Workshop on boundary elements and adaptivity
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Optimal convergence rates for goal-oriented adaptivity

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joint work with

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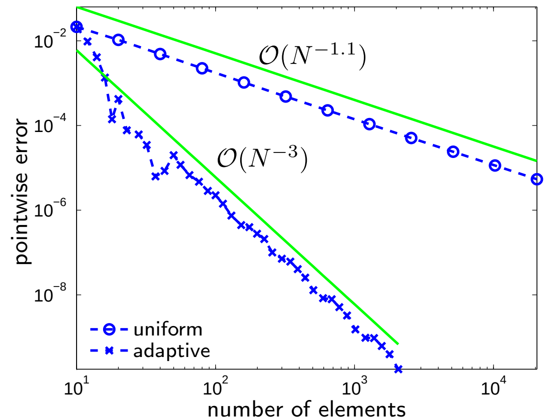
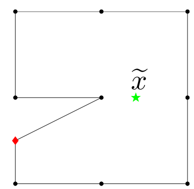
FWF Der Wissenschaftsfonds.

Outline

What is all about?

GOAL: For $\widetilde{x} \in \Omega$, compute $u(\widetilde{x})$, where $-\Delta u = 0$, $u|_\Gamma = g$





Model problem

- $\Omega \subset \mathbb{R}^d$ bounded Lipschitz domain, $d = 2, 3$
- $\Gamma := \partial\Omega$

Laplace-Dirichlet problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in} \quad \Omega \\ u &= g \quad \text{on} \quad \Gamma \end{aligned}$$

- simple-layer operator

$$V\phi(\widetilde{x}) := \int_\Gamma G(\widetilde{x}, y)\phi(y)dy$$

- double-layer operator

$$Kg(\widetilde{x}) := \int_\Gamma \partial_{n(y)} G(\widetilde{x}, y)g(y)dy$$

Representation formula

$$u(\widetilde{x}) = V\partial_n u(\widetilde{x}) - Kg(\widetilde{x}) \quad \text{for all} \quad \widetilde{x} \in \Omega$$

Galerkin BEM

- normal derivative $\phi = \partial_n u$ obtained by

$$V\phi = (K + 1/2)g =: f \quad \text{on } \Gamma$$

- mesh $\mathcal{T}_\star$ of Γ
- $\mathcal{P}^p(\mathcal{T}_\star)$ space of piecewise polynomials of degree $\leq p$

Galerkin approximation of $V\phi = f$

For given $f \in H^{1/2}(\Gamma)$, find $\Phi_\star \in \mathcal{P}^p(\mathcal{T}_\star)$ s.t.

$$\langle V\Phi_\star \,,\, X_\star \rangle_{L^2(\Gamma)} = \langle f \,,\, X_\star \rangle_{L^2(\Gamma)} \quad \text{for all } X_\star \in \mathcal{P}^p(\mathcal{T}_\star)$$

- Lax-Milgram $\implies$ exists unique solution $\Phi_\star$

Main idea 1/3

- approximate solution $u = V\phi - Kg$ by

$$u \approx u_\ell := V\Phi_\ell - Kg \quad \text{in} \quad \Omega$$

- for fixed $\widetilde{x} \in \Omega$

$$u(\widetilde{x}) - u_\ell(\widetilde{x}) = V(\phi - \Phi_\ell)(\widetilde{x}) = \langle G(\widetilde{x},\cdot)\,,\,\phi - \Phi_\ell\rangle$$

- Galerkin orthogonality

$$\langle V\Psi_\ell\,,\,\phi - \Phi_\ell\rangle = 0 \quad \text{for all } \Psi_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)$$

$\Longrightarrow$ obtain

$$u(\widetilde{x}) - u_\ell(\widetilde{x}) = \langle G(\widetilde{x},\cdot) - V\Psi_\ell\,,\,\phi - \Phi_\ell\rangle$$

Main idea 2/3

- for fixed $\widetilde{x} \in \Omega$ and all $\Psi_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)$

$$u(\widetilde{x}) - u_\ell(\widetilde{x}) = \langle G(\widetilde{x},\cdot) - V\Psi_\ell\,,\,\phi - \Phi_\ell\rangle$$

Dual problem

- consider auxiliary problem

$$V\psi(\cdot) = G(\widetilde{x},\cdot)$$

- find Galerkin approximation $\Psi_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)$ s.t.

$$\langle V\Psi_\ell\,,\,X_\ell\rangle_{L^2(\Gamma)} = \langle G(\widetilde{x},\cdot)\,,\,X_\ell\rangle_{L^2(\Gamma)} \quad \text{for all } X_\ell \in \mathcal{P}^p(\mathcal{T}_\star)$$

Main idea 3/3

- for fixed $\widetilde{x} \in \Omega$ and $\Psi_\ell \approx \psi = V^{-1}G(\widetilde{x},\cdot)$

$$u(\widetilde{x}) - u_\ell(\widetilde{x}) = \langle G(\widetilde{x},\cdot) - V\Psi_\ell\,,\,\phi - \Phi_\ell\rangle = \langle V(\psi - \Psi_\ell)\,,\,\phi - \Phi_\ell\rangle$$

- with energy norm $|\!|\!|\, z \,|\!|\!|^2 = \langle Vz\,,\,z\rangle \simeq \|z\|^2_{H^{-1/2}(\Gamma)}$

$$|u(\widetilde{x}) - u_\ell(\widetilde{x})| \le |\!|\!|\, \psi - \Psi_\ell \,|\!|\!|\; |\!|\!|\, \phi - \Phi_\ell \,|\!|\!|$$

- with appropriate error estimators

$$|u(\widetilde{x}) - u_\ell(\widetilde{x})| \le |\!|\!|\, \psi - \Psi_\ell \,|\!|\!|\; |\!|\!|\, \phi - \Phi_\ell \,|\!|\!| \lesssim \eta_{\psi,\ell}\eta_{\phi,\ell}$$

Weighted-residual error estimator

- $z \in \{\phi, \psi\}$ with Galerkin approximation $Z_\star \in \mathcal{P}^p(\mathcal{T}_\star)$
- $\eta_{z,\star}(T)^2 := |T|^{1/(d-1)}\,\|\nabla(f - VZ_\star)\|^2_{L^2(T)}$ for $T \in \mathcal{T}_\star$
- define $\eta_{z,\star} := \Big(\sum_{T\in\mathcal{T}_\star} \eta_{z,\star}(T)^2\Big)^{1/2}$
- $\eta_{z,\star}$ is reliable i.e., $|\!|\!|\, z - Z_\star \,|\!|\!| \le C_{\rm rel}\,\eta_{z,\star}$

- Carstensen, Stephan: Math. Comp. 64 (1995)
- Carstensen: Math. Comp. 65 (1996)
- Carstensen, Maischak, Stephan: Numer. Math. 90 (2001)

Solve - Estimate - Mark - Refine

- Input: initial mesh $\mathcal{T}_0$ and adaptivity parameter $0 < \theta \leq 1$

For all $\ell = 0, 1, 2, 3, \ldots$ iterate (Becker et al. '11)

1. compute approximations Φ_ℓ and Ψ_ℓ
2. compute indicators $\eta_{\phi,\ell}(T)$ and $\eta_{\psi,\ell}(T)$ for all $T \in \mathcal{T}_\ell$
3. assemble $\rho_\ell(T)^2 := \eta_{\phi,\ell}(T)^2 \eta_{\psi,\ell}^2 + \eta_{\phi,\ell}^2 \eta_{\psi,\ell}(T)^2$
4. find (minimal) set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t.

$$\theta \rho_\ell^2 \leq \sum_{T \in \mathcal{M}_\ell} \rho_\ell(T)^2$$

5. refine (at least) marked elements $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$

- Output: approximations Φ_ℓ, Ψ_ℓ and $\eta_{\phi,\ell}, \eta_{\psi,\ell}$ for all $\ell \in \mathbb{N}$

Becker, Estecahandy, Trujillo: SINUM 49 (2011)

Separate marking

- Input: initial mesh $\mathcal{T}_0$ and adaptivity parameter $0 < \theta \leq 1$

For all $\ell = 0, 1, 2, 3, \ldots$ iterate (Mommer, Stevenson '09)

1. compute approximation Φ_ℓ and Ψ_ℓ
2. compute indicators $\eta_{\phi,\ell}(T)$ and $\eta_{\psi,\ell}(T)$ for all $T \in \mathcal{T}_\ell$
3. find (minimal) sets $\mathcal{M}_{\phi,\ell}, \mathcal{M}_{\psi,\ell} \subseteq \mathcal{T}_\ell$ s.t.

$$\theta \eta_{\phi,\ell}^2 \leq \sum_{T \in \mathcal{M}_{\phi,\ell}} \eta_{\phi,\ell}(T)^2 \quad \text{and} \quad \theta \eta_{\psi,\ell}^2 \leq \sum_{T \in \mathcal{M}_{\psi,\ell}} \eta_{\psi,\ell}(T)^2$$

4. choose $\mathcal{M}_\ell \in \{\mathcal{M}_{\phi,\ell}, \mathcal{M}_{\psi,\ell}\}$ s. t. $\#\mathcal{M}_\ell = \min\{\#\mathcal{M}_{\phi,\ell}, \#\mathcal{M}_{\psi,\ell}\}$
5. refine (at least) marked elements $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$

- Output: approximations Φ_ℓ, Ψ_ℓ and $\eta_{\phi,\ell}, \eta_{\psi,\ell}$ for all $\ell \in \mathbb{N}$

Mommer, Stevenson: SINUM 47 (2009)

Approximation class

- for $s > 0$, define $z \in \mathbb{A}_s$ by

$$\|z\|_{\mathbb{A}_s} := \sup_{N \in \mathbb{N}_0} \Big((N+1)^s \min_{\#\mathcal{T}_\star - \#\mathcal{T}_0 \leq N} \eta_{z,\star} \Big) < \infty$$

- $z \in \mathbb{A}_s \iff \eta_{z,\star} = \mathcal{O}\Big((\#\mathcal{T}_\star - \#\mathcal{T}_0)^{-s} \Big)$ for optimal meshes

- main result: ABEM meshes guarantee

$$\forall s, t > 0 \quad \Big[\phi \in \mathbb{A}_s, \psi \in \mathbb{A}_t \implies \eta_{\phi,\ell}\eta_{\psi,\ell} = \mathcal{O}\Big((\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-(s+t)} \Big) \Big]$$

- consequence:
  - no gain to use other mesh for dual problem
  - no gain to use higher-order polynomials for dual problem

Optimal convergence

Theorem (Feischl, Führer, Gantner, H., Praetorius 16)

- $\forall 0 < \theta \leq 1 \quad \exists 0 < q_{\text{lin}} < 1 \quad \exists C_{\text{lin}} > 0 \quad \forall \ell, n \geq 0$
$$\eta_{\phi,\ell+n}\eta_{\psi,\ell+n} \leq C_{\text{lin}}\, q_{\text{lin}}^n\, \eta_{\phi,\ell}\eta_{\psi,\ell}$$
- $\forall 0 < \theta \ll 1 \quad \forall s,t > 0$ with $(\phi,\psi) \in \mathbb{A}_s \times \mathbb{A}_t \quad \exists C_{\text{opt}} \quad \forall \ell \geq 0$
$$\eta_{\phi,\ell}\eta_{\psi,\ell} \leq C_{\text{opt}}\, \|\phi\|_{\mathbb{A}_s}\|\psi\|_{\mathbb{A}_t}\, (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-(s+t)}$$

- algorithm realizes all possible algebraic rates $s+t$
- reliability guarantees
$$|u(\widetilde{x}) - u_\ell(\widetilde{x})| \lesssim \eta_{\phi,\ell}\eta_{\psi,\ell} \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-(s+t)}$$

Feischl, Führer, Gantner, H., Praetorius: Numer. Math., 132 (2016)

- proof follows ideas of Mommer & Stevenson
  - but avoids FEM, efficiency, bisec5-refinement
  - and gives first rigorous proof for the BET algorithm
  - also generalizes FEM works beyond Poisson model problem
- analysis only based on properties of estimator
- analysis fits into axiomatic framework
- same axioms as for standard adaptivity

Mommer, Stevenson: SINUM 47 (2009)
Carstensen, Feischl, Page, Praetorius: Comp. Math. Appl. 67 (2014)
Becker, Estecahandy, Trujillo: SINUM 49 (2011)

Axioms

- $z \in \{\phi,\psi\}$ with Galerkin approximation $Z \in \{\Phi,\Psi\}$
- for all $\mathcal{T}_\star, \mathcal{T}_\bullet$ with $\mathcal{T}_\star$ is refinement of $\mathcal{T}_\bullet$

(A1) stability on non-refined elements
$$|\eta_{z,\star}(\mathcal{T}_\bullet \cap \mathcal{T}_\star) - \eta_{z,\bullet}(\mathcal{T}_\bullet \cap \mathcal{T}_\star)| \leq C_{\text{stb}} |\!|\!|\, Z_\bullet - Z_\star \,|\!|\!|$$

(A2) reduction on refined elements
$$\eta_{z,\star}(\mathcal{T}_\star \setminus \mathcal{T}_\bullet)^2 \leq q_{\text{red}}\, \eta_{z,\bullet}(\mathcal{T}_\bullet \setminus \mathcal{T}_\star)^2 + C_{\text{red}}\, |\!|\!|\, Z_\bullet - Z_\star \,|\!|\!|^2$$

(A3) discrete reliability
$$|\!|\!|\, Z_\bullet - Z_\star \,|\!|\!| \leq C_{\text{rel}}\, \eta_{z,\bullet}(\mathcal{R}_z(\mathcal{T}_\bullet, \mathcal{T}_\star))$$

with $\mathcal{T}_\bullet \setminus \mathcal{T}_\star \subseteq \mathcal{R}_z(\mathcal{T}_\bullet,\mathcal{T}_\star)$ and $\#\mathcal{R}_z(\mathcal{T}_\bullet,\mathcal{T}_\star) \leq C_{\text{rel}}\, \#(\mathcal{T}_\bullet \setminus \mathcal{T}_\star)$

Linear convergence

- $z \in \{\phi,\psi\}$ and $\ell \in \mathbb{N}$ with mesh $\mathcal{T}_\ell$
- suppose $\ell \leq j_1 < \cdots < j_k < \ell + n$ with
$$\theta\eta_{z,j_m}^2 \leq \eta_{z,j_m}^2(\mathcal{T}_{j_m} \setminus \mathcal{T}_{j_{m+1}}) \quad \text{for all} \quad m = 1,\dots,k$$

$\Longrightarrow$ there exist $C_{\text{conv}} > 0$ and $0 < q_{\text{conv}} < 1$ such that
$$\eta_{z,\ell+n}^2 \leq C_{\text{lin}} q_{\text{conv}}^k \eta_{z,\ell}^2.$$

Linear convergence

- for all $\ell \in \mathbb{N}$, algorithm guarantees

$$\theta \eta_{\phi,\ell}^2 \le \sum_{T \in \mathcal{M}_\ell} \eta_{\phi,\ell}(T)^2 \quad \text{or} \quad \theta \eta_{\psi,\ell}^2 \le \sum_{T \in \mathcal{M}_\ell} \eta_{\psi,\ell}(T)^2$$

$\Longrightarrow$ for n steps:
  - k-times Dörfler Marking for $\eta_{\phi,\ell}$
  - $(n-k)$-times Dörfler marking for $\eta_{\psi,\ell}$

$\Longrightarrow$ $\eta_{\phi,\ell+n}^2 \le C_{\rm lin} q_{\rm conv}^k \eta_{\phi,\ell}^2$ and $\eta_{\psi,\ell+n}^2 \le C_{\rm lin} q_{\rm conv}^{(n-k)} \eta_{\psi,\ell}^2$

$\Longrightarrow$ for $q_{\rm lin} = q_{\rm conv}^{1/2}$, obtain linear convergence

$$\eta_{\phi,\ell+n}\eta_{\psi,\ell+n} \le C_{\rm lin} q_{\rm lin}^n \eta_{\phi,\ell}\eta_{\psi,\ell}$$

Control of $\#\mathcal{M}_\ell$ 1/3

Lemma

$\forall 0 < \theta \ll 1 \ \ \forall s,t > 0$ with $(\phi,\psi) \in \mathbb{A}_s \times \mathbb{A}_t \ \ \exists C_2, C_3 > 0$

$$\#\mathcal{M}_\ell \le C_2 \big(C_3 \|\phi\|_{\mathbb{A}_s} \|\psi\|_{\mathbb{A}_t}\big)^{1/(s+t)} (\eta_{\phi,\ell}\eta_{\psi,\ell})^{-1/(s+t)}$$

- for each $\kappa > 0$, there exists $C_1 > 0$ and a refinement $\widehat{\mathcal{T}}_\ell$ of $\mathcal{T}_\ell$ s.t.

$$\#\widehat{\mathcal{T}}_\ell - \#\mathcal{T}_\ell \le 2\big(C_1 \kappa^{-1/2} \|\phi\|_{\mathbb{A}_s} \|\psi\|_{\mathbb{A}_t}\big)^{1/(s+t)} (\eta_{\phi,\ell}\eta_{\psi,\ell})^{-1/(s+t)}$$

$$\widehat{\eta}_{\phi,\ell}^2 \widehat{\eta}_{\psi,\ell}^2 \le \kappa \ \eta_{\phi,\ell}^2 \eta_{\psi,\ell}^2$$

$\Longrightarrow$ $\widehat{\eta}_{\phi,\ell}^2 \le \kappa^{1/2} \eta_{\phi,\ell}^2$ or $\widehat{\eta}_{\psi,\ell}^2 \le \kappa^{1/2} \eta_{\psi,\ell}^2$

Control of $\#\mathcal{M}_\ell$ 2/3

- $\widehat{\eta}_{\phi,\ell}^2 \le \kappa^{1/2} \eta_{\phi,\ell}^2$ or $\widehat{\eta}_{\psi,\ell}^2 \le \kappa^{1/2} \eta_{\psi,\ell}^2$

- optimality of Dörfler marking for $z \in \{\phi, \psi\}$

$$\widehat{\eta}_{z,\ell}^2 \le \kappa^{1/2} \eta_{z,\ell}^2 \quad \Longrightarrow \quad \theta \eta_{z,\ell}^2 \le \eta_{z,\ell}\big(\mathcal{R}_z(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)\big)^2$$

- $\mathcal{R}_z(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)$ is the set of refined elements from (A3)

$\Longrightarrow$ Dörfler marking for
  - ϕ with set $\mathcal{R}_\phi(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)$ or
  - ψ with set $\mathcal{R}_\psi(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)$

Control of $\#\mathcal{M}_\ell$ 3/3

- minimality of $\mathcal{M}_\ell$ implies

$$\begin{aligned}\#\mathcal{M}_\ell &= \min\{\#\mathcal{M}_{\phi,\ell}, \#\mathcal{M}_{\psi,\ell}\} \\ &\le C_{\rm mark} \max\{\#\mathcal{R}_\phi(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell), \#\mathcal{R}_\psi(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)\} \\ &\le C_{\rm mark} C_{\rm rel} \#(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell)\end{aligned}$$

- recall: $\#\widehat{\mathcal{T}}_\ell - \#\mathcal{T}_\ell \le 2\big(C_1 \kappa^{-1/2} \|\phi\|_{\mathbb{A}_s} \|\psi\|_{\mathbb{A}_t}\big)^{1/(s+t)}$

- define $C_2 = 2 C_{\rm mark} C_{\rm rel}$ and $C_3 = C_1 \kappa^{-1/2}$

- use $\#(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell) \le \#\widehat{\mathcal{T}}_\ell - \#\mathcal{T}_\ell$ to obtain

$$\#\mathcal{M}_\ell \le C_2 \big(C_3 \|\phi\|_{\mathbb{A}_s} \|\psi\|_{\mathbb{A}_t}\big)^{1/(s+t)} (\eta_{\phi,\ell}\eta_{\psi,\ell})^{-1/(s+t)}$$

Proof of main theorem

- linear convergence

$$\eta_{\phi,\ell+n}\eta_{\psi,\ell+n} \le C_{\mathrm{lin}} q_{\mathrm{lin}}^n \eta_{\phi,\ell}\eta_{\psi,\ell}$$

- control of $\#\mathcal{M}_\ell$

$$\#\mathcal{M}_\ell \le C_2\big(C_3\|\phi\|_{\mathbb{A}_s}\|\psi\|_{\mathbb{A}_t}\big)^{1/(s+t)}(\eta_{\phi,\ell}\eta_{\psi,\ell})^{-1/(s+t)}$$

- mesh-closure estimate

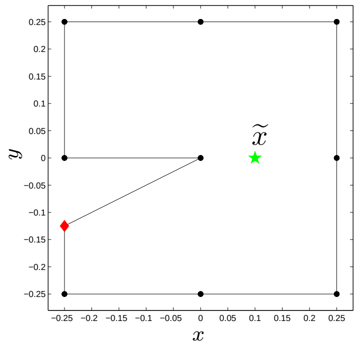
$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C_{\mathrm{mesh}}\sum_{j=0}^{\ell-1}\#\mathcal{M}_j$$

$$\Longrightarrow \quad \eta_{\phi,\ell}\eta_{\psi,\ell} \le C_{\mathrm{opt}}\ \|\phi\|_{\mathbb{A}_s}\|\psi\|_{\mathbb{A}_t}\,(\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-(s+t)}$$

Z-shaped domain in 2D

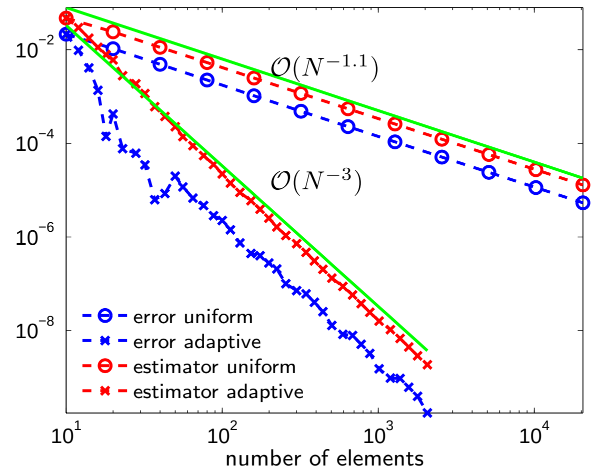
- evaluation point $\widetilde{x} = (0.1, 0)$
- exact PDE solution $u(x,y)$

$$u(x,y) = r^{\pi/\alpha}\cos\Big(\frac{\pi}{\alpha}\varphi\Big)$$



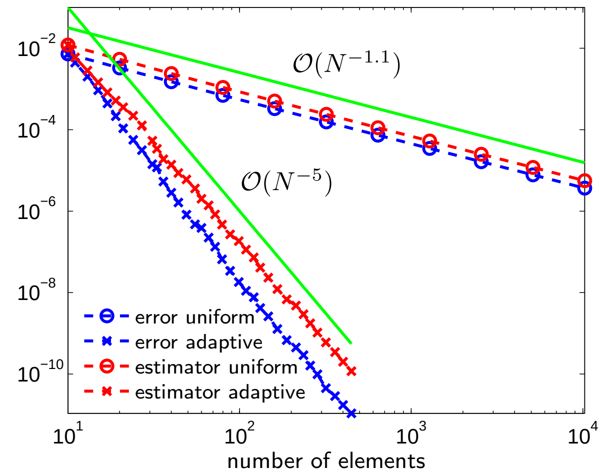
- GOAL:
  - $\|\phi - \Phi_\ell\|_{H^{-1/2}} \lesssim h^{(p+1)+1/2} \simeq N^{-(p+3/2)}$ for $\Phi_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)$
  - aim for $|u(\widetilde{x}) - u_\ell(\widetilde{x})| \lesssim N^{-(2p+3)}$

$|u(\widetilde{x}) - u_\ell(\widetilde{x})|$ for p=0



- $|u(\widetilde{x}) - u_\ell(\widetilde{x})|$ and $\eta_{\phi,\ell}\eta_{\psi,\ell}$ realize optimal rate $\mathcal{O}(N^{-3})$

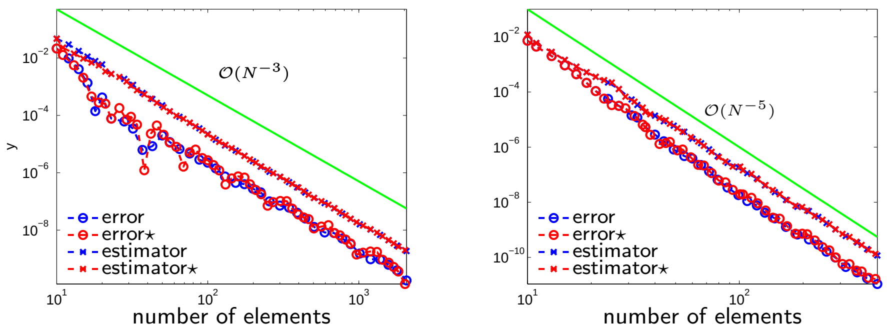
$|u(\widetilde{x}) - u_\ell(\widetilde{x})|$ for p=1



- $|u(\widetilde{x}) - u_\ell(\widetilde{x})|$ and $\eta_{\phi,\ell}\eta_{\psi,\ell}$ realize optimal rate $\mathcal{O}(N^{-5})$

Marking strategies

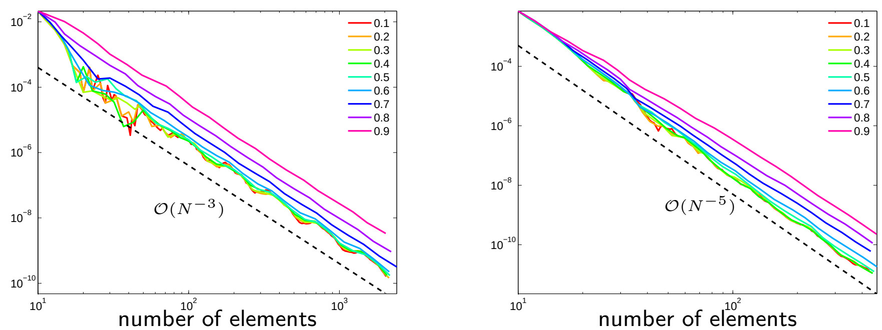
$p = 0$ $p = 1$



- ⋆ . . . . . . separate marking (Mommer & Stevenson)
- both marking strategies realize optimal convergence rate

Adaptivity parameter $0 < \theta \leq 1$

$p = 0$ $p = 1$

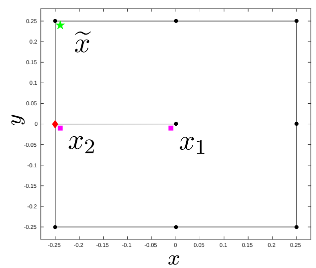


- rate is stable in θ, while $0 < \theta \ll 1$ in the analysis

Almost-slit domain in 2D

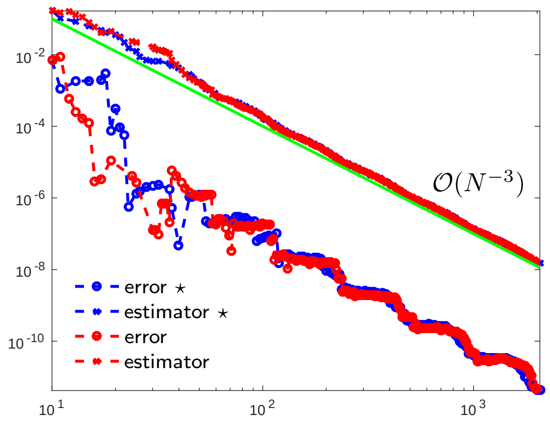
- evaluation point
  $\widetilde{x} = (-0.24, 0.24)$
- exact PDE solution $u(x, y)$

$$u(x, y) = r^{\pi/\alpha} \cos\left(\frac{\pi}{\alpha}\varphi\right)$$



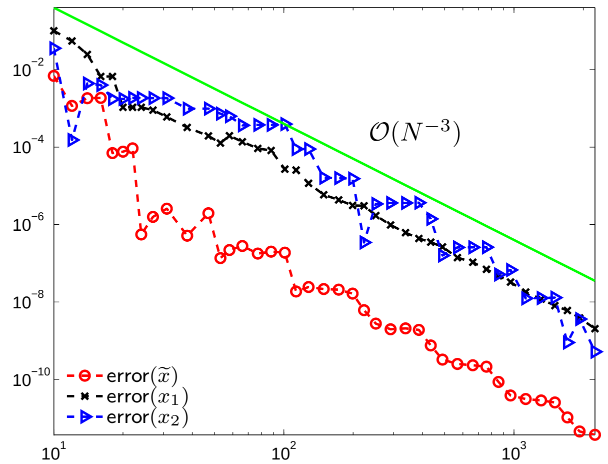
- add. points $x_1 = (-0.01, -0.01)$ and $x_2 = (-0.24, -0.01)$

$|u(\widetilde{x}) - u_\ell(\widetilde{x})|$ for p=0



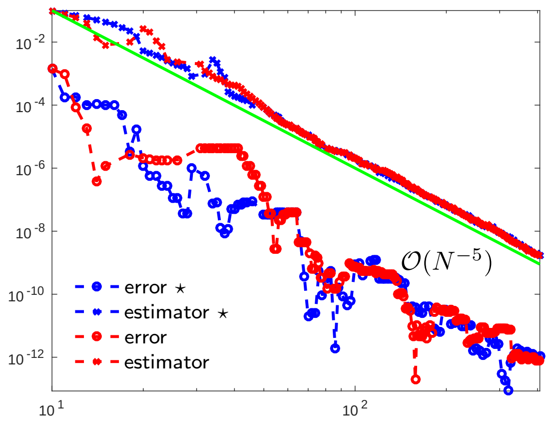
- ⋆ . . . . . . separate marking (Mommer & Stevenson)
- both marking strategies realize optimal convergence rate $\mathcal{O}(N^{-3})$

Additional evalutation points for p=0



- optimal rate $\mathcal{O}(N^{-3})$ for points errors at x_1 and x_2

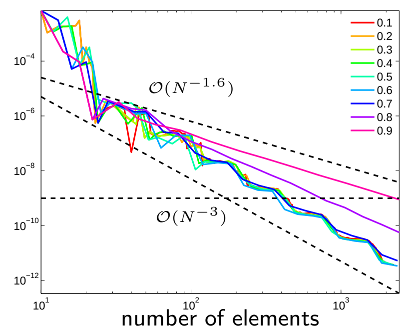
$|u(\widetilde{x}) - u_\ell(\widetilde{x})|$ for p=1



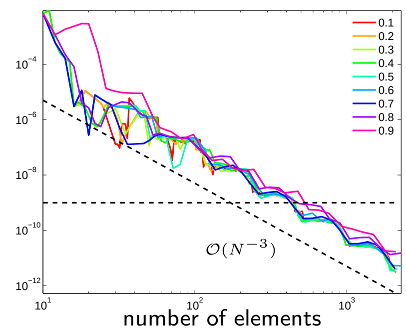
- ⋆ . . . . . . separate marking (Mommer & Stevenson)
- both marking strategies realize optimal convergence rate $\mathcal{O}(N^{-5})$

Adaptivity parameter $0 < \theta \leq 1$ for $p = 0$

combined marking



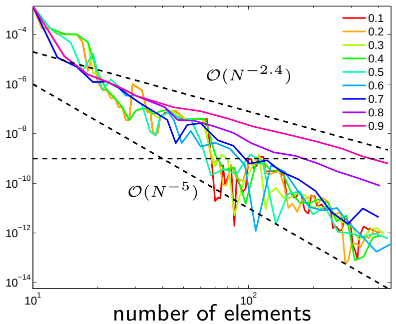
separate marking



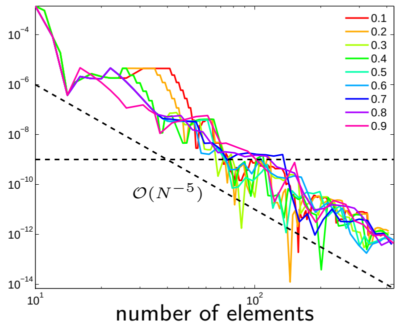
- rate of separate marking (MS) is stable in θ
- rate of combined marking (BET) degenerates for $\theta \geq 0.7$

Adaptivity parameter $0 < \theta \leq 1$ for $p = 1$

combined marking



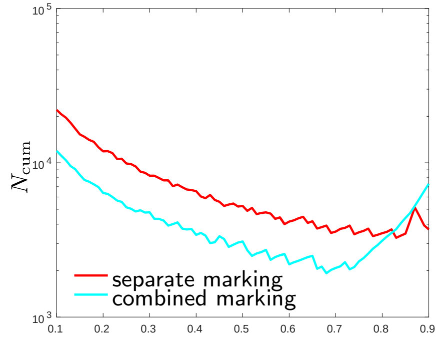
separate marking



- rate of separate marking (MS) is stable in θ
- rate of combined marking (BET) degenerates for $\theta \geq 0.7$

What is the best θ ?

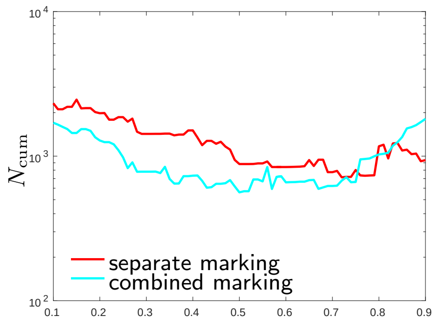
- $p = 0$
- $\texttt{error} \leq 10^{-9}$
- $N_{\text{cum}} := \sum_{j=0}^{\ell} \#\mathcal{T}_j$



- combined marking is slightly better for $\theta \leq 0.7$
- algorithm works best for $0.5 \leq \theta \leq 0.7$

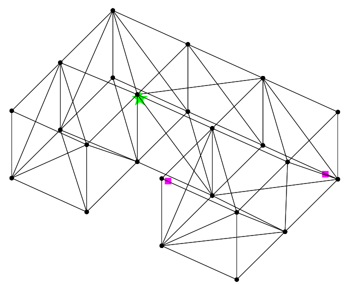
What is the best θ ?

- $p = 1$
- $\texttt{error} \leq 10^{-9}$
- $N_{\text{cum}} := \sum_{j=0}^{\ell} \#\mathcal{T}_j$



- combined marking is slightly better for $\theta \leq 0.7$
- algorithm works best for $0.5 \leq \theta \leq 0.7$

U-shape domain in 3D



- evaluation point $\widetilde{x} = (-0.05, 0, 0.9)$
- exact PDE solution $u(x, y, z)$ in polar coordinates

$$u(x, y, z) = z r_1^{2/3} \cos(2\varphi_1/3) + r_2^{2/3} \cos(2\varphi_2/3)$$

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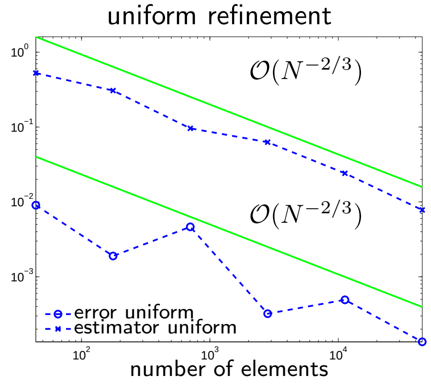

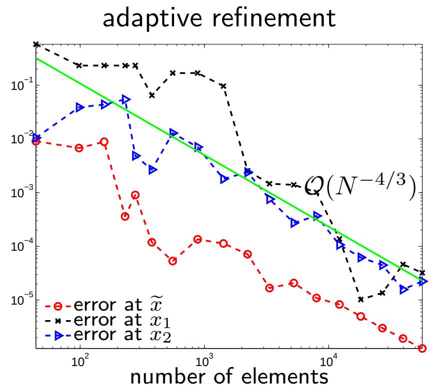

- algorithm leads to optimal convergence for point-error at $\widetilde{x}$
- also optimal convergence for point-errors at x_1, x_2

Summary

- GOAL: For $\widetilde{x} \in \Omega$, compute $u(\widetilde{x})$, where $-\Delta u = 0$, $u|_\Gamma = g$
- in each adaptive step: Solve

Primal problem

$$V\phi = (K + 1/2)g$$

Dual problem

$$V\psi = G(\widetilde{x}, \cdot)$$

- error bounded by estimator product

$$|u(\widetilde{x}) - u_\ell(\widetilde{x})| \leq \| \psi - \Psi_\ell \| \| \phi - \Phi_\ell \| \lesssim \eta_{\psi,\ell}\eta_{\phi,\ell}$$

- get optimal rate for the estimator product

$$|u(\widetilde{x}) - u_\ell(\widetilde{x})| \lesssim \eta_{\phi,\ell}\eta_{\psi,\ell} \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-(s+t)}$$

- concept also applies to goal-orientated AFEM
- extends work of Mommer & Stevenson!

Thanks for listening

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