Optimal Control Models of Renewable Energy Production Under Fluctuating Supply

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Abstract. The probably biggest challenge for climate change mitigation is to find a secure low-carbon energy supply, which especially is difficult as the supply of renewable sources underlies strong volatility and storage possibilities are limited. We therefore consider the energy sector of a small country that optimizes a portfolio consisting of fossil and/or renewable energy to cover a given energy demand, considering seasonal fluctuations in renewable energy generation. By solving these non-autonomous optimal control models with infinite horizon, we investigate the impact of fossil energy prices on the annual optimal portfolio composition shown by the obtained periodic solutions.

Keywords: Optimal control · Nonlinear dynamical systems · Resources and environment · Renewable energy

1 Introduction

With a constantly increasing world-wide energy demand, the progressively obvious impacts of climate change and the energy sector as the main source of green house gas emission, the possibly biggest challenge of the 21st century is to find a low-carbon, secure and sustainable energy supply. Renewable energy generation is already carried out, but technology and policy efforts are not yet sufficient. Besides the high costs and the limited storage possibilities the possibly biggest problem is the fluctuating supply of renewable sources.

To address this issue we investigate the decision of an energy sector in a small country that optimizes a portfolio consisting of fossil and renewable energy. We assume that this energy sector has full information about the energy demand that has to be covered, which is postulated to be stationary, as done in [3], but instead of assuming that the energy demand is dependent on the GDP of the country (see also [2]) and on the electricity price, we follow [8] and consider the energy demand to be exogenous. Given this demand as well as the mentioned seasonal fluctuations and the fossil energy price, the energy sector optimizes its

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portfolio to find the most cost-effective solution. Following [1] we focus especially on solar energy and omit storage completely, so that the generated energy has to be used immediately or is lost.

Due to the seasonal fluctuations this optimal control problem with one state and two controls exhibits a particular mathematical property by being non-autonomous. We solve this problem by applying Pontryagin's Maximum Principle, but instead of the usual steady-state analysis of autonomous approaches we are looking for a periodic solution that solves the non-autonomous canonical system, which makes the problem numerically sophisticated.

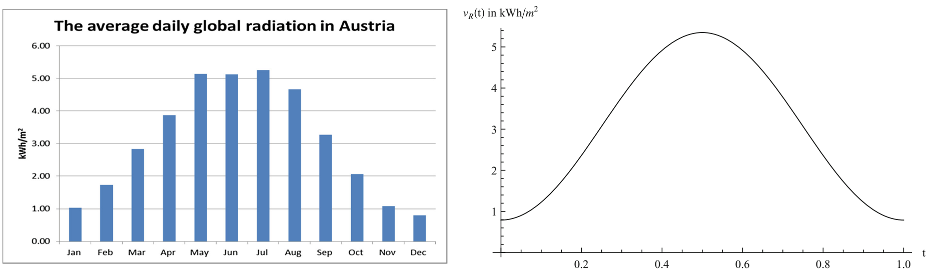
2 The Model

While fossil energy is assumed to be constantly available and imported for the price p_F, the supply of renewable energy fluctuates over time but harvesting is for free and the generation is possible within the country. To do so, however, investments for proper energy generation capital are necessary. One important implication of the (small) size of the country is that the energy sector is assumed to be a price taker, which means that its decision does not impact the market prices.

We especially focus in this paper on solar energy as renewable resource. Figure 1a shows the average global radiation per month in Austria. One can clearly observe the seasonal differences underlining the challenge of a constant renewable energy supply over the whole year. To include such seasonal fluctuations[1] in our model, we use a deterministic time-dependent function

$$v_R(t) = \nu \sin(t\pi)^2 + \tau,$$

which can be seen in Fig. 1. The parameter τ defines the minimal supply in winter and ν is the maximal increment during summer. The necessary capital



(a) (b)

Fig. 1. (a) Average global radiation per month in Austria. (b) Deterministic function to describe the varying global radiation over one year.

[1] Note that we only consider annual fluctuations and do not include daily fluctuations from day to night nor changes due to weather conditions. To get reasonable parameter values we used Austrian data for the estimation (cf. [9]).

$K_S(t)$ in form of photovoltaic (PV) cells is accumulated by investments $I_S(t)$ and depreciates at a rate δ_S which later on will be set to $\delta_S = 0.03$, implying that a PV cell has a lifetime of about 33 years. With the current capital stock and the given global radiation, renewable energy is generated as in Eq. (1b), where η is the degree of efficiency, which for common PV cells is about 20 %. Note that this function explicitly depends on time t which makes the problem non-autonomous. As the required energy demand E that has to be covered is well known, it is postulated that the demand has to be satisfied completely with the portfolio of fossil, $E_F(t)$, and renewable, $E_S(K_S(t), t)$, energy. This means that shortfalls are not allowed while surpluses are in general possible but are simply lost as saving options do not exist. This balance is included in the model by the mixed path constraint in Eq. (1a). Given this restriction and the current market price for fossil energy, the energy sector searches for the most cost-effective solution by maximizing its profit as shown in Eq. (1), where p is the electricity price. Note that we distinguish between linear investment and quadratic adjustment costs, where the latter arise from installation efforts.

Summing up, we consider a non-autonomous optimal control model with infinite horizon, two controls describing the capital investments and the imported fossil energy, and one state for the capital stock,

$$\max_{E_F(t),\, I_S(t)} \int_0^\infty e^{-rt}\Big(pE - I_S(t)\big(b + cI_S(t)\big) - p_F E_F(t)\Big)\, dt \tag{1}$$

$$\text{s.t.:}\quad \dot{K}_S(t) = I_S(t) - \delta_S K_S(t)$$

$$E_F(t) + E_S\big(K_S(t), t\big) - E \geq 0 \tag{1a}$$

$$E_S\big(K_S(t), t\big) = \big(\nu \sin(t\pi)^2 + \tau\big) K_S(t)\eta \tag{1b}$$

$$E_F(t), I_S(t) \geq 0,$$

where the discount rate r and the parameters b and c are positive constants.

3 Solution

3.1 Canonical System and Necessary First Order Conditions

Let $(K_S^*(t), I_S^*(t), E_F^*(t))$ be an optimal solution of the control problem in Eq. (1), then, according to the maximum principle for infinite time horizon problems (cf. [4]), there exists a continuous and piecewise continuously differentiable function $\lambda(t) \in \mathbb{R}$ satisfying

$$\mathscr{L}(K_S^*(t), I_S^*(t), E_F^*(t), \lambda(t), t) = \max_{I_S(t), E_F(t)} \mathscr{L}(K_S^*(t), I_S(t), E_F(t), \lambda(t), t)$$

where $\mathscr{L}$ defines the Lagrangian which reads as

$$\begin{aligned}\mathscr{L}(K_S(t), I_S(t), E_F(t), \lambda(t), t) = {} & pE - bI_S(t) - cI_S(t)^2 - p_F E_F(t) \\ & + \lambda(t)(I_S(t) - \delta_S K_S(t)) + \mu_1(t)(E_F(t) \\ & + K_S(t)\eta(\nu \sin(t\pi)^2 + \tau) - E) \\ & + \mu_2(t)E_F(t) + \mu_3(t)I_S(t),\end{aligned}$$

with $\mu_1(t), \mu_2(t), \mu_3(t)$ being the Lagrange multipliers for the mixed path constraint and the non-negativity conditions. Further on, at each point where the controls are continuous

$$\dot{\lambda}(t) = r\lambda(t) - \frac{\partial \mathscr{L}(K_S(t), I_S(t), E_F(t), \lambda(t), t)}{\partial K_S}$$

is given and the complementary slackness conditions

$$\begin{aligned} \mu_1(t)\left(E_F^*(t) + E_S^*\left(K_S^*(t), t\right) - E\right) = 0, &\quad \mu_1(t) \geq 0, \\ \mu_2(t)E_F^*(t) = 0, &\quad \mu_2(t) \geq 0, \\ \mu_3(t)I_S^*(t) = 0, &\quad \mu_3(t) \geq 0, \end{aligned}$$

have to be satisfied. Hence, the necessary first order conditions and the adjoint equation are given as follows:

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial E_F(t)} &= -p_F + \mu_1(t) + \mu_2(t) = 0 \\ \frac{\partial \mathscr{L}}{\partial I_S(t)} &= -b - 2cI_S(t) + \lambda(t) + \mu_3(t) = 0 \Leftrightarrow I_S(t) = \frac{\lambda(t) + \mu_3(t) - b}{2c} \\ \dot{\lambda}(t) = r\lambda(t) - \frac{\partial \mathscr{L}}{\partial K_S(t)} &= (r + \delta_S)\lambda(t) - \mu_1(t)\eta(\nu \sin(t\pi)^2 + \tau). \end{aligned}$$

Looking for an interior solution with both controls $I_S(t), E_F(t) > 0$ and the mixed-path constraint of (1a) satisfied with strict inequality, it can be shown that such a solution never can be optimal as costs could be reduced by lowering the amount of fossil energy until the mixed path constraint is satisfied with equality, which makes surpluses in fossil energy inefficient. Hence, we focus for the following analysis on the three boundary cases, which are: the *fossil case* with zero investments[2], $E_F(t) > 0$, $I_S(t) = 0$ and $E_F(t) + E_S(K_S(t), t) - E = 0$; the *mixed case* where both types of energy are used for the coverage, $E_F(t)$, $I_S(t) > 0$ and $E_F(t) + E_S(K_S(t), t) - E = 0$; and finally the *renewable case*, where only renewable energy is used to cover the demand, $E_F(t) = 0$, $I_S(t) > 0$ and $E_S(K_S(t), t) - E > 0$. Inserting the corresponding values of the controls and Lagrange multipliers yields the canonical systems for these boundary cases:

$$\dot{K}_S(t) = A - \delta_S K_S(t), \text{ with } A = \begin{cases} 0, & \text{fossil case,} \\ \frac{\lambda(t)-b}{2c}, & \text{mixed and renewable case,} \end{cases} \tag{2}$$

$$\dot{\lambda}(t) = (r + \delta_S)\lambda(t) - B, \text{ with } B = \begin{cases} p_F\eta(\nu \sin(t\pi)^2 + \tau), & \text{fossil and mixed case,} \\ 0, & \text{renewable case.} \end{cases} \tag{3}$$

In what follows, we refer to these canonical systems as $\dot{K}_S(t) = f^K(t, K_S(t), \lambda(t))$ and $\dot{\lambda}(t) = f^\lambda(t, \lambda(t))$.

[2] If the initial capital stock along the fossil solution arc is zero, the whole energy demand is covered with fossil energy. If, however, the initial capital stock is positive, also renewable energy contributes to the coverage of the energy demand, nevertheless at a decreasing rate as no investments are done and depreciation reduces the stock.

3.2 Periodic Solution

As the canonical system in (2)–(3) is non-autonomous we have to find a trajectory with the property to be hyperbolic. Detailed theory about the existence, the computation and the manifolds of such distinguished hyperbolic trajectories can be found, e.g., in [5], [7], or [6]. Due to the periodicity of the dynamics candidates for the long-run optimal solution of the problem in (1) are periodic solutions with the period length of one year. In order to find such a periodic solution of the canonical system numerically, we first determine the instantaneous equilibrium points $K_S^{IEP}(t)$ and $\lambda^{IEP}(t)$ (cf. [5]) by setting $(\dot{K}_S, \dot{\lambda})(t) = (0, 0)$, and then solve the following boundary value problem using these instantaneous equilibrium points as starting function,

$$\dot{K}_S(t) = f^K(t, K_S(t), \lambda(t)), \qquad \text{with } K_S(0) = K_S^{IEP}(0) \text{ and } K_S(1) = K_S(0),$$
$$\dot{\lambda}(t) = f^\lambda(t, \lambda(t)), \qquad \text{with } \lambda(0) = \lambda^{IEP}(0) \text{ and } \lambda(1) = \lambda(0).$$

Solving this BVP yields the periodic solution $(K_S^*(t), \lambda^*(t))$ that lies completely within one of the three boundary cases. However, it can happen that the solution at some point leaves the current admissible area before the course of the period of one year is completed. In this case one has to switch to the corresponding canonical system to get a periodic solution existing of several arcs. Therefore, a multi-point boundary value problem has to be solved. At each point of time where the constraints of the current region are violated a switch to the proper region happens, meaning that the corresponding canonical system is used to continue the solution. For n switching times $\tau_0 := 0 < \tau_1 < \tau_2 < \cdots < \tau_{n-1} < \tau_n < 1 =: \tau_{n+1}$, one has calculate $n+1$ arcs, for which the continuity at each switching time has to be guaranteed. We introduce an index $a_i \in \{1, 2, 3\}$ that distinguishes the canonical systems for the fossil, the mixed and the renewable case, respectively, for each arc i with $i = 1, \ldots, n+1$. If n switches are necessary along the periodic solution and we use for simplicity the notation

$$\dot{K}_{S_i}(t) = f_{a_i}^K(t, K_{S_i}(t), \lambda_i(t)), \quad t \in [\tau_{i-1}, \tau_i], \qquad i = 1, \ldots, n+1, \tag{4}$$
$$\dot{\lambda}_i(t) = f_{a_i}^\lambda(t, \lambda_i(t)), \quad t \in [\tau_{i-1}, \tau_i], \qquad a_i \in \{1, 2, 3\}, \tag{5}$$

for the corresponding canonical system at arc i, it has to hold that $a_i \neq a_{i-1}$ and $|a_i - a_{i-1}| = 1$, which means that switches only can happen between fossil/mixed or mixed/renewable cases. For the numerical solution of the system for each arc i we use a time transformation so that it can be solved with fixed time intervals. This means that, in order to solve an equation

$$\dot{x}(t) = f(t, x(t)), \quad t \in [\tau_{i-1}, \tau_i],\ i = 1, \ldots, n+1,\ \tau_0 = 0,\ \tau_{n+1} = 1$$

as in (4)–(5), we are looking for a time transformation $t = T(s)$ so that

$$\dot{y}(s) = \tilde{f}(s, y(s)), \quad s \in [i-1, i], \text{ with } y(s) = x(T(s)).$$

It turns out that the linear transformation $T(s) = (\tau_i - \tau_{i-1})(s - i + 1) + \tau_{i-1}$ satisfies the required conditions. Hence, in terms of the original dynamics this yields

$$\dot{x}(s) = \frac{dx(T(s))}{ds} = \frac{dx(T(s))}{dT}\frac{dT(s)}{ds} = f(s, x(s))(\tau_i - \tau_{i-1}).$$

Using this transformation, we have to solve for $i = 1, \ldots, n+1$, $j = 1, \ldots, n$, $s \in [i-1, i]$, $\tau_0 = 0$, $\tau_{n+1} = 1$ the multi-point boundary problem

$$\begin{aligned}
&\dot{K}_{S_i}(s) = (\tau_i - \tau_{i-1}) f^K_{a_i}(T(s), K_{S_i}(s), \lambda_i(s)), \quad \dot{\lambda}_i(s) = (\tau_i - \tau_{i-1}) f^\lambda_{a_i}(T(s), \lambda_i(s)),\\
&\left(K_{S_j}(\tau_j), \lambda_j(\tau_j)\right) = \left(K_{S_{j+1}}(\tau_j), \lambda_{j+1}(\tau_j)\right), \quad (K_{S_n}(1), \lambda_n(1)) = (K_{S_1}(0), \lambda_1(0)),\\
&(K_{S_1}(0), \lambda_1(0)) = \left(K_S^{IEP}(0), \lambda^{IEP}(0)\right).
\end{aligned} \tag{6}$$

Equation (6) ensures that the continuity in state and controls at each switch is given and, as a periodic solution is calculated, the beginning and the endpoint coincide. The following Eq. (7) finally guarantees the necessary condition that the Lagrangian is continuous as well, which depends on the involved regions as well as on the direction of the switch and is given for $j = 1, \ldots, n$ as

$$0 = c(a_j, a_{j+1}) = \begin{cases} b - \lambda_j(\tau_j), & \text{if } a_j = 1,\ a_{j+1} = 2,\\ \frac{\lambda_j(\tau_j) - b}{2c}, & \text{if } a_j = 2,\ a_{j+1} = 1,\\ E_S(K_{S_j}(\tau_j), \tau_j) - E, & \text{if } a_j = 2,\ a_{j+1} = 3,\\ E_F(\tau_j), & \text{if } a_j = 3,\ a_{j+1} = 2. \end{cases} \tag{7}$$

The periodic solution that solves this BVP then is given as

$$(K_S^*(t), \lambda^*(t)) = \left(\left(K^*_{S_1}(t), \lambda^*_1(t)\right)_{0 \le t < \tau_1}, \left(K^*_{S_2}(t), \lambda^*_2(t)\right)_{\tau_1 \le t < \tau_2}, \ldots, \left(K^*_{S_n}(t), \lambda^*_n(t)\right)_{\tau_{n-1} \le t < \tau_n = 1}\right).$$

Calculating the eigenvalues of the monodromy matrix for the obtained periodic solution reflects the stability, which here are given as $e_1 = e^{-\delta_S}$ and $e_2 = e^{r+\delta_S}$. As $\delta_S < 1$ always is satisfied, one can see that $e_1 < 1$ holds. For reasonable values of the discount rate and the depreciation rate it further is supposed that $r+\delta_S<1$ which implies that $e_2 > 1$ in these cases. As the Jacobian is independent of the state and the control variable, this means that every periodic solution that we can find within one of the boundary regions is of saddle-type and, as no eigenvalue $e_i = 1$ occurs, it is a hyperbolic cycle which guarantees that the behavior of the system near this periodic solution can be fully described by its linearisation (see [4]).

Table 1. Parameter values used for the numerical analysis.

Interpretation	Parameter	Value	Interpretation	Parameter	Value
Investment costs	b	0.6	Discount rate	r	0.04
Adjustment costs	c	0.3	Depreciation rate	δ_S	0.03
Energy demand	E	1053.82	Maximal radiation increment	ν	4.56
Electricity price	p	0.1	Degree of efficiency	η	0.2
Fossil energy price	p_F	0.08	Minimal radiation in winter	τ	0.79

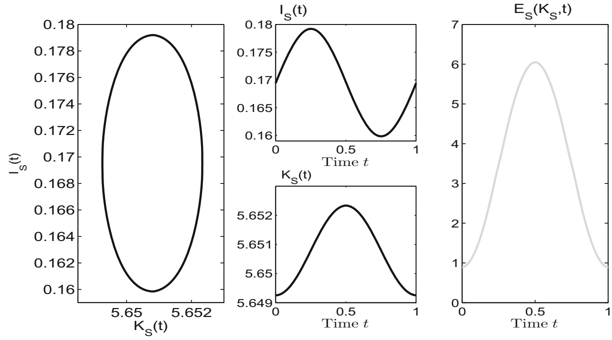


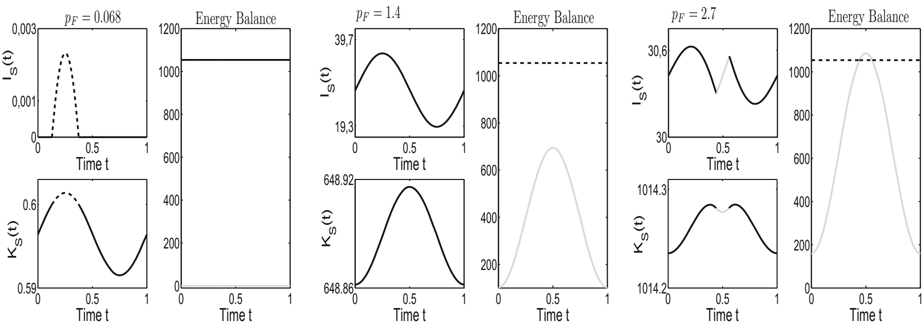
Fig. 2. Periodic solution (left box), time paths for investments and capital over one year (two boxes in the middle) and renewable energy generation (right box) for a fossil energy price of $p_F = 0.08$.

4 Results

For the following numerical analysis, we use the parameter values summarized in Table 1. Figure 2 shows the long-run optimal periodic solution for this parameter value set which corresponds to the mixed case where both types of energy are used. While the initial capital stock in winter is quite low, it increases and peaks during summer due to investments to accumulate new or maintain already existing capital. Note, that the peak is exactly where also the global radiation is maximal and hence, the generation of renewable energy reaches a peak during this time as well. The investments, however, start to decline again even before this period because a further increase of the capital stock in autumn would not be beneficial due to the declining radiation. Therefore, the capital stock decreases again after the summer peak and renewable energy generation goes down. The proportion of renewable energy in this scenario's portfolio with only 0.6 % is very low, but this comes from the fact that fossil energy with $p_F = 0.08$ is really cheap and hence high investments in renewable energy are simply too costly.

5 Sensitivity Analysis

As the previous scenario has shown, not much is invested in renewable energy in case of a low fossil energy price. This aspect raises the question how the portfolio composition will change if fossil energy gets more expensive. We therefore investigate in this section the impact of the fossil energy price on the long-run optimal portfolio solution by increasing the price step by step and then using numerical continuation. Figure 3 shows the results for different values of p_F. The two boxes on the left hand side contain the time paths for investments $I_S(t)$ and capital stock $K_S(t)$, respectively, while the box on the right hand side depicts the composition of the energy portfolio with renewable energy shown as gray line, fossil energy as black line and the energy demand as black dashed line. While for a very low price (below $p_F = 0.06785$) fossil energy is so cheap that the whole energy demand is covered with fossil energy, meaning that no investments are done and, consequently, no capital is accumulated, renewable energy very soon is used as additional energy source for the portfolio if the fossil energy price increases (see Fig. 3a). Here, a very interesting aspect can be observed. Due to the high global radiation in summer and the low fossil energy price, it is only worthwhile to do investments in the first half of the year to increase renewable energy capital (or to do some maintenance to have it in a good condition) in order to optimally utilize this productive period. During the rest of the year, however, investments are again set to zero as a high capital stock would not be cost effective. The periodic solution for this scenario therefore consists of two arcs, the first one with positive (black dashed line in Fig. 3a) and the second one with zero investments (black line in Fig. 3a). Note that the contribution of renewable energy to cover the demand still is very low and hence the line for fossil energy and the energy demand basically coincide. The price interval for which this kind of result can be seen is, however, very small, $p_F \in [0.06785, 0.06897]$. For a higher fossil energy price, investments are done over the whole year but still with a peak before summer, the generation of renewable energy increases and the additional fossil energy amount during the summer period is reduced. During the winter period, however, fossil energy still is required. Figure 3b shows the long-run optimal solution for $p_F = 1.4$, which completely corresponds to the mixed case. At an even higher fossil energy price of $p_F = 2.1025$, the renewable energy generation is so high that it reaches the demand at the peak in summer. This is a certain point of interest because here a switch to the complete renewable case happens. Being only an osculation point at the beginning, it develops to an interval if the price goes further up. In this interval, which always is around the maximum of global radiation, the energy demand can be fully covered with renewable energy and no additional fossil energy is needed. Figure 3c shows this



(a) $p_F = 0.068$ (b) $p_F = 1.4$ (c) $p_F = 2.7$

Fig. 3. Periodic solution for (a) $p_F = 0.068$: two arcs given by the mixed solution as dashed line and the fossil solution as solid line, (b) $p_F = 1.4$: mixed solution over the whole year, (c) $p_F = 2.7$: two arcs given by the mixed solution as black solid line and the renewable solution as gray solid line.

scenario for a fossil energy price of $p_F = 2.7$. The energy portfolio in the right box shows that surpluses are generated during summer which are lost as no storage possibilities exist. The periodic solution in this scenario again consists of two arcs, the mixed solution arc (black line) and the renewable solution arc (gray line). The interval in which renewable energy is sufficient to cover the energy demand increases the further the fossil energy price goes up. However, it turns out that this happens at a decreasing speed, and during winter fossil energy still is necessary to cover the shortfalls, even if the fossil energy price is already unreasonably high.

6 Conclusions

We have investigated in this paper the impact of the fossil energy price on the optimal portfolio composition consisting of fossil and renewable (solar) energy in a small country. We postulated that the supply of the renewable source is seasonally varying, the energy demand is well known and constant over the year.

Sensitivity analysis with respect to the fossil energy price p_F showed that a higher fossil energy price indeed is an incentive for more investments in renewable energy capital. However, an autarkic solar energy supply is not possible. While independence of fossil energy can be achieved during some time interval in summer in which global radiation is high and even surpluses can be generated, the shortfalls in winter always have to be covered with fossil energy no matter how high the fossil energy price is.

These results underline that the non-constant supply is one of the major challenges of renewable energy generation. This not only concerns solar energy but also other types of renewable energy like wind and water. Hence, probably a combined portfolio of several types of renewable energy could compensate for the fluctuations of each other and enable a more or less constant supply. Such a portfolio is exactly what we want to study in some model extension in the near future. In addition, also the effect of learning by doing on investment costs and efficiency as well as a time-dependent energy demand will be a special matter of interest.

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