangle^{\uparrow}$ anyways, even without being forced by the $\downarrow$ modifier. Moreover, observe, that *your* extended payoff of the successor state amounts to $\langle \bot, \Gamma' \mid \Delta, F{\downarrow}_{N''} \rangle^{\uparrow}$ in both cases.

$[(-F){\uparrow}_N, \Gamma' \mid \Delta]$: By Shapiro's definition of the '$-$' operator, a formula $-F$ is true or indefinite at N iff it is true for some sharpening of N. Observe that *you* have a winning strategy for $[(-F){\uparrow}_N, \Gamma' \mid \Delta]$ iff there exists a sharpening $N' \succeq N$ such that *you* have a winning strategy for $[(-F)_{N'}, \Gamma' \mid \Delta]$: After *you* choose $N' \succeq N$, the dialogue rule from *your* point of view proceeds exactly as for *your* unmarked assertion of $(-F)_{N'}$—for *your* winning strategy it is irrelevant whether a world is chosen by *Nature* and *me*, or just by *me*, since *you* must be prepared for all of *Nature* and *my* choices.

$[\Gamma \mid \Delta', (F \Rightarrow G)_N]$: If *you* attack *my* assertion of $(F \Rightarrow G)_N$ then the succeeding game state will be $[\Gamma, F{\downarrow_{N''}} \mid \Delta', G{\downarrow_{N''}}]$ where $N'' \succeq N'$ is chosen by *you* after $N' \succeq N$ is chosen by *Nature* or $[\Gamma \mid \Delta']$. Thus we need to show that $\langle \Gamma \mid \Delta', (F \Rightarrow G)_N \rangle^{\uparrow} < 0$ iff for all sharpenings $N' \succeq N$ there exists a sharpening $N'' \succeq N'$ such that $\langle \Gamma, F{\downarrow_{N''}} \mid \Delta', G{\downarrow_{N''}} \rangle^{\uparrow} < 0$ or $\langle \Gamma \mid \Delta' \rangle^{\downarrow} < 0$. By induction we then conclude that $\langle \Gamma \mid \Delta', (F \Rightarrow G)_N \rangle^{\uparrow} < 0$ iff *you* have a winning strategy for $[\Gamma \mid \Delta', (F \Rightarrow G)_N]$.

If $F \Rightarrow G$ is true or indefinite at N, then $\langle \Gamma \mid \Delta', (F \Rightarrow G)_N \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow}$. Moreover, observe that *you* cannot, for all choices of $N' \succeq N$ by *Nature* find $N'' \succeq N'$ such that $\langle \Gamma, F{\downarrow_{N''}} \mid \Delta', G{\downarrow_{N''}} \rangle^{\uparrow} < \langle \Gamma \mid \Delta' \rangle^{\uparrow}$. This would entail, that *you* always could find a sharpening N'' such that F is true, but G is not true at N'', but then $F \Rightarrow G$ would be false at N. Therefore, if *you* can find $N'' \succeq N'$ for all $N' \succeq N$,such that $\langle \Gamma, F{\downarrow_{N''}} \mid \Delta', G{\downarrow_{N''}} \rangle^{\uparrow} < 0$, then also $\langle \Gamma \mid \Delta' \rangle^{\uparrow} = \langle \Gamma \mid \Delta', (F \Rightarrow G)_N \rangle^{\uparrow} < 0$.

If $F \Rightarrow G$ is false N, then $\langle \Gamma \mid \Delta', (F \Rightarrow G)_N \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} - 1$ and $F \Rightarrow G$ is also false for all sharpenings $N' \succeq N$. Thus, there always exists a sharpening $N'' \succeq N'$ where F is true and G is not true, and we have $\langle \Gamma, F{\downarrow_{N''}} \mid \Delta', G{\downarrow_{N''}} \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} - 1 = \langle \Gamma \mid \Delta', (-F)_N \rangle^{\uparrow}$. Therefore $\langle \Gamma \mid \Delta', (-F)_N \rangle^{\uparrow} < 0$ iff for all $N' \succeq N$ there exists $N'' \succeq N'$ such that $\langle \Gamma, F{\downarrow_{N''}} \mid \Delta', G{\downarrow_{N''}} \rangle^{\uparrow} < 0$ or $\langle \Gamma \mid \Delta' \rangle^{\uparrow} < 0$.

$[\Gamma \mid \Delta', (F \Rightarrow G){\uparrow_N}]$: Observe that this case is equal to *your* attack on my unmodified assertion of $F \Rightarrow G$ as above. Since *I* am asserting $F \Rightarrow G$, the formula is evaluated optimistically when calculating $\langle \Gamma \mid \Delta', (F \Rightarrow G){\uparrow_N} \rangle^{\uparrow}$ anyways, even without being forced by the $\uparrow$ modifier. Moreover, observe, that *your* extended payoff of the successor state amounts to $\langle \Gamma, F{\downarrow_{N''}} \mid \Delta', G{\downarrow_{N''}} \rangle^{\uparrow}$ or $\langle \Gamma \mid \Delta' \rangle^{\uparrow}$ in both cases.

$[\Gamma \mid \Delta', (F \Rightarrow G){\downarrow_N}]$: If *you* attack *my* assertion of $(F \Rightarrow G){\downarrow_N}$ then the succeeding game state will be $[\Gamma, F{\downarrow_{N'}} \mid \Delta', G{\downarrow_{N'}}]$ where $N' \succeq N$ is chosen by *you* or $[\Gamma \mid \Delta']$. Thus we need to show that $\langle \Gamma \mid \Delta', (F \Rightarrow G){\downarrow_N} \rangle^{\uparrow} < 0$ iff for some sharpening $N' \succeq N$ we have $\langle \Gamma, F{\downarrow_{N'}} \mid \Delta', G{\downarrow_{N'}} \rangle^{\uparrow} < 0$ or $\langle \Gamma \mid \Delta \rangle^{\uparrow} < 0$. By induction we then conclude that $\langle \Gamma \mid \Delta', (F \Rightarrow G){\downarrow_N} \rangle^{\uparrow} < 0$ iff *you* have a winning strategy for $[\Gamma \mid \Delta', (F \Rightarrow G){\downarrow_N}]$.

If $F \Rightarrow G$ is true at N then $\langle \Gamma \mid \Delta', (F \Rightarrow G)_N \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow}$. Moreover, no matter which $N' \succeq N$ *you* choose, we will always have $\langle \Gamma, F{\downarrow_{N'}} \mid \Delta', G{\downarrow_{N'}} \rangle^{\uparrow} \geq \langle \Gamma \mid \Delta' \rangle^{\uparrow}$: If both F and G are true or not true at N' we have $\langle \Gamma, F{\downarrow_{N'}} \mid \Delta', G{\downarrow_{N'}} \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow}$ and if F is not true, but G is true at N' we have $\langle \Gamma, F{\downarrow_{N'}} \mid \Delta', G{\downarrow_{N'}} \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} + 1$.

Thus, $\langle \Gamma \mid \Delta', (F \Rightarrow G){\downarrow_N} \rangle^{\uparrow} < 0$ iff $\langle \Gamma \mid \Delta \rangle^{\uparrow} < 0$, and, moreover, whenever we have $\langle \Gamma, F{\downarrow_{N'}} \mid \Delta', G{\downarrow_{N'}} \rangle^{\uparrow} < 0$ for all $N' \succeq N$, then also $\langle \Gamma \mid \Delta' \rangle^{\uparrow} < 0$.

If $F \Rightarrow G$ is false or indefinite at N then $\langle \Gamma \mid \Delta', (F \Rightarrow G){\downarrow_N} \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} - 1$. Moreover, F is true, while G is not true at some sharpening $N' \succeq N$. Thus if *you* choose such an N', we have $\langle \Gamma, F{\downarrow_{N'}} \mid \Delta', G{\downarrow_{N'}} \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} - 1 = \langle \Gamma \mid \Delta', (F \Rightarrow G){\downarrow_N} \rangle^{\uparrow}$ and, obviously, there is no choice of N' which would decrease *my* (and thus increase *your*) payoff even more. As above, we obtain that $\langle \Gamma \mid \Delta', (F \Rightarrow G){\downarrow_N} \rangle^{\uparrow} < 0$ iff *you* can find a sharpening N' such that $\langle \Gamma, F{\downarrow_{N'}} \mid \Delta', G{\downarrow_{N'}} \rangle^{\uparrow} < 0$ or $\langle \Gamma \mid \Delta' \rangle^{\uparrow} < 0$.

$[(F \Rightarrow G)_N, \Gamma' \mid \Delta]$: If *I* attack *your* assertion of $(F \Rightarrow G)_N$ then the succeeding game state will be $[G{\downarrow_{N''}}, \Gamma' \mid \Delta, F{\downarrow_{N''}}]$ or $[\Gamma' \mid \Delta]$ where $N'' \succeq N'$ is chosen by *me* after $N' \succeq N$ is chosen by *Nature*. Thus we need to show that $\langle (F \Rightarrow G)_N, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$ iff $\langle G{\downarrow_{N''}}, \Gamma' \mid \Delta, F{\downarrow_{N''}} \rangle^{\uparrow} < 0$ for all sharpenings $N'' \succeq N$ and also $\langle \Gamma' \mid \Delta \rangle^{\uparrow} < 0$. By induction we then conclude that $\langle (F \Rightarrow G)_N, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$ iff *you* have a winning strategy for $[(F \Rightarrow G)_N, \Gamma' \mid \Delta]$.

If $F \Rightarrow G$ is true at the sharpening N then $\langle (F \Rightarrow G)_N, \Gamma' \mid \Delta \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow}$ holds, thus $\langle (F \Rightarrow G)_N, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$ iff $\langle \Gamma' \mid \Delta \rangle^{\uparrow} < 0$. Moreover, for all sharpenings $N'' \succeq N$ we have $\langle G{\downarrow_{N''}}, \Gamma' \mid \Delta, F{\downarrow_{N''}} \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow}$ if F and G are both true or not true at N'' and we have $\langle G{\downarrow_{N''}}, \Gamma' \mid \Delta, F{\downarrow_{N''}} \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow} - 1$ if F is not true at N'', but G is. Therefore, if $\langle (F \Rightarrow G)_N, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$, then also $\langle G{\downarrow_{N''}}, \Gamma' \mid \Delta, F{\downarrow_{N''}} \rangle^{\uparrow} < 0$ for all of *my* and choices for N''.

If $F \Rightarrow G$ is either false or indefinite at N then $\langle (F \Rightarrow G)_N, \Gamma' \mid \Delta \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow} + 1$. For all of *Nature*'s and *my* choices for $N'' \succeq N$ we have $\langle G{\downarrow_{N''}}, \Gamma' \mid \Delta, F{\downarrow_{N''}} \rangle^{\uparrow} \leq \langle (F \Rightarrow G)_N, \Gamma' \mid \Delta \rangle^{\uparrow}$: If F and G are both true or both not true at N'' we have $\langle G{\downarrow_{N''}}, \Gamma' \mid \Delta, F{\downarrow_{N''}} \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow}$, otherwise if F is true, but G is not true we have $\langle G{\downarrow_{N''}}, \Gamma' \mid \Delta, F{\downarrow_{N''}} \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow} + 1$, and if F is not true, but G is true we have $\langle G{\downarrow_{N''}}, \Gamma' \mid \Delta, F{\downarrow_{N''}} \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow} - 1$. Therefore, if $\langle (F \Rightarrow G)_N, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$ then also $\langle G{\downarrow_{N''}}, \Gamma' \mid \Delta, F{\downarrow_{N''}} \rangle^{\uparrow} < 0$ for all $N'' \succeq N$. On the other hand, as $F \Rightarrow G$ is not true at N, there exists a sharpening $N'' \succeq N$ such that F is true at N'', but G is not. We have $\langle G{\downarrow_{N''}}, \Gamma' \mid \Delta, F{\downarrow_{N''}} \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow} + 1 = \langle (F \Rightarrow G)_N, \Gamma' \mid \Delta \rangle^{\uparrow}$. Therefore, if $\langle G{\downarrow_{N''}}, \Gamma' \mid \Delta, F{\downarrow_{N''}} \rangle^{\uparrow} < 0$ for all $N'' \succeq N$, then also $\langle (F \Rightarrow G)_N, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$.

$[(F \Rightarrow G){\downarrow_N}, \Gamma' \mid \Delta]$: Observe that this case is equal to *my* attack on *your* unmodified assertion of $F \Rightarrow G$ as above. Since *you* are asserting $F \Rightarrow G$, the formula is evaluated pessimistically when calculating $\langle (F \Rightarrow G){\downarrow_N}, \Gamma' \mid \Delta \rangle^{\uparrow}$ anyways, even without being forced by the $\downarrow$ modifier. Moreover, observe, that *your* extended payoff of the successor state amounts to $\langle G{\downarrow_{N''}}, \Gamma' \mid \Delta, F{\downarrow_{N''}} \rangle^{\uparrow}$ or $\langle \Gamma' \mid \Delta \rangle^{\uparrow}$ in both cases.

$[(F \Rightarrow G)\!\uparrow_N, \Gamma' \mid \Delta]$: By Shapiro's definition of the '$F \Rightarrow G$' operator, a formula $F \Rightarrow G$ is true or indefinite at N iff it is true for some sharpening of N. Observe that *you* have a winning strategy for $[(F \Rightarrow G)\!\uparrow_N, \Gamma' \mid \Delta]$ iff there exists a sharpening $N' \succeq N$ such that *you* have a winning strategy for $[(-F)_{N'}, \Gamma' \mid \Delta]$: After *you* choose $N' \succeq N$, the dialogue rule from *your* point of view proceeds exactly as for *your* unmarked assertion of $(F \Rightarrow G)_{N'}$—for *your* winning strategy it is irrelevant whether a world is chosen by *Nature* and *me*, or just by *me*, since *you* must be prepared for all of *Nature* and *my* choices.

$[\Gamma \mid \Delta', (Ax\ F(x))_N]$: If *you* attack *my* assertion of $(Ax\ F(x))_N$ then the succeeding game state will be $[\Gamma \mid \Delta', F(c)\!\downarrow_{N'''}]$ where first *Nature* chooses $N' \succeq N$, then *you* choose c and $N'' \succeq N'$, and *I* choose $N''' \succeq N''$. Thus we need to show that $\langle \Gamma \mid \Delta', (Ax\ F(x))_N \rangle^{\uparrow} < 0$ iff for all sharpenings $N' \succeq N$ there exists a constants c and a sharpening $N'' \succeq N'$ such that for all sharpenings $N''' \succeq N''$ we have $\langle \Gamma \mid \Delta', F(c)\!\downarrow_{N'''} \rangle^{\uparrow} < 0$. By induction we then conclude that $\langle \Gamma \mid \Delta', (Ax\ F(x))_N \rangle^{\uparrow} < 0$ iff *you* have a winning strategy for $[\Gamma \mid \Delta', (Ax\ F(x))_N]$.

If $Ax\ F(x)$ is true or indefinite at N, then $\langle \Gamma \mid \Delta', (Ax\ F(x))_N \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow}$ and the formula is also true at some sharpening $N' \succeq N$. If *Nature* chooses this world N', then, no matter which constant c *you* choose, $F(c)$ will be forced at N'. This means, that for all of *your* choices of $N'' \succeq N'$, *I* can find $N''' \succeq N''$ such that $F(c)$ is true at N''' and we obtain $\langle \Gamma \mid \Delta', F(c)\!\downarrow_{N'''} \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} = \langle \Gamma \mid \Delta', (Ax\ F(x))_N \rangle^{\uparrow}$. If *Nature* chooses another sharpening $N' \succeq N$ where the formula is not true, then *your* payoff might even be higher.

If $Ax\ F(x)$ is false at N, then $\langle \Gamma \mid \Delta', (Ax\ F(x))_N \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} - 1$ and the formula is true at no sharpening $N' \succeq N$. Thus, no matter which sharpening $N' \succeq N$ *Nature* chooses, *you* can always find a constant c such that $F(c)$ is not forced at N'. Hence, you can find $N'' \succeq N'$ such that for none of *my* choices of $N''' \succeq N''$ the formula $F(c)$ is true. We obtain $\langle \Gamma \mid \Delta', F(c)\!\downarrow_{N'''} \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} - 1 = \langle \Gamma \mid \Delta', (Ax\ F(x))_N \rangle^{\uparrow}$.

Summarizing, we see that in both cases we have $\langle \Gamma \mid \Delta', (Ax\ F(x))_N \rangle^{\uparrow} < 0$ iff for all sharpenings $N' \succeq N$ there exists a constants c and a sharpening $N'' \succeq N'$ such that for all sharpenings $N''' \succeq N''$ we have $\langle \Gamma \mid \Delta', F(c)\!\downarrow_{N'''} \rangle^{\uparrow} < 0$.

$[\Gamma \mid \Delta', (Ax\ F(x))\!\uparrow_N]$: Observe that this case is equal to *your* attack on my unmodified assertion of $Ax\ F(x)$ as above. Since *I* am asserting $Ax\ F(x)$, the formula is evaluated optimistically when calculating $\langle \Gamma \mid \Delta', (Ax\ F(x))\!\uparrow_N \rangle^{\uparrow}$ anyways, even without being forced by the $\uparrow$ modifier. Moreover, observe, that *your* extended payoff of the successor state amounts to $\langle \Gamma \mid \Delta', F(c)\!\downarrow_{N'''} \rangle^{\uparrow}$ in both cases.

$[\Gamma \mid \Delta', (Ax\ F(x)){\downarrow_N}]$: If *you* attack *my* assertion of $(Ax\ F(x)){\downarrow_N}$ then the succeeding game state will be $[F(c){\downarrow_{N''}}, \Gamma' \mid \Delta]$ where $N'' \succeq N'$ is chosen by *me*, after *you* have chosen a sharpening $N' \succeq N$ and a constant c. Thus we need to show that $\langle \Gamma{\downarrow}\ \Delta', (Ax\ F(x))_N \mid < \rangle^{\uparrow} 0$ iff there exists a sharpening $N' \succeq N$ and a constant c such that for all sharpenings $N'' \succeq N'$ we have $\langle \Gamma \mid \Delta', F(c){\downarrow_{N''}} \rangle^{\uparrow} < 0$. By induction we then conclude that $\langle \Gamma \mid \Delta', (Ax\ F(x)){\downarrow_N} \rangle^{\uparrow} < 0$ iff *you* have a winning strategy for $[\Gamma \mid \Delta', (Ax\ F(x)){\downarrow_N}]$.

If $Ax\ F(x)$ is true at N then $\langle \Gamma \mid \Delta', (Ax\ F(x)){\downarrow_N} \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\downarrow}$. Moreover, no matter which constant c *you* choose, $F(c)$ will always be forced at N. Thus, also no matter which sharpening $N' \succeq N$ *you* choose, *I* can always find a sharpening $N'' \succeq N'$ such that $F(c)$ is true at N'' and thus $\langle \Gamma \mid \Delta', F(c){\downarrow_{N''}} \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} = \langle \Gamma \mid \Delta', (Ax\ F(x))_N \rangle^{\uparrow}$, and if *I* choose another sharpening $N'' \succeq N'$ *your* payoff is even higher.

If $Ax\ F(x)$ is false or indefinite at N then $\langle \Gamma \mid \Delta', (Ax\ F(x)){\downarrow_N} \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} - 1$. Moreover, there exists a sharpening $N' \succeq N$ where $F(c)$ is not forced for some constant c. If *you* choose this sharpening N' and constant c, then *I* cannot find a sharpening $N'' \succeq N'$ such that $F(c)$ is true at N'' and thus $\langle \Gamma \mid \Delta', F(c){\downarrow_{N''}} \rangle^{\uparrow} = \langle \Gamma \mid \Delta' \rangle^{\uparrow} - 1 = \langle \Gamma \mid \Delta', (Ax\ F(x)){\downarrow_N} \rangle^{\uparrow}$.

Summarizing, in both cases we obtain that $\langle \Gamma \mid \Delta', (Ax\ F(x))_N \rangle^{\uparrow} < 0$ iff *you* can find a sharpening $N' \succeq N$ and a constant c such that for all of *my* choices of $N'' \succeq N'$ we have $\langle \Gamma \mid \Delta', F(c){\downarrow_{N''}} \rangle^{\uparrow} \geq 0$.

$[(Ax\ F(x))_N, \Gamma' \mid \Delta]$: If *I* attack *your* assertion of $(Ax\ F(x))_N$ then the succeeding game state will be $[F(c){\downarrow_{N'''}}, \Gamma' \mid \Delta]$ where first *Nature* chooses $N' \succeq N$, then *I* choose c and $N'' \succeq N'$, and *you* choose $N''' \succeq N''$. Thus we need to show that $\langle (Ax\ F(x))_N, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$ iff for all sharpenings $N'' \succeq N$ and constants c there exists a sharpening $N''' \succeq N''$ such that we have $\langle F(c){\downarrow_{N'''}}, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$.

If $Ax\ F(x)$ is true at N then $\langle (Ax\ F(x))_N, \Gamma' \mid \Delta \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow}$ and $F(c)$ is forced at N for all constants c. Hence, for all of *my* and *Nature*'s choices for c and $N'' \succeq N$ *you* can always find $N''' \succeq N''$ such that $F(c)$ is true at N''' and thus $\langle F(c){\downarrow_{N'''}}, \Gamma' \mid \Delta \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\downarrow} = \langle (Ax\ F(x))_N, \Gamma' \mid \Delta \rangle^{\uparrow}$.

If $Ax\ F(x)$ is false or indefinite at N then $\langle (Ax\ F(x))_N, \Gamma' \mid \Delta \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow} + 1$ and there exists c such that $F(c)$ is not forced at N. Thus, after *Nature* and *me* choosing c and $N'' \succeq N$, *you* cannot always find $N''' \succeq N''$ such that $F(c)$ is true at N''' and we have $\langle F(c){\downarrow_{N'''}}, \Gamma' \mid \Delta \rangle^{\uparrow} = \langle \Gamma' \mid \Delta \rangle^{\uparrow} + 1 = \langle (Ax\ F(x))_N, \Gamma' \mid \Delta \rangle^{\uparrow}$.

Summarizing, in both cases we obtain that $\langle (Ax\ F(x))_N, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$ iff for all of *Nature*'s and *my* choices of $N'' \succeq N$ and c, *you* can find a sharpening $N''' \succeq N''$ such that we have $\langle F(c){\downarrow}_{N''}, \Gamma' \mid \Delta \rangle^{\uparrow} < 0$.

$[(Ax\ F(x)){\downarrow}_N, \Gamma' \mid \Delta]$: Observe that this case is equal to *my* attack on *your* unmodified assertion of $Ax\ F(x)$ as above. Since *you* are asserting $Ax\ F(x)$, the formula is evaluated pessimistically when calculating $\langle (Ax\ F(x)){\downarrow}_N, \Gamma' \mid \Delta \rangle^{\uparrow}$ anyways, even without being forced by the $\downarrow$ modifier. Moreover, observe, that *your* extended payoff of the successor state amounts to $\langle F(c){\downarrow}_{N'''}, \Gamma' \mid \Delta \rangle^{\uparrow}$ in both cases.

$[(Ax\ F(x)){\uparrow}_N, \Gamma' \mid \Delta]$: By Shapiro's definition of the 'A' operator, a formula $Ax\ F(x)$ is true or indefinite at N iff it is true for some sharpening of N. Observe that *you* have a winning strategy for $[(Ax\ F(x)){\uparrow}_N, \Gamma' \mid \Delta]$ iff there exists a sharpening $N' \succeq N$ such that *you* have a winning strategy for $[(Ax\ F(x))_{N'}, \Gamma' \mid \Delta]$: After *you* choose $N' \succeq N$, the dialogue rule from *your* point of view proceeds exactly as for *your* unmarked assertion of $(Ax\ F(x))_{N'}$—for *your* winning strategy it is irrelevant whether a world is chosen by *Nature* and *me*, or just by *me*, since *you* must be prepared for all of *Nature* and *my* choices.

$[\Gamma \mid \Delta', (Ex\ F(x))_N]$, $[\Gamma \mid \Delta', (Ex\ F(x)){\downarrow}_N]$, $[\Gamma \mid \Delta', (Ex\ F(x)){\uparrow}_N]$, $[(Ex\ F(x))_N, \Gamma' \mid \Delta]$, $[(Ex\ F(x)){\downarrow}_N, \Gamma' \mid \Delta]$, $[(Ex\ F(x)){\uparrow}_N, \Gamma' \mid \Delta]$: Again, these cases are analogous to the respective attacks on $Ax\ F(x)$. Note that the only difference in the dialogue rules is the player selecting the constant c.

□

Finally, Theorem 12 is updated to refer to the particular world N:

Theorem 14. *Consider the game $\mathcal{G} = [\ \mid F_N]$ starting with* my *assertion of F at N.*

(i) I *have a winning strategy for $\mathcal{G}$ iff F is* true*,*

(ii) you *have a winning strategy for $\mathcal{G}$ iff F is* false*, and*

(iii) *neither of us has a winning strategy for $\mathcal{G}$ iff F is* indefinite

at the world N.

Proof. The theorem follows from Lemma 5 analogously to the proof of Theorem 12. □

5.4 Relation to Barker's Approach

Shapiro's approach "Vagueness in Context" and Barker's account of the "Dynamics of Vagueness" (see Chapter 4) may seem fundamentally different and incompatible at the first glance: For Barker, vague predicates apply to an individual *up to some degree* measured on an appropriate scale. Remember that 'degrees of applicability' here are not to be confused with degrees of truth as, e.g., promoted by fuzzy logics. For example, the height of people might be measured in centimeters, while their weight is measured in kilograms. Deciding whether an individual counts as 'tall' (or as 'heavy') then amounts to comparing these degrees with some threshold value locally in each possible world of the given context. Shapiro's approach, on the other hand, does not mention degrees of applicability at all, but models vague predicates by partial interpretations linked by a precisification order. In this section we investigate how these approaches can be related to each other. As it turns out, for situations which can equally be modeled in both approaches, they will both predict that exactly the same propositions are validated after a context update. Models for one approach which do not have a counterpart for the other approach, indicate situations where one approach is more expressive than the other. However, we still have to give a precise definition for what it means that situation can 'equally be modeled' using Barker's and Shapiro's approaches. Already von Stechow [91], analyzing comparative expressions like 'John is taller than Mary', observed that delineation based approaches like Kamp [47] and Lewis [60] (see also Section 1.4) are in fact closely related to degree-based approaches: The predicate, e.g., 'taller than' may be considered as being defined in terms of degrees of height. This analysis however has not been extended to (dynamic) contextual accounts of vagueness like Barker's and Shapiro's.

Essentially based on the article [77], we first show how contexts, according to Shapiro as well as to Barker, can be transformed to sets of classical interpretations as an intermediate representation for contexts, intuitively capturing their 'propositional content'. Recall from Section 5.1 that Shapiro defines a context as a tuple $\langle W, N_0\rangle_{Sh}$ where W denotes the space of possible worlds with the root $N_0 \in W$. Barker does not explicitly keep track of the initial context: After a context update, the worlds not surviving the update are just deleted. In order to keep track of these worlds we consider a initial context C_0 and denote such a model by $\langle \mathcal{P}(C_0), C_0\rangle_B$. Furthermore we will denote the set of (first-order) formulas as *Form* built from the relevant (vague and non-vague) predicates $\mathcal{R}$, relevant individuals $\mathcal{U}$ and the usual logical connectives and quantifiers. Moreover, for $\phi \in$ we write $W, N_0 \models_{Sh} \phi$ iff ϕ is forced at N_0 in W, since Shapiro's favoured notion of truth is defined in terms of forcing. Observe that Shapiro's language is richer than *Form*, containing the operator 'D' denoting definite truth and also other, non-local, quantifiers, but nevertheless contains *Form* as a subset. Similarly, for Barker, we write $C \models_B \phi$ iff ϕ is true at all worlds $w \in C$, i.e., if all worlds $w \in C$ are classical models of ϕ. Observe that, in both cases, we may have that neither $W, N_0 \models_{Sh} \phi$ nor $W, N_0 \models_{Sh} \neg\phi$ holds and also neither $C \models_B \phi$ nor $C \models_B \neg\phi$, respectively.

For Shapiro's approach, we obtain an intermediate representation for contexts by mapping a possible world, i.e., a partial valuation, to all its complete sharpenings. Remember that the relation '$\succeq$' forms a partial order on the set W of possible worlds.

Definition 24. Let W be a set of possible worlds and, $N \in W$ a partial interpretation. Then the translation from N to a set of classical worlds, denoted as ${}^{W}T_{sh}(N)$, maps N to the set of all complete sharpenings of N in W.

$$ {}^{W}T_{sh}(N) =_{df} \{N' \in W : N' \succeq N, N' \text{ is complete}\} $$

Remember that Shapiro does not require that there exist complete sharpenings for all possible worlds and, indeed, for typical Sorites situations the tolerance principle forbids complete precisifications (see Section 1.3). However, turning to Barker's notion of context, we see that there all possible worlds are completely precise in the sense that for each vague predicate under consideration each world has a precise threshold value and for each individual a degree to which that predicate applies to it. Therefore, it can be identified with a complete interpretation of the vague predicates in question. As there is no analogue to Shapiro's 'jumps' to another branch in the context space, by a suitable series of statements it is always possible to obtain a context where all vague predicates are completely precisified. We therefore from now on stipulate that Shapiro's contexts (i.e., partial interpretations) always include completions to classical worlds. If this is not the case in a model, then there exists no equivalent counterpart using Barker's approach—in this respect Shapiro's approach is more expressive than Barker's. We thus have

$$ {}^{W}T_{sh}(N) \models_B \phi \text{ iff } W, N \models_{Sh} \phi $$

as now ϕ being forced at N in W collapses to ϕ being true in all complete sharpenings of N.

Observe that the mapping ${}^{W}T_{sh}$ is in general not reversible. Assume two vague predicates 'tall' and 'heavy' with the penumbral connections stating that the individuals under consideration are of normal stature, i.e., nobody can be both tall and not heavy (nor not tall but heavy) at the same time (but everybody can be tall with his state of heaviness left undecided, nevertheless). In Shapiro's notation these restrictions are formalized as

$$ D(\forall x \neg(\mathbf{tall}(x) \wedge \neg\mathbf{heavy}(x)) \text{ and} $$
$$ D(\forall x \neg(\neg\mathbf{tall}(x) \wedge \mathbf{heavy}(x)) $$

where the 'D'-operator ensures that both formulas are *determinately* true, so they are forced at the base of the context. Since we stipulated that there always exist complete precisifications,

$$
\begin{tikzcd}[column sep=tiny]
 & & & N_0 : \{\} \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\
N_1 : \{\mathbf{tall}(\mathbf{a})\} \arrow[dr] & & N_2 : \{\mathbf{heavy}(\mathbf{a})\} \arrow[dl] & & N_3 : \{\neg\mathbf{tall}(\mathbf{a})\} \arrow[dr] & & N_4 : \{\neg\mathbf{heavy}(\mathbf{a})\} \arrow[dl] \\
 & N_5 : \{\mathbf{tall}(\mathbf{a}), \mathbf{heavy}(\mathbf{a})\} & & & & N_6 : \{\neg\mathbf{tall}(\mathbf{a}), \neg\mathbf{heavy}(\mathbf{a})\} &
\end{tikzcd}
$$

Figure 5.5: An example model $\langle W, N_0 \rangle_{Sh}$

forcing here coincides with supertruth as used in supervaluationist approaches. Furthermore, assume that only one individual a is under consideration. Figure 5.5 shows the structure of the according frame $\langle W, N_0 \rangle_{Sh}$. The mapping ${}^{W}T_{sh}$, according to Definition 24, is given by

	${}^{W}T_{sh}$
N_0	$\{N_5, N_6\}$
N_1	$\{N_5\}$
N_2	$\{N_5\}$
N_3	$\{N_6\}$

	${}^{W}T_{sh}$
N_4	$\{N_6\}$
N_5	$\{N_5\}$
N_6	$\{N_6\}$.

When mapping a partial interpretation to a set of classical interpretations, we thus lose, e.g., the information whether the original world was N_1, N_2, or N_5. However, Shapiro argues that an appropriate notion of *truth at a possible world* is defined in terms of forcing and exactly the same formulas are forced at those worlds (and analogously for N_3, N_4, and N_6). For any possible world N we denote by $[N]^{W}_{Sh}$ the set of worlds in W which are mapped to the same set of classical interpretations:

$$[N]^{W}_{Sh} =_{df} \{N' \in W : {}^{W}T_{sh}(N') = {}^{W}T_{sh}(N)\}$$

thus providing a partition of the context space. Intuitively, all worlds in $[N]^{W}_{Sh}$ validates the same propositions, i.e., the same formulas of *Form* are forced there.

Conversely, the mapping ${}^{W}T^{-1}_{sh}$ maps sets of classical worlds to sets of possible worlds and satisfies $N \in {}^{W}T^{-1}_{sh}({}^{W}T_{sh}(N))$ for all N in W. We call the mapping ${}^{W}T^{-1}_{sh}$ the *preimage* to ${}^{W}T_{sh}$, but keep in mind that ${}^{W}T_{sh}$ in general is not reversible and therefore ${}^{W}T^{-1}_{sh}$ returns sets of contexts instead of a single one. Sticking to the above example we have ${}^{W}T^{-1}_{sh}(\{N_5, N_6\}) = \{N_0\}$, ${}^{W}T^{-1}_{sh}(\{N_5\}) = \{N_1, p_2, N_5\}$, and ${}^{W}T^{-1}_{sh}(\{N_6\}) = \{N_3, N_4, N_6\}$.

In general, for an arbitrary frame $\langle W, N_0 \rangle$ it is computed by

$$
\begin{aligned}
{}^{W}T_{sh}^{-1}(S) = \{w \in W : & \forall s \in S.\ s \succeq w \wedge \\
& \forall w' \in W.\ (w' \succeq w \wedge w' \text{ is complete}) \rightarrow w' \in S\}.
\end{aligned}
$$

For Barker's approach, contexts can be mapped to sets of classical worlds in an obvious way: Consider a set C of possible worlds according to Barker and a particular world $w \in C$. Thus, e.g., for the vague predicate 'tall' the expression $\mathbf{d}(w)(^{\uparrow}\textit{tall})$ denotes the local standard of tallness at w, and an individual a is 'tall' in w if and only if w survives the associated context update and $w \in [\![\textit{tall}]\!](a)(C)$ holds. Each world w can straightforwardly be identified with a complete (classical) interpretation by evaluating the relevant predicates in this manner. The following mapping T_b can be seen as an *abstraction* of Barker's notion of context away from concrete degree values:

Definition 25. Let C be a context according to Barker. Then the translation of C to complete interpretations $T_b(C)$ is defined as

$$
\begin{aligned}
& T_b(C) = \{s(w) : w \in C\} \text{ where } s(w) \text{ denotes a classical interpretation such that} \\
& \qquad R(u_1, ..., u_n) \text{ is true at } s(w) \text{ iff } w \in [\![R]\!](u_1, ..., u_n)(C) \text{ and} \\
& \qquad R(u_1, ..., u_n) \text{ is false at } s(w) \text{ iff } w \notin [\![R]\!](u_1, ..., u_n)(C)
\end{aligned}
$$

for all individuals $u_1, ..., u_n \in \mathcal{U}$ and predicates $R \in \mathcal{R}$ with arity n.

Observe that Barker considers only unary predicates, but his formalism can straightforwardly be generalized to arbitrary predicates: For the n-ary predicate R the associated element filter $[\![R]\!]$ just takes an n tuple as its first argument. As above, the mapping T_b induces a partition of the context space by

$$\big[C\big]_B^{C_0} =_{df} \{C' \in \mathcal{P}(C_0) : T_b(C') = T_b(C)\}.$$

Analogously as for Shapiro, we denote the inverse mapping to T_b as ${}^{C_0}T_b^{-1}$: where, above, we needed to refer to the overall frame $\langle W, M \rangle_{Sh}$ in the definition of ${}^{W}T_{sh}^{-1}$, the mapping ${}^{C_0}T_b^{-1}$ is relative to the fixed initial context C_0. We find the appropriate subset C of C_0, i.e., a more precise context, given a set of classical worlds S by taking those possible worlds in C_0 which are sanctioned by some interpretation $s \in S$:

$${}^{C_0}T_b^{-1}(S) =_{df} \{w \in C_0 : \exists s' \in S. s' = s(w)\}$$

where $s(w)$ is defined as above in Definition 25.

We introduce the notion of *corresponding models*. Two models $\langle W, N_0 \rangle_{Sh}$ and $\langle \mathcal{P}(C_0), C_0 \rangle_B$ correspond to each other if they contain the same underlying assumptions: Penumbral connections in W must implicitly be enforced in C_0 and also vice versa.

Definition 26. Let $\langle W, N_0 \rangle_{Sh}$ and $\langle \mathcal{P}(C_0), C_0 \rangle_B$ be two models according to Shapiro and Barker, respectively. They are called *corresponding models* iff the following conditions are met:

- They agree on the set $\mathcal{R}$ of relevant predicates and $\mathcal{U}$ of relevant individuals,
- for each $N \in W$ there exists $C \in \mathcal{P}(C_0)$ such that ${}^{W}T_{sh}(N) = T_b(C)$, and
- for each $C \in \mathcal{P}(C_0)$ there exists $N \in W$ such that $T_b(C) = {}^{W}T_{sh}(N)$.

The notion of *corresponding models* of the two different approaches intuitively means that the same situation is modeled both using Barker's and Shapiro's approaches making exactly the same assumptions. More precisely, the requirements ensure that neither the penumbral connections on the partial interpretations are too strong, i.e., they forbid possible contexts in $\mathcal{P}(C_0)$, nor does the initial set of possible worlds C_0 implicitly contain restrictions which are not reflected in the penumbral connections.

These requirements also imply that for each vague predicate under consideration the scale implicitly given in Barker's model is expressed in $\langle W, N_0 \rangle_{Sh}$ via penumbral connections: if according to Barker, some individual is regarded as tall in all possible worlds, another individual with a higher degree of height is automatically also regarded as tall. If a model $\langle W, N_0 \rangle_{Sh}$ according to Shapiro fails to sanction such relationships via penumbral connections, then there will be possible worlds which cannot be translated to an according context as in Barker. Barker simply stipulates that degrees range over an appropriate scale. There are few restrictions on the scale—it must be a partial order obeying monotonicity with respect to the predicates in question—but its exact type is not described *inside* the model. For all of his examples Barker uses linear scales, like people's height. Contrarily, for Shapiro there is a priori no such externally defined scale structure, instead it can be defined inside the model itself by the means of penumbral connections. We use 'tall' as an example for an adjective denoting a degree on a linear scale. Other scale structures, however, can be modeled analogously. The binary precise predicates $\mathbf{taller\_than}$ and $\mathbf{as\_tall\_as}$ and the unary vague predicate $\mathbf{tall}$ can be characterized as follows:

$$
\begin{array}{rl}
(NV_1) & D(\forall x \forall y.(\mathbf{as\_tall\_as}(x,y) \vee \neg \mathbf{as\_tall\_as}(x,y)) \\
(RE_1) & D(\forall x.\mathbf{as\_tall\_as}(x,x)) \\
(TR_1) & D(\forall x \forall y \forall z.(\mathbf{as\_tall\_as}(x,y) \wedge \mathbf{as\_tall\_as}(y,z)) \\
 & \to \mathbf{as\_tall\_as}(x,z)) \\
(SY_1) & D(\forall x \forall y.(\mathbf{as\_tall\_as}(x,y) \to \mathbf{as\_tall\_as}(y,x)) \\
(NV_2) & D(\forall x \forall y.(\mathbf{taller\_than}(x,y) \vee \neg \mathbf{taller\_than}(x,y)) \\
(TR_2) & D(\forall x \forall y \forall z.(\mathbf{taller\_than}(x,y) \wedge \mathbf{taller\_than}(y,z)) \\
 & \to \mathbf{taller\_than}(x,z)) \\
(TI_2) & D(\forall x \forall y.\mathbf{taller\_than}(x,y) \vee \mathbf{taller\_than}(y,x) \vee \mathbf{as\_tall\_as}(x,y)) \\
(i) & D(\forall x \forall y.(\mathbf{tall}(x) \wedge \mathbf{taller\_than}(y,x)) \to \mathbf{tall}(y)) \\
(ii) & D(\forall x \forall y.(\neg \mathbf{tall}(x) \wedge \mathbf{taller\_than}(x,y)) \to \neg \mathbf{tall}(y))
\end{array}
$$

Properties (NV_1) and (NV_2) here ensure that both **as_tall_as** and $\mathbf{taller\_than}$ are non-vague; at each partial interpretation any pair of individuals is either in their extension or anti-extension. This goes in accordance with Barker [5] arguing that comparative clauses like 'taller than' do not involve any vagueness. Properties (RE_1), (SY_1), and (TR_1) ensure that **as_tall_as** is indeed an equivalence relation by postulating reflexivity, symmetry, and transitivity. Similarly, properties (TR_2),(TI_2) ensure that $\mathbf{taller\_than}$ is a strict total order by postulating that it is transitive and trichotomous. Finally, properties (i) and (ii) ensure that the vague predicate $\mathbf{tall}$ respects this total ordering. Rooij [89] also gives similar axiomatizations for other orderings motivated by taking the positive adjective, e.g. 'tall', as a starting point and observing how 'tall' behaves with respect to comparison classes. These orderings can be characterized by penumbral connections analogously as described here for a linear ordering.

Consider again the partitions $\left[\,\cdot\,\right]^{W}_{Sh}$ of the model $\langle W, N_0\rangle_{Sh}$ and $\left[\,\cdot\,\right]^{C_0}_{B}$ of $\langle \mathcal{P}(C_0), C_0\rangle_B$. Definition 26 essentially states that the intermediate representation obtained from a context in one model can also be reached from the other. Since these two partitions are defined in terms of that intermediate representation, we indeed have:

Proposition 7. *For two corresponding models $\langle W, N_0\rangle_{Sh}$ and $\langle \mathcal{P}(C_0), C_0\rangle_B$ there exists a bijection between the partition blocks $\left[\,\cdot\,\right]^{W}_{Sh}$ and $\left[\,\cdot\,\right]^{C_0}_{B}$ induced by the functions*

$$
{}^{C_0}T_b^{-1} \circ {}^{W}T_{sh} \qquad \textit{and} \qquad {}^{W}T_{sh}^{-1} \circ T_b.
$$

Proof. Let $N \in W$ be a partial interpretation. By the definition of $\left[\,\cdot\,\right]^{W}_{Sh}$ all elements of $\left[N\right]^{W}_{Sh}$ are mapped to the same set of classical interpretations S. By Definition 26, in a corresponding model $\langle \mathcal{P}(C_0), C_0 \rangle_B$ there exists $C \in \mathcal{P}(C_0)$ such that $T_b(C) = S$ and also vice versa. Thus the images of the two homomorphisms induced by $\big[\,\cdot\,\big]^W_{Sh}$ and $\big[\,\cdot\,\big]^{C_0}_B$ coincide and the fundamental homomorphism theorem states that both partitions are isomorphic. For any context $N' \in \big[N\big]^W_{Sh}$ the mapping ${}^{C_0}T_b^{-1} \circ {}^W T_{sh}$ yields the corresponding set of contexts in $\mathcal{P}(C_0)$ and vice versa for any context $C' \in \big[C\big]^{C_0}_B$. $\square$

If two contexts N and C are connected by this bijection, i.e., by $N \in ({}^W T_{sh}^{-1} \circ T_b)(C)$ and $C \in ({}^{C_0}T_b^{-1} \circ {}^W T_{sh})(N)$, we write $\big[N\big]^W_{Sh} \simeq \big[C\big]^{C_0}_B$ By the construction, all contexts $N' \in \big[N\big]^W_{Sh}$, and all contexts $C' \in \big[C\big]^{C_0}_B$, since they map to the same set of classical interpretations, validate the same (first order) formulas. Hence for all $\phi \in$ *Form* we have $W, N' \models_{Sh} \phi$ iff $C' \models_B \phi$.

Remark. There exists an (antitone) Galois connection $\langle \psi; \phi \rangle$ (see, e.g., [31]) between the sets $\mathcal{P}(C_0)$ and W induced by the functions ψ and ϕ defined as

$$\psi : \mathcal{P}(\mathcal{P}(C_0)) \to \mathcal{P}(W) \qquad \mathfrak{C} \mapsto \{N \in W : \forall C \in \mathfrak{C}.\, T_b(C) =^W T_{sh}(N)\} \qquad \text{and}$$
$$\phi : \mathcal{P}(W) \to \mathcal{P}(\mathcal{P}(C_0)) \qquad \mathfrak{N} \mapsto \{C \in \mathcal{P}(C_0) : \forall N \in \mathfrak{N}.\, {}^W T_{sh}(N) = T_b(C)\}.$$

This Galois connection characterizes the partitions $\big[\,\cdot\,\big]^W_{Sh}$ and $\big[\,\cdot\,\big]^{C_0}_B$ as follows:

Proposition 8. *For two corresponding models $\langle W, N_0 \rangle_{Sh}$ and $\langle \mathcal{P}(C_0), C_0 \rangle_B$ and the Galois connection $\langle \psi; \phi \rangle$ the partitions $\big[\,\cdot\,\big]^W_{Sh}$ and $\big[\,\cdot\,\big]^{C_0}_B$ coincide with the closure operators $\psi \circ \phi$ and $\phi \circ \psi$.*

Proof. Observe that the definition of ψ and ϕ amounts to

$$\psi(\mathfrak{C}) = \bigcap_{C \in \mathfrak{C}} \psi(\{C\}) = \bigcap_{C \in \mathfrak{C}} \big\{N \in W : T_b(C) =^W T_{sh}(N)\big\} = \bigcap_{C \in \mathfrak{C}} ({}^W T_{sh}^{-1} \circ T_b)(C) \qquad \text{and}$$
$$\phi(\mathfrak{N}) = \bigcap_{N \in \mathfrak{N}} \phi(\{N\}) = \bigcap_{N \in \mathfrak{N}} \big\{C \in \mathcal{P}(C_0) :^W T_{sh}(N) = T_b(C)\big\} = \bigcap_{N \in \mathfrak{N}} ({}^{C_0}T_b^{-1} \circ^W T_{sh})(N).$$

The induced closure operators $\phi \circ \psi$ and $\psi \circ \phi$ applied to singleton sets are computed as follows:

$$(\psi \circ \phi)(\{N\}) = \bigcap_{C \in \phi(\{N\})} \psi(C) = \bigcap_{C \in ({}^{C_0}T_b^{-1} \circ {}^W T_{sh})(\{N\})} ({}^W T_{sh}^{-1} \circ T_b)(C) \qquad \text{and}$$
$$(\phi \circ \psi)(\{C\}) = \bigcap_{N \in \psi(\{C\})} \psi(N) = \bigcap_{N \in ({}^W T_{sh}^{-1} \circ T_b)(\{C\})} ({}^{C_0}T_b^{-1} \circ^W T_{sh})(N)$$

Since ${}^{C_0}T_b^{-1}$ and ${}^W T_{sh}^{-1}$ are the preimages to T_b and to ${}^W T_{sh}$, respectively, (and since N and C belong to corresponding models) this amounts to

$$(\psi \circ \phi)(\{N\}) = ({}^W T_{sh}^{-1} \circ^W T_{sh})(\{N\}) = \big[N\big]^W_{Sh} \text{ and}$$
$$(\phi \circ \psi)(\{C\}) = ({}^{C_0}T_b^{-1} \circ T_b)(\{N\}) = \big[C\big]^{C_0}_B.$$

$\square$

Assume now that a situation is described by two corresponding models $\langle \mathcal{P}(C_0), C_0 \rangle_B$ and $\langle W, N_0 \rangle_{Sh}$, in the first case using a scale based model in the sense of Barker and in the second case using a delineation model in the sense of Shapiro, where the scales used in Barker's model are accordingly encoded as penumbral connections. By Definition 26 we have $\big[C_0\big]_B^{C_0} \simeq \big[M\big]_{Sh}^{W}$. We now analyze how context update with information conveyed by a statement $\phi \in$ *Form* takes place in these two models. There are three cases:

- ϕ conveys no new information, i.e., both $W, N_0 \models_{Sh} \phi$ and $C_0 \models_B \phi$ holds. In this case for both models the current context remains unchanged.

- ϕ conveys conflicting information, i.e., $W, N_0 \models_{Sh} \neg\phi$ and $C_0 \models_B \neg\phi$. In this case, according to Barker's model the whole context is filtered out. Shapiro's approach here differs significantly: A jump (see Section 5.1) occurs to another context within W where ϕ holds, but other propositions are possibly invalidated—handling conflicting information is surely one one aspect where Shapiro's approach is superior to Barker's.

- ϕ is not yet settled as true or as false in the current context: Neither $W, N_0 \models_{Sh} \phi$ nor $W, N_0 \models_{Sh} \neg\phi$ and neither $C_0 \models_B \phi$ nor $C_0 \models_B \neg\phi$ hold. The current context, in both approaches, is updated to new contexts N_1 and C_1, respectively which both validate ϕ. Our claim that both approaches predict the same statements to be validated after the updated amounts to showing that for the succeeding contexts N_1 and C_1 the relation $\big[N_1\big]_{Sh}^{W} \simeq \big[C_1\big]_B^{C_0}$ still holds. The argument can also be iterated to show that also after a series of context updates, both models yield contexts validating the same propositions.

Theorem 15. *Let $\langle W, N_0 \rangle_{Sh}$ and $\langle \mathcal{P}(C_0), C_0 \rangle_B$ be two corresponding models. Consider a formula $\phi \in$ Form which is still unsettled in N_0 and in C_0. Then, after a context update with ϕ resulting in the contexts N_1 and C_1, the relation $\big[N_1\big]_{Sh}^{W} \simeq \big[C_1\big]_B^{C_0}$ holds.*

Proof. Definition 26 entails for two corresponding models $\langle W, N_0 \rangle_{Sh}$ and $\langle \mathcal{P}(C_0), C_0 \rangle_B$ that $\big[N_0\big]_{Sh}^{W} \simeq \big[C_0\big]_B^{C_0}$. After the update with ϕ both $W, N_1 \models_{Sh} \phi$ and $W, N_1 \models_{Sh} \phi$ hold. Assume that $\big[N_1\big]_{Sh}^{W} \not\simeq \big[C_1\big]_B^{C_0}$. Then there exists a proposition ψ such that $W, N_1 \models_{Sh} \psi$ but not $C_1 \models_B \psi$. (Note that, alternatively, if a proposition is validated by C_1, but refuted by N_1, its negation provides a suitable witness.) Since $\langle W, N_0 \rangle_{Sh}$ and $\langle \mathcal{P}(C_0), C_0 \rangle_B$ are corresponding models, there exists a sharpening $N' \succeq N_0$ such that $W, N' \models_{Sh} \phi$ but not $W, N' \models_{Sh} \psi$.

However, since $W, N_1 \models_B \psi$ (and since ψ is not a logical consequence of ψ) there exists penumbral connection in $\langle W, N_0 \rangle_{Sh}$ which ensures that ψ holds after an update with ϕ in N_0. This penumbral connection is violated at N' and therefore such a partial interpretation cannot exist in W. Hence, there exists no such formula ψ as assumed above and we have $[N_1]^W_{Sh} \not\simeq [C_1]^{C_0}_B$. $\square$

CHAPTER 6

Conclusion

6.1 Summary

In this thesis we have seen how the phenomenon of vagueness can be examined from certain linguistic, philosophical, and logical points of view and how these different kinds of approaches can be related to each other. As primary tool for analysis we have used evaluation games in the tradition of Giles's game. These games aim to provide *meaning* to a formal system; e.g., for fuzzy logics they model how truth values and truth functions can be interpreted.

As demonstrated in Chapter 2 Giles's game [33] for Łukasiewicz logic $\mathbf{Ł}_\infty$ can be regarded as a generalization of Hintikka and Sandu's evaluation game for classical logic [41]. Their game can straightforwardly be extended to a many-valued setting to yield a characterization of weak Łukasiewicz logic $\mathbf{Ł}^w$. As a further (and bolder) step, one may consider game states where both players may assert multisets of formulas instead of just one formula asserted by one player at each point in the game as in Hintikka and Sandu's game. Together with an evaluation function mapping final game states to truth values (which Giles motivates by means of a betting scheme) we arrive at Giles's game. Giles's choices for dialogue rules and the evaluation scheme are not chosen arbitrarily: Section 2.4 explores which other logics can be modeled within Giles's general setup by variating dialogue rules and evaluation function. We provide so-called *payoff principles* ensuring that the game still defines *some* logic and show that even under these very general conditions only logics that are closely related to Łukasiewicz logic emerge, such as Abelian logic, or Cancellative Hoop Logic.

Chapter 3 extends Giles's game to fuzzy quantifiers [62] like 'about a half' or 'at least a third', called proportional quantifiers. While for the classical existential and universal quantifiers witnesses are chosen by the respective player, we here introduce the notion of *ran-*

domly selected witnesses, i.e., witnesses selected by *Nature* as a third, neutral player instead. This enables the construction of a plethora of new dialogue rules. We single out two families of quantifiers, *deliberate choice quantifiers* and *blind choice quantifiers*. All randomly chosen constants are revealed to the players either before or after they have chosen their attack and defense moves, respectively. We show how the above examples of vague expressions can be modeled as deliberate and blind choice quantifiers. As it turns out, this subtle design decision has a substantial impact on the type of the resulting truth functions: deliberate choice quantifiers correspond to polynomial, generally non-linear, truth functions, which do not immediately yield adequate truth functions for the natural language quantifiers in questions. We demonstrate how these can be obtained by introducing an additional operator modifying the deliberate choice quantifier. Blind choice quantifiers, however, correspond to piecewise linear truth functions with integer coefficients and, vice versa, such a function always can directly be modeled by a blind choice quantifier. In the literature on fuzzy quantification often certain truth functions are assumed without further justification. The extensions of Giles's game presented in this chapter provide a game-theoretic justification for some of these truth functions. For blind choice quantifiers we we show that all piecewise linear functions can be modeled by blind choice quantifiers, while for deliberate choice quantifiers we demonstrate how candidate functions, e.g., for the natural language quantifier 'about half' can be classified by few discrete parameters.

Chapter 4 revisits Chris Barker's account of vagueness [5] as a prototypical example of a linguistic, scale-based approach. Barker explicitly models the context in which the conversation takes place and how this context is changed by a vague statement. His model comprises both epistemic uncertainty and vagueness. We point out some technical problems with Barker's presentation and show how they can be resolved. Furthermore we show how to formulate propositional logical operators within this approach. Observing how the context size changes when updating with a vague statement allows us to associate a truth value with this statement. The resulting logic is not truth functional: the truth value of a compound statement cannot be computed only from the truth values of its components. However, we can give tight bounds for this value and, as it turns out, for the extremal cases the truth value is computed by a set of well known functions: truth functions used in the realm of t-norm based fuzzy logics. Thus, this observation connects these two different kinds of approaches to vagueness. We single out the class of so-called *saturated contexts* which ensure that there are no hidden dependencies between vague predicates (and also that epistemic uncertainty and vagueness are kept independent of each other). Within such saturated contexts truth values of compound statements can be computed in a truth functional manner and, again, co-t-norms emerge as truth functions.

Finally, Chapter 5 presents Stewart Shapiro's account of 'Vagueness in Context' [82] as an example of how vagueness is modeled in philosophy. We show how to adapt Giles's game to

Shapiro's approach. According to Shapiro, locally, for a fixed context, statements are evaluated using strong Kleene logic. We therefore first present a corresponding variant of Giles's game: in contrast to the original game, it now may occur that none of the two players has a winning strategy—in this case the formula is indefinite at the given context. We then proceed by extending this game to cover also non-local features of Shapiro's approach: assertions are always relative to precisification points and dialogue rules may allow the players to change these points. We furthermore allow the players to strengthen or weaken their assertions. Adding these two new features to Giles's game allow us to formulate dialogue rules for all connectives defined by Shapiro.

We furthermore explore connections between Shapiro's and Barker's models. At the first glance they are fundamentally different: Shapiro uses partial valuations and his approach can be counted as delineation-based, while Barker uses scales and degrees for modeling vague predicates. Nevertheless, a correspondence between such kinds of concepts has already been hinted at by Arnim von Stechow in a static setting [91]. We prove that for situations which can be modeled both using Shapiro's and Barker's approaches, these are equivalent in the following sense: after context updates with a sequence of vague statements, both approaches validate the same first order formulas. Thus from this inferential point of view, i.e., comparing which conclusions can be drawn from an initial situation and consecutive updates, the two approaches share much common ground. The differences only pertain to which situations can be modeled conveying exactly the same information as well as the way they handle conflicting information. To our best knowledge such a formal comparison of approaches to vagueness is novel to the literature; moreover it may leverage practical applications of these models.

6.2 Further work

Due to the interdisciplinary nature of this thesis, there are many possible starting points for promising further work. As two particular examples we pick out the following endeavors combining the different approaches to vagueness examined in this thesis with other ones.

Chris Kennedy [50,51] provides an extensive linguistic analysis of the semantics of gradable predicates. In most cases these coincide with *vague* predicates under consideration in this thesis (c.f. Section 1.4). Kennedy elaborates how the scale structure associated with such predicates enables drawing inferences. Take as an example the predicate 'wet' associated with a closed scale and consider the following statement: 'The floor is wetter then the table'. From this we can usually conclude that the floor is *not dry* (with 'dry' as the antonym of 'wet'). However, considering another predicate with an open scale like 'tall' does not enable this kind of reasoning: 'John is taller than Jack' does not entail anything about either John or Jack being regarded as tall or as small. More complex situations for drawing such inference arise when also taking negation into account. This kind of reasoning has so far not been incorporated into logical approaches to vagueness. Taking such a logical approach and extending it by inferences based on the scale structures of the predicates in question—or even further, extending Giles's game in this direction—would considerably increase its ability to model "real-world reasoning".

Alice Kyburg and Michael Moreau [56] introduced (yet) another approach to vagueness, which could provide a foundation for new computational models of vagueness. It shares many similarities with Shapiro's account of vagueness in context covered in this thesis, and, in fact, Shapiro's notion of truth can be recovered in this framework by lifting the requirement of complete precisifications. Interestingly, such a move arguably matches closer Kyburg and Moreau's intentions of how points in the precisification space are to be interpreted: namely as possible states in a conversation.

Both aforementioned approaches model context update with conflicting information. Shapiro regards such updates as jumps in the current frame—a tree-like structure covering all possible worlds. Such an update leads to a new context containing not only the new data, but also invalidating old information, (c.f. Section 5.1). Shapiro however is rather vague about which possible world exactly will be reached by such a jump. His only criterion is that, although retracting already established information, the speaker should still be regarded as "competent". Kyburg and Moreau model context update with conflicting information in a more fine-grained way. They advise the use of *belief revision theory* to perform updates. The idea is to view the context as a knowledge base and to perform a context update by revising the current context with the new information. The AGM postulates [2] then provide a set of rules which must be respected by a valid revision operator. From this point of view, their approach can be seen as a refinement of Shapiro's, describing in more detail how to arrive at the new context. Although belief revision theory still gives more stringent criteria for valid updates, also Kyburg and Moreau, like Shapiro, do not give an explicit mechanism. Indeed, we believe that such a procedure must take into account aspects of scale-based approaches like Barker's or Kennedy's in order to decide between different valid ways of updating the context. Kyburg and Moreau describe such a situation with two pigs, where one is borderline fat and in front of the feeder and the other one is definitely fat, but slightly off to one side. Uttering the statement 'The fat pig in front of the feeder won a prize' selects one of these pigs, but for deciding which, one has to compare the gradable predicates 'fat' and 'in front of the feeder'. Such an update mechanism could be based on Alan Bale's work [4] investigating methods to compare gradable predicates which are measured on different scales. The resulting model would demonstrate how features of delineation- and scale-based approaches can fruitfully be incorporated in computational models of vagueness.

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