Chapter 3

Time-mean Turbulent Shear Flows: Classical Modelling — Asymptotic Analysis — New Perspectives

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This contribution gives an introduction to the description of time-averaged single-phase turbulent flows where the largeness of the typical Reynolds numbers at play is their essential characteristic. Hence, the full Navier–Stokes equations form the starting point, and the viewpoint is a most rigorous asymptotic one. Emphasis is placed on the aspects of modelling the unclosed terms in the accordingly Reynolds-averaged equations when governing slender shear flows. Such represent the natural manifestation of turbulence as triggered internally in laminar shear layers by the no-slip condition to be satisfied at rigid walls rather than by free-stream turbulence, neglected here. Given the inherent closure problem associated with the separation and interaction of the variety of spatial/temporal scales involved, this focus allows for a surprisingly deep understanding of turbulent flows resorting to formal asymptotic techniques under the premise of a minimum of reliable assumptions. These are motivated by physical intuition and/or based on classical findings of the statistical theory of locally isotropic turbulence. Intrinsic differences to the analysis of related problems dealing with laminar high-Reynolds-number flows are highlighted. Finally, the crucial aspects of numerical simulation of turbulent flows are considered for the staggered levels of filtering, ranging from a most complete resolution to full averaging.

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1. Introduction

Without doubt, turbulence is one of the most fascinating and likewise vital fields of modern fluid dynamics, if not still going to be the most challenging of all in classical physics. Needless to say, gaining a deep understanding of turbulent flows is also of tremendous importance because of their omnipresence in various engineering applications.

1.1. *Prerequisites, objectives, scope, further reading*

The chapter is motivated by the exciting challenge addressed above and thus intends to highlight *some selected* intriguing phenomena associated with slender shear flows and topics of the current research from a most rigorous (asymptotic) viewpoint, some in a previously unappreciated manner. To this end, we assume a suitably (globally) defined Reynolds number, Re, to take on arbitrarily large values. Hence, the goal is definitely *not* to provide an overview on the entire subject (which is a "mission impossible" at all) but to encourage the interested reader to delve deeper into specific topics. He/she is expected to be familiar with fundamentals of fluid mechanics (including turbulence), dimensional analysis and perturbation methods.

We complete these introductory aspects by pointing to some classical and modern textbooks on turbulence as definite references. (This list must remain incomplete given the fast development of the field; to a certain extent, it naturally factors in the author's personal interests and approach and his integration in the Continental evolution of modern fluid dynamics and what is acknowledged as "Viennese school of asymptotics".)

Reference 1 provides a definitely pioneering and classical overview on the essence of shear-flow turbulence in a Newtonian fluid, appealing newcomers. Their specific peculiarities in terms of scaling laws are considered in Refs. 2 and 3, the latter taking up an asymptotic viewpoint and also giving an comprehensive survey on the commonly adopted turbulence models. More general and recent approaches are provided by Refs. 4–6; modelling aspects are dealt with extensively in Refs. 7–9. The biographic (non-asymptotic) view in Ref. 10, albeit probably not attracting the novice, deserves attention as well. Some

further exposures and reviews of more specific topics are cited in due place.

1.2. *Preliminaries*

The enormous difficulties we are facing in a most rigorous treatment of turbulent flows — theoretically and/or numerically — lie in a specific property of the underlying Navier–Stokes equations (NSE) in suitable non-dimensional form, thus entered by Re as the essential parameter: for unsteady flow, the subtle interplay between the nonlinear convective and the viscous term, at first considered as of $O(\nu_r)$ with the non-dimensional reference viscosity $\nu_r := Re^{-1}$, entails a cascade of temporal/spatial scales involved: these range from the largest ones fixing Re and here of $O(1)$, subsequently referred to as global ones, to, on the other extreme, the smallest ones responsible for the conversion of the work exerted by internal viscous forces into internal energy, the so-called viscous dissipation. The well-known little mnemonic rhyme by the renowned physicist and meteorologist L. F. Richardson from ca. 1930 condenses nicely this process associated with separation of scales:

> *Big eddies have little eddies/and little eddies have smaller eddies/that feed on their vorticity/and so on to viscosity.*

As a crucial observation, this not only becomes the more pronounced the larger Re is — which justifies an asymptotic approach — but cannot be inferred *a piorily* from inspection of the NSE when subjected to decent initial/boundary conditions (ICs/BCs). In other words, it represents an inherent property of the NSE and originates in the latest stages of laminar–turbulent transition controlled by the presence of (rigid) surfaces which is not fully understood yet; for some tantalising clues, though speculative in nature, see Ref. 13 (plus the references to preceding work therein).

Notably, nowadays also the inverse energy cascade (from smaller to larger scales) is a source of present debates in some circumstances (e.g., in the understanding of turbulent particle agglomeration).

Here, we shall not delve deeper in the matter of hydrodynamic instabilities and the transition process in the limit of large values

Table 1. Lower thresholds of Re for turbulence in typical flows.

Flow Case	Re **Exceeding**	**Reference Velocity**	**Reference Length**
Developed circular-pipe and (open-) channel flow	2400	Flow-area-averaged	Hydraulic diameter
Flat-plate BL	300,000	External-flow speed	Leading-edge distance
Entire BL on circular cylinder in cross flow	10^6	Unperturbed speed	Diameter

of Re; some remarks are given below in connection with boundaries. Our concern is with an already fully developed turbulent flow, i.e., broadband turbulence. This is characterised by the in-time-and-space simultaneous presence of all scales at play (in contrast to the observed dominant/isolated ones governing its history as a transitional flow). Despite the aforementioned inherent difficulties, asymptotic methods provide the proper and powerful means for gaining a deep understanding, at least for the Reynolds- or, equivalently, time-averaged slender shear flow, as they have proven likewise for several decades in the context of laminar and transitional flows. The ultimate goal of this endeavour is to predict the key features and the structure of turbulent flows in the formal limit $Re \to \infty$. Let us note the magnitudes of Re exceeding of which renders developed turbulent shear flows in some important situations and for perfectly smooth surfaces: see Table 1.

1.3. *Governing equations: a critical view*

Although the extremely large shear rates typical of turbulent flows raise doubts whether continuum theory provides the proper framework for their description, this classical approach in connection with the constitutive law for a Newtonian fluid lay the foundations for a most rational turbulence research. This viewpoint is supported by the vast majority of experimental data available, and no convincing arguments have yet led to a strikingly different accepted one. Here it is noteworthy that relaxation times on a molecular level are still

much smaller than the smallest time scales at play and the local Knudsen number $Kn := \hat{l}_p/\hat{l}_t$, formed with the the free path length of the molecules $\hat{l}_p$ and the smallest turbulent (macroscopic) length scale $\hat{l}_p$, is sufficiently small. Then the flow can be safely considered as being locally in thermodynamic equilibrium; the common Stokes hypothesis of vanishing bulk viscosity applies.

Hence, our starting point is formed by the continuity equation

$$\partial_t \rho + \partial_j(\rho u_j) = 0 \tag{1}$$

and the conventional Navier–Stokes equations

$$\rho(\partial_t u_i + u_j \partial_j u_i) = -\partial_i p - \rho g_i + \nu_r \left[\partial_{jj} u_i - \frac{\partial_{ij} u_j}{3}\right], \tag{2}$$

for a single-phase Newtonian fluid having uniform dynamic viscosity (which can be regarded a very weak restriction). Here and subsequently, the following conventions and prerequisites are adopted if not stated otherwise. A Cartesian coordinate system and the corresponding covariant Einstein notation are advantageously used $(i, j, k = 1, 2, 3)$; with t being the time, x_i, $u_i(x_j, t)$ and $-g_i$ denote the components of the position vector, the flow velocity, and a body force (especially gravity, most relevant for free-surface flows), respectively, $p(x_j, t)$ is the fluid pressure and $\rho(x_j, t)$ its density; ∂_i (∂_{ij}) indicates first (second) derivatives with respect to x_i (and x_j); all quantities are made appropriately non-dimensional as mentioned above, e.g., u_i, p, ρ with the values taken from the uniform parallel flow assumed infinitely far upstream of a solid obstacle, x_i with a dimension typical of the latter, and t according to x_i and u_i. Furthermore, their dimensional forms are indicated by hats indicate and dependences on Re not stated explicitly.

Thus any external unsteady external forcing of the flow, imposing naturally one or a multitude of further reference time scales, shall be discarded. However, by the last assumption also the occurrence of free-stream turbulence acting on boundary or free shear layers is excluded, but in many aerodynamic problems of practical relevance the atmosphere around the flight vehicle of interest can in fact be viewed as sufficiently quiet.

The thermal energy equation and an equation of state extend Eqs. (1), (2) to a closed system of governing equations determining the unknowns u_i, p, ρ when supplemented with appropriate BCs, satisfied at walls and for $|x_i| \to \infty$ (upstream), and ICs for $t = 0$, say. However, here our concern is chiefly with Eqs. (1), (2) as we are mostly interested in fundamental properties of turbulent flows which already ensue from the study of incompressible ones of constant density: these are characterised by $\rho \equiv 1$ in Eqs. (1), (2), which thus decouple from and can be treated independently of the energy equation.

We are consequently led to the following self-consistent picture: the flow is assumed to have undergone an intrinsic process of laminar–turbulent transition further upstream due to the presence of solid walls — for external flows, in a boundary layer (BL) emerging for sufficiently large values of Re. Free-stream turbulence causes unstable Klebanoff modes in a shear layer and thus *by-pass* transition, but convectively unstable Tollmien–Schlichting (TS) waves trigger the classical, prevalent routes to shear-layer turbulence on a rather long scale. This scenario due to instabilities (having ubiquitous sources) and/or the receptivity of imposed disturbances, as by surface roughness and/or acoustic waves present in the external flow contrasts with short-scale transition as by reattachment of marginally separated BLs.

Insofar, the presence of walls is crucial, not only for attached turbulent BLs as also granting the existence of free/separated turbulent shear flows as forming nozzles, etc. further upstream. When coinciding with the x_1-axis, say, a wall provokes the usual kinematic no-slip/penetration conditions at

$$x_2 = 0: \quad u_1 = u_3 = 0, \quad u_2 \text{ prescribed}, \tag{3}$$

either by strict impermeability of the rigid wall ($u_2 = 0$) or a (given) rate of suction/blowing. According to Eq. (3) and for slender turbulent shear layers, let subsequently $(x, u) := (x_1, u_1)$, characterise the mean or streamwise flow direction, $(y, v) := (x_2, u_2)$ the one perpendicular to the mean shear or in wall-normal direction, and $(z, w) := (x_3, u_3)$ the spanwise one. Since curvature effects do not

have an important influence on such flows, these are neglected from the outset. The coordinates then can also be interpreted as natural ones, e.g., x as tangential to a curved mean-flow streamline or a curved surface, typically a two-dimensional (2D) one with generatrices parallel to the z-direction. Such scenarios are tacitly assumed hereafter. They include axisymmetric BL flows; the extension to axisymmetric free jets with x_1 as the axial, x_2 the radial and x_3 the circumferential coordinate is straightforward.

A few words deserve also to be left on the validity of the no-slip condition. It can be argued for liquids by adhesion forces but is less obvious for gases. However, our scales are still so large such that continuum mechanics applies even in the immediate vicinity of the wall. As a consequence of diffuse reflection at a surface, the velocity distribution of the reflected fluid particles becomes statistically independent of that of the incident ones when the spatial scale typical of averaging is much larger than $\hat{l}_p$ but much smaller than $\hat{l}_t$. Therefore, on the latter scale the no-slip condition is observed. For an ideal gas in a state of equipartition, the dynamic viscosity can be expressed as $\hat{\mu} = \hat{\rho}\hat{c}\hat{l}_p/3$ with $\hat{c}$ being the macroscopically observed particle speed, under the basic assumption $Kn = \hat{l}_f/\hat{l}_m \ll 1$ hence the speed of sound, here evaluated at the wall. Furthermore, it is noted that c and ρ undergo changes independent of Re in the x_2-direction. With λ_w denoting the ratio of $\hat{l}_m$ and the global reference length and Ma the Mach number at the reference state, the last condition is then cast in the convenient form

$$\frac{Ma}{(Re\lambda_w)} \ll 1 \quad \text{for } Re \gg 1. \tag{4}$$

Strictly speaking, it is obligatory to check this criterion in the analysis of BLs driven by external flows at large Mach numbers. However, it is expected to be met throughout for the usually enormous values of Re and λ_w known to be of $O(1/\ln Re)$.[13,14]

1.4. *What exactly is shear-flow turbulence?*

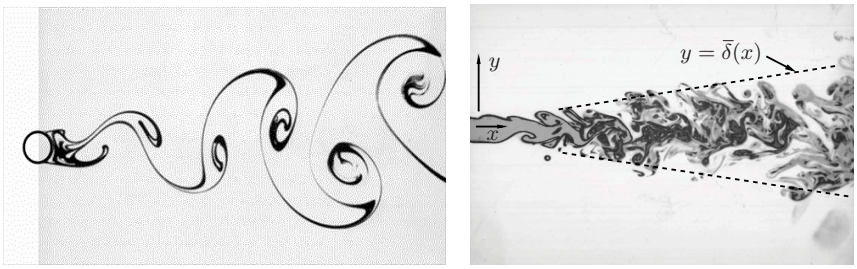
The above introductory view on turbulence motivates the following — for our purposes adequately complete and precise —

characterisation of a turbulent flow. Developed turbulent flows:

(i) Appear naturally for sufficiently large values of Re,
(ii) Are intrinsically stochastic in time and space (keyword: deterministic chaos in mechanical systems) as a consequence of the viscous forces at play howsoever large Re is,
(iii) Are three-dimensional (associated with vortex stretching).

Items (ii) and (iii) have their origins in the instability scenarios an originally even nominally 2D and steady laminar flow has undergone. Specifically, property (ii) implies that turbulent flows are vortex flows where the dimensions of vortices span the aforementioned range of scales. Issue (iii) is associated with vortex stretching as a typical feature of turbulent flows (we exclude exceptional cases of degenerate 2D turbulence).

In this respect, a typical example of a non-turbulent flow is the famous von Kármán vortex street forming already for $Re = O(10^3)$ in the wake of plane flow past a closed bluff body (Fig. 1(a)): here the underlying vortex shedding in the separated shear layers is characterised by a well-defined Strouhal number, and three-dimensionality is still poor, so the typical features of turbulence are not present for such rather moderate values of Re.



(a) Von Kàrmaǹ's vortex street visualised[12] (©Dr. Sadatoshi Taneda in Lieu of The Parabolic Press, modified/reprinted by permission).

(b) Axisymmetric immersed water jet with fractal edge, visualised by laser-induced fluorescence technique[11] (©American Institute of Physics (AIP), modified/reprinted by permission).

Fig. 1. Non-turbulent though highly unsteady (a) versus fully turbulent shear layers (b).

In striking contrast with laminar shear layers having a typical width of $O(Re^{-1/2})$ as a result of the straightforward analysis of Eqs. (1) and (2), such a definite scaling cannot be given at this stage for developed turbulent ones. As an instructive typical example, the jet depicted in Fig. 1(b) has a numerically small but finite opening angle however large Re becomes when formed with a distance from its virtual origin $x = y = 0$ where it is viewed as fully turbulent ($x = 1$). This is due to their inertia-driven character and the minor importance of the viscous term in Eq. (2), unlike in the shear-layer balance typical of laminar flows. However, turbulence is in fact in most circumstances confined to a rather slender region having a rather abrupt edge, in the current setting measured by the width $\overline{\delta}(x)$, say. As recognised in Fig. 1(b), the interface separating the turbulent from the entrained (here solely induced) irrotational external flow is a sharp one as the transition towards it not driven by viscous diffusion like in laminar shear layers.

This motivates us to give the:

Definition 1 (slender turbulent shear flow). We speak of a *slender turbulent shear flow* if its width $\overline{\delta}$ satisfies

$$\overline{\delta} \ll 1 \quad \text{and} \quad \frac{\mathrm{d}\overline{\delta}}{\mathrm{d}x} \ll 1, \tag{5}$$

on the global scale x sufficiently far downstream of the transition process.

Albeit quite simple and intuitive, this characterisation proves powerful due to its generality. This allows for remarkable progress by formal asymptotic methods as outlined in Sec. 4. Marked shear $\partial_y u$ across the layer emerges as a pure consequence. Shear flows comprise wall-bounded and separating BLs and free shear flows as separated BLs, jets, mixing layers, wakes behind obstacles and (forced and) buoyancy-driven plumes.

2. On Averaging and Modelling

The classical averaging technique means decomposing the flow into a nominally steady background flow and its turbulent fluctuations

about that mean flow, the former studied on the basis of the accordingly Reynolds-averaged NSE (RANS). In most cases, the nominal flow is not only steady but also 2D, which represents a decisive simplification. It is then independent of z (as mentioned wherever appropriate in the following).

2.1. *Principles of averaging*

In order to "prove" consistency of the classical closure ideas with the flow structure in the high-*Rey*-limit in an hitherto unappreciated manner, the basic ingredients to the common averaging strategies have to be condensed.

2.1.1. *Conventional Reynolds-averaging and ergodicity*

By adopting the usual notation, we then decompose any (tensorial) flow quantity, here represented by $\mathcal{Q}$, in the form

$$\mathcal{Q}(x_i,t) = \overline{\mathcal{Q}}(x_i) + \mathcal{Q}'(x_i,t) \quad (\text{2D flow:} \quad \partial_z \overline{\mathcal{Q}} = 0). \tag{6}$$

Here the mean contribution $\overline{\mathcal{Q}}$ is either interpreted statistically, i.e., in terms of ensemble averaging, or expressed via typical time-averaging:

$$\overline{\mathcal{Q}} := \int\limits_{\mathcal{Q}} \mathcal{Q}\,\mathrm{PDF}(\mathcal{Q})\,\mathrm{d}\mathcal{Q} = \lim_{\Delta t\to\infty} \frac{1}{T}\int\limits_0^T \mathcal{Q}(x_i,t+\theta)\,\mathrm{d}\theta. \tag{7}$$

Here PDF stands for the probability density function; more precisely, the first integral in Eq. (7) has to be taken as an Lebesgue measure, and for a stationary process, in the second case implied by the limit process (provided it exists), the celebrated ergodicity theorem (law of large numbers) guarantees the equivalence of both representations. Finally, the relations in Eq. (7) *define* $\mathcal{Q}'$ and give the basic results $\overline{\overline{\mathcal{Q}}} = \overline{\mathcal{Q}}$, $\overline{\mathcal{Q}'} = 0$. We now identify $\overline{\delta}$ in Eq. (5) with the (half) width of the time-mean shear layer. Note that *filtering* the NSE means a (*Re*-dependent) *finite* filter width T.

As a fundamental finding, $\overline{\mathcal{Q}}$ is interpreted equivalently as expectation or time-mean value. Here Jensen's inequality $\phi(\overline{\mathcal{Q}}) \le \overline{\phi(\mathcal{Q})}$ for some convex function ϕ is noteworthy, generalising the basic

finding $\overline{\mathcal{Q}}^2 \leq \overline{\mathcal{Q}^2}$. We furthermore recall the nth (statistical) *central moment* $\overline{\mathcal{Q}'^n}$ for some integer $n \geq 1$ and note the generalised Cauchy–Schwarz inequality

$$\overline{\prod_{i=1}^{i=n} \mathcal{Q}_i} \leq \prod_{i=1}^{i=n} \overline{\mathcal{Q}_i^n}^{\frac{1}{n}} \quad (n = 2^m, \;\; m \geq 1) \tag{8}$$

involving $2m$ quantities $\mathcal{Q}_i$.

2.1.2. *Favre-averaging*

Classical Reynolds-averaging is generalised by writing

$$\mathcal{Q} = \widetilde{\mathcal{Q}} + \mathcal{Q}'', \quad \widetilde{\mathcal{Q}} := \frac{\overline{\rho \mathcal{Q}}}{\overline{\rho}}. \tag{9}$$

Now Reynolds-averaging Eq. (1) yields its laminar-like form for steady flow

$$\partial_j(\overline{\rho}\widetilde{u}_j) = 0. \tag{10}$$

Hence, this well-known Favre- or density-weighted averaging has proven useful for compressible flows, and we will adopt it subsequently. We accordingly find that $\widetilde{\widetilde{\mathcal{Q}}} = \widetilde{\mathcal{Q}}$, $\widetilde{\mathcal{Q}''} = 0$. By noting that $\overline{\widetilde{\mathcal{Q}}} = \overline{\mathcal{Q}}$ and $\overline{\widetilde{\mathcal{Q}}} = \widetilde{\mathcal{Q}}$, one obtains $\widetilde{\mathcal{Q}} = \overline{\mathcal{Q}} + \widetilde{\mathcal{Q}'}$, $\overline{\mathcal{Q}} = \widetilde{\mathcal{Q}} + \overline{\mathcal{Q}''}$, giving the relationship $\widetilde{\mathcal{Q}'} = -\overline{\mathcal{Q}''} \not\equiv 0$ between the fluctuations $\mathcal{Q}'$ and those introduced via Eq. (9).

Favre-averaging contains Reynolds-averaging as special case for vanishing density fluctuations ρ'. This is noticed in connection with some further important relationships, notations and rules explained next.

Let us remind the Steiner translation theorem $\widetilde{\mathcal{Q}_1\mathcal{Q}_2} = \widetilde{\mathcal{Q}_1}\widetilde{\mathcal{Q}_2} + \widetilde{\mathcal{Q}_1''\mathcal{Q}_2''}$, that $\widetilde{\mathcal{Q}_i\mathcal{Q}_j}$ is a *covariance* or *cross-correlation* for $i \neq j$ and a *variance* or *auto-correlation* for $i = j$, and the usual standard deviation $\sigma_{\mathcal{Q}} := \left(\widetilde{\mathcal{Q}^2} - (\widetilde{\mathcal{Q}})^2\right)^{1/2} = \widetilde{\mathcal{Q}''^2}^{1/2}$ or RMS value of $\mathcal{Q}''$, being a proper measure for its average magnitude, i.e., the one typical for the predominant fractions of time (where averaged flow quantities can be observed). Hence, by the quite valuable consequence

$$\widetilde{\mathcal{Q}_1''\mathcal{Q}_2''} \leq \sigma_{\mathcal{Q}_1}\sigma_{\mathcal{Q}_2}, \tag{11}$$

of Eq. (8), the once estimated order of magnitudes of the auto-correlations bound those of the cross-correlations.

Moreover, as most important for averaging the governing equations, one readily deduces the following rules: For the time derivatives, $\partial_t \mathcal{Q} = \partial_t \mathcal{Q}' = (\partial_t \mathcal{Q})' = \partial_t \mathcal{Q}''$ such that $\overline{\partial_t \mathcal{Q}} = \partial_t \overline{\mathcal{Q}} = \partial_t \overline{\mathcal{Q}'} = 0$ but $\partial_t Q'' \not\equiv (\partial_t Q)''$, $\widetilde{\partial_t \mathcal{Q}} = \widetilde{\partial_t \mathcal{Q}''} = \widetilde{\partial_t \mathcal{Q}'} = -\overline{\partial_t \mathcal{Q}''} = \overline{\rho' \partial_t \mathcal{Q}'}/\overline{\rho} \not\equiv 0$, and

$$\overline{\rho' \mathcal{Q}''} = \overline{(\rho - \overline{\rho})\mathcal{Q}''} = -\overline{\rho}\,\overline{\mathcal{Q}''}\ (\not\equiv 0); \tag{12}$$

for the spatial ones $\partial_i \overline{\mathcal{Q}} = \overline{\partial_i \mathcal{Q}}$, $\partial_i \widetilde{\mathcal{Q}} = \widetilde{\partial_i \mathcal{Q}}$. The latter express the filtering of the small spatial scales/wavelengths characteristic of the turbulent fluctuations, potentially reducing the order of magnitude of the gradients.

Let us next give the following:

Definition 2 (fully turbulent and near-wall regions). Let l_2 and λ_2 denote the locally largest (time-mean or global) scale and the smallest scale (wavelength), respectively, of the turbulent dynamics for the x_2-direction:
the *fully turbulent* regime emerges at scale separation, implying $\lambda_2 \ll l_2$;
the *near-wall* or *viscous sublayer* adjacent to a wall at the collapse $\lambda_2 \sim l_2$.

This just expresses the apparent necessity to distinguish between at least two flow regions in the high-Re limit.

The first regime sufficiently remote from a wall is characterised by

$$|\partial_i \overline{\mathcal{Q}_1' \mathcal{Q}_2'}| \ll |\overline{\mathcal{Q}_1 \partial_i \mathcal{Q}_2'}|, \quad |\partial_i \widetilde{\mathcal{Q}_1' \mathcal{Q}_2'}| \ll |\widetilde{\mathcal{Q}_1 \partial_i \mathcal{Q}_2'}| \quad \text{for } Re \gg 1. \tag{13}$$

Specified for slender shear flows, this yields typical estimates for those being

$$\text{weakly 2D:} \quad |u'| \ll |\widetilde{u}| = O(1), \ \ |v| \ll 1, \ \ |w| \ll 1, \ \ \text{so} \ \ |\widetilde{u_i'' u_j''}| \ll 1; \tag{14}$$

$$\text{strictly 2D:} \quad |w'| \ll 1, \ \ \widetilde{w} \equiv \widetilde{u'' w''} \equiv \widetilde{v'' w''} \equiv 0. \tag{15}$$

Equations (5) and (14), (15) enable further progress by virtue of a formal asymptotic analysis, by heavy exploitation of Eqs. (11) and (13). The collapse of y-scales in the near-wall regime entails that of

the velocity scales, and it turns out that there the classical near-wall scaling applies as a consequence of Reynolds-averaging Eq. (2).

2.2. *RANS and higher-moments transport equations*

Reynolds-averaging of Eq. (2) written in conservative form with the aid of Eqs. (1) and subject to (9) yields the RANS or Reynolds equations

$$\overline{\rho}\widetilde{u}_j\partial_j\widetilde{u}_i = -\partial_i\overline{p} - \overline{\rho}g_i + \partial_j\overline{\tau}_{ji} + \nu_r\left[\partial_{jj}\overline{u}_i - \partial_{ij}\frac{\overline{u}_j}{3}\right], \;\; \overline{\tau}_{ij} := -\overline{\rho}\widetilde{u_i''u_j''}. \tag{16}$$

The last double-correlation, equal to $-\overline{\rho\, u_i'u_j'}$ according to Eqs. (7) and (9), represents the Reynolds stress tensor. Together with Eq. (10) and the correspondingly averaged further governing equations, these equations govern the mean flow but represent an unclosed system, first with regard to the new unknown $\overline{\tau}_{ij}$ — which can be viewed as the central element of what is referred to as the *turbulence closure problem*. Let us recall this intrinsic property of the RANS, where specifying them for incompressible flows ($\rho \equiv 1,\ \rho' \equiv 0$) has Eqs. (10), (16) decouple from the correspondingly averaged thermal energy equation, so that problem reduces to modelling $\overline{\tau}_{ii}$.

The equations governing the fluctuations ρ' and u_i'',

$$\partial_t\rho' + \partial_j(\overline{\rho}u_j'' + \rho'\widetilde{u}_j + \rho'u_j'') = 0, \tag{17}$$

$$\rho(\partial_t u_i'' + u_j\partial_j u_i'' + u_j''\partial_j\widetilde{u}_i) + \rho'\widetilde{u}_j\partial_j\widetilde{u}_i = -\partial_i p' - \rho' g_i - \partial_j\overline{\tau}_{ji} + \nu_r\left[\partial_{jj}u_i'' - \frac{\partial_{ij}u_j''}{3}\right], \tag{18}$$

complement Eqs. (10) and (16) to give Eqs. (1) and (2), respectively. We introduce the operator $\mathcal{D}_f$ so as to rewrite the left-hand side of Eq. (18), the contribution of fluctuations to ρ times the total derivative of u_i, as $\mathcal{D}_f\{u_i\}$. Then the expression $\overline{u_j''\mathcal{D}_f\{u_i\} + u_i''\mathcal{D}_f\{u_j\}}$ yields the so-called Reynolds-stress equations (RSE) or transport equations for the unclosed terms $\widetilde{u_i''u_j''}$. More specifically, these are obtained by rewriting Eq. (18) in conservative form with the aid

of (1) and taking into account Eq. (12) and Eq. (17) with $\partial_j\overline{(\rho u_j'')} = 0$. We finally have

$$\overline{\rho}(\partial_t+\widetilde{u}_k\partial_k)\,\widetilde{u_i''u_j''} = R_{ij}+R_{ji},\quad R_{ij} := P_{ij}+S_{ij}^p+D_{ij}^t+D_{ij}^\nu-\overline{\varepsilon_{ij}^p}. \tag{19}$$

Herein the tensors at the right-hand side are conveniently distinguished as

$$\textit{turbulent production}\;\; P_{ij} := \overline{\tau}_{ik}\partial_k\overline{u}_j + \overline{u_i''}(\overline{\rho}\,\widetilde{u}_k\partial_k\widetilde{u}_j - \partial_k\overline{\tau}_{kj} + \overline{\rho}g_j), \tag{20}$$

$$\textit{pressure-shear terms}\;\; S_{ij}^p := \overline{p'\partial_i u_j''}, \tag{21}$$

$$\textit{turbulent diffusion}\;\; D_{ij}^t := -\frac{\partial_k(\overline{\rho}\,\widetilde{u_i''u_j''u_k''})}{2} - \partial_i\overline{u_j''p'}, \tag{22}$$

$$\textit{viscous diffusion}\;\; D_{ij}^\nu := \nu_r\left(\frac{\partial_{kk}\overline{u_i''u_j''}}{2} - \frac{\partial_i\overline{u_j''\partial_k u_k''}}{3}\right), \tag{23}$$

$$\textit{turbulent dissipation}\;\; \varepsilon_{ij}^p := \nu_r\left[(\partial_k u_i'')(\partial_k u_j'') - (\partial_i u_j'')\frac{(\partial_k u_k'')}{3}\right]. \tag{24}$$

It is noted that $\overline{u_i''} \equiv 0$ in Eq. (20) indicates incompressible flow.

Generally spoken, the left-hand side of a transport equations for any averaged quantity exhibits the convective operator $\widetilde{u}_j\partial_j$, the right-hand side "diffusive" terms, written in divergence/gradient form, where the viscous ones are those proportional to ν_r, hence further "dissipative" terms also proportional to ν_r, and the remaining "source" or so-called production terms. However, a physical interpretation is only admissible for Eq. (19) as this represents the budget of the share of specific mechanical power exerted by a fluid particle due to the velocity fluctuations u_i'', u_j''. Accordingly, ε_{ij}^p is frequently termed *pseudo*-dissipation: this notation more appropriately matches its physical origin as D_{ij}^ν includes the complementary contribution to dissipation by internal viscous forces (positive by the second law of thermodynamics). Also, typical of these equations are the products of ρ with Favre-averaged terms, as a result of Eq. (9), and that time derivatives vanish identically only in the incompressible-flow limit. However, this is otherwise obtained via Eq. (9) for the sake of consistency with Eqs. (10), (16) for nominally steady flow.

Equation (19) not only contain triple correlations, cf. Eq. (22), but also double-correlations involving p', cf. Eqs. (21) and (22), and finally such involving only gradients $\partial_i u'_j$, cf. Eq. (24). This reflects the impossibility to formulate a closed systems of equations by considering Eqs. (10), (16), plus any finite number of arbitrary moments of the NSE, Eq. (2), obtained by multiplying Eq. (18) with terms involving u''_j and subsequent Reynolds-averaging. As an obvious weakness of all types of averaging or filtering Eq. (2), this just means the irretrievable loss of information about the stochastic small-scale dynamics, solely returned in modelled form. Further insight is accomplished, however, by considering the trace of Eq. (19). The resultant budget of the specific turbulent kinetic energy K, the so-called $\widetilde{K}$-equation, serves as the starting point for all considerations on "solving rationally" the closure problem in the limit $Re \to \infty$:

$$\overline{\rho}(\partial_t + \widetilde{u}_j \partial_j)\,\widetilde{K} = R_{ii} = P + S^p + D^t + D^\nu - \overline{\varepsilon^p}, \quad K := \frac{u''_i u''_i}{2}. \tag{25}$$

Here the scalar counterparts to the quantities introduced in Eqs. (20)–(24)

$$P := \overline{\tau}_{ij}\partial_j \overline{u}_i + \overline{u''_i}(\overline{\rho}\,\widetilde{u}_j \partial_j \widetilde{u}_i - \partial_j \overline{\tau}_{ji} + \overline{\rho} g_i), \tag{26}$$

$$S^p := \overline{p' \partial_i u''_i}, \tag{27}$$

$$D^t := -\partial_i(\overline{\rho}\,\widetilde{K u''_i} + \overline{u''_i p'}), \tag{28}$$

$$D^\nu := \nu_r \partial_i \left(\partial_i \overline{K} - \frac{\overline{u''_i \partial_j u''_j}}{3} \right), \tag{29}$$

$$\varepsilon^p := \nu_r \left[(\partial_k u''_i)(\partial_k u''_i) - \frac{(\partial_i u''_i)^2}{3} \right], \tag{30}$$

reduce to their well-known standard form in the limit of incompressibility, where Eq. (17) simplifies to $\partial_i u'_i \equiv 0$ ($u'_i \equiv u''_i$).

The structure of Eq. (25) suggests to gain further information by considering $2\overline{(\partial_i u''_j)\partial_i \mathcal{D}_f\{u_j\}}$. A procedure analogous to that leading to the $\widetilde{K}$-equation then yields the so-called $\overline{\varepsilon^p}$-equation, here only

specified for incompressible flow for the sake of conciseness:

$$\begin{aligned}\frac{\overline{u}_j\partial_j\overline{\varepsilon^p}}{\nu_t} = & -2[\overline{(\partial_k u_i')(\partial_k u_j')} + \overline{(\partial_i u_k')(\partial_j u_k')}]\partial_j\overline{u}_i - 2\overline{u_k'\partial_j u_i'}\,\partial_{jk}\overline{u}_i \\ & + \partial_i[\partial_i\overline{\varepsilon^p} - \overline{u_i'\varepsilon^p} - 2\overline{(\partial_j u_i')(\partial_j p')}] - 2\overline{(\partial_i u_j')(\partial_i u_k')(\partial_j u_k')} \\ & -2\nu_t\overline{(\partial_{ij}u_k')(\partial_{ij}u_k')}. \end{aligned} \tag{31}$$

The structure of this equation resembles that of Eqs. (19) and (25). However, only the $\widetilde{K}$-equation is susceptible to a physical interpretation. As exemplifying the associated difficulties categorised above in view of the individual terms in Eq. (31), its last term means a "dissipation of dissipation".

Equations (19), (25) can be simplified further for shear flows by virtue of Eqs. (11), (13). A first important conclusion is drawn for firmly attached BLs by inspection analysis, which indicates that the flow regimes introduced by Def. 2 are characterised by

$$Re^{-1}|\partial_i\widetilde{u}_j| \ll |\overline{\tau}_{ij}|, \quad |D^{\nu}_{[ij]}| \ll |\varepsilon^p_{[ij]}| \quad \text{for } Re \gg 1 \tag{32}$$

and
$Re^{-1}|\partial_i\widetilde{u}_j|/|\overline{\tau}_{ij}| = O(1)$, $|D^{\nu}_{[ij]}|/|\varepsilon^p_{[ij]}| = O(1)$, respectively. Hence, ν_r enters Eqs. (25), (19) predominantly via the turbulent dissipation in the fully turbulent region, which enables a drastic simplification of the closure problem for this regime. On the other hand, asymptotic analysis shows that the near-wall time-mean flow exhibits universal properties due to the negligibly small effects of inertia there, which extremely alleviates its treatment; for an outline see Ref. 14 (and the references to pioneering work therein), and Ref. 15. Its scaling is obviously recovered by the classical one,

$$\begin{aligned}&\frac{\widetilde{u}}{\widetilde{u}_\tau} = u^+(x,y^+) = O(1), \quad y^+ := \frac{y}{\overline{\delta}_\nu}, \\ &\overline{\delta}_\nu := \frac{1}{\widetilde{u}_\tau Re}, \quad \widetilde{u}_\tau := \sqrt{\nu_r\frac{\partial\widetilde{u}}{\partial y}\Big|_{y=0}}. \end{aligned} \tag{33}$$

It is based upon the aforementioned equal order of magnitudes of the molecular and Reynolds shear stresses and the resulting dominant balance of their sum with the wall shear stress. Hence, the local

skin-friction velocity $\widetilde{u}_\tau$ provides the suitable velocity scale for the near-wall region.

It is therefore sufficient to restrict the considerations on modelling to the fully turbulent flow. Also, closing $\overline{\tau}_{ij}$ represents the core problem, envisaged next. Closing other relevant quantities then is a subordinate task accomplished in a straightforward manner as far as those enter an asymptotically correct leading-order flow description. The central step is to critically review and thereby substantiate the classical Boussinesq hypothesis.

2.3. *A promising view on the Boussinesq ansatz*

Let us apply the common decomposition of $\partial_i u_j$ into its symmetric and anti-symmetric part, i.e., the rate-of-deformation or strain-rate tensor $S_{ij} := (\partial_i u_j + \partial_j u_i)/2$ accounting for stretching and volumetric dilatation of the particles, and the rotational contribution $\Omega_{ij} := (\partial_i u_j - \partial_j u_i)/2$ by the vorticity, $\varpi := \epsilon_{ijk}\partial_j u_k = \epsilon_{ijk}\Omega_{ij}$ (Levi-Cività symbol ϵ_{ijk}). Now consider a tensor Σ_{ij} depending on the local state of deformation of the fluid considered, i.e., the velocity gradient $\partial_i u_j$ and possibly higher derivatives in an Eulerian frame of reference. As a fundamental finding of continuum mechanics,[16] two consecutive kinematic statements can be made on a sole isotropic dependence on $\partial_i u_j$, i.e., one invariant against reflections and rotations of the coordinate system (Kronecker delta δ_{ij}):

Theorem 1 (Rivlin–Ericksen representation theorem). *(a) Isotropy requires Σ_{ij} to be a function of S_{ij} solely;*
(b)for a symmetric tensor $\Sigma_{ij} \equiv \Sigma_{ji}$ having the irreducible invariants $I_1 = S_{ii}$, $I_2 = (S_{ii}S_{jj} - S_{ij}S_{ij})/2$, $I_3 = \det S_{ij}$, any isotropic dependence on S_{ij} is of the generic quadratic form

$$\Sigma_{ij} = \nu_0\,\delta_{ij} + \nu_1 S_{ij} + \nu_2 S_{ik}S_{kj} \quad \textit{with } \nu_{0,1,2} = \nu_{1,2,3}(I_j), \tag{34}$$

being some so-called structure functions.

If Σ_{ij} is identified with the Cauchy stress tensor (symmetry is entailed by the Boltzmann axiom), the linear two-parameter constitutive law for a Newtonian fluid represents the simplest conceivable prototype of such a relationship (here $\nu_1 = (\nu_b - 2\nu_r/3)S_{ii}$ with ν_b

being the (kinematic) bulk viscosity, $\nu_2 = 2\nu_r$, $\nu_3 = 0$). Bearing this in mind, the question arises to which extent the deep interrelation (34) allows for a putative dual by setting $\Sigma_{ij} = -\widetilde{u_i'' u_j''}$. The following considerations on the crucial issues of such a relationship (locality, stress–strain-type relation, linearity, local isotropy) guide us in establishing a Reynolds-stress closure in this spirit. This is then found fully consistent with the characteristics of high-Re turbulence.

Locality. This might be questionable at a first glance, given upstream influences on the flow and its history. However, a turbulent shear flow adjusts quite rapidly to local conditions as long as changes in the external-flow conditions or the surface topography are sufficiently smooth. Contrarily, short-scale disturbances (typically, by sudden changes in the wall roughness, individual wall-mounted obstacles topography or shock-impingement on a BL) provoke a distinctly slow recovery of the flow. However, even this phenomenon can be traced back merely to inertial effects, inasmuch as the anticipated scale separation underlying the modelling is eradicated only locally. Moreover, a turbulent flow "forgets" rather quickly the particular mechanism of laminar–turbulent transition as it manifests itself in a generic manner independent of this and, like in laminar flow, it is the ellipticity of the NSE which accounts for further non-local effects non-locality. Finally, the extensions of Theorem 1 including history effects account for viscous relaxation but in Newtonian turbulent flows these are purely inertia-driven. As a conclusion, locality in the Reynolds-stress–strain relationship does not pose a seriously troublesome issue.

A pure stress–strain relationship. In fact, any dependence of $\overline{\tau}_{ij}$ on higher-order gradients of $\widetilde{u}_i$ would increase the order of the RANS compared to that of the underlying NSE and thus raise an inconsistency, mostly in terms of the BCs given by Eq. (3) to be satisfied.

Linearity. As for locality, nonlinear inertia terms are an essential feature of the RANS, so that ν_2 in Eq. (34) is virtually set to zero. Then the nonlinearities involving the small-scale motion are coped

with though by the proper modelling of ν_0 and ν_1, which then must not depend on the (averaged) invariants $\widetilde{I}_1$, $\widetilde{I}_2$, $\widetilde{I}_3$ for the sake of consistency. As the original NSE have only quadratic nonlinearities and for consistency with the preceding point raised, these structure function are required to depend on double-correlations involving first derivatives of u_i'' solely.

Isotropy. In the high-Re limit and for the associated smallness of the smallest scales identified in a turbulent flow, Kolmogorov's hypothesis of *local isotropy*[17] seems a reasonable. Here, this is put in a different context: the relationship Eq. (34) for $\Sigma_{ij} = \overline{\tau}_{ij}/\overline{\rho}$ is a locally isotropic one but not the tensor $\overline{\tau}_{ij}$ itself (which would imply the much more restrictive global isotropy). As of great practical value, we cast this in the

Proposition 1 (Local isotropy). *In the limit $Re \to \infty$, for a turbulent slender shear layer all components of $\overline{\tau}_{ij}$ scale in the same manner [equality in Eq. (11)] : some gauge function $\gamma(Re; y)$ gives $\lim_{Re\to\infty} \overline{\tau}_{ij}/\gamma = O(1)$.*

By recalling the hydrostatic stress contribution $\overline{\tau}_{ii}/3 = -2\overline{\rho}\widetilde{K}/3$, we arrive at the well-known Boussinesq ansatz for the deviatoric stresses:

$$\overline{\tau}_{ij} = \nu_0\, \delta_{ij} + \nu_1 \widetilde{S}_{ij}, \quad \nu_0 = -\frac{2(\widetilde{K} + \nu_t \partial_i \widetilde{u}_i)}{3}, \;\; \nu_1 = 2\nu_t. \tag{35}$$

This is seen as the "turbulent" counterpart to the phenomenological relationship for a Newtonian fluid. It introduces the so-called (kinematic) eddy or turbulent viscosity ν_t. In accordance with the above considerations, it is assumed to depend isotropically on the averaged motion on the microscopic scales (the turbulent fine structure) and on its molecular counterpart (herewith on the thermodynamic state), the perturbation parameter ν_r, solely. The kinematic dependence then can only involve $\widetilde{K}$ and $\widetilde{(\partial_i u_j'')(\partial_i u_j'')}$ in the limit $Re \to \infty$. Dimensional analysis gives

$$\frac{\hat{\nu}_t\, \overline{\widehat{\varepsilon^p}}}{\overline{\widehat{\widetilde{K}}}^{\,2}} \sim \Pi\left(\frac{\hat{\nu}_t}{\hat{\nu}_r}\right). \tag{36}$$

From here on, density variations are neglected. Furthermore, the estimates in Eq. (32) characteristic of the fully turbulent region imply $\nu_t/\nu_r \to 0$ as $Re \to \infty$ there, so that the function Π in Eq. (36) tends to a constant c_ν, say. This rationale recovers the widely-accepted, deceptively simple formula for the fully turbulent flow regime, as an asymptotic one on a sound basis:

$$\nu_t \sim \frac{c_\nu \overline{K}^2}{\overline{\varepsilon^p}} (c_\nu \approx 0.09) \quad \text{for } Re \gg 1. \tag{37}$$

Hence, the empirical value of c_ν is an asymptotic property of the NSE.

In analogy to the above dimensional considerations, one may alternatively introduce a turbulent length scale measuring the diameter of the largest eddies, the so-called mixing length ℓ to be attributed to Prandtl.[18] With his choice of a new constant c_P motivated by empirical observations, one then writes

$$\nu_t \sim c_P\, \ell \sqrt{\overline{K}} \quad (c_P := 0.55). \tag{38}$$

This is fully equivalent with Kolmogorov's similarity hypothesis[17] whereby $\hat{\nu}_t$ is expressed as product of a length and velocity typical of the turbulent motion, given by $\hat{\ell}$ and $\overline{\hat{K}}^{1/2}$, respectively. Finally, combining Eqs. (37) and (38) yields the famous (experimentally confirmed) Prandtl–Kolmogorov formula.[3,18]

$$\ell = \frac{c_\varepsilon \overline{K}^{\frac{3}{2}}}{\overline{\varepsilon^p}} \left(c_\varepsilon := \frac{c_\nu}{c_P} \approx 0.168 \right). \tag{39}$$

Notably, at this stage no assertions on the variations of the quantities ν_t, $\overline{K}$, $\overline{\varepsilon^p}$, ℓ with Re are made.

Most of the commonly adopted turbulence closures rely on the Boussinesq ansatz, or in combination with one of Eqs. (37)–(39) as outlined next.

2.4. *Categorisation of closures*

Available closures are (roughly) divided into the following different families. For their most salient and widespread members we refer to Refs. 7–9; for shear layer approximations according to Eqs. (13),

(14), (32) to Ref. 3. Here Eq. (16) reduces to the least-degenerate form

$$\overline{\rho}(\widetilde{u}\,\partial_x\widetilde{u}+\widetilde{v}\,\partial_y\widetilde{u})\sim-\partial_x\overline{p}-\overline{\rho}g_i+\partial_y\overline{\tau},\quad \partial_y\overline{p}\sim 0,\quad \overline{\tau}:=\overline{\tau}_{12}\sim\nu_t\,\partial_y\overline{u}. \tag{40}$$

2.4.1. *Incomplete closures*

Incomplete models rely on directly formulating algebraic expressions or ordinary differential equations (ODEs) with respect to x either for ν_t or ℓ.

Algebraic (zero-equation) closures. These simplest models only resort to the continuity and Reynolds equations, Eqs. (10) and (16) as here ν_t is ad hoc expressed in terms of $\overline{S_{ij}}$. Insofar, these are essentially nonlinear and *non-rational* models. This is so because such an approach does not necessarily comply with the constitutive-type relationship (37) but also as they predict a physically unacceptable, diffusive transition from the turbulent to the external, mainly irrotational flow. The most popular eddy-viscosity-based algebraic closures are the Cebeci–Smith and Baldwin–Lomax models.

The classical picture of turbulent BLs predicts a predominantly irrotational flow in their fully turbulent region with the weak vortical perturbations associated with the Reynolds shear stress (cf. Sec. 4.2.1). Irrotational mean-flow components in the i-th direction are described by $\partial_i\varphi$ with a steady scalar potential φ. As Eq. (10) then gives $\partial_{ii}\varphi=0$ for incompressible flow and the mean shear rates read $\overline{S_{ij}}=\partial_{ij}\varphi$, we have $\partial_j\overline{\tau}_{ji}\propto\partial_{ijj}\varphi=0$ in Eq. (16), in agreement with that flow structure. Indeed, its consistency with Eqs. (35), (36) is also guaranteed by the higher-order models below when exposed to the underlying asymptotic expansions.

One-equation closures. Equation (37) suggests to supplement the basic equations Eqs. (10), (16) with the transport equations (19), (25), (31) for $\overline{\tau}_{ij}$ and, as entering Eq. (37), the turbulent kinetic energy and dissipation, respectively. Transport equations for triple-correlations are neglected (in view of the quadratic nonlinearities of

the NSE). In the aforementioned equations, they together with correlations involving p' contribute to the turbulent diffusion terms and are modelled as such ad hoc in analogy to their viscous counterparts: in the closed transport equation for some passive scaler $\overline{\mathcal{Q}}$ (convected with $\overline{u}_i$), they become $\partial_i[(\nu_t/Pr_{\mathcal{Q}})\,\partial_i\overline{\mathcal{Q}}\,]$ with some turbulent Prandtl number $Pr_{\mathcal{Q}}$ forming a further empirical input, often assumed as constant. For instance, closing D^t in Eqs. (25), (28) in this manner by identifying $\mathcal{Q}$ with $\overline{K}$ gives

$$\overline{u}_j\partial_j\overline{K} = \overline{\tau}_{ij}\partial_j\overline{u}_i + \partial_i\left[\left(\frac{\nu_r+\nu_t}{Pr_{\overline{K}}}\right)\partial_i\overline{K}\right] - \frac{c_\varepsilon\overline{K}^{\frac{3}{2}}}{\ell} \tag{41}$$

for incompressible flow. One-equation closures only employ Eqs. (10), (16), and the modelled $\overline{K}$-equation (41) as ℓ is modelled independently.

The algebraic mixing-length closures, as widely employed for BL calculations, are, correctly speaking as demonstrated in Sec. 4.1, asymptotically reduced one-equation closures. These in addition adopt the $\widetilde{K}$-equation. The same holds for the refinements involving ODEs, which are accordingly often referred to as "one-half"-equation or "one-one-half"-equation closures. A popular, interesting member of this class is the Johnson–King "non-equilibrium" model, which seeks $\max\overline{\tau}(x)$ across the BL. To this end, in its fully turbulent main portion one expresses $\widetilde{K}$ as proportional to $\overline{\tau}$ in the $\widetilde{K}$-equation to obtain $\nu_t = \max(\overline{\tau}/\partial_y\overline{u})(x)$; cf. Eqs. (40) and (41).

2.4.2. *Complete closures*

Complete models also adopt a modelled form of the $\overline{\varepsilon^p}$-equation (31) or, equivalently, of the transport equation for some scalar

$$Z := \overline{K}^q\ell^r \propto \overline{K}^{q+\frac{3r}{2}}\,\overline{\varepsilon^p}^{\,-r}, \quad q,r\in\mathbb{Q} \quad \text{(otherwise largely arbitrary)} \tag{42}$$

or even the RSE, Eq. (19), as outlined above: no longer specific models for ν_t or ℓ are required.

Two-equation closures. One formally obtains the model equation for Z by means of the following "recipe":

(1) "multiply" Eq. (41) with $Z/\overline{K}$ while differential operators are ignored;
(2) replace the arising factor $\overline{\varepsilon^p}/\overline{K}$ by $\overline{K}/\nu_t$, according to Eq. (37), so as to model consistently the dissipative term as e.g., the last one in Eq. (31);
(3) insert proportionality constants and a further turbulent Prandtl number at the appropriate places.

This yields

$$\overline{u}_k\partial_k Z = C_{P,Z}\left(\frac{Z}{\overline{K}}\right)\overline{\tau}_{ij}\partial_j\left[\overline{u}_i + \partial_i\left(\frac{\nu_r+\nu_t}{Pr_Z}\right)\partial_i Z\right] - \frac{C_{\varepsilon,Z}\overline{K}}{\nu_t}, \tag{43}$$

containing the two model constants $C_{P,Z}$, $C_{Z,\varepsilon}$ representative for the arising production and dissipation terms and the Prandtl number Pr_Z to be modelled as accounting for the associated turbulent diffusivity. The most pervasive combinations of n and m, namely leading the commonly adopted two-equation closures, are given in Table 2. (In Rotta's ℓ-model ℓ is redefined as the "natural" length scale $\ell := \int_0^\infty \overline{u_i'(x_j,t)u_i'(x_j+\xi_j,t)}\,\mathrm{d}\xi_j/(2\overline{K})$.)

Three-equation or Reynolds-stress equation closures. Finally, the so-called three-equation closures even discard the Boussinesq ansatz in favour of supplementing Eqs. (25), (31) with the six scalar RSE, Eqs. (19)–(24), now truncated as $u_i' \equiv u_i''$, $\partial_i u_i' \equiv 0$:

$$-\overline{u}_k\partial_k\overline{\tau}_{ij} = \overline{\tau}_{ik}\partial_k\overline{u}_j + \overline{\tau}_{jk}\partial_k\overline{u}_i + S^p_{ij} + S^p_{ji} + D^t_{ij} + D^t_{ji} - \nu_r\,\partial_{kk}\overline{\tau}_{ij} - 2\overline{\varepsilon^p_{ij}}. \tag{44}$$

Table 2. Most popular two-equation models (see Eq. (42) for q, r).

q	r	Denotation of Z	$\varepsilon^p \propto$	Authors (see Refs. 3, 7–9)
3/2	−1	ε^p	Z	Chou, Jones, Launder
1/2	−1	ω	$Z\overline{K}$	Kolmogorov, Wilcox
1	−2	ω^2	$z^{1/2}\overline{K}$	Spalding
1	1	n/a	$Z^{-1}\overline{K}^{5/2}$	Rotta, Spalding
0 (1)	1	ℓ $(\overline{K}\ell)$	$Z^{-1}\overline{K}^{3/2}$ $(z^{-1}\overline{K}^{5/2})$	Rotta

Here the remaining unclosed terms S^p_{ij}, D^t_{ij}, $\overline{\varepsilon^p_{ij}}$ are consequently modelled in terms of $\overline{\tau}_{ij}$, $\overline{K}$, $\overline{\varepsilon^p}$ as follows. At first, we note that the pressure fluctuations give non-symmetric contributions to S^p_{ij}, but only symmetric tensors of 12 model parameters $c_p^{ij} = c_p^{ji}$ and $Pr_\tau^{ij} = Pr_\tau^{ji}$ are introduced in

$$S^p_{ij} + S^p_{ji} = -c_p^{(ij)} \overline{K}\, \overline{S_{ij}}, \quad D^t_{ij} + D^t_{ji} = -\partial_k \left[\left(\frac{\nu_t}{Pr_\tau^{(ij)}} \right) \partial_k \overline{\tau}_{ij} \right]. \quad (45)$$

A further tensor of six constants $c_\varepsilon^{ij} = c_\varepsilon^{ji}$ is needed to express

$$\overline{\varepsilon^p_{ij}} = \frac{\delta_{ij} \overline{\varepsilon^p} (1 - 2c_\varepsilon^{(ij)})}{3} - \frac{c_\varepsilon^{(ij)} \overline{\varepsilon^p}\, \overline{\tau}_{ij}}{\overline{K}}, \quad (46)$$

such that taking the trace of Eq. (44) now recovers twice Eq. (41).

Any potential benefit of avoiding the Boussinesq hypothesis in the RSE closure is definitely impaired by the myriad of adjustable model constants.

2.5. *Some critical aspects*

Some remarks shall be devoted to two popular extensions of the two-equation models. At first, models involving nonlinear extensions of the linear Boussinesq hypothesis in terms of an explicit dependence of ν_t on $\widetilde{S}_{ii}$ and $\widetilde{I}_2$ have gained awareness. The Menter two-equation or shear stress transport (SST) model pertains to this family as the currently probably most relevant representative. Here the BL approximation of a shear-dependence of ν_t shall improve modelling of the blending between the fully turbulent and the near-wall region. Secondly, there are specific features concerning modelling the terms that account for compressible effects. The Menter baseline (BSL) model is a quite popular member of the class of models coping with strongly compressible flows. Also, including compressibility in the eddy-viscosity closure has attracted attention. However, this has led to numerical instabilities and no definite enhancement over the standard BSL model been achieved so far (here the last word has not been spoken yet).[8] Both the SST and the BSL model have proven superior over the classical two-equation closures and highly successful for flows in a wide range of engineering applications, in particular

such undergoing gross separation. They are reviewed in Ref. 9 and still developed further.

Non-locality in also accounted for by considering correlations $\widetilde{\mathcal{Q}_1''\mathcal{Q}_2''}$ where $\mathcal{Q}_1''$, $\mathcal{Q}_2''$ are calculated at different positions x_i. However, establishing reasonable closures of the so arising two-point correlations has proven a much less viable task compared to modelling of the conventional one-point-correlations (which originate in the locality of the NSE).

Despite these more recent modelling activities, the popularity and undeniable success of the classical Boussinesq formulation, Eq. (35), for high-Re turbulence resorts to its theoretical foundation, revealed in Kolmogorov's, Prandtl's and coworkers' seminal work as forerunners. The often mentioned shortcoming of this ansatz when applied to even conventional BL flows but with sudden changes of external conditions, as addressed in the context of locality above, is critical at least locally where Eq. (35) predicts zero Reynolds shear stress at zero mean shear rate, i.e., for local maxima of $\overline{u}$. This is definitely in doubt as in conflict with experimental evidence. The same situation emerges at the onset of backflow by mean separation. However, since the associated failure of the *rationally* founded linear model is tied in with the local breakdown of the cascade of disjunct scales anticipated by Eqs. (13) and (14), attempts to establish *non-rational* nonlinear ones cope only insufficiently with it. We elucidate this next.

3. Comments on "Turbulence Asymptotics"

The central challenge in a fully rational (model-free or *ab initio*) treatment of high-Re turbulence lies in the simultaneous presence of all scales.

Let us consider Eqs. (17) and (18), which describe the fluctuations provided the mean field is given. Usual multiple-scales and homogenisation techniques are appropriate for the asymptotic treatment of several spatial/temporal scales as long as a hierarchy of problems can be established and the dependence of some quantity on the small scales entails a solvability condition regarding the

long-scale behaviour in a lower-order approximation. However, such an approach must fail in broadband turbulence as the coefficients in the associated equations resort in the background flow: dependences on global scales involve in all approximations the averaged dependence on all the smaller ones, which requires a sophisticated method of *beyond-all-order* asymptotics — which is not available.

The best one can do here is to resort to Eq. (13) at each level of approximation, expressing the equivalence of complete time-averaging and filtering all scales but the global ones. However, this is not so little. One just must be aware of a specific peculiarity: order of magnitudes might be reduced by averaging, so that averaged equations, containing all scales, are "fuller" than the individual ones that arise by expanding the NSE. An associated theoretical framework,[19] however, has not proven convincingly superior compared to the asymptotic concept pursued here and based on Def. 1, Prop. 1, Eqs. (13)–(15), (32), (33) and the scaling arguments concerning the fully turbulent part of BLs given in Sec. 4.2.1.

Now let us refine the ideas of exploiting the unsteady-flow scaling: Because of Eq. (32) and since $\sigma_{u_i'} = O(\gamma^{1/2})$ by Prop. 1 and $\overline{\varepsilon^p} = O(1)$ at the maximum in Eq. (25), the possibly smallest scales at play for most of the time are of $O(\sqrt{\nu_r \gamma})$. Since the presence of a wall and thus $\widetilde{u}_\tau$ in Eq. (33) provides a lower bound of the amplitudes $\sigma_{u_i'}$ associated with the smallest scales in the fully turbulent regime, there the viscous term in Eq. (18) of $O(\gamma^{-1/2})$ is comparatively small. These considerations hold also for $\varepsilon^p = o(1)$. Hence, the leading-order approximation of Eq. (18) just implies a Rayleigh stage governing those smallest scales carried by the mean flow,

$$\partial_t u_i' + \overline{u}_j \partial_j u_i' \sim -\partial_i p' \quad \left[= O\left(\sqrt{\frac{\gamma}{\nu_r}}\right)\right], \tag{47}$$

(compressibility has no significant effect here). Thus, self-sustained turbulence means that the unstable Rayleigh waves are damped on the associated longer scales. That is, for $\overline{u}_i$ taken as prescribed, a formal multiple-scales approach might deepen our insight into the dynamics of the fluctuations, at least for the aforementioned portions

of t where the averaged equations allow for identifying distinct scales. This ties in well with the (intensely debated) doubts on the validity of unsteady-BL theory[20] as severe small-scale mechanisms are potentially overlooked in a shear layer setting: the suggested "race" between modal instabilities (TS and Rayleigh stage) and inherently nonlinear ones leading to blow-ups (TS scale) looses its criticality for finite values of Re, damping those sufficiently. Filtering such small scales finally provokes an ill-posedness of the equations governing the longer-scale dynamics, but this does not render the overall asymptotic approach invalid. For a recent discussion see Braun & Scheichl[21] (triple-deck/TS scales), Cassel & Conlisk.[21] In case of developed turbulent flow, finally the impossibility manifests to determine the mean flow on the largest scales $[x = O(1),\ y = O(\overline{\delta})]$ in a finite number of steps, i.e., a hierarchical asymptotic concept.

The last statement agrees with another feature typical of turbulence:

Observation 1 (Non-interchangeable limits). A solution of Eqs. (1), (2) (and appropriate ICs/BCs) does in general *not* converge to that of the corresponding Euler equations (and the identical ICs/BCs) as $Re \to \infty$.

In fact, rather the truncated Euler equations (47) hold. A prominent candidate for an exception are classically scaled (firmly attached) turbulent BLs in the fully turbulent main portion of which the steady, imposed potential flow predominates (Sec. 4.2.1). As Ω_{ij} is dual to the vorticity, one finds $\overline{\varpi'^2} \equiv 2\,\overline{\Omega'_{ij}\Omega'_{ij}} \equiv \overline{(\partial_i u'_j)(\partial_i u'_j)} - \overline{(\partial_i u'_j)(\partial_j u'_i)}$. The last term equals $-\partial_{ij}\overline{u'_i u'_j}$ with the aid of Eq. (17), thus it is negligibly small by Eq. (13). By these expressions, the Helmholtz's vortex theorem not only prevents the generation of vorticity in an inviscid flow but any small-scale fluctuations. This simple but nonetheless outstanding finding corroborates the splitting into two mainly inviscid flow regions initiating scale separation: an external, predominantly irrotational and fluctuation-free one, reigned by the full Euler equations, and a turbulent shear layer, characterised by Eqs. (5), (47) and crucially Re-dependent small scales.

4. The Asymptotic Framework of Turbulent Shear Flows

Before skipping down to a topical insight into turbulent BLs, we envisage free shear layers. These are easier to deal with as they lack wall binding.

4.1. *Free slender shear flows reappraised*

For the original asymptotic analysis of free shear layers (and the associated near-field close to a nozzle, etc.), solely resorting to (14) and the empirical finding (5), we refer to Ref. 22, for a review cf. Ref. 3.

Due to the absence of wall binding, the viscous term in Eq. (16) is of subordinate importance across the whole shear layer. In turn, ultimately developed turbulence means Re-independence of $\overline{\tau}$ as $Re \to \infty$ and Eq. (40) retained in full (apart from the body-force term). We are therefore concerned with an ad hoc scaling $\overline{\delta} = O(\alpha)$, $\overline{\tau} = O(\alpha)$ where the small but finite so-called slenderness parameter α represents the principal perturbation parameter aside from Re. Our earlier analysis gives $\overline{K} = O(\alpha)$, $\sigma_{u_i'} = O(\alpha^{1/2})$, hence γ is identified with α in the main portion of the shear layer according to Prop. 1. We then have $P \sim \overline{\tau}\partial_y\overline{u} = O(1)$ in Eq. (25), and the only candidate to enter the so reduced $\overline{K}$-equation apart from $\overline{\varepsilon^p}$ is the dominant approximation of D^t, here $-\partial_y\overline{v'p'}$, see Eq. (28). As this quantity drops out by integration across the shear layer, $\varepsilon^p = O(1)$ is confirmed and the bracketed scaling in Eq. (47) applies. Moreover, Eqs. (1), (2) give $-(\partial_i u_j)(\partial_j u_i) \equiv \Omega_{ij}\Omega_{ij} - S_{ij}S_{ij} = \partial_{ii}p$, posing a "Poisson problem" for p' to leading order subject to homogeneous boundary conditions imposed at the edges of the turbulent region. This yields $p' = O(\alpha)$, $D^t \sim -\partial_y\overline{v'(K+p')} = O(\alpha^{1/2})$ rather than $p' = O(\alpha^{1/2})$, $D^t = O(1)$, which reduces Eq. (25) finally to the balance

$$\overline{\tau}\,\partial_y\overline{u} \sim \overline{\varepsilon^p}\ \ [= O(1)]. \tag{48}$$

Townsend coined the notion *structural equilibrium*[2] for shear flows where Eq. (48) applies. From the asymptotic viewpoint, it only holds

for a large velocity deficit with respect to the non-turbulent region, i.e., for $\partial_y \overline{u} = O(1/\overline{\delta})$, although frequently also assigned to the overlap between the fully turbulent and the near-wall region in BLs.[3] However, diffusion is at play both in the latter region, where Eq. (25) stays fully intact, and in the particular layer on its top in a multi-tiered BL.

Noticing the shear layer approximation $\tau \sim \nu_t\, \partial_y \overline{u}$ of Eq. (35) and substituting Eqs. (38), (39) into Eq. (48) reveals the mixing-length closure

$$\nu_t \sim (c_\ell \ell)^2 |\partial_y \overline{u}|, \quad \overline{\tau} \sim (c_\ell \ell)^2 \partial_y \overline{u} |\partial_y \overline{u}| \quad \left(c_\ell := \frac{c_P}{c_\nu^{1/4}} \approx 1.004 \right). \tag{49}$$

Considering the exchange of turbulent momentum across mean shear on a "mixing" length scale ℓ yields this expression for ν_t independently.[18] Hence, algebraic mixing-length closures for $\overline{\tau}$ are seen as asymptotically correct one-equation closures. This underpins their strength and the recommendation to discard typical eddy-viscosity closures in favour of suitably modelling an $O(1)$-function l as $\ell/\overline{\delta}(x) \sim \alpha^{1/2}\, l(\eta)$ with $\eta := y/\overline{\delta}$. One typically finds

$$\alpha \approx 0.1 \quad \text{(free shear layers)}, \quad c_l := \alpha^{\frac{1}{2}} \approx 0.085 \quad \text{(BLs)}. \tag{50}$$

This completes the empirical input by proposing fixed real-world values α should take on. We remark that all commonly used algebraic mixing-length closures employ a value of c_l close to that in Eq. (50), but the most popular ad hoc (diffusive) eddy-viscosity-based algebraic Cebeci–Smith and Baldwin–Lomax models propose $\alpha = 0.0168$.[3] As a most salient result, we find that $\overline{\delta} \gg \ell = O(\alpha^{3/2})$ and $\nu_t = O(\alpha^2)$, but this is only compatible with Eqs. (36)–(38) if the constants therein are taken as of $O(1)$. However, the value of c_ν is reliable for BLs but definitely larger for free shear flows, so the numbers in Eq. (50) can be confidently viewed as small.

Assuming $l(1) > 0$ predicts a physically reliable abrupt edge of the shear layer and implies $\overline{\tau} = \partial_Y \overline{\tau} = \partial_Y \overline{u} = 0$ there and thus continuously differentiable flow quantities there. This allows for a sufficiently smooth patching with the external flow. This obviates the need to consider a passive overlayer (of width $\alpha^{3/2}$ as there convection and

diffusion are retained in Eq. (25)) that accounts for intermittency.[22] The latter can be taken care of properly though by multiplying the formula for ν_t in Eq. (49) with well-known Klebanoff's empirical intermittency probability function $I_K(\eta) := 1/(1 + 5.5\,\eta^6)$,[3] which improves the computation of the flow near the shear-layer edge and lessens the otherwise often over-predicted values of $\overline{\tau}$ to more realistic ones.

As $\overline{\tau}$ changes sign in a free shear layer, let this take place at $\eta = 0$ (axis of free jet, see Fig. 1(b), or dividing streamline in a mixing layer). There also P vanishes, $\overline{\tau}$ varies linearly with η, and $\overline{u}$ must be regular. As ℓ in Eq. (49) would behave singularly there, an inner region is necessitated where also D^t enters Eq. (48). From Eqs. (28), (38) one infers that $\overline{\varepsilon^p} = O(\alpha^{3/2}/\ell)$ and $D^t = O(\alpha^{3/2}/y)$, so that the inner layer emerges for $\eta = O(\ell/\alpha)$ or $y = O(\overline{\delta}_i)$, say. There $\overline{\tau} = O(\overline{\delta}_i)$ implies $\partial_y \overline{u} = O\big(\overline{\delta}_i^{\,-1/2}\big)$. As $\partial_y \overline{u}$ must vary with η^a where a is some constant, say, in the overlap conjoining both layers, there l must vary with $\eta^{3/2-a}$ sub-linearly ($a > 1/2$); see Eq. (49). The $\overline{K}$-equation does not provide any further information here. However, Prandtl's original mixing-length concept outlined above Eq. (38) is of avail here, which then does not change its order of magnitude in a free shear flow as a wall is absent: $r = 0$, i.e., $\ell = O(\alpha^{3/2})$ in the entire layer, giving $\overline{\delta}_i = O(\alpha^{3/2})$ and $\gamma = O(\alpha^{3/4})$ in its inner part: Fig. 1(b), Table 3.

It is physically evident that the freely moving large eddies determine the inner length scale, entering the mean-flow description where Eq. (48) degenerates, e.g., due to symmetry of a jet flow with respect to its centreline. There $\overline{u}$ exhibits a cuspidal singularity (perturbing the finite centreline speed), so that the $\overline{u}$-variations

Table 3. Scaling of $O(1)$-deficit shear flow summarised (Landau symbols omitted).

	Thickness	$\boldsymbol{\ell}$	$\boldsymbol{\nu_t}$	$\boldsymbol{\tau_{ij}}$, $\boldsymbol{\overline{K}}$	$\boldsymbol{P}$, $\boldsymbol{\overline{\varepsilon^p}}$	$\boldsymbol{D^t}$	$\boldsymbol{Pr_{\overline{K}}}$
outer (main) layer:	α	$\alpha^{3/2}$	α^2	α	1	$\alpha^{1/2}$	$\alpha^{3/2}$
inner (diffusive) layer:	$\alpha^{3/2}$	$\alpha^{3/2}$	$\alpha^{9/4}$	$\alpha^{3/2}$	$\alpha^{3/4}$	$\alpha^{3/4}$	1

are of $O(\overline{\delta}_i^{1/2})$ in the inner layer. There turbulent diffusion accounts for the aforementioned regular behaviour of all flow quantities and Eqs. (36)–(38) hold as well; for modelling aspects see Ref. 22. It is noted that α is also a measure for the (Re-independent) entrainment of the flow and $Pr_{\overline{K}}$ must be taken as small in the main layer.

Incompressibility of the shear flow is effectively allowed by its slenderness to the asymptotic accuracy considered. By introducing a streamfunction ψ to satisfy Eq. (10), we then write for both free layers and BLs

$$[\overline{u}, \overline{v}] = [\partial_y \psi, -\partial_x \psi], \quad [\psi, \overline{\tau}] = [u_r(x)\, \overline{\delta}(x) f(x,\eta),\, u_r(x)^2 s(x,\eta)]. \tag{51}$$

Herein the reference speed u_r of $O(1)$ is either that along the centre-line (jet, pressure-free) or that imposed by an external potential flow (wake, mixing layer, BL), in the latter case subsequently denoted as u_e. Expanding

$$[f, s] \sim [F, \alpha S] + \cdots, \quad \overline{\delta} \sim \alpha\, \Delta(x) + \cdots, \tag{52}$$

yields the leading-order shear-layer approximation of Eq. (16) for free flows:

$$(\mathrm{d}_x \ln u_r)(F'^2 - I) - [\mathrm{d}_x \ln(u_r \Delta)] F F'' + F' \partial_x F' - F'' \partial_x F = \frac{S'}{\Delta}. \tag{53}$$

Here and hereafter in this context, d_x stands for x-derivatives and primes for η-derivatives. The indicator function I equals 1 if $u_r = u_e$ and 0 otherwise. Matching the main with the inner layer and the external flow raises the BCs $F(0) = S(0) = 0$ and $F'(1) = 0$ $(I = 0)$ or $F'(1) = 1$ $(I = 1)$, $S(1) = 0$, respectively; the first can also be accepted from a pure kinematic point of view, cf. Eq. (3). These and a suitable model for l, see Eq. (49), supplement Eq. (53) to a parabolic eigenvalue problem with eigenvalues $\Delta(x)$ in terms of a non-trivial (vorticity-affected) solution $F' \not\equiv 1$, $S \not\equiv 0$.

The specification of the arising problem governing the flow in the main layer for $I = 1$ is straightforward. Free flows are self-preserving $(\partial_x F' \equiv \partial_x S \equiv 0)$ if $u_r(x)$ satisfies a power law $u_r \propto x^m$ with some constant exponent m, giving $\Delta \propto dx$ with $d\ (> 0)$ being an eigenvalue. This holds in any case for $I = 0$ due to matching the flow quantities with those of the transitional flow as $x \to 0_+$. Here the most

interesting case is the free (here planar) jet, cf. Fig. 1(b), described by the classical Schlichting problem[3]: $mF'^2 - (m+1)FF'' = S'/d$ subject to the homogeneous BCs, integration between $\eta = 0$ and 1 gives $m = -1/2$ for the second eigenvalue m governing the centre-line speed u_r and expressing conservation of axial momentum. As the solution F does not induce an external flow, higher-order terms in Eq. (52) originate in the flow history for $x \ll 1$. The validity of its self-similar leading-order structure for $x \gg 1$ and the associated variation of the momentum by entrainment are still under debate.

4.2. *Turbulent boundary layers: some exciting novelties*

At first, the asymptotic structure of (initially attached) turbulent BLs allows for a powerful categorisation in terms of the magnitude of the streamwise velocity defect related to the imposed surface speed $u_e(x)$ in the fully turbulent region[14]: $\Delta_u := 1 - \widetilde{u}(x,y)/u_e(x)$. Secondly, distinguishing between *slightly underdeveloped* and fully developed turbulence yields substantial analytical progress when it comes to the challenging problems of turbulent marginal and massive separation.

4.2.1. *Classification*

Let the fundamental velocity scale $\widetilde{u}_\tau$ serve as a reference scale. Then the following statements about the internal BL structure can be made.

Observation 2 (BL scaling). In the limit $Re \to \infty$, each sublayer in a turbulent BL is characterised by an intrinsic velocity scale, u_t with $u_t \to 0$, that governs the turbulent dynamics such that $\sigma_{u_i''} = O(u_t)$.

This follows also intuitively from Eq. (14) and Prop. 1. Hence, u_t is also a local measure for Δ_u *and* u_t^2 for $\overline{\tau}$. Consider a sublayer of wall-normal extent δ_s, say. Since the shear-layer approximation of Eq. (16) gives $\overline{\rho}\,\widetilde{u}\,\partial_x\widetilde{u} + \cdots \sim \partial_y\overline{\tau}$, we have $u_e u_t/x \sim u_t^2/\delta_y$ by order-of-magnitude analysis. For x properly scaled such that $u_e = O(1)$,

this implies $\delta_s = O(u_t)$ also and a time-mean vorticity of $O(i)$ as this is approximated by $u_e \partial_y \Delta_u$. Reverting to Eq. (33), we find the total shear stress as nearly constant across the near-wall sublayer by matching it with $\overline{\tau}$ in the small-deficit layer located on top of the latter. Hence, u_t is identified with $\widetilde{u}_\tau$ there, and matching $y\,\partial_y \widetilde{u}$ confirms the famous, widely accepted logarithmic law of the wall: $u^+ \sim \kappa^{-1} \ln y+ + O(1)$ $(y^+ \to \infty)$; κ is the von Kármán constant.

In turn, the well-known skin-friction law $\widetilde{u}_\tau \sim \kappa / \ln Re$ as $Re \to \infty$ arises from matching $\widetilde{u}$, so that the sublayer scalings, Eq. (33), are completed by

$$\overline{\delta} = O\left(\frac{1}{\ln Re}\right), \quad \overline{\delta}_\nu = O\left(\frac{\ln Re}{Re}\right). \tag{54}$$

This finally gives the smallest scales determined by $\widetilde{u}_\tau$ and $\overline{\delta}_\nu$.[15] One then finds that those coincide with the celebrated Kolmogorov scales, whereas these cannot be identified in the fully turbulent regime. (They merely follow from Kolmogorov's first similarity hypothesis.[17])

These findings allow for the first classification of turbulent BLs:

(I) *Classical two-tiered*: $\Delta_u = O(\widetilde{u}_\tau)$, single velocity scale $\widetilde{u}_\tau$;
(II) *Three-tiered*: $\Delta_u = O(u_t)$, second turbulent velocity scale $u_t := \widetilde{u}_\tau^{2/3}$;
(III) *Four-tiered*: $\Delta_u \ll 1$, second velocity scale u_t of $O(\Delta_u)$;
(IV) *Four-tiered*: $\Delta_u = O(1)$, further velocity scales u_t of $O(1)$.

The situations (I), (II) are both associated with so-called small-deficit BLs and allow for a rigorous asymptotic analysis in the limit $Re \to \infty$. The main-layer equations follow from expanding Eq. (53) with $s = O(\Delta_u^2)$.

The occurrence of an $O(1)$-velocity deficit as proposed by (IV) can only be understood if the slenderness of the BL is also measured by a slenderness parameter α taken as small, as for free shear flows. The rationale of Sec. 4.1 applies also here, and a justification is again by experimental evidence and the closure constants, Eq. (49). But then the structure of a wake with $\Delta_u = O(1)$ applies to such a BL as far as the largest part of the fully turbulent regime is concerned. Hence, this wake region is two-tiered and gives rise to a smaller velocity scale of $O(\alpha^{3/4})$. Here an appealing physical interpretation is possible:

the presence of the wall is felt by the largest eddies in that wake region, increasingly with shrinking of the y-scale but just down to their diameter of $O(\alpha^{3/2})$ where their motion is blocked. The further near-wall region then exhibits a small-defect structure as applying in case (I): $\widetilde{u}_\tau$ is the typical velocity scale but the deficit now around the $O(1)$-surface slip exerted by the wake flow. Thus, the BL is four-tiered: inner and outer layers, defect layer, viscous sublayer.

The "intermediate" category (III) then provides the missing link between (II) and (IV): in (IV), a genuine two-perturbation analysis based on α and Re is adopted following (III), where we the first time refrain from considering the BL scaling strictly in the limit $Re \to \infty$ but u_t just as small.

The level of turbulence intensity in a shear layer is advantageously epitomised by the so-called turbulence level parameter T_u,[3] here specified as an actual reference value of $\overline{K}$ related to that which would apply if turbulence was fully developed. One then characterises the BL flow as

(a) *Fully developed*: $T_u = 1$ (as considered so far);
(b) *Slightly underdeveloped*: $T_u \to 0$ as $Re \to \infty$.

This concept put forward by Neish & Smith[23] is of paramount importance when it comes to breakaway separation of the BL.

4.2.2. *Addressing turbulent separation: where are we now?*

There are two types of mean-flow separation from a perfectly smooth surface that can be described in a self-consistent manner. At first, a suitably controlled smooth adverse pressure gradient imposed on the BL by a fully attached external flow leads to so-called marginal or BL-internal separation, characterised by closed zones of reverse flow. Secondly, gross or breakaway separation means massively separated flow. As a well-known result of potential flow-theory, here the imposed pressure varies proportionally to $(x_s - x)^{1/2}$ immediately upstream of the separation point, $(x, y) = (x_s, 0)$, and regularly downstream of it.

Small-deficit BLs are fully insensitive to smooth adverse pressure gradients in terms of their tendency to separate. That is, they remain

firmly attached as their fully turbulent main portion is in fact a predominantly inviscid, irrotational one. Hence, a description of that type of separation, which more precisely means states of marginal or BL-internal separation, has to consider an initially attached four-tiered large-deficit BL.[14] However, underdeveloped small-deficit BLs appear quite naturally as a result of laminar–turbulent transition close to a stagnation point of the external potential flow on a surface More precisely, the local viscous-inviscid interaction process regularising the above singularity at $x = x_s$ fixes the dependence $T_u(Re)$ as $\overline{\delta}_\nu/\overline{\delta}$ must vary predominantly algebraically with Re rather than exponentially as in Eq. (54). We are thus led to the following possibilities:

(A) $T_u \ll 1$: case (I) above, applies to massive separation *only*;
(B) $T_u = O(1)$: case (IV) above, applies to marginal separation.

Concerning (A), the line of research is initiated by Ref. 23 and its status quo covered by Scheichl.[21]

Let us complete this survey by focussing on the largely unappreciated case (B) applied to massive separation. This proves physically attractive since it predicts fully developed turbulence already upstream of separation, expected to take place (sufficiently far) downstream of it. Scenario (A), however, then has to imply a kind of *secondary* or "turbulent-turbulent" transition towards this ultimate state, but such a mechanism has not been detected so far theoretically in the high-Re limit, neither for attached nor separated BLs. Furthermore, the Re-independent eddy viscosity in case (B) predicts separated shear layers belonging to the class of massive ones addressed in Sec. 4.1 but with the difference that these separate the free-stream flow from the weakly reversing/recirculating flow in the open/closed eddy emerging due to large-scale separation. They are pressure-driven and "carry the frozen state" of their near-wall structure at separation near their boundary with the latter flow region. Self-consistent ideas on the structure of the flow past an obstacle on the global scale, for both cases (A) and (B), are preliminary yet.

In striking contrast to laminar BLs, where ICs are well-defined, e.g., by the existence of a stagnation point and give rise to a well-posed parabolic problem, the situation for turbulent ones is more

awkward as correct ICs have to be found by matching with the time-mean transitional flow. For the (2D) flow past an obstacle, the region of transition shrinks to a point, coinciding with the front stagnation point $x = y = 0$, say, as $Re \to \infty$. Hence, first the singular structure of Eq. (53) for $x \to 0_+$ has to be elucidated, implied by $u_e \sim cx + O(x^2)$ in this limit with some constant $c > 0$. We accordingly expand $[F, S] \sim [F_0, S_0](\eta) - [\gamma_F(x)F_1(\eta), \gamma_S(x)S_1(\eta)] + \cdots$ with $(\gamma_F, \gamma_S) \to (0_+, 0_+)$. Providing a flavour of the method of balancing at first largely unknown gauge functions (γ_1, Δ), as typically adopted in such situations, on the basis of this particular important case proves quite instructive.

Order-of-magnitude analysis of Eq. (53) shows that its left-hand balances its right-hand side in leading order if $\Delta \sim dx$ $(x \to 0_+)$:

$$F_0'^2 - 1 - 2F_0F_0'' = \frac{S_0'}{d}, \quad F_0(0) = S_0(0) = 0, \;\; F_0'(1) - 1 = S_0(1) = 0. \tag{55}$$

We additionally have in mind that $S_0 = l^2 F_0''|F_0''|$ and require forward flow: $F_0' > 0$ $(F_0 \geq 0)$. There are three scenarios: $F'(0) > 1$, $F'(0) < 1$, $F'(0) = 1$. The first/second refers to an overshooting-jet-/wake-type velocity profile with $S > 0/S < 0$ and $F'' > 0/F'' < 0$ for sufficiently small values of η. This means at least one local maximum/minimum $F_0'(\eta^*) > / < 1$ for some $\eta^* \in]0, 1[$ where F_0'', S_0 change sign. But then S_0 exhibits a local maximum/minimum for some $0 < \eta < \eta^*$, implying $S'(\eta^*) < / > 0$, contradicting Eq. (55) (supported by attempts to solve this problem numerically).

We hence have to accept only the third possibility of the trivial (potential-flow) solution $F_0 \equiv \eta$, $S_0 \equiv 0$, associated with a velocity deficit of $O(\gamma_F)$ and $\gamma_S = \gamma_F^2$. By integration with respect to η, one finally verifies

$$c\eta F_1' - [c + \Delta\,(x\gamma_1)^{-2}\mathrm{d}_x(x^2\gamma_1)]F_1 \sim S_1, \;\; c := \lim_{x\to 0_+}(x\gamma_1)^{-1}\mathrm{d}_x(x\Delta), \tag{56}$$

and $F_1 = S_1 = 0$ $(\eta = 0)$, $F_1' = S_1 = 0$ $(\eta = 1)$. Therefore, any non-trivial form of F_1, S_1 requires both c to be positive and the bracketed term in Eq. (56) to vanish. However, since the second added therein is non-negative, we again face a contradiction: the problem posed by Eq. (53) cannot be solved with ICs provided for arbitrarily small

values of x. Hence, current activities focus on spontaneous secondary transition by a loss of parabolicity of the small-deficit equations referring to case (A).

5. Some Exercises

(1) Derive Eq. (16) from Eqs. (1) and (2).
(2) Derive Eqs. (25) and (31) from Eqs. (17) and (18).
(3) Outline the main ideas underlying the Re-independent time-mean scalings of turbulent shear layers having a large velocity deficit.
(4) Large-deficit BL immediately upstream of separation: show that the surface slip $u_e(x)F'(x,1)$ vanishes like $(-X)^{1/4}$ as $X := x - x_s \to 0_-$ for $u_e(x_s) > 0$ and $u_e(x) - u_e(x_s)$ vanishing like $(-X)^{1/2}$. Hint: consider the essential sublayer where $Y = O[(-X)^{1/3}]$ and all terms of Eq. (53) are retained, then show by matching that the solution proceeds downstream uniquely (*advanced*).
(5) Show that the initial-value problem for the BL posed by Eq. (53), $[F,T](x,0) = [0,0]$, $F'(x,1) - 1 = T(x,1) = 0$ and appropriate initial conditions for F, T, Δ prescribed for $x = 0$ is well-posed (strictly parabolic). Hint: here the essential sublayer is given by $Y = O(x^{1/3})$ as $x \to 0_+$ (*advanced*).

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