A half-normal limit distribution scheme

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1 Limit theorem

Let $c(x) = \sum_n c_n x^n$ be a generating function with non-negative coefficients, and $c(x,u) = \sum c_{nk} x^n u^k$ be the bivariate generating function (BGF) where a parameter of interest has been marked, i.e. $c(x,1) = c(x)$. We define a sequence of random variables $X_n, n \geq 1$ by

$$\mathbb{P}[X_n = k] = \frac{c_{nk}}{c_n} = \frac{[x^n u^k]c(x,u)}{[x^n]c(x,1)}.$$

Our goal is to identify the limit distribution of X_n when n tends to infinity.

In [2, Theorems 1-3] Drmota and Soria show that the limit distribution of X_n is either Gaussian, Rayleigh or a convolution of both under certain conditions and proper rescaling. We extend these results to conditions implying a half-normal distribution.

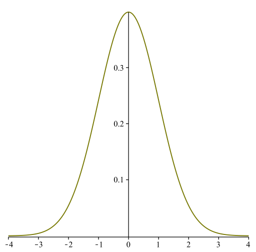

Figure 1: Normal: $\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), x \in \mathbb{R}$

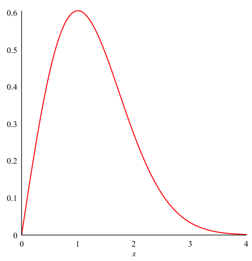

Figure 2: Rayleigh: $x \exp\left(-\frac{x^2}{2}\right), x \geq 0$

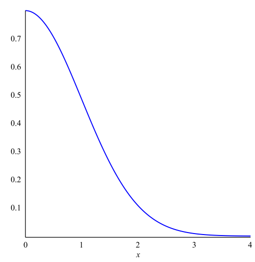

Figure 3: Half-normal: $\sqrt{\frac{2}{\pi}} \exp\left(-\frac{x^2}{2}\right), x \geq 0$

The technical conditions are given by Hypothesis [H] from [2]. We define Hypothesis [H'] as [H] except that $h(\rho,1) > 0$ is dropped. The most important condition is an algebraic singularity of the square-root type:

$$\frac{1}{c(x,u)} = g(x,u) + h(x,u)\sqrt{1 - \frac{x}{\rho(u)}},$$

for $|u-1| < \varepsilon$ and $|x - \rho(u)| < \varepsilon$, $\arg(x - \rho(u)) \neq 0$, where $\varepsilon > 0$ is some fixed real number, and $g(x,u), h(x,u)$, and $\rho(u)$ are analytic functions.

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Theorem 1. *Let $c(x,u)$ be a BGF satisfying [H']. If $\rho(u) = \rho = const$ for $|u-1| < \varepsilon$, $g_x(\rho,1) \neq 0$, $h_u(\rho,1) \neq 0$, and $h(\rho,1) = g_u(\rho,1) = g_{uu}(\rho,1) = 0$, then the sequence of random variables X_n has a half-normal limit distribution, i.e.*

$$\frac{X_n}{\sqrt{n}} \xrightarrow{d} \mathcal{H}(\sigma),$$

where $\sigma = \sqrt{2}\frac{h_u(\rho,1)}{\rho g_x(\rho,1)}$, and $\mathcal{H}(\sigma)$ has density $\frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$ for $x \geq 0$.

Note that we can also derive the asymptotic forms of the moments, and a strong limit theorem.

2 Applications to lattice paths

The motivation of this work arose in the study of a new lattice path model: the reflection-absorption model [1]. In the case of the absorption model for the final altitude of meanders for drift 0 the above theorem shows the appearance of a half-normal distribution.

However, a variant of Theorem 1 also applies to other parameters of lattice paths:

- Returns to zero of simple aperiodic walks

  The step set of such walks is given by $\{(1,s_1),\ldots,(1,s_k)\}$ with $\gcd(s_2-s_1,\ldots,s_k-s_1) = 1$. A return to zero is a point of altitude 0 after the starting point.

- Sign changes of weighted Motzkin walks

  The step set of such walks is given by $\{(1,-1),(1,0),(1,1)\}$. It changes sign if it moves from strictly above the x-axis to strictly below, or vice versa.

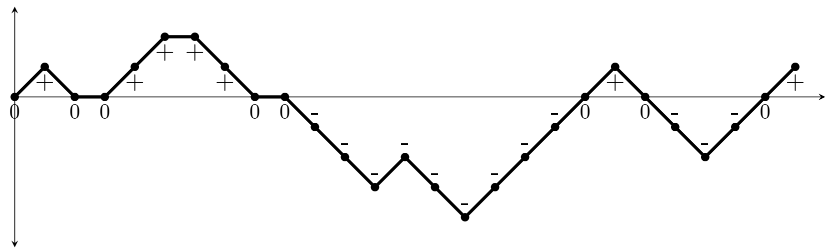

Figure 4: Motzkin walk with 7 returns to zero and 4 sign changes

References

[1] Cyril Banderier and Michael Wallner. Some reflections on directed lattice paths. *Prob., Comb. and Asymp. Methods for the Analysis of Algorithms*, page 25, 2014.

[2] Michael Drmota and Michèle Soria. Images and preimages in random mappings. *SIAM J. Discrete Math.*, 10(2):246–269, 1997.