# Formulating the perfectly matched layer as a control optimization problem

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Abstract

An automated approach to emulate the absorbing properties of a perfectly matched layer (PML) in wave equations is presented. Instead of applying the coordinate stretching to obtain a modified PML wave equation, a feedback boundary controller is parameterized. The set of unknown control parameters is obtained through genetic optimization by minimizing the error between the wave equation with additional feedback controller and the desired damped fundamental solution at certain frequency pairs. With this approach the time-consuming task of constructing a PML, especially for complex wave-like equations like the moving Euler-Bernoulli beam, is automated and it leads to an easy-to-implement and computationally efficient alternative.

*Keywords: Absorbing boundary conditions, Genetic optimization, Euler-Bernoulli beam*

## 1. Introduction

In many applications where an unbounded solution of a wave-like equation is desired, the problem occurs that due to limited computational capabilities the domain has to be truncated at some point. To let the solution of this confined domain approximate the free-wave propagation, boundary conditions with absorbing properties have to be applied. The work by Engquist and Majda [1] addressed this issue and absorbing boundary conditions (ABCs) were derived which worked well under certain circumstances. The technique to surround the computational domain with a perfectly matched layer was first described by Berenger [2] for the absorption of electromagnetic waves. The idea of the perfectly matched layer was later extended and applied to other wave propagation problems, both in a split or un-split field formulation [3] [4].
The key idea of the perfectly matched layer is that if the fundamental solution of a wave equation is evaluated along a complex coordinate an additional damping is gained. This can be easily shown by investigating the one-dimensional wave equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} = c^2 \frac{\partial^2 w(x,t)}{\partial x^2} \tag{1}$$

with its fundamental solution

$$w(x,t) = e^{i\omega_x x} e^{i\omega_t t} \tag{2}$$

where $\omega_x$ is the so called wavenumber or spatial frequency and $\omega_t$ the angular frequency. If this fundamental solution is evaluated along a contour that is stretched into the complex plane $\tilde{x} = x + if(x)$, Eq. (2) can be rewritten as

$$w(\tilde{x},t) = e^{i\omega_x (x+if(x))} e^{i\omega_t t} = e^{-\omega_x f(x)} \underbrace{e^{i\omega_x x} e^{i\omega_t t}}_{w(x,t)}. \tag{3}$$

Note, that when $f(x)$ is zero the original fundamental solution is obtained, whereas if $f(x) > 0$ an exponential decay is added. The wave equation with respect to its complex coordinate is then transformed back to its real-valued coordinate using

$$\partial\tilde{x} = \big(1 + i\frac{df(x)}{dx}\big)\partial x \longrightarrow \frac{\partial}{\partial\tilde{x}} = \frac{\partial}{\big(1 + i\frac{df(x)}{dx}\big)\partial x} \tag{4}$$

This transformation is a tedious task especially when spatial derivatives of higher orders are involved as, for example, in the Euler-Bernoulli beam equation and usually involves using several auxiliary variables which increases the computational effort.

## 2. PML as a control optimization problem

For demonstration purposes, again the scalar wave equation (1) is considered for deriving the PML as a control optimization problem. As it is shown later, this method can easily be adapted for controlling different, more complex, wave-like equations such as the Euler-Bernoulli beam equation. Discretizing the scalar wave equation using central finite difference approximations on a uniform grid results in

$$w_n^{j+1} - 2w_n^j + w_n^{j-1} = c^2 \frac{\Delta t^2}{\Delta x^2}(w_{n+1}^j - 2w_n^j + w_{n-1}^j) \tag{5}$$

which can be aggregated for every node into a discrete state-space system

$$\boldsymbol{x}^{j+1} = \boldsymbol{A}\boldsymbol{x}^j \tag{6}$$

where $\boldsymbol{x}^j = [\boldsymbol{w}^j, \boldsymbol{w}^{j-1}]^{\mathrm{T}}$ is the solution vector at the discrete time $j\Delta t$. The fundamental solution becomes

$$w_n^j = e^{i\omega_x \Delta x n} e^{i\omega_t \Delta t j}. \tag{7}$$

Inserting (7) into the discretized wave equation (5) results in the so-called dispersion relation which expresses the dependency between $\omega_x$ and $\omega_t$. There exist infinitely many $\{\omega_x, \omega_t\}$-pairs but the magnitudes of $\omega_x\Delta x$ and $\omega_t\Delta t$ can be confined between $[-\pi, +\pi]$. Higher magnitudes can not be resolved by the grid. To control the system so that it has a reflection-less exponential decay of the solution inside a layer surrounding the computational domain, a state feedback controller is added.

$$\boldsymbol{x}^{j+1} = \boldsymbol{A}\boldsymbol{x}^j + \boldsymbol{B}\boldsymbol{K}\boldsymbol{x}^j \tag{8}$$

where the control matrix $\boldsymbol{K}$ is defined to have diagonal substructures of the form

$$\boldsymbol{K} = \begin{pmatrix} k_1 & & & k_{p+1} & & \\ & k_2 & & & k_{p+2} & \\ & & \ddots & & & \ddots \\ & & k_p & & & k_{2p} \end{pmatrix} \tag{9}$$

where $p$ is the number of nodes that are inside the damping layer. The input matrix $\boldsymbol{B}$ distributes the control input to the corresponding nodes of the damping layer.
The desired behavior can be analytically given for a single fundamental wave, e.g. a single frequency pair $\{\omega_x, \omega_t\}$, by evaluating the discrete form of (3). For $f(x)$, a function that is zero inside the computational domain, increasing with second or third order within the damping layer and continuous at the interface is preferable. Let this desired fundamental solution be denoted as $\boldsymbol{w}^j_{\text{fund}}(\omega_x, \omega_t, j)$ where $\{\omega_x, \omega_t\} \subset \Omega$ is a certain frequency pair and $\Omega$ a set containing a finite number of pairs.
Eq. (6) is initialized with $\boldsymbol{x}^1 = [\boldsymbol{w}^1_{\text{fund}}, \boldsymbol{w}^0_{\text{fund}}]$ and continued for certain amount of time steps $j_{max}$. The error between the fundamental solution and the controlled state space system is aggregated over time and the frequency set $\Omega$ to form the objective function

$$J(\boldsymbol{K}) = \sum_{\Omega} \sum_{j=2}^{j_{\max}} \left| \boldsymbol{w}^j_{\text{fund}}(\omega_x, \omega_t, j) - \boldsymbol{w}^j(\boldsymbol{K}) \right|_2^2 \tag{10}$$

The objective function is then minimized using a genetic algorithm to obtain the optimal control matrix $\boldsymbol{K}$. To address stability of the controlled damping layer, the eigenvalues of the state space system (8) are evaluated during the optimization and destabilizing controllers are penalized with a multiplicative weighting term.

## 3. Resulting controller for moving Euler-Bernoulli beam

The procedure described above is used to find a state feedback controller that emulates PML properties on one side for the moving Euler-Bernoulli beam equation

$$\rho A \ddot{w} = -EI w'''' + (T - \rho A v^2) w'' + 2v\rho A \dot{w}' \tag{11}$$

where $\rho A$ is the mass per unit length, $EI$ the bending stiffness, $T$ the tensile force and $v$ the speed of the moving coordinate. Figure 1 shows the normalized phase velocity for the non-moving and the moving Euler-Bernoulli beam. Substantial dispersion occurs due to the bending stiffness. Furthermore, the two branches for left and right going waves are not symmetrical for the moving Euler-Bernoulli beam.

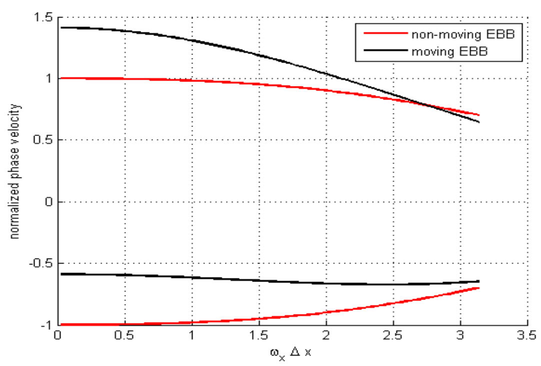


Figure 1: Normalized phase velocity for left and right going waves over the spatial frequency The phase velocity for the non-moving EBB (black) is symmetric around zero whereas the branches for the moving EBB (red) are tilted.

The parameters used are shown in Table 1. When discretizing to obtain the state space system the spatial grid size was set to $\Delta x = 0.4$ [m] and the temporal grid size to $\Delta t = 7 * 10^{-4}$ [s].

Table 1: Simulation Parameters

| Parameter | Symbol | Value |
|---|---|---|
| mass per unit length | $\rho A$ | 1.35 [kg/m] |
| bending stiffness (contact) | $EI$ | 150 [Nm$^2$] |
| tensile force (contact) | $T$ | 20 [kN] |
| speed | $v$ | 50 [m/s] |

The state feedback controller actuates $p = 10$ nodes . For the two outer nodes of the damping layer Dirichlet boundary conditions are applied. The set $\Omega$ consists of 10 pairs where $\omega_x \Delta x$ is equidistantly spaced between 0 and $\pi$ and the corresponding $\omega_t \Delta t$ are calculated from the dispersion relation. The maximum number of time steps for which the objective function is evaluated is set to $j_{max} = 100$. The satisfactory performance of the optimized feedback controller is illustrated in Figure 2. No significant reflections into the domain are produced.

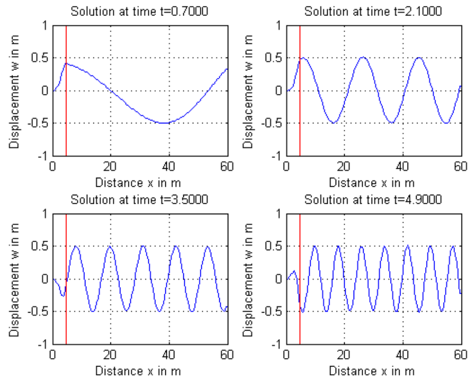


Figure 2: The optimized feedback controller is applied on the left boundary. A frequency sweep excites the right boundary. No reflections back into the computational domain are visible

## 4. Conclusion

In this work the construction of a PML is described as an optimization problem to obtain a feedback controller. The method is applied for the moving Euler-Bernoulli beam and it was shown in numerical results that a high absorption is achieved. The procedure is highly automated, and the mathematical effort for the user is reduced to determining the dispersion relation instead of performing the original PML transformation which is a task especially tedious with wave-like equations of high orders of (mixed) derivatives.

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