PROSOLVABILITY CRITERIA AND PROPERTIES OF THE PROSOLVABLE RADICAL VIA SYLOW SEQUENCES

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To the memory of O.V. Mel'nikov

Abstract. We extend a finite group solvability criterion of J.G. Thompson, based on his classification of finite minimal simple groups, to a prosolvability criterion. Moreover, we generalize to the profinite setting subsequent developments of Thompson's criterion by G. Kaplan and the second author, which recast it in terms of properties of sequences of Sylow subgroups and their products. This generalization also encompasses a possible characterization of the prosolvable radical whose scope of validity is still open even for finite groups. We prove that if this characterization is valid for finite groups, then it carries through to profinite groups.

1. Introduction

The most prominent class of *Mel'nikov formations* (or *NE-formations*, see [12]), that is, formations also closed under taking normal subgroups and extensions (see [3, ch.17]), is the class of all prosolvable groups. Recall that a profinite group G is prosolvable if it is the inverse limit of an inverse system of finite solvable groups, or, equivalently, if G/N is solvable for any $N \lhd_O G$ (N is an open normal subgroup of G). In this paper we present new prosolvability criteria which are based upon and generalize a solvability criterion for finite groups, conjectured already by G. Miller ([11]) and P. Hall ([4]), and first proved by J.G. Thompson ([14, Corollary 3]) as a corollary to his classification of the minimal simple finite groups.

We now state our main results. In order to keep the introduction at a manageable size, some details are postponed to the relevant sections. We begin with the generalization of [14, Corollary 3] to profinite groups.

Definition 1.1. *Let G be a profinite group. A triple $(a, b, c) \in G^3$ will be called a* Thompson triple *(in short a* T-triple*) of G, if $(a, b, c) \neq (1, 1, 1)$, a, b, c are coprime in pairs, and $abc = 1$. An element $g \in G$ will be called a Thompson factor if g appears in some T-triple of G.*

Definition 1.2. *Let G be a profinite group, and let $N \lhd G$ be closed. A triple $t \in G^3$ will be called a T-triple modulo N of G if tN^3 is a T-triple of G/N.*

Prosolvability is characterized by the non-existence of T-triples.

Theorem 1.3. *Let G be a profinite group, and let $M \lhd_O G$ be such that G/M is non-solvable. Then G has a T-triple which is also a T-triple modulo M. It follows that G is prosolvable if and only if G has no T-triples.*

More can be said about the connection between Thompson factors and prosolvability. For this we need the following known result which can be derived from [12, Prop. 2.2.1].

Proposition 1.4. *Let G be a profinite group, and let $\{N_i | i \in I\}$ be any family of closed normal subgroups of G such that G/N_i is prosolvable for any $i \in I$. Then $G/(\cap_{i\in I} N_i)$ is prosolvable. It follows that if $\{N_i | i \in I_{\max}\}$ is the family of all closed normal subgroups of G such that G/N_i is prosolvable for any $i \in I_{\max}$, then $S(G) := \cap_{i\in I_{\max}} N_i$ satisfies that $G/S(G)$ is prosolvable and for any $N \trianglelefteq G$ such that G/N is prosolvable, we have $S(G) \leq N$.*

For any profinite group G, the subgroup $S(G)$ defined in Proposition 1.4 will be called the *prosolvable residual* of G. It is a closed normal subgroup of G. For a finite G the prosolvable residual of G coincides with the solvable residual of G.

Theorem 1.5. *Let G be a profinite group. Then $S(G)$ is equal to the subgroup generated by all Thompson factors of G.*

The finite group precursor of Theorem 1.5 is [8, Theorem 9]. Because of differences in formulation, and since the finite group case is needed for proving Theorem 1.3, we state and prove it separately, as Lemma 3.4.

Another solvability criterion for finite groups whose proof employs Thompson's T-triple solvability criterion makes use of the concept of a complete Sylow sequence. Let $G \neq 1$ be a finite group and let $\pi(G)$ be the set of prime divisors of $|G|$. A complete Sylow sequence $\mathcal{P}$ of G is a sequence of $|\pi(G)|$ Sylow subgroups of G, each of which is associated with a different prime. The product $\Pi(\mathcal{P})$ of $\mathcal{P}$ is the setwise product of the subgroups in $\mathcal{P}$ as ordered in $\mathcal{P}$. Section 4 presents generalizations of these concepts for profinite groups and proves the following prosolvability criteria, relying on and generalizing [5, Theorem A] (see also [1]).

Theorem 1.6. *Let G be a profinite group. Then the following are equivalent:*
(1) G is prosolvable.
(2) For every permutation τ of the set of all primes and for every complete Sylow sequence $\mathcal{P}$ of type τ it holds that $G = \Pi(\mathcal{P})$.
(3) Fix an arbitrary permutation τ of the set of all primes. Then for every complete Sylow sequence $\mathcal{P}$ of type τ it holds that $G = \Pi(\mathcal{P})$.

Let G be a finite group, $\mathcal{P} = (P_1, ..., P_m)$ any complete Sylow sequence of G, where P_i is a Sylow p_i-subgroup of G ($p_i \in \pi(G)$ for all $1 \leq i \leq m$), and $g \in G$. Then the multiplicity of g in $\mathcal{P}$, denoted $m_{\mathcal{P}}(g)$, is the number of possible factorizations of g in $\mathcal{P}$, namely:

$$m_{\mathcal{P}}(g) := |\{(g_1, ..., g_m) | g_i \in P_i, \forall 1 \leq i \leq m \text{ and } g_1 \cdots g_m = g\}|.$$

Since $|G| = |P_1| \cdots |P_m|$, an elementary counting argument shows that $G = \Pi(\mathcal{P})$ if and only if $m_{\mathcal{P}}(g) = 1$ for all $g \in G$. Hence, the non-negative integers $m_{\mathcal{P}}(g)$ tell if the group is solvable or not. In Section 5 we show that upon a suitable choice of a definition of $m_{\mathcal{P}}(g)$ for profinite groups, we have:

Proposition 1.7. *Let G be a profinite group, and let $\mathcal{P} = \left(G_{\tau(p)}\right)_{p\in \mathrm{Pr}}$ be a complete Sylow sequence of type τ. Then $G = \Pi(\mathcal{P})$ if and only if $m_{\mathcal{P}}(g) = 1$ for all $g \in G$.*

Corollary 1.8. *Combining Theorem 1.6 and Proposition 1.7, G is prosolvable if and only if there exists an ordering τ of the primes such that $m_{\mathcal{P}}(g) = 1$ for every complete Sylow sequence of G and every $g \in G$.*

Our final topic concerns the prosolvable radical of a profinite group. In order to introduce it we begin with a finite group G. If $N_1, N_2 \trianglelefteq G$ are solvable, then $N_1N_2 \trianglelefteq G$ is solvable and consequently, the product of all normal solvable subgroups of G is the unique largest (with respect to inclusion) normal solvable subgroup of G. This subgroup is called the solvable radical of G and will be denoted $R(G)$. The following proposition generalizes this to profinite groups.

Proposition 1.9. *Let G be a profinite group. Then G has a unique normal prosolvable subgroup which contains any normal prosolvable subgroup of G. Furthermore, this subgroup is closed and characteristic.*

For any profinite group G, the subgroup whose existence is proved in Proposition 1.9, will be called the *prosolvable radical* of G and will be denoted $R(G)$. Note that $R(G)$ is the solvable radical of G if G is finite.

In [5] it was observed that the intersection of all complete sylow products of a finite group G, denoted $H(G)$, is a characteristic subgroup of G which contains the solvable radical of G and has a number of properties in common with it ([5, Theorem B]). In fact, it is an open question, whether or not $H(G) = R(G)$ for every finite group G. As far as we know, there is no published counterexample. Further evidence in favour of $H(G) = R(G)$ can be found in [5],[6],[7],[9].

In section 6 we define an appropriate generalization of $H(G)$ to the profinite case. This involves the introduction of an opposite Sylow sequence, denoted $\mathcal{P}^{op}$, for every complete Sylow sequence $\mathcal{P}$. We show that the generalization of $H(G)$ satisfies the same basic properties as in the finite case (compare the following theorem with [5, Theorem B]).

Theorem 1.10. *Let G be a profinite group, and let τ be any permutation of $\mathbb{N}$. Then:*

(a) *$H_\tau(G)$ and therefore $H(G)$, is a closed characteristic subgroup of G.*
(b) *Let $\mathcal{P}$ be a complete Sylow sequence of G of type τ. Then each of $\Pi(\mathcal{P})$ and $\Pi(\mathcal{P}^{op})$ is a union of cosets of $H_\tau(G)$.*
(c) *$R(G) = R(H(G))$, and in particular, $R(G)$ is a characteristic subgroup of $H(G)$.*
(d) *$H_\tau(G/H_\tau(G)) = 1$. In particular, $R(G/H_\tau(G)) = 1$.*
(e) *$H(G/H(G)) = 1$. In particular, $R(G/H(G)) = 1$.*

It would be extremely interesting if one can utilize some profinite technique in order to shed new light on the $H(G) = R(G)$ question. Here we observe that a positive answer for finite groups implies a positive answer for profinite groups.

Proposition 1.11. *Let G be a profinite group. If $H(G/N) = R(G/N)$ for every open normal subgroup N, then $H(G) = R(G)$.*

2. Basic notions and notations

Let G be a profinite group. For any subset S of G we denote by $cl(S)$ the usual set topological closure of S (the union of S with the set of all its limit points). The following fact is well-known.

Claim 2.1. *Let G be a profinite group. Let H be a subgroup of G. Then $cl(H)$ is also a subgroup of G.*

If H is a subgroup of G we'll denote $\overline{H} = cl(H)$. It follows from the claim above that $\overline{H}$ is the smallest (with respect to inclusion) closed subgroup of G, containing

H. For any subset S of G, the abstract group generated by S is denoted $\langle S\rangle$, and we shall say that S topologically generates $\overline{\langle S\rangle}$.

Unless otherwise stated, an automorphism α of a profinite group will be assumed to be both an abstract group automorphism and a continuous map. As is well-known, this implies that α is a homeomorphism. Moreover, if H is a closed subgroup of G then so is $\alpha(H)$, and if $N \triangleleft_O G$ then $\alpha(N) \triangleleft_O G$. Hence indices and orders are invariant under automorphisms. It follows that if P is a p-Sylow subgroup of G, then $\alpha(P)$ is also a p-Sylow subgroup of G ([15, Definition 2.2.1]). A subgroup $H \leq G$ will be called characteristic if it is invariant under any automorphism of G.

A filter base in G is any collection $\mathcal{N}$ of normal subgroups of G such that for any $N_1, N_2 \in \mathcal{N}$ there exists $N_3 \in \mathcal{N}$ with $N_3 \leq N_1 \cap N_2$. Suppose that $\mathcal{N}$ is a filter base in G, H is a closed subgroup of G, and K is a closed normal subgroup of G. Then:

(1). $\mathcal{N}_H := \{H \cap N | N \in \mathcal{N}\}$ is a filter base in H. If all members of $\mathcal{N}$ are closed, then so are all members of $\mathcal{N}_H$, and if $\bigcap\limits_{N\in\mathcal{N}} N = 1$ then $\bigcap\limits_{N\in\mathcal{N}_H} N = 1$.

(2). If all members of $\mathcal{N}$ are closed then $\mathcal{N}_{NK} := \{KN | N \in \mathcal{N}\}$ is a filter base in G of closed normal subgroups containing K. If $\bigcap\limits_{N\in\mathcal{N}} N = 1$ then $\bigcap\limits_{N\in\mathcal{N}_K} N = K$ (see proof of [15, Theorem 1.2.5]). Furthermore $\mathcal{N}_{G/K} := \{KN/K | N \in \mathcal{N}\}$ is a filter base in G/K of closed normal subgroups, and $\bigcap\limits_{N\in\mathcal{N}} N = 1$ implies $\bigcap\limits_{N\in\mathcal{N}_{G/K}} N = 1$.

(3). Let $\mathcal{N}$ be the set of all normal open subgroups of G. Then $\mathcal{N}$ is a filter base of G. By [15, Theorem 1.2.3], we get $\bigcap\limits_{N\in\mathcal{N}} N = 1$. Henceforth we denote this filter base of G by $\mathcal{N}^{(G)}$. Furthermore, let M be any normal open subgroup of G. Then, by (1) above, $\left(\mathcal{N}^{(G)}\right)_M$ is a filter base in M which satisfies $\bigcap\limits_{N\in\left(\mathcal{N}^{(G)}\right)_M} N = 1$. But $\left(\mathcal{N}^{(G)}\right)_M$ is precisely the set of all normal open subgroups of G which are contained in M, and hence it is also a filter base in G.

Lemma 2.2. *Let G be a profinite group. Let $A, B \subseteq G$. If $AN/N = BN/N$ for every $N \in \mathcal{N}^{(G)}$ then $cl(A) = cl(B)$. Thus if A and B are closed then $A = B$.*

Proof. From the assumption of the lemma it follows that $AN = BN$ for every $N \in \mathcal{N}^{(G)}$. Now, by [15, Proposition 0.3.3.(c)], we get:

$$cl(A) = \bigcap_{N\in\mathcal{N}^{(G)}} AN = \bigcap_{N\in\mathcal{N}^{(G)}} BN = cl(B).$$

∎

We denote $\Pr := (p_i)_{i\in\mathbb{N}}$ the set of all primes with the natural ordering, where $\mathbb{N}$ is the set of all natural numbers, so $p_1 = 2$, $p_2 = 3$ etc. For a profinite group G we let $\pi(G)$ denote the set of primes such that G possesses a nontrivial Sylow p-subgroup. Two elements $x, y \in G$ are *coprime* if $\pi(\overline{\langle x\rangle}) \cap \pi(\overline{\langle y\rangle}) = \emptyset$. The proof of the next result is straightforward.

Lemma 2.3. *Let G be a profinite group and let $x \in G$. Let $\mathcal{N}$ be a filter base of open normal subgroups of G.*

(i) Two elements $x, y \in G$ have coprime orders if and only if for every $N \in \mathcal{N}$, the elements xN and yN of G/N have coprime orders.

(ii) If $x, y \in G$ have coprime orders and K is a closed normal subgroup of G, then xK and yK (as elements of G/K) have coprime orders.

We end up this section with proofs of Proposition 1.4 (existence of the prosolvable residual) and of Proposition 1.9 (existence of the prosolvable radical).

Proof of Proposition 1.4. Since prosolvability is inherited by Cartesian products, $C := Cr\,(G/N_i | i \in I)$ is prosolvable. Define $\phi : G \to C$ by $\phi\,(g)_i = gN_i$, for every $g \in G$ and every $i \in I$. Now ϕ is a group homomorphism and ϕ is continuous since its composition with each projection map $C \to G/N_i$ is continuous. Since G is compact and ϕ continuous, $\phi\,(G)$ is compact, and hence (C is Hausdorff) closed. Therefore $\phi\,(G)$ is prosolvable. On the other hand, $\phi\,(G) \cong G/\ker(\phi)$. Since $\ker(\phi) = \cap_{i\in I} N_i$, we get that $G/\left(\cap_{i\in I} N_i\right)$ is prosolvable. ∎

Proof of Proposition 1.9. Let G be a profinite group. For any $N \in \mathcal{N}^{(G)}$ let R_N be the inverse image of $R(G/N)$ in G, with respect to the natural map $G \to G/N$. Set $R := \bigcap_{N\in\mathcal{N}^{(G)}} R_N$. Since $R \leq R_N$ for all $N \in \mathcal{N}^{(G)}$, we have that $RN/N \leq R(G/N)$ is solvable. Hence R is prosolvable. Now let K be any normal prosolvable subgroup of G. Then, for all $N \in \mathcal{N}^{(G)}$, $KN/N \leq R(G/N)$ whence $KN \leq R_N$. Therefore $K = \bigcap_{N\in\mathcal{N}^{(G)}} KN \leq \bigcap_{N\in\mathcal{N}^{(G)}} R_N = R$. Thus R contains every normal prosolvable subgroup of G.

Let $N \in \mathcal{N}^{(G)}$. Since the natural map $G \to G/N$ is continuous, and every subgroup of the finite group G/N is closed, R_N is closed and hence R is closed. Finally, if α is an automorphism of G, and K is a normal prosolvable subgroup of G then $\alpha\,(K)$ is also normal and prosolvable. Therefore R is characteristic. ∎

3. Existence of T-triples in non-prosolvable groups

In this section we prove Theorem 1.3 which characterizes prosolvable groups in terms of T-triples (Definition 1.1).

Observe that if $(a, b, c) \in G^3$ is a T-triple then, in fact, $(a, b, c) \in (G - \{1\})^3$. Moreover, since G^3 can be viewed as the direct product of three copies of G, and hence is itself a profinite group, for any $t = (a, b, c) \in G^3$ and $M \subseteq G$, we shall use the notation $tM^3 := (aM, bM, cM)$. Also note that $(a, b, c) \in G^3$ is a T-triple if and only if (c, a, b) and $\left(c^{-1}, b^{-1}, a^{-1}\right)$ are T-triples.

The difficult part of [14, Corollary 3] is to show that a finite non-solvable group must possess a T-triple. We begin by generalizing the easy part.

Lemma 3.1. *Let G be a prosolvable group. Then G has no T-triple.*

Proof. Let $t = (a, b, c) \in G^3$, be such that a, b, c are coprime in pairs and $abc = 1$. Let $N \in \mathcal{N}^{(G)}$ be arbitrary. Then, since G is prosolvable, G/N is a finite solvable group, and hence has no T-triple. Now aN, bN, cN are pairwise coprime by Lemma 2.3 (ii), and $(aN)\,(bN)\,(cN) = (abc)\,N = 1_{G/N}$. Hence, the fact that G/N has no T-triple implies $(aN, bN, cN) = (N, N, N)$, from which it follows that if $x \in \{a, b, c\}$ then $x \in N$. Since N is arbitrary, we get that $x \in \{a, b, c\}$ implies $x \in \bigcap\limits_{N\in\mathcal{N}^{(G)}} N = 1$, so $t = (1, 1, 1)$ proving that G has no T-triple. ∎

For the first claim of Theorem 1.3 we need a few lemmas.

Lemma 3.2. *Let G be a profinite group, and let $t = (a, b, c) \in G^3$ be a T-triple in G. Let $M \lhd G$ be closed. Then either $t \in M^3$ or $t \in (G - M)^3$. Moreover, t is a T-triple modulo M of G if and only if $t \in (G - M)^3$.*

Proof. Suppose that $t = (a, b, c) \notin M^3$, and assume by contradiction that one of a, b, c is in M. If $c \in M$, then $abc = 1$ implies $ab \in M$ whence $a^{-1}M = bM$. But then, $a^{-1}M$ and bM (viewed as elements of G/M) are coprime, by Lemma 2.3 (ii) and the fact that a^{-1} and b are coprime. This implies $a^{-1}M = bM = M$, whence $a, b \in M$, contradicting $t \notin M^3$. A similar argument applies if we assume $b \in M$ or $a \in M$. Thus $t \in (G - M)^3$.

If $t \in M^3$ then $tM^3 := (aM, bM, cM) = (M, M, M)$ is not a T-triple in G/M. If $t \in (G - M)^3$ then tM^3 is a T-triple in G/M, since aM, bM, cM are pairwise coprime and $(aM)(bM)(cM) = M$ (see proof of Lemma 3.1). ∎

Lemma 3.3. *Let G be a profinite group, $N, K \lhd G$ closed and $N \leq K$. Suppose that $t = (a, b, c)$ is T-triple modulo N of G such that $t \in (G - K)^3$. Then t is also a T-triple modulo K of G.*

Proof. Since $G/K \cong (G/N)/(K/N)$, the fact that aN, bN and cN are coprime in pairs implies (Lemma 2.3 (ii)) that also aK, bK and cK are coprime in pairs. Moreover $(aN)(bN)(cN) = N$ gives $abc \in N \leq K$ so $(aK)(bK)(cK) = K$. Finally, $t \in (G - K)^3$ implies that aK, bK and cK are non-trivial. ∎

The last ingredient needed for the proof of Theorem 1.3 is the finite group case of Theorem 1.5 (see Section 1). For this and for the general case of Theorem 1.5, it is convenient to denote by $T(G)$ the subgroup which is topologically generated by the set of all Thompson factors in the profinite group G. Note that $T(G)$ is a closed characteristic subgroup of G.

Lemma 3.4. *Let G be a finite group. Then $T(G) = S(G)$. Furthermore, if $M \trianglelefteq G$ and G/M is not solvable then there exists a T-triple t of G such that $t \in (G - M)^3$, and so, by Lemma 3.2, t is a T-triple modulo M of G .*

Proof. First we show $T(G) \leq S(G)$. Suppose by contradiction that $t = (a, b, c)$ is a T-triple in G such that $\{a, b, c\} \nsubseteq S(G)$. Then, by Lemma 3.2, $tS(G)^3$ is a T-triple in $G/S(G)$ - a contradiction since $G/S(G)$ is a finite solvable group.

In order to prove $S(G) \leq T(G)$, it is sufficient to prove that $G/T(G)$ is solvable. Let M be a minimal supplement to $T(G)$ in G, Then $G = T(G)M$, and $T(G) \cap M$ is nilpotent (see [13, Exercise 618, p.271]). We get $G/T(G) \cong M/T(G) \cap M$, and therefore $G/T(G)$ is solvable if and only if M is solvable. Assuming that M is not solvable there exists a T-triple t of M. But clearly t is also a T-triple of $T(G)$ so t is a T-triple of $T(G) \cap M$ which is solvable - a contradiction.

Finally, if $M \trianglelefteq G$ and G/M is not solvable, we get $T(G) = S(G) \nleq M$. Hence there exists a T-triple t of G such that $t \notin M^3$. By Lemma 3.2, $t \in (G - M)^3$. ∎

Proof of Theorem 1.3. Let $\mathcal{N}$ be the set of all normal open subgroups of G which are contained in M. Then (see Section 2, (3) where this set is denoted $\left(\mathcal{N}^{(G)}\right)_M$) $\mathcal{N}$ is a filter base in G satisfying $\bigcap_{N \in \mathcal{N}} N = 1$. Let $N \in \mathcal{N}$. Then G/N is a finite group, and since $N \leq M$ we get that M/N is a normal subgroup of G/N, and, by an isomorphism theorem, $(G/N)/(M/N) \cong G/M$. Since G/M is non-solvable then so is G/N. By Lemma 3.4, G/N has a T-triple which is also a T-triple modulo M/N. Let $t_N := (a_N, b_N, c_N) \in G^3$ be such that $t_N N^3$ is a T-triple of G/N which is also a T-triple modulo M/N (no claim is made that t_N is a T-triple of G). It follows that t_N is a T-triple modulo N of G, and $t_N \in (G - M)^3$.

Observe that the assignment $N \longmapsto t_N$ defines a function $\mathcal{N} \to (G - M)^3$ which is a topological net since $\mathcal{N}$ is a filter base. Moreover, because M is open in G, the subset $(G - M)^3$ is closed in G^3 and since G^3 is compact so is $(G - M)^3$. Therefore the net defined by $N \longmapsto t_N$ has at least one cluster point. Choose $t = (a, b, c)$ to be such a cluster point. Now let $K \in \mathcal{N}$ be arbitrary. Then tK^3 is an open neighborhood of t in G^3, and hence $tK^3 \cap (G - M)^3$ is an open neighborhood of t in $(G - M)^3$. Therefore, by definition of a cluster point, there exists $N \in \mathcal{N}$, such that $N \leq K$ and $t_N \in tK^3 \cap (G - M)^3$. From $t_N \in tK^3$ we get

$$t_N K^3 = tK^3. \tag{3.1}$$

Furthermore, $N \leq K$, the fact that t_N is a T-triple modulo N of G, and $t_N \in (G - M)^3 \subseteq (G - K)^3$ (since $K \leq M$), imply, by Lemma 3.3, that t_N is a T-triple modulo K of G. Hence, by Equation 3.1, t is a T-triple modulo K of G. Thus we have proved that t is a T-triple modulo K of G for all $K \in \mathcal{N}$. This implies that aK, bK, cK are coprime in pairs for all $K \in \mathcal{N}$, and hence, by Lemma 2.3 (i), a, b, c are coprime in pairs. We also get that $abc \in K$ for all $K \in \mathcal{N}$. Since $\bigcap\limits_{K \in \mathcal{N}} K = 1$, this implies $abc = 1$. Finally, $t = (a, b, c) \in (G - M)^3$. Hence t is a T-triple of G which is also a T-triple modulo M of G. Combining what we proved here with Lemma 3.1 shows that G is prosolvable if and only if G has no T-triples. ∎

Proof of Theorem 1.5. First we show $T(G) \leq S(G)$. Suppose by contradiction that $t = (a, b, c)$ is a T-triple of G such that $\{a, b, c\} \nsubseteq S(G)$. Then, by Lemma 3.2, $tS(G)^3$ is a T-triple of $G/S(G)$. But, by definition, $G/S(G)$ is prosolvable, so we get a contradiction with Lemma 3.1.

To prove $S(G) \leq T(G)$, we show that $G/T(G)$ is prosolvable. Suppose by contradiction that $G/T(G)$ is not prosolvable. Then there exists $N \lhd_O (G/T(G))$ such that $(G/T(G))/N$ is non-solvable. Thus $M := T(G)N$ is an open normal subgroup of G and G/M is non-solvable. By Theorem 1.3, G has a T-triple t which is also a T-triple modulo M. Hence tM^3 is a T-triple of G/M - a contradiction. ∎

4. Complete Sylow Sequences

Here we define complete Sylow sequences and products and prove Theorem 1.6.

Definition 4.1. *Let G be a profinite group. Fix a permutation τ of $\mathbb{N}$. A complete Sylow sequence of G of type τ is a sequence $\mathcal{P} = \left(\left(G_{p_{\tau(1)}}, ..., G_{p_{\tau(i)}}\right)\right)_{i \in \mathbb{N}}$, where for each $p \in \Pr$, G_p is a p-Sylow subgroup of G (if $p \notin \pi(G)$ then $G_p = \{1\}$). The product of the sequence $\mathcal{P}$ is defined by*

$$\Pi(\mathcal{P}) := cl\left(\bigcup_{i=1}^{\infty} G_{p_{\tau(1)}} \cdots G_{p_{\tau(i)}}\right),$$

where the product $P_{\tau(p_1)} \cdots P_{\tau(p_i)}$ is the setwise product. If $\mathcal{P} = \left(\left(G_{p_{\tau(1)}}, ..., G_{p_{\tau(i)}}\right)\right)_{i \in \mathbb{N}}$, is a given complete Sylow sequence of G of type τ then the opposite sequence to $\mathcal{P}$, denoted $\mathcal{P}^{op}$, is defined by $\mathcal{P}^{op} := \left(\left(G_{p_{\tau(i)}}, ..., G_{p_{\tau(1)}}\right)\right)_{i \in \mathbb{N}}$, and its product is

$$\Pi(\mathcal{P}^{op}) := cl\left(\bigcup_{i=1}^{\infty} G_{p_{\tau(i)}} \cdots G_{p_{\tau(1)}}\right).$$

Remark 4.2. *a. We will allow ourselves to regard τ also as a permutation of* Pr*, via $\tau(p) = p_{\tau(i)}$ for $p \in \Pr$, where i is uniquely determined by $p = p_i$, and write, when convenient, $\mathcal{P} = \left(G_{\tau(p)}\right)_{p \in \Pr}$ for $\mathcal{P} = \left(\left(G_{p_{\tau(1)}}, ..., G_{p_{\tau(i)}}\right)\right)_{i \in \mathbb{N}}$.*

b. The choice of working simultaneously with all primes is out of notational convenience. We could have limited the definition to primes in $\pi(G)$, since primes outside $\pi(G)$ do not really matter. For this reason, if $\pi(G)$ is finite, modulo the harmless addition of trivial Sylows, Definition 4.1 agrees with the finite group definition given in Section 1 and we will freely use the latter in this case. Furthermore, note that if $\pi(G)$ is finite, an opposite Sylow sequence is essentially just a Sylow sequence while if $\pi(G)$ is infinite it has to be introduced as an independent object.

For any non-empty subset S of a group G denote $S^{-1} := \left\{s^{-1} | s \in S\right\}$.

Lemma 4.3. *Let G be a profinite group. Let $\mathcal{P} = \left(\left(G_{p_{\tau(1)}}, ..., G_{p_{\tau(i)}}\right)\right)_{i \in \mathbb{N}}$ be a complete Sylow sequence of G of type τ. Then:*

(a) Let N be a closed normal subgroup of G. Then $\mathcal{P}_{G/N} := \left(G_{p_{\tau(p)}} N/N\right)_{p \in \Pr}$ is a complete Sylow sequence of G of type τ, and $\Pi(\mathcal{P}) N/N = \Pi\left(\mathcal{P}_{G/N}\right)$.

(b) $\Pi(\mathcal{P}) = G$ if and only if $\Pi\left(\mathcal{P}_{G/N}\right) = G/N$ for every $N \in \mathcal{N}^{(G)}$.

(c) $\Pi(\mathcal{P})^{-1} = \Pi(\mathcal{P}^{op})$.

Proof. (a) Let $\varphi_N : G \to G/N$ be the natural projection. For any $p \in \Pr$ we have that $\varphi_N(G_p) = G_p N/N$ is a Sylow p-subgroup of G/N ([15, Proposition 2.2.3(b)]). Hence φ_N maps $\mathcal{P}$ to $\mathcal{P}_{G/N}$. Moreover, since φ_N is a group homomorphism, $\varphi_N\left(G_{p_{\tau(1)}} \cdots G_{p_{\tau(i)}}\right) = (G/N)_{p_{\tau(1)}} \cdots (G/N)_{p_{\tau(i)}}$. By continuity of φ_N, we have $\varphi_N(cl(S)) \subseteq cl(\varphi_N(S))$ for any $S \subseteq G$. Applying this for $S = \bigcup_{i=1}^{\infty} G_{p_{\tau(1)}} \cdots G_{p_{\tau(i)}}$, we get $\varphi_N(\Pi(\mathcal{P})) \subseteq \Pi\left(\mathcal{P}_{G/N}\right)$. Now, by [15, Lemma 0.1.2(a)-(c)], $\varphi_N(\Pi(\mathcal{P}))$ is closed and hence we obtain $\varphi_N(\Pi(\mathcal{P})) = \Pi\left(\mathcal{P}_{G/N}\right)$.

(b) If $\Pi(\mathcal{P}) = G$ then, by (a), $\Pi(\mathcal{P}) N/N = \Pi\left(\mathcal{P}_{G/N}\right) = G/N$ for every $N \in \mathcal{N}^{(G)}$. In the other direction suppose that $\Pi\left(\mathcal{P}_{G/N}\right) = G/N$ for every $N \in \mathcal{N}^{(G)}$. Then, by (a), $\Pi(\mathcal{P}) N/N = G/N$ for every $N \in \mathcal{N}^{(G)}$. Using Lemma 2.2 with $A = \Pi(\mathcal{P})$ and $B = G$ (both are closed) gives $\Pi(\mathcal{P}) = G$.

(c) Set $S := \bigcup_{i=1}^{\infty} G_{p_{\tau(1)}} \cdots G_{p_{\tau(i)}}$. For any $i \in \mathbb{N}$ we have $\left(G_{p_{\tau(1)}} \cdots G_{p_{\tau(i)}}\right)^{-1} = G_{p_{\tau(i)}} \cdots G_{p_{\tau(1)}}$. Hence, by definition, $cl\left(S^{-1}\right) = \Pi(\mathcal{P}^{op})$. On the other hand, $S \subseteq \Pi(\mathcal{P})$, so we have $S^{-1} \subseteq (\Pi(\mathcal{P}))^{-1}$, and since the map $G \to G$ defined by $g \longmapsto g^{-1}$ is a homeomorphism, $(cl(S))^{-1} = cl\left(S^{-1}\right)$. Therefore

$$(\Pi(\mathcal{P}))^{-1} = (cl(S))^{-1} = cl\left(S^{-1}\right) = \Pi(\mathcal{P}^{op}).$$

∎

Proof of Theorem 1.6. (1)-(3) are equivalent for any finite group G (for a finite group prosolvability is the same as solvability) by [5, Theorem A]. Now the claim follows from Lemma 4.3 (b). ∎

5. Sylow factorizations in a profinite group

In this section we define the Sylow multiplicity of a given element in a given Sylow sequence and prove Proposition 1.7.

Definition 5.1. *Let G be a profinite group, let $g \in G$ and let $\mathcal{P} = \left(G_{\tau(p)}\right)_{p\in\mathrm{Pr}}$ be a complete Sylow sequence of type τ. An element sequence in $\mathcal{P}$ is a sequence $\left(g_{\tau(p)}\right)_{p\in\mathrm{Pr}}$ where for each $p \in \mathrm{Pr}$, $g_{\tau(p)} \in G_{\tau(p)}$. A factorization of g in $\mathcal{P}$ is an element sequence $\left(g_{\tau(p)}\right)_{p\in\mathrm{Pr}}$ in $\mathcal{P}$ such that $g = \lim_{k\to\infty}\left(g_{\tau(p_1)}\cdots g_{\tau(p_k)}\right)$. We denote by $M_{\mathcal{P}}(g)$ the set of all factorizations of g in $\mathcal{P}$, and by $m_{\mathcal{P}}(g)$ the cardinality of $M_{\mathcal{P}}(g)$.*

Remark 5.2. *In the notations of Definition 5.1, let $\left(g_{\tau(p)}\right)_{p\in\mathrm{Pr}}$ be any element sequence in $\mathcal{P}$. Then, since G is Hausdorff, $\left(g_{\tau(p)}\right)_{p\in\mathrm{Pr}}$ has at most one limit, and if the limit exists, it belongs to $\Pi(\mathcal{P})$, since $\Pi(\mathcal{P})$ is closed. It follows that if $m_{\mathcal{P}}(g) = 1$ for every $g \in G$ then $G = \Pi(\mathcal{P})$.*

Lemma 5.3. *Let G be a profinite group, and let $\mathcal{P} = \left(G_{\tau(p)}\right)_{p\in\mathrm{Pr}}$ be a complete Sylow sequence of type τ. Let $g \in \Pi(\mathcal{P})$. Then g has a factorization in $\mathcal{P}$.*

Proof. Let $Cr\left(G_{\tau(p)}\right)$ be the Cartesian product of all of the Sylow subgroups of G which appear in the complete Sylow sequence $\mathcal{P}$. We view $Cr\left(G_{\tau(p)}\right)$ as a topological space with the product topology (each Sylow subgroup is a closed subgroup of G). We define a net in $Cr\left(G_{\tau(p)}\right)$ which is based on $\mathcal{N}^{(G)}$, in the following way. First note that since $\mathcal{N}^{(G)}$ is a filter base it is a directed system with respect to the partial order relation defined by: $N_1 \leq_D N_2$ if and only if $N_2 \leq N_1$ (for every $N_1, N_2 \in \mathcal{N}^{(G)}$ we have $N_1 \cap N_2 \in \mathcal{N}^{(G)}$ and $N_1 \leq_D N_1 \cap N_2$ and $N_2 \leq_D N_1 \cap N_2$). By Lemma 4.3 (a), $\mathcal{P}_{G/N} := \left(G_{\tau(p)}N/N\right)_{p\in\mathrm{Pr}}$ is a complete Sylow sequence of type τ of the finite group G/N, and $\Pi(\mathcal{P})N/N = \Pi\left(\mathcal{P}_{G/N}\right)$, for any $N \in \mathcal{N}^{(G)}$. Thus, there exists some positive integer k_N (which depends on N) such that $\Pi(\mathcal{P})N/N = \left(G_{\tau(p_1)}N/N\right)\cdots\left(G_{\tau\left(p_{k_N}\right)}N/N\right)$ and hence there exist elements $g_{i,N} \in G_{\tau(p_i)}$, $1 \leq i \leq k_N$, such that $gN = g_{1,N}\cdots g_{k_N,N}N$. We define our net by associating to $N \in \mathcal{N}^{(G)}$ the element $(g_{1,N}, \ldots, g_{k_N,N}, 1, 1, \ldots) \in Cr\left(G_{\tau(p)}\right)$. Since each Sylow subgroup is closed and hence compact, $Cr\left(G_{\tau(p)}\right)$ is compact, and therefore the net we have just defined has at least one cluster point. Let $(g_i)_{i\in\mathbb{N}}$ be such a cluster point, where $g_i \in G_{\tau(p_i)}$ for all i. We claim that $g = \lim_{k\to\infty}(g_1\cdots g_k)$. In order to prove this, let U be an open neighborhood of g. We have to show that there is some positive integer k_0 such that for all $k \geq k_0$ it holds that $g_{\tau(p_1)}\cdots g_{\tau(p_k)} \in U$. By [15, Proposition 0.3.3(a)], U is a union of cosets of open normal subgroups of G. Hence there exists $N \in \mathcal{N}^{(G)}$ such that g belongs to a coset of N. This coset is clearly equal to gN. Since $(g_i)_{i\in\mathbb{N}}$ is a cluster point for the net we have defined, there exists $N_1 \in \mathcal{N}^{(G)}$, $N \leq_D N_1$ (which is equivalent to $N_1 \leq N$) such that $g_iN_1 = g_{i,N_1}N_1$ for each i. Since $gN_1 = g_{1,N_1}\cdots g_{k_{N_1},N_1}N_1$, we get $gN_1 = g_1\cdots g_{k_{N_1}}N_1$. Moreover, for all $k > k_{N_1}$ we have $g_kN_1 = N_1$, so $gN_1 = g_1\cdots g_kN_1$, for all $k \geq k_{N_1}$, or equivalently, $g_1\cdots g_k \in gN_1 \subseteq gN \subseteq U$ for all $k \geq k_{N_1}$. ∎

Proof of Proposition 1.7. If $m_{\mathcal{P}}(g) = 1$ for all $g \in G$ then $G = \Pi(\mathcal{P})$ by Remark 5.2. To prove the other direction suppose that $G = \Pi(\mathcal{P})$. Then, by Lemma 5.3, $m_{\mathcal{P}}(g) \geq 1$ for all $g \in G$. It remains to show that if $\lim_{k\to\infty}(g_1\cdots g_k) = \lim_{k\to\infty}(g_1'\cdots g_k')$, where $g_i, g_i' \in G_{\tau(p_i)}$ for all $i \geq 1$ then $g_i = g_i'$ for all $i \geq 1$. By definition of a limit the equality of the two limits implies that for each $N \in \mathcal{N}^{(G)}$ there exists some positive integer k_0 such that for all $k \geq k_0$ we have

$(g_1 N)\cdots(g_k N) = (g'_1 N)\cdots(g'_k N)$. The factorizations on both sides of the last equality are Sylow factorizations in the Sylow sequence $\mathcal{P}_{G/N}$ of the finite group G/N which satisfies $G/N = \Pi\left(\mathcal{P}_{G/N}\right)$. The last equality implies $m_{\mathcal{P}_{G/N}}(x) = 1$ for all $x \in G/N$, and therefore $g_i N = g'_i N$ for all $i \geq 1$. Equivalently, $g_i^{-1} g'_i \in N$ for all $i \geq 1$. Thus, for all $i \geq 1$,

$$g_i^{-1} g'_i \in \bigcap_{N \in \mathcal{N}(G)} N = 1,$$

and therefore $g_i = g'_i$ for all $i \geq 1$. ∎

6. The prosolvable radical

Here we consider the relationship between the prosolvable radical of G and the intersection of all complete Sylow products of a profinite group G and their opposites, proving Theorem 1.10 and Proposition 1.11. We begin with some basic properties of the prosolvable radical.

The next Lemma is a consequence of prosolvable groups forming a Mel'nikov formation and its proof follows from [12, Prop. 2.2.1].

Lemma 6.1. *Let G be a profinite group, and let K be a closed normal subgroup of G such that both K and G/K are prosolvable. Then G is prosolvable.*

Corollary 6.2. *Let G be a profinite group. Then $R(G/R(G)) = 1$.*

Proof. Suppose by contradiction that $R(G/R(G)) > 1$. Let K be the inverse image of $R(G/N)$ in G with respect to the natural map $G \to G/N$. Then $R(G) < K \trianglelefteq G$, and since both $K/R(G)$ and $R(G)$ are prosolvable, K is prosolvable by Lemma 6.1, contradicting $K > R(G)$. ∎

Lemma 6.3. *Let G be a profinite group, and let N be a closed normal subgroup of G. Then:*

(a) $R(N) = R(G) \cap N$.
(b) $R(G)N/N \leq R(G/N)$.

Proof. (a) Using the properties of $R(G)$ we get that $R(G) \cap N$ is a closed prosolvable normal subgroup of N. Therefore $R(G) \cap N \leq R(N)$. Suppose, by contradiction, that $R(G) \cap N < R(N)$. Then $R(N)/N \cap R(G)$ is a non-trivial, closed, normal, prosolvable group of $N/N \cap R(G)$. But $N/N \cap R(G)$ and $R(G)N/R(G)$ are isomorphic as profinite groups, therefore, $R(R(G)N/R(G))$ is non-trivial. Since $R(R(G)N/R(G))$ is characteristic, it is normal in $G/R(G)$ and hence $R(G/R(G))$ is non-trivial, in contradiction to Corollary 6.2.

(b) $R(G)N/N \trianglelefteq G/N$, and $R(G)N/N \cong R(G)/N \cap R(G)$ which is prosolvable. Hence $R(G)N/N \leq R(G/N)$. ∎

Note that in Lemma 6.3(b) we can have $R(G)N/N < R(G/N)$, e.g., $G = Sym(5)$, $N = Alt(5)$.

The following notion was introduced in [2], and the discussion which ensues is based on its application to Sylow sequences of finite groups in [5].

Definition 6.4. *Let S be a non-empty subset of a group G.* The (left) kernel *of S, denoted by $K_L(S)$, is defined by:*

$$K_L(S) := \{g \in G \mid gS = S\}.$$

It is easy to check that $K_L(S)$ is a subgroup of G, and that S is a union of right cosets of $K_L(S)$. Furthermore, if $1 \in S$ then $K_L(S) \subseteq S$. Also, for any $g \in G$ we have $K_L(S^g) = (K_L(S))^g$. For the discussion of left kernels of Sylow products the following characterization of kernels is useful.

Lemma 6.5. *Let S be a non-empty subset of a group G. Then*

$$K_L(S) = \left(\bigcap_{a \in S} Sa^{-1}\right) \cap \left(\bigcap_{a \in S} aS^{-1}\right). \tag{6.1}$$

It follows that if G is a topological group and S is closed then $K_L(S)$ is closed.

Proof. Let $a \in S$. Then $g \in Sa^{-1}$ if and only if there exists $s \in S$ such that $g = sa^{-1}$ which is equivalent to $ga = s \in S$. Hence $g \in \bigcap\limits_{a \in S} Sa^{-1}$ if and only if $ga \in S$ for every $a \in S$ which is equivalent to $gS \subseteq S$. Similarly, $g \in \bigcap\limits_{a \in S} aS^{-1}$, if and only if $S \subseteq gS$. Therefore, $g \in \left(\bigcap\limits_{a \in S} Sa^{-1}\right) \cap \left(\bigcap\limits_{a \in S} aS^{-1}\right)$ if and only if $S \subseteq gS \subseteq S$ which is equivalent to $gS = S$. This proves Equation 6.1.

If G is is a topological group and S is closed, then by continuity of the group operations, Sa^{-1} and aS^{-1} are closed for any $a \in G$, and hence $K_L(S)$ is closed, being the intersection of closed subsets. ∎

Remark 6.6. *If G is finite we have $K_L(S) = \bigcap\limits_{a \in S} Sa^{-1} = \bigcap\limits_{a \in S} aS^{-1}$.*

Definition 6.7. *Let G be a profinite group, and let $\mathcal{P} = ((P_1, ..., P_i))_{i \in \mathbb{N}}$ be a complete Sylow sequence of G of type τ (so P_i is a $p_{\tau(i)}$-Sylow subgroup of G for all $i \in \mathbb{N}$). Let $(a_i)_{i \in \mathbb{N}}$ be an element sequence in $\mathcal{P}$ such that $\lim_{i \to \infty} a_1 a_2 \cdots a_i$ exists. Then $\mathcal{P}\left(a_i^{-1}\right)_{i \in \mathbb{N}} = ((R_1, ..., R_i))_{i \in \mathbb{N}}$ is the complete Sylow sequence of G of type τ which is defined by:*

$$R_1 := P_1, \; R_i := P_i^{a_{i-1}^{-1} \cdots a_1^{-1}} \; \forall i \geq 2.$$

We denote the set of all Sylow sequences of G of the form $\mathcal{P}\left(a_i^{-1}\right)_{i \in \mathbb{N}}$ where $\lim_{i \to \infty} a_1 a_2 \cdots a_i$ exists, by $R(\mathcal{P})$, and $R(\mathcal{P})^{op} := \{Q^{op} | Q \in R(\mathcal{P})\}$.

Lemma 6.8. *Let G be a profinite group, and let $\mathcal{P} = (P_1, ..., P_i)_{i \in \mathbb{N}}$ be a complete Sylow sequence of G of type τ. Let $(a_i)_{i \in \mathbb{N}}$ be an element sequence in $\mathcal{P}$ such that the limit $a := \lim_{i \to \infty} a_1 a_2 \cdots a_i$ exists. Set $\mathcal{Q} = \mathcal{P}\left(a_i^{-1}\right)_{i \in \mathbb{N}}$. Then $\Pi(\mathcal{Q}) = \Pi(\mathcal{P})\, a^{-1}$, and $\Pi(\mathcal{Q}^{op}) = a\Pi(\mathcal{P})^{-1}$.*

Proof. Both $\Pi(\mathcal{Q})$, and $\Pi(\mathcal{P})\, a^{-1}$ are closed, hence it suffices, by Lemma 2.2, to prove that $\Pi(\mathcal{Q})N/N = \Pi(\mathcal{P})\, a^{-1}N/N$ for every $N \in \mathcal{N}^{(G)}$. Let $N \in \mathcal{N}^{(G)}$ be arbitrary. Let $\varphi_N : G \to G/N$ be the natural projection. By Lemma 4.3(a), $\varphi_N(\mathcal{Q}) = \mathcal{Q}_{G/N}$ and $\varphi_N(\mathcal{P}) = \mathcal{P}_{G/N}$ are complete Sylow sequences of the finite group G/N, and $\Pi(\mathcal{Q})N/N = \Pi(\mathcal{Q}_{G/N})$, $\Pi(\mathcal{P})\, N/N = \Pi\left(\mathcal{P}_{G/N}\right)$. Therefore, $\Pi(\mathcal{P})\, a^{-1}N/N = \Pi\left(\mathcal{P}_{G/N}\right)\left(a^{-1}N\right)$. Moreover, by finiteness of G/N, there exist distinct positive integers $i_1, ..., i_m$ such that $\pi(G/N) = \{p_{i_1}, ..., p_{i_m}\}$. Hence, $\varphi_N(a_i) = 1$ for all i with $p_i \notin \pi(G/N)$ and $\varphi_N(a) = \varphi_N(a_{i_1}) \cdots \varphi_N(a_{i_m}) = (a_{i_1}N) \cdots (a_{i_m}N)$ is a Sylow factorization of $\varphi_N(a)$ in $\varphi_N(\mathcal{P})$. We get $\Pi(\mathcal{Q}_{G/N}) = \Pi\left(\mathcal{P}_{G/N}\right)\left(a^{-1}N\right)$ by [5, Lemma 8].

For $\Pi(\mathcal{Q}^{op}) = a\Pi(\mathcal{P})^{-1}$, we use Lemma 4.3 (c), and the first part of the proof:

$$\Pi(\mathcal{Q}^{op}) = \left(\Pi\left(\mathcal{P}\left(a_i^{-1}\right)_{i\in\mathbb{N}}\right)\right)^{-1} = \left(\Pi(\mathcal{P})\, a^{-1}\right)^{-1} = a\Pi(\mathcal{P})^{-1}.$$

∎

Proposition 6.9. *Let G be a profinite group, and let $\mathcal{P}$ be a complete Sylow sequence of G. Then*

$$K_L(\Pi(\mathcal{P})) = \bigcap_{\mathcal{Q}\in R(\mathcal{P})\cup R(\mathcal{P})^{op}} \Pi(\mathcal{Q}).$$

Proof. By Lemma 6.5 it would suffice to prove that

$$(*)\qquad \left(\bigcap_{a\in\Pi(\mathcal{P})} \Pi(\mathcal{P})\, a^{-1}\right) \cap \left(\bigcap_{a\in\Pi(\mathcal{P})} a\Pi(\mathcal{P})^{-1}\right) = \bigcap_{\mathcal{Q}\in R(\mathcal{P})\cup R(\mathcal{P})^{op}} \Pi(\mathcal{Q}).$$

By Lemma 6.8, for any $\mathcal{Q} \in R(\mathcal{P})\cup R(\mathcal{P})^{op}$, there exists $a \in \Pi(\mathcal{P})$ (see Remark 5.2) such that either $\Pi(\mathcal{Q}) = \Pi(\mathcal{P})\, a^{-1}$ or $\Pi(\mathcal{Q}) = a\Pi(\mathcal{P})^{-1}$. Conversely, if $a \in \Pi(\mathcal{P})$, then by Lemma 5.3, a has a factorization $(a_i)_{i\in\mathbb{N}}$ in $\mathcal{P} = ((P_1, ..., P_i))_{i\in\mathbb{N}}$, such that $a = \lim_{i\to\infty} a_1a_2\cdots a_i$ and hence $\mathcal{Q} = \mathcal{P}\left(a_i^{-1}\right)_{i\in\mathbb{N}} \in R(\mathcal{P})$, and $\mathcal{Q}^{op} \in R(\mathcal{P})^{op}$. By Lemma 6.8, $\Pi(\mathcal{Q}) = \Pi(\mathcal{P})\, a^{-1}$, and $\Pi(\mathcal{Q}^{op}) = a\Pi(\mathcal{P})^{-1}$. This proves (*). ∎

Now we can define the main object of interest in this section.

Definition 6.10. *Let G be a profinite group, and let τ be any permutation of $\mathbb{N}$. Denote by $CSS(G)$ ($CSS_\tau(G)$) the set of all complete Sylow sequences of G (of type τ). Then*

$$\begin{aligned} H_\tau(G) &:= \bigcap_{\mathcal{P}\in CSS_\tau(G)} (\Pi(\mathcal{P})\cap\Pi(\mathcal{P}^{op})) \\ H(G) &:= \bigcap_{\mathcal{P}\in CSS(G)} (\Pi(\mathcal{P})\cap\Pi(\mathcal{P}^{op})). \end{aligned}$$

Note that $H(G) = \cap_\tau H_\tau(G)$.

Proof of Theorem 1.10. (a) By definition of $H_\tau(G)$ and Proposition 6.9 we get:

$$\begin{aligned} H_\tau(G) &= \bigcap_{\mathcal{P}\in CSS_\tau(G)} (\Pi(\mathcal{P})\cap\Pi(\mathcal{P}^{op})) = \bigcap_{\mathcal{P}\in CSS_\tau(G)}\ \bigcap_{\mathcal{Q}\in R(\mathcal{P})\cup R(\mathcal{P})^{op}} \Pi(\mathcal{Q}) = \\ &= \bigcap_{\mathcal{P}\in CSS_\tau(G)} K_L(\Pi(\mathcal{P})). \end{aligned}$$

Thus $H_\tau(G)$ is a closed subgroup of G. Let α be a continuous automorphism of G. Using the fact that the image of a p-Sylow subgroup under α is also a p-Sylow subgroup, it follows that $\Pi(\mathcal{P})^\alpha = \Pi(\mathcal{P}^\alpha)$ where, if $\mathcal{P} = ((P_1, ..., P_i))_{i\in\mathbb{N}}$ then $\mathcal{P}^\alpha := ((P_1^\alpha, ..., P_i^\alpha))_{i\in\mathbb{N}}$. Similarly, $\Pi(\mathcal{P}^{op})^\alpha = \Pi((\mathcal{P}^{op})^\alpha)$. Moreover, if $\mathcal{P}$ is of type τ then so is $\mathcal{P}^\alpha$, and $(\mathcal{P}^{op})^\alpha = (\mathcal{P}^\alpha)^{op}$. Hence α induces a bijection $CSS_\tau(G) \to CSS_\tau(G)$. It follows that $H_\tau(G)$ is characteristic.

(b) $\Pi(\mathcal{P})$ is a union of right cosets of $K_L(\Pi(\mathcal{P}))$ (see the remarks following Definition 6.4). Consequently, $\Pi(\mathcal{P}^{op}) = \Pi(\mathcal{P})^{-1}$ is a union of left cosets of $K_L(\Pi(\mathcal{P}))$. Since by definition $H_\tau(G) \subseteq K_L(\Pi(\mathcal{P}))$, and by (a) $H_\tau(G) \trianglelefteq G$, we get that $K_L(\Pi(\mathcal{P}))$ is a union of cosets of $H_\tau(G)$, and the claim follows.

(c) Let P be a p-Sylow subgroup of G. By [15, Proposition 2.2.3(a)], $R(G) \cap P$ is a p-Sylow subgroup of $R(G)$. It follows that if $\mathcal{P} = ((P_1, ..., P_i))_{i\in\mathbb{N}}$ is any complete Sylow sequence of G, then $\mathcal{Q} = ((Q_1, ..., Q_i))_{i\in\mathbb{N}}$, where $Q_i = R(G) \cap P_i$, is a complete Sylow sequence of $R(G)$. Now, since $R(G)$ is prosolvable, we get by Theorem 1.6 that $R(G) = \Pi(\mathcal{Q}) = \Pi(\mathcal{Q})^{-1} = \Pi(\mathcal{Q}^{op})$. Clearly, $\Pi(\mathcal{Q}) \subseteq \Pi(\mathcal{P})$, and $\Pi(\mathcal{Q}^{op}) \subseteq \Pi(\mathcal{P}^{op})$, so $R(G) \subseteq \Pi(\mathcal{P})\cap\Pi(\mathcal{P}^{op})$ for every $\mathcal{P} \in CSS(G)$. This implies $R(G) \leq H(G)$. Hence $R(G) \leq R(H(G))$. But $R(H(G))$ is characteristic in $H(G)$, hence $R(H(G)) \leq R(G)$, and equality follows.

(d) For all $K \leq G$, set $\widehat{K} := KH_\tau(G)/H_\tau(G)$. Note that $\widehat{G}$ is profinite since $H_\tau(G)$ is a normal closed subgroup of G. Under the natural mapping $G \to \widehat{G}$, each Sylow sequence $\mathcal{P}$ is mapped to a unique Sylow sequence of $\widehat{G}$, of the same type, which we denote $\widehat{\mathcal{P}}$, and $\Pi(\mathcal{P})$ is mapped onto $\Pi\left(\widehat{\mathcal{P}}\right)$ (Lemma 4.3(a)). Hence:

$$H_\tau\left(\widehat{G}\right) = \bigcap_{\mathcal{Q}\in CSS_\tau\left(\widehat{G}\right)} (\Pi(Q) \cap \Pi(Q^{op})) = \bigcap_{\mathcal{P}\in CSS_\tau(G)} \left(\Pi(\widehat{\mathcal{P}}) \cap \Pi(\widehat{\mathcal{P}}^{op})\right).$$

Let $\mathcal{P} = \left(G_{\tau(p)}\right)_{p\in\Pr} \in CSS_\tau(G)$. Using:

$$\begin{aligned}\widehat{G}_{p_{\tau(1)}} \cdots \widehat{G}_{p_{\tau(i)}} &= \left(G_{p_{\tau(1)}} H_\tau(G)\right)/H_\tau(G) \cdots \left(G_{p_{\tau(i)}} H_\tau(G)\right)/H_\tau(G) = \\ &= \left(G_{p_{\tau(1)}} \cdots G_{p_{\tau(i)}} H_\tau(G)\right)/H_\tau(G),\end{aligned}$$

we get:

$$\begin{aligned}\Pi\left(\widehat{\mathcal{P}}\right) &= cl\left(\bigcup_{i=1}^{\infty} \widehat{G}_{p_{\tau(1)}} \cdots \widehat{G}_{p_{\tau(i)}}\right) = cl\left(\bigcup_{i=1}^{\infty} \left(G_{p_{\tau(1)}} \cdots G_{p_{\tau(i)}} H_\tau(G)\right)/H_\tau(G)\right) = \\ &= \left(cl\left(\bigcup_{i=1}^{\infty} G_{p_{\tau(1)}} \cdots G_{p_{\tau(i)}}\right)\right) H_\tau(G)/H_\tau(G) = \Pi(\mathcal{P}) H_\tau(G)/H_\tau(G),\end{aligned}$$

where we have also used the fact that images of closed sets under $G \to \widehat{G}$ are closed, that $H_\tau(G)$ is closed and Lemma 2.2. Part (b) and $H_\tau(G) \trianglelefteq G$ give:

$$\Pi(\mathcal{P}) H_\tau(G) = H_\tau(G) \Pi(\mathcal{P}) = \Pi(\mathcal{P}).$$

Hence $\Pi\left(\widehat{\mathcal{P}}\right) = \Pi(\mathcal{P})/H_\tau(G)$, and also $\Pi\left(\widehat{\mathcal{P}}^{op}\right) = \Pi(\mathcal{P}^{op})/H_\tau(G)$. We get:

$$\begin{aligned}H_\tau\left(\widehat{G}\right) &= \bigcap_{\mathcal{P}\in CSS_\tau(G)} \left(\Pi(\widehat{\mathcal{P}}) \cap \Pi(\widehat{\mathcal{P}}^{op})\right) \\ &= \bigcap_{\mathcal{P}\in CSS_\tau(G)} \left((\Pi(\mathcal{P})/H_\tau(G)) \cap (\Pi(\mathcal{P}^{op})/H_\tau(G))\right) \\ &= \bigcap_{\mathcal{P}\in CSS_\tau(G)} (\Pi(\mathcal{P}) \cap \Pi(\mathcal{P}^{op}))/H_\tau(G) = H_\tau(G)/H_\tau(G) = 1.\end{aligned}$$

Finally, $R(G/H_\tau(G)) = 1$ follows immediately from $R(G/H_\tau(G)) \leq H(G/H_\tau(G))$.

(e) Essentially the same proof as (d). ∎

Lemma 6.11. *Let G be a profinite group, K a closed normal subgroup of a group G, and let τ a permutation of $\mathbb{N}$. Then $H_\tau(G)K/K \leq H_\tau(G/K)$, and consequently $H(G)K/K \leq H(G/K)$.*

Proof. Let $\mathcal{Q} \in CSS_\tau(G/K)$, and let $\mathcal{P} \in CSS_\tau(G)$ be such that $\mathcal{P}$ is mapped to $\mathcal{Q}$ under the natural map $G \to G/K$ (see proof of Theorem 1.10(d)). As is shown in the proof of Theorem 1.10(d), $H_\tau(G) \subseteq \Pi(\mathcal{P})$ and $H_\tau(G) \subseteq \Pi(\mathcal{P}^{op})$. Therefore $H_\tau(G)K \subseteq (\Pi(\mathcal{P}) \cap \Pi(\mathcal{P}^{op}))$. Using the same reasoning as in the proof of Theorem 1.10(d), this implies $H_\tau(G)K/K \subseteq (\Pi(\mathcal{Q}) \cap \Pi(\mathcal{Q}^{op}))$. Since $\mathcal{Q}$ is arbitrary, it follows that $H_\tau(G)K/K \leq H_\tau(G/K)$.

From the last claim we get $H(G)K/K \leq H_\tau(G/K)$ for any permutation τ of $\mathbb{N}$, since $H(G) \leq H_\tau(G)$. Taking the intersection over all possible τ, yields $H(G)K/K \leq H(G/K)$. ∎

Proof of Proposition 1.11. Set $\mathcal{N}_{H(G)} := \left\{N \cap H(G) \,|\, N \in \mathcal{N}^{(G)}\right\}$. To prove our claim it will suffice to prove that $H(G)$ is prosolvable, and for this it is sufficient to prove that $H(G)/N$ is solvable for any $N \in \mathcal{N}_{H(G)}$. Let $N \in \mathcal{N}_{H(G)}$ be arbitrary. Let $N_1 \in \mathcal{N}^{(G)}$ be such that $N = N_1 \cap H(G)$. Then, by Lemma 6.11, we have:

$$H(G)/N \cong H(G)N_1/N_1 \leq H(G/N_1) = R(G/N_1),$$

proving that $H(G)/N$ is solvable. ∎

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