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1 : 3-Resonance in a Hopf-Hopf bifurcation

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We investigate the bifurcating solutions at a Hopf-Hopf interaction point with an internal $1 : 3$ resonance. It turns out, that the transitions from single to mixed modes can be described by Duffing or Mathieu szenarios.

1 Normal Form equations and investigation of the primary solution branches

If in some dynamical system, like the model of a fluid conveying tube, three parameters are varied, Hopf-Hopf mode interactions with internal low-order resonances may occur, which can lead to very interesting system dynamics. In this article we have a closer look at the $1 : 3$-resonance, in which the resonance terms appear at third degree and therefore have the same order of influence as the leading terms in the non-resonant system. After shortly summarizing the results for a Hopf-Hopf mode interaction without resonance, we will investigate the existence and stability of the primary solution branches and study the transition to mixed-mode solutions.

After transforming the original system at the bifurcation point to Jordan Normal Form, reducing the system to the Center Manifold, applying nonlinear Normal Form and unfolding of the linearized system, we obtain the complex differential equations ([1, 2])

$$\dot{z}_1 = (\lambda + i\omega + A_1|z_1|^2 + A_2|z_2|^2)z_1 + A_3\overline{z}_1^2 z_2, \tag{1}$$

$$\dot{z}_2 = (\mu + 3i\omega + i\delta + A_4|z_1|^2 + A_5|z_2|^2)z_2 + A_6 z_1^3, \tag{2}$$

where λ, μ and δ are the unfolding parameters and the complex valued coefficients $A_j = c_j + id_j$ are obtained from the nonlinear contributions of the original equations.

1.1 Bifurcation szenario for the non-resonant Hopf-Hopf bifurcation

If the frequencies ω_j at the Hopf-Hopf interaction are not close to a low order resonance, also the terms $A_3\overline{z}_1^2$ and $A_6 z_1^3$ can be eliminated by the Normal Form method. In polar coordinates $z_j = r_j \exp(i\varphi_j)$ the Normal Form equations become

$$\dot{r}_1 = (\lambda + c_1 r_1^2 + c_2 r_2^2)r_1, \qquad \dot{r}_2 = (\mu + c_4 r_1^2 + c_5 r_2^2)r_2, \tag{3}$$

$$\dot{\varphi}_1 = (\omega_1 + d_1 r_1^2 + d_2 r_2^2), \qquad \dot{\varphi}_2 = (\omega_2 + d_4 r_1^2 + d_5 r_2^2). \tag{4}$$

The angles φ_j have completely disappeared from the equations for the radii and need not be taken into account during the further investigation, because the frequencies ω_j are of order 1.

Besides the trivial solution with eigenvalues λ and μ the equations (3) have the equilibria listed in Table 1. The mixed

Table 1: Nontrivial solution branches of the non-resonant Hopf-Hopf interaction

Type		Branching equation	Eigenvalues σ_i resp. matrix	
Mode 1	$r_1 \neq 0, r_2 = 0$	$\lambda + c_1 r_1^2 = 0$	$\sigma_1 = 2c_1 r_1^2,$	$\sigma_2 = \mu + c_4 r_1^2$
Mode 2	$r_1 = 0, r_2 \neq 0$	$\mu + c_5 r_2^2 = 0$	$\sigma_1 = \lambda + c_2 r_2^2,$	$\sigma_2 = 2c_5 r_2^2$
Mixed mode	$r_1 \neq 0, r_2 \neq 0$	$\lambda + c_1 r_1^2 + c_2 r_2^2 = 0$ $\mu + c_4 r_1^2 + c_5 r_2^2 = 0$	Eigenvalues of	$2\begin{pmatrix} c_1 r_1^2 & c_2 r_1 r_2 \\ c_4 r_1 r_2 & c_5 r_2^2 \end{pmatrix}$

mode branch bifurcates from the primary solutions along the rays

$$(\lambda, \mu) = -(c_1, c_4) r_1^2, \quad \text{and } (\lambda, \mu) = -(c_2, c_5) r_2^2$$

by pitchfork bifurcations.

If the coefficients c_1 and c_5 have different signs and $c_1 c_5 - c_2 c_4 > 0$, a tertiary branch of slow periodic solutions bifurcates from the mixed mode solutions. At the same parameter values a heteroclinic orbit connecting the single modes exists. In order to calculate the branching behaviour for the periodic solutions and distinguish it from the heteroclinic orbit, higher order terms in the bifurcation equations are needed.

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1.2 The slow solution branch in the resonant case

Due to the presence of the term $A_6 z_1^3$ in (2) no pure slow mode can exist, there will always be a contribution of $O(|z_1|^3)$ in the second mode. Taking care of this term by applying a center manifold reduction to the fast equation (2), which is possible as long as $\mu + i\delta + A_4|z_1|^2$ is sufficiently far from 0, we find that this term only contributes high order terms to the slow equation (1). Therefore we recover the slow oscillation from the non-resonant case and observe that outside the critical region it has the same branching behaviour. But its stability with respect to oscillations in the second mode is quite different: If $z_1(t)$ oscillates like $r_1 \exp(i\Omega t)$, the term $A_6 z_1^3$ acts like an external Duffing-like excitation in (2). With $\Omega = \omega + d_1 r_1^2$ we find that the primary resonance in this equation occurs when

$$3\omega + \delta + d_4 r_1^2 = 3(\omega + d_1 r_1^2) \qquad \text{and} \qquad \mu + c_4 r_1^2 = 0.$$

Introducing for fixed values of r_1 the shifted “parameters”

$$\tilde{\delta} = \delta + (d_4 - 3d_1) r_1^2 \qquad \text{and} \qquad \tilde{\mu} = \mu + c_4 r_1^2,$$

and setting $z_1 = r_1 \exp(i\Omega t)$ and $z_2 = w_2 \exp(3i\Omega t)$, we obtain the autonomous differential equation

$$\dot{w}_2 = (\tilde{\mu} + i\tilde{\delta} + A_5|w_2|^2) w_2 + A_6 r_1^3, \tag{5}$$

whose stationary values satisfy the Duffing-like equation

$$\Big((\tilde{\mu} + c_5 r_2^2)^2 + (\tilde{\delta} + d_5 r_2^2)^2 \Big) r_2^2 = |A_6|^2 r_1^6. \tag{6}$$

For given values of r_1 and r_2 the possible values of $\tilde{\mu}$ and $\tilde{\delta}$ lie on circles with radius $|A_6| r_1^3 / r_2$ and center at $(-c_5 r_2^2, -d_5 r_2^2)$. The backbone of this family of solutions, which is obtained by setting $A_6 = 0$, corresponds to the mixed-mode solution of the non-resonant case.

These stationary solutions correspond to slow periodic oscillations of the full system (1,2) and approximate both the mode-1 solution and the mixed mode solution of the non-resonant case, as long as the contribution of the slow mode is dominant. We observe, that the pitchfork bifurcation from the slow mode to the mixed mode in the non-resonant case is governed by a Duffing szenario in the 1 : 3-resonance.

1.3 The fast solution branch in the resonant case

If we set $z_1 = 0$, (1) is fulfilled and for the fast mode we obtain the same branch equation as in the non-resonant case. The solution oscillates with constant amplitude r_2 and frequency $\Omega_2 = 3\omega + \delta + d_5 r_2^2$.

The stability of $z_1 = 0$ is lost, when $\lambda + c_2 r_2^2 = 0$. Since in that case the imaginary linear part $\Omega_1 = \omega + d_2 r_2^2 \neq 0$, we would expect a Hopf bifurcation to occur. But the situation is somewhat more difficult, because the term $A_3 \overline{z}_1^2 z_2$ forces an oscillation close to the slow eigenfrequency, therefore we have to deal with a nonlinear Mathieu szenario. In order to investigate the bifurcation in this case we assume $|z_1| \ll |z_2|$ and that the influence of z_1 on the motion of z_2 can be neglected.

We introduce the new parameters

$$\tilde{\lambda} = \lambda + c_2 r_2^2 \qquad \text{and} \qquad \Delta = \Omega_1 - \Omega_2/3 = -\delta/3 + (d_2 - d_5/3) r_2^2 \tag{7}$$

and set $z_1 = w_1 \exp(\Omega_2 t/3)$. Then w_1 satisfies the autonomous equation

$$\dot{w}_1 = (\tilde{\lambda} + i\Delta + A_1|w_1|^2) w_1 + A_3 \overline{w}_1^2 r_2 \tag{8}$$

The steady states are obtained from the equation

$$(\tilde{\lambda} + c_1 r_1^2)^2 + (\Delta + d_1 r_1^2)^2 = |A_3|^2 r_1^2 r_2^2, \tag{9}$$

which for fixed r_1 and r_2 describes a circle with radius $|A_3| r_1 r_2$ and center $-(c_1, d_1) r_1^2$ in the $(\tilde{\lambda}, \Delta)$-plane. The backbone curve of this solution set, given by $A_3 = 0$, coincides again with the mixed-mode solution of the non-resonant case. If $\Delta = 0$, a pair of slow periodic oscillations bifurcates from the fast solution branch. Otherwise first a quasiperiodic oscillation with frequencies Ω_2 and $\Omega_1 + d_1 r_1^2$ is found. For larger amplitudes of r_1 the two oscillations can synchronize into the frequency ratio 1 : 3.

Due to the parametric excitation term $A_3 \overline{z}_1^2 z_2$ the pitchfork bifurcation from the fast mode in the non-resonant case changes to a Mathieu-like transition in the resonant case.

References

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