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Receptivity and non-uniqueness of turbulent boundary layer flows

Bernhard Scheichl[1,2,*]

[1] Vienna University of Technology, Institute of Fluid Mechanics and Heat Transfer, Getreidemarkt 9, 1060 Vienna, Austria
[2] AC²T research GmbH, Viktor-Kaplan-Straße 2C, 2700 Wiener Neustadt, Austria

The classical theory of firmly attached time-mean turbulent boundary layers, exhibiting fully developed broadband turbulence, in the limit of large Reynolds numbers, Re, is extended in two ways, where the focus lies on their stability and receptivity. At first, the description of non-unique equilibrium flows having a so-called moderately large velocity deficit is generalised as weak spanwise variations and unsteadiness is taken into account. Here the time period of filtering the flow tends to infinity such that conventional Reynolds averaging is obtained as $Re \to \infty$. Secondly, the method of matched asymptotic expansions is supplemented with a suitable multiple-scales approach so as to capture the global effect of pronounced surface waviness on the time-mean velocity profile of the boundary layer in a self-consistent manner in terms of a memory decaying exponentially from the surface. Remarkably, the results of the asymptotic analysis are qualitatively independent of any turbulence closure.

1 Introductory remarks: the problems considered

In the following, we consider incompressible and nominally two-dimensional fully developed turbulent BL flow driven by an external potential flow (that does not impose any free-stream turbulence). All lengths and velocities and thus the time t are non-dimensional with values typical of the external flow, then characterised by a sufficiently large Reynolds number Re. In a curvilinear coordinate system (x, y, z) with x being the streamwise direction, the surface is nominally considered at $y = 0$.

The stability of such a BL can be viewed as its behaviour when complete Reynolds- or time-averaging is slightly relaxed such that a time scale depending on $\Re$ forms. This allows for extending the classical asymptotic flow description of two-tiered where the principal perturbation parameter is given by $\gamma := u_\tau/u_e \sim \kappa/\ln Re$: here κ, u_τ and $u_e(x)$ are respectively the von Kármán constant, the local wall friction velocity, and the wall speed exerted by the potential flow.

Another phenomenon not addressed by the classical analysis though topical as of great practical interest concerns a more savage form of distributed surface roughness. This is not a crucial issue in laminar flows with the following exceptions: external (internal) layers or BLs so thin that their wall-normal extent compares with the average asperity height (for internal ones, then usually the application of lubrication theory is justified). However, such a situation strongly promotes laminar–turbulent transition, and, in analogy to the microfluidics context we address by the above internal flows, surface roughness has a much more pronounced effect on wall friction when the asperity elements protrude distinctly into or beyond the viscous sublayer of a turbulent flow. (In the latter case, the sublayer thickness is even dictated by the typical micro-scale associated with technical roughness.) So far, only this limiting form of roughness is well-understood, at least in a semi-empirical manner, on the global x-scale. In the present scenario we associate with *receptivity* of a turbulent BL, surface waviness even modifies the turbulent flow to leading order in its outer main region, so that the local surface curvature dominates its inner parts.

The aforementioned slow variation of a turbulent BL with time in its *response* to marked roughness points to future research.

2 Convective bifurcation instability of non-unique flows

Non-uniqueness arising in the conventional steady-state description of a turbulent BL predicted by its generalisation put forward in [1] has the most attractive and sensitive impact on the extended analysis taking into account its temporal stability. Contrasting with the classical scaling of the BL predicting a small streamwise velocity deficit of $O(\gamma)$ in its main layer, the specific situation of non-unique equilibrium BLs requires a moderately larger deficit of $O(\epsilon)$ with $\epsilon := \gamma^{2/3}$. For BLs attaining quasi-equilibrium in their main tier, the defect shape factor G is expanded as $G \sim \epsilon c D + \cdots$ where the universal coefficient c of $O(1)$ accounts for nonlinear effects of convection and the difference of the Reynolds normal stresses. Hence, the quantity D of $O(1)$ measures the actual magnitude of the velocity defect. Introducing correspondingly long fractions of t in the perturbation analysis of the partially averaged Navier–Stokes equations requires secular-term elimination in the equation governing the defect to second order, having its temporal reorganisation in leading order described by the evolution equation

$$D\,\partial_\theta D = 1 - 9\mu(\theta,\xi)D^2 + rD^3, \quad D = D(\theta,\xi), \quad \theta := \ln t/(\ln Re)^{2/3}, \quad \xi := t - x^{4/3}. \tag{1}$$

Here the function μ governs the external power-law flow: $u_e \propto x^m$, $m = -1/3 + \epsilon\mu$. Straining of t is necessitated by the self-similar structure of the base flow. This gives rise to a turning-point bifurcation in the (μ, D)-plane, described by the

* Corresponding author: e-mail bernhard.scheichl@tuwien.ac.at, phone +43 1 58801 32225, fax +43 1 58801 9 32225

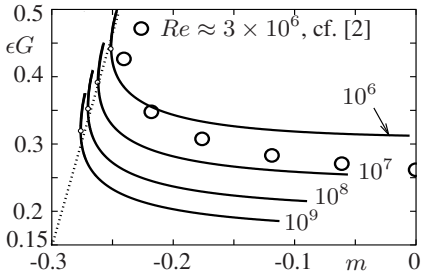

Fig. 1: Equilibrium-flow chart (from [1]).

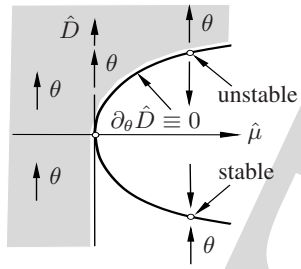

Fig. 2: Bifurcation (shaded: blow-ups).

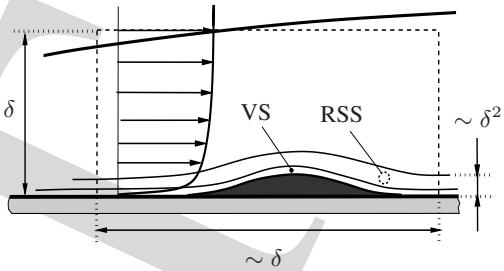

Fig. 3: Local three-tiered splitting of the BL.

steady-state version of (1). Figure 1 resorts to numerical solutions of the BL problem closed with a mixing length model (zero normal stresses), compared with (rarely available) experimental data. The dotted line $\mu \sim rD/6$ connecting the turning points is well confirmed. According to (1), the lower/upper branch is convectively stable/unstable. Moreover, finite-time blow-ups are predicted in the unstable regime known in different context, cf. [3], but not yet understood for fully turbulent flow.

Expanding $Dr^{1/3}/2^{1/3} \sim 1 + \sigma^{1/2}\hat{D}(\xi, \hat{\theta}, Z) + \cdots$ and setting $6\mu r^{-2/3} = 2^{1/3}[1 + \sigma\hat{\mu}(\hat{\theta}, \xi)]$, $\hat{\theta} \propto \sigma^{1/2}\theta$ with some small parameter σ yields the bifurcation scenario close to the turning point shown in Fig. 2. Here weak spanwise flow variations are considered by virtue of the stretched variable $Z := z/\Lambda$, $\Lambda := \epsilon/\sigma^{1/2} \gg 1$, and some $O(1)$-coefficient $\alpha(\xi)$ depending on the external flow modifies (1) to an extended Fisher's or FKPP equation $\partial_\theta \hat{D} - \alpha\, \partial_{ZZ}\hat{D} = \hat{D}^2 - \hat{\mu}$: here we again refer to [3].

3 From individual protuberances to distributed surface roughness

Let δ denote the local BL thickness. As demonstrated first in [4], an isolated surface irregularity (dent/hump) of streamwise and wall-normal extents of $O(\delta)$ and $O(\delta^{3/2})$, respectively, yields the most generic (least-degenerate) local modification of the classical BL structure. The so prevalent Euler stage agrees largely with that found in [5], there originating in the square-root singularity encountered by $u_e(x)$ immediately upstream of gross separation. Such a three-tiered splitting of the BL was also proposed in the (incomplete) study of turbulent flow past a sharp trailing edge at an angle of incidence of $O(1)$; cf. [6]. Hence, we extend this most generic description of a locally distorted subsonic turbulent BLs by assuming a wavy surface $y = \gamma^{3/2}S(x, X)$, $X := x/\gamma^{3/2}$. The n-th mode of the Fourier decomposition $S = \sum_{n=1}^{\infty} a_n(x)\sin(\lambda_n X)$, $\lambda_n := n/[2\pi k(x)]$ of the shape function S carries a slowly varying amplitude a_n and wavenumber k_n ($k_1 \equiv 1$). Here the origin $x = 0$ refers to a streamwise position where the BL is fully turbulent but is otherwise chosen arbitrarily.

The structure of the BL when distorted by a single surface-mounted hump is sketched in Fig. 3 with the inner part consisting of the viscous and the Reynolds stress sublayer (VS, RSS). Denote ψ the streamfunction and τ^{ij} $(i, j = x, y)$ the components of the Reynolds stress tensor, two-scale expansions govern the small-deficit flow in the main layer: $\eta := y/\delta = O(1)$,

$$\psi/(u_e\delta) - \eta \sim \gamma^{1/2}\Psi_1 - \gamma F_1 + \gamma^{3/2}\Psi_2 - \gamma^2 F_2 + O(\gamma^{5/2}), \quad \tau^{ij}/(u_e\gamma)^2 \sim T_1^{ij} + O(\gamma^{1/2}), \quad \delta/\gamma \sim \Delta_1(x) + O(\gamma^{1/2}). \quad (2)$$

Herein $\Psi_1(x, X, \eta)$ describes the potential flow induced by the individual asperity element of the quasi-periodic wavetrain S at a particular position x, $\Psi_l(x, X, \eta)$ $(i = 2, 3, \ldots)$ and $[F_l, T_l^{ij}](x, X, \eta)$, $(l = 1, 2, \ldots)$ the response it causes in higher orders by nonlinear convection and the turbulent contributions to the vortical BL flow, respectively. We stress that any X-dependence of the function Δ_1 dominating in the according expansion of δ would render the assumption of a slender BL invalid.

One proceeds by plugging (2) into the Reynolds equations and expanding by considering the conditions of matching with the Reynolds stress sublayer in terms of Prandtl's transposition theorem. This yields $(\partial_{XX} + \partial_{\eta\eta})\Psi_1 = 0$ subject to $\Psi_1(x, X, 0) = -S$, $\Psi_1(x, X, \infty) = 0$, thus $\Psi_1 = -\sum_{n=1}^{\infty} a_n \exp(-\lambda_n\eta)\sin(\lambda_n X)$, and $\partial_X[F_1, T_1^{ij}] \equiv [0, 0]$. We arrive at a classical homogenisation approach as averaging the problem describing the small-scale waviness of F_2 yields its solvability condition governing the sought long-scale variation of F_l and Δ_1, i.e. the accretion of the BL by entrainment as $\mathrm{d}\Delta_1/\mathrm{d}x > 0$, not appreciated in the forerunner studies initiated by [7]. The X-variation of Ψ_2 corresponds to a local thickening of δ of $O(\gamma^{1/2})$ and an out-of-phase streamline displacement on top of the RSS due to the pressure rise leeward of the crest of the surface undulation, predicted by the form of Ψ_1. Finally, the inertia-driven coupling between Ψ_1 and Ψ_2 modifies the classical long-scale problem for F_l, Δ_1 in terms of a novel exponentially decaying memory of F_1. This is currently under investigation.

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