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THE WIGNER DISTRIBUTION — A TOOL FOR TIME-FREQUENCY SIGNAL ANALYSIS

PART I: CONTINUOUS-TIME SIGNALS

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Abstract

The paper deals with the Wigner distribution which is a signal transformation that is particularly suited for the time-frequency analysis of nonstationary signals. Several of its properties are summarized. The usefulness of the concept is indicated in some examples. This first part deals with the Wigner distribution of continuous-time signals.

1. Introduction

In many cases of practical importance one is interested in having a mixed time-frequency representation of a signal. This is particularly so if the signal under investigation is nonstationary, which means that at different instants the signal has a different "frequency content". Such an approach is very popular, for example, in the analysis of speech signals. More than 30 years ago a vivid interest in the visualisation of speech led to the development of the sound spectrograph which is a device that produces an intensity pattern that is found to be characteristic for the frequency content of the speech as a function of time [1]). These spectrograms are nowadays computed in a much more sophisticated way and still form a useful tool of speech analysis [2,3,4]).

A second example is that of a piece of music for which De Bruijn [5]) has remarked the following.

> For example, if f represents a piece of music, then the composer does not produce f itself; he does not even define it. He may try to prescribe the exact frequency and the exact time interval of a note (although the uncertainty principle says that he can never be completely successful in this effort), but he does not try to prescribe the phase. The composer does not deal with f; it is only the gramophone company which produces and sells an f. On the other hand, the composer certainly does not want to describe the Fourier transform. This Fourier transform is very useful for solving mathematical and physical problems, but it gives an absolutely unreadable picture of the given piece of music.
>
> What the composer really does, or thinks he does, or should think he does, is something entirely different from describing either f or $\mathcal{F}f$. Instead, he constructs a function of two variables. The variables are the time and the frequency, the function describes the intensity of the sound. He describes the function by a complicated set of dots on score paper. His way of describing time is slightly different from what a mathematician would do, but certainly vertical lines denote constant time, and horizontal lines denote constant frequency.

Although all signals for which such a time-frequency analysis is desired are nonstationary, all methods for such an analysis that have been used extensively are based on the assumption that on a short-time basis the signals are stationary. This has the important drawback that the length of the assumed short-time stationarity determines the frequency resolution which can be obtained. To increase the frequency resolution one has to take a longer measurement interval (window), which means that nonstationarities occurring during this interval will be smeared out in time and frequency.

A time-frequency characterization of a signal that overcomes this drawback is the Wigner distribution [6,7]). This signal transformation has some important properties that make it an ideal tool for time-frequency signal analysis. Although the concept as such is not new, it is little known in the area of signal processing.

The concept of the Wigner distribution (WD) was introduced in 1932 by Wigner [6]) in the context of quantum mechanics. Although the concept was reintroduced in 1948 by Ville [8]), this time for signal analysis, it has received little attention in this field. A review of the history of the WD has been given by De Bruijn [7]), who also gave a mathematical basis for this new signal transformation. Recently the WD has obtained considerable attention in optics [13–15]).

In our opinion the properties of the WD justify more widespread knowledge of this concept, notably in the area of signal analysis and processing. Therefore in this paper the emphasis is mainly on properties, and not on mathematical subtleties. Also we will use sometimes a notation which is slightly different from that in ref. 7 but which, in our opinion, is more adapted to engineering practice. In this first part the WD for continuous time signals is considered. In a companion paper the WD for discrete time signals will be introduced.

In section 2 of this paper the definition of the WD and some of its properties are given. The concept is illustrated with several examples in sec. 3. The effect of linear signal operations, like modulation and filtering, on the WD is the subject of sec. 4. Like the Fourier transform the WD requires the signal to be known for all time. As a first step towards a more realistic approach the issue of windowing of the signals is considered in this section.

For analysis of narrow-band signals the analytic signal has been introduced [8]) and shown to be of considerable importance. Therefore in sec. 5 the WD for analytic signals is discussed. Roughly speaking, the WD of a signal can be interpreted as a function that indicates the distribution of the signal energy over time and frequency. However, this is not strictly true because, for example, the WD is not always positive. As pointed out by De Bruijn [7]),

suitably taken averages of the WD will always be positive. Furthermore he has shown that this property of the WD is intimately connected to Heisenberg's uncertainty relation, which is reflected in the behaviour of the WD. He has also indicated that the consideration of the moments of the WD gives a coarse characterization of the energy distribution over time and frequency.

In contrast to De Bruijn's approach, who only considers the global moments of the WD (i.e. they are taken over the whole time-frequency plane), we will consider locally defined moments as well. This will be done in sec. 6. It is shown there that the average frequency of the WD at a certain instant is equal to the instantaneous frequency of the signal, and the average time at a certain frequency is equal to the group delay. This demonstrates that these local moments are useful characterizations of the WD.

In section 7 the WD is considered for band-limited signals. It is shown that the WD can be expressed easily in the samples of the signal if these samples are taken at a rate higher than twice the Nyquist rate of the signal. This expression forms the basis for the discussion of the WD of discrete-time signals, which is the topic of part II.

2. The Wigner distribution for continuous-time signals

2.1. *Preliminaries*

In this section we will consider (in general complex) continuous-time functions defined for all time: $f(t) \in \mathsf{C}$, $t \in \mathsf{R}$.

The (Fourier) spectrum of these signals is given by

$$F(\omega) = (\mathscr{F}f)(\omega) = \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t \tag{2.1.a}$$

where $\mathscr{F}$ denotes the operation of Fourier transformation. The inversion formula corresponding to (2.1.a) is given by

$$f(t) = (\mathscr{F}^{-1}F)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,\mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}\omega. \tag{2.1.b}$$

For the moment we assume that we are dealing with well-behaved signals (quadratically integrable or smooth [7]) so that all signal operations can be performed in the ordinary sense. If this assumption is not true, we will consider the signals as generalized functions. For more details we refer to ref. 7.

In subsequent discussions it will turn out to be useful to work with inner products and norms of functions. For two signals f and g the inner product is defined by

$$(f, g) = \int_{-\infty}^{\infty} f(t)\, g^*(t)\, \mathrm{d}t \tag{2.2.a}$$

while the inner product of two spectra F and G is defined *) similarly apart from a factor $1/2\pi$:

$$(F, G) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, G^*(\omega)\, \mathrm{d}\omega. \tag{2.2.b}$$

The asterisk denotes complex conjugation. As the norm of a signal f we take

$$||f|| = (f, f)^{\frac{1}{2}} \tag{2.3.a}$$

and similarly for the spectrum

$$||F|| = (F, F)^{\frac{1}{2}}. \tag{2.3.b}$$

With this notation Parseval's theorem [10]) then reads

$$(f, g) = (F, G). \tag{2.4}$$

Several more operations on signals will be introduced:
the shift operation

$$(\mathscr{S}_\tau f)(t) = f(t - \tau), \tag{2.5}$$

the (complex) modulation

$$(\mathscr{M}_\Omega f)(t) = \mathrm{e}^{\mathrm{j}\Omega t} f(t), \tag{2.6}$$

differentiation

$$(\mathscr{D} f)(t) = \frac{1}{\mathrm{j}} f'(t), \tag{2.7}$$

multiplication by the running variable

$$(\mathscr{Q} f)(t) = t f(t) \tag{2.8}$$

and reversal of the running variable

$$(\mathscr{R} f)(t) = f(-t). \tag{2.9}$$

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*) In most of the mathematical literature signals and spectra are treated in an identical way and a more symmetrical definition of the Fourier transform is used [7]). Because in engineering practice a non-symmetrical definition of the Fourier transform is common we prefer to extend this asymmetry to the definition of inner products, norms and later also to the Wigner distribution.

The unusual definition of the operation $\mathscr{D}$ for differentiation has been taken because in this case the inner product

$$(\mathscr{D}f, f) = \frac{1}{\mathrm{j}} \int_{-\infty}^{\infty} f'(t) f^*(t)\, \mathrm{d}t$$

is real.

When presenting a new concept like the Wigner distribution there are in general two approaches. The first, which has been followed by Ville [8]) and Mark [9]), starts with some heuristic considerations about the properties which such a time-frequency representation is supposed to have. Subsequently, from these properties a suitable form for the Wigner distribution is derived.

Following De Bruijn [7]) we will adopt an axiomatic approach that starts with a rather formal definition. The usefulness of the concept is then indicated by deriving the corresponding properties. Such an approach is more lucid but has the disadvantage that it does not immediately reveal the uniqueness of the concept, i.e. other definitions might have led to concepts with similar or even more desirable properties. Therefore, in part III of this paper the connection between the Wigner distribution and other time-frequency signal representations will be discussed. It will be shown there that many more such representations exist but they can all be derived from the Wigner distribution.

2.2. *Definition of the Wigner distribution*

The cross-Wigner distribution of two signals f and g is defined by [7]):

$$W_{f,g}(t, \omega) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}\omega\tau} f(t + \tau/2)\, g^*(t - \tau/2)\, \mathrm{d}\tau. \tag{2.10}$$

The auto-Wigner distribution of a signal is then given by

$$W_f(t, \omega) = W_{f,f}(t, \omega) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}\omega\tau} f(t + \tau/2) f^*(t - \tau/2)\, \mathrm{d}\tau. \tag{2.11}$$

When no confusion can arise we will call both functions a Wigner distribution (WD).

The above definitions are given for time functions. Its usefulness as a time-frequency characterization of signals is underlined by the fact that a similar expression also exists for the spectra.

If we define the WD for the spectra F and G by

$$W_{F,G}(\omega, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{j}\xi t} F(\omega + \xi/2)\, G^*(\omega - \xi/2)\, \mathrm{d}\xi \tag{2.12}$$

we obtain the important relation

$$W_{F,G}(\omega, t) = W_{f,g}(t, \omega). \tag{2.13}$$

This means that the Wigner distribution of the spectra of two signals can simply be determined from that of the time functions by an interchange of frequency and time variables. This illustrates the symmetry between time and frequency domain definitions.

The form of the WD is reminiscent of that of the ambiguity function of a signal [11]). In fact the ambiguity function can be obtained from the WD by a two-dimensional Fourier transformation and scaling with respect to the signal power.

2.3. *Properties of the Wigner distribution*

In this subsection a number of properties of the WD are given. Proofs are omitted because these properties either follow directly from the definition or the proofs can be found in ref. 7.

2.3.1.

For any two signals f and g we have

$$W_{f,g}(t, \omega) = W^*_{g,f}(t, \omega). \tag{2.14}$$

Hence the WD of any (real or complex) function will be real:

$$W_f(t, \omega) = W_f^*(t, \omega). \tag{2.15}$$

Moreover, the WD of a real signal is an even function of the frequency:

$$W_f(t, \omega) = W_{f^*}(t, -\omega). \tag{2.16}$$

2.3.2.

A time shift in both signals corresponds to a time shift of the WD:

$$W_{\mathscr{S}_\tau f,\, \mathscr{S}_\tau g}(t, \omega) = W_{f,g}(t - \tau, \omega). \tag{2.17}$$

2.3.3.

Modulating both signals f and g with $e^{j\Omega t}$ results in a frequency shift of the WD:

$$W_{\mathscr{M}_\Omega f,\, \mathscr{M}_\Omega g}(t, \omega) = W_{f,g}(t, \omega - \Omega). \tag{2.18}$$

2.3.4.

Combining (2.17) and (2.18) yields

$$W_{\mathscr{M}_\Omega \mathscr{S}_\tau f,\, \mathscr{M}_\Omega \mathscr{S}_\tau g}(t, \omega) = W_{\mathscr{S}_\tau \mathscr{M}_\Omega f,\, \mathscr{S}_\tau \mathscr{M}_\Omega g}(t, \omega) = W_{f,g}(t - \tau, \omega - \Omega). \tag{2.19}$$

This result can be used to express the WD by

$$W_{f,g}(t, \omega) = W_{\mathcal{S}_{-t}\mathcal{M}_{-\omega}f, \mathcal{S}_{-t}\mathcal{M}_{-\omega}g}(0, 0). \tag{2.20}$$

Using the definition of the inner product in (2.3) and of the operator $\mathcal{R}$ in (2.10) we get

$$W_{f,g}(0, 0) = 2\,(f, \mathcal{R}g). \tag{2.21}$$

From (2.20) and (2.21) it then follows that

$$W_{f,g}(t, \omega) = 2\,(\mathcal{S}_{-t}\,\mathcal{M}_{-\omega}f, \mathcal{R}\,\mathcal{S}_{-t}\,\mathcal{M}_{-\omega}g). \tag{2.22}$$

The value of the WD at a certain time t and frequency ω can thus be determined by the inner product of the shifted and modulated signals, the second of which has undergone a time reversal.

2.3.5.

The WD of two signals is a bilinear functional of f and g. This means that the WD of the sum of two signals is not simply the sum of the WD's of the signals. From the definition it follows that

$$W_{f_1+f_2,g_1+g_2}(t, \omega) = W_{f_1,g_1}(t, \omega) + W_{f_1,g_2}(t, \omega) + W_{f_2,g_1}(t, \omega) + W_{f_2,g_2}(t, \omega) \tag{2.23}$$

and as a special case

$$W_{f+g}(t, \omega) = W_f(t, \omega) + W_g(t, \omega) + 2\,\mathrm{Re}\,W_{f,g}(t, \omega). \tag{2.24}$$

2.3.6.

The product of the WD with t can be expressed as a sum of two WD's according to

$$2t\,W_{f,g}(t, \omega) = W_{\mathcal{Q}f,g}(t, \omega) + W_{f,\mathcal{Q}g}(t, \omega), \tag{2.25}$$

where the operator $\mathcal{Q}$ is defined in (2.8). Similarly the multiplication of the WD by ω can be expressed as

$$2\omega\,W_{f,g}(t, \omega) = W_{\mathcal{D}f,g}(t, \omega) + W_{f,\mathcal{D}g}(t, \omega), \tag{2.26}$$

where the differentiation operator $\mathcal{D}$ is defined in (2.7).

2.3.7.

According to (2.10) the WD is the spectrum of the signal $f(t + \tau/2)\,g^*(t - \tau/2)$ considered as a function of τ with t as a fixed parameter. Therefore the inverse Fourier transform yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega\tau}\, W_{f,g}(t, \omega)\, d\omega = f(t + \tau/2)\, g^*(t - \tau/2),$$

which can be written in the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(t_1-t_2)}\, W_{f,g}\left(\frac{t_1 + t_2}{2}, \omega\right) d\omega = f(t_1)\, g^*(t_2). \tag{2.27}$$

Two special cases of this equation deserve special attention.

(i) $t_1 = t_2 = t$ in (2.27) yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} W_{f,g}(t, \omega)\, d\omega = f(t)\, g^*(t) \tag{2.28}$$

and in particular

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} W_f(t, \omega)\, d\omega = |f(t)|^2. \tag{2.29}$$

This means that the integral of the WD over the frequency variable at a certain time t yields the instantaneous signal power at that time.

(ii) $t_1 = t$, $t_2 = 0$ in (2.27) gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, W_{f,g}(t/2, \omega)\, d\omega = f(t)\, g^*(0). \tag{2.30}$$

This relation is very remarkable. It shows that $f(t)$ can be recovered from the WD at time $t/2$ by the inverse Fourier transform up to the constant factor $g^*(0)$. A similar relation holds for $g(t)$.

2.3.8.

Integration of eq. (2.28) with respect to time yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{f,g}(t, \omega)\, dt\, d\omega = \int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt = (f, g) \tag{2.31}$$

and of eq. (2.29)

$$\frac{1}{2\pi}\int_{t_a}^{t_b}\left[\int_{-\infty}^{\infty} W_{f,g}(t,\omega)\,\mathrm{d}\omega\right]\mathrm{d}t = \int_{t_a}^{t_b} |f(t)|^2\,\mathrm{d}t. \tag{2.32}$$

This relation shows that the integral of the WD of the signal f over the infinite vertical strip $-\infty < \omega < \infty$, $t_a < t < t_b$ is equal to the energy contained in $f(t)$ in the time interval $t_a < t < t_b$. The total energy in f is therefore given by the integral of the WD over the whole plane (t, ω):

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(t,\omega)\,\mathrm{d}t\,\mathrm{d}\omega = \int_{-\infty}^{\infty} |f(t)|^2\,\mathrm{d}t = (f,f) = ||f||^2. \tag{2.33}$$

2.3.9.

A similar discussion as in 2.3.7 applies to (2.12), but in this case $W_{F,G}(\omega, t)$ is the inverse Fourier transform of $F(\omega + \xi/2)\,G^*(\omega - \xi/2)$, considered as a function of ξ. Hence,

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}\xi t}\, W_{F,G}(\omega, t)\,\mathrm{d}t = F(\omega + \xi/2)\,G^*(\omega - \xi/2). \tag{2.34}$$

Using (2.13) and changing the variables yields

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}(\omega_1-\omega_2)t}\, W_{f,g}\left(t, \frac{\omega_1 + \omega_2}{2}\right)\mathrm{d}t = F(\omega_1)\,G^*(\omega_2). \tag{2.35}$$

Equation (2.35) shows that the Fourier transformation of the WD with respect to time gives information on the spectra of f and g. Two special cases of (2.35) are of interest.

(i) $\omega_1 = \omega_2 = \omega$ yields

$$\int_{-\infty}^{\infty} W_{f,g}(t,\omega)\,\mathrm{d}t = F(\omega)\,G^*(\omega) \tag{2.36}$$

and in particular

$$\int_{-\infty}^{\infty} W_f(t,\omega)\,\mathrm{d}t = |F(\omega)|^2. \tag{2.37}$$

This means that the integral of the WD over the time variable at a certain frequency ω yields the energy density spectrum of f at this frequency.

(ii) $\omega_1 = \omega, \omega_2 = 0$ in eq. (2.35) gives

$$\int_{-\infty}^{\infty} e^{-j\omega t}\, W_{f,g}\,(t, \omega/2)\mathrm{d}t = F(\omega)\, G^*(0). \tag{2.38}$$

This relation shows that the spectrum of f can be recovered from the WD by the Fourier transform of the WD at frequency $\omega/2$, up to the constant factor $G^*(0)$.

2.3.10.

Integration of eq. (2.36) with respect to frequency yields

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{f,g}(t, \omega)\, \mathrm{d}t\, \mathrm{d}\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, G^*(\omega)\, \mathrm{d}\omega = (F, G), \tag{2.39}$$

which is identical to eq. (2.31).
Integration of (2.37) over a finite frequency interval gives

$$\frac{1}{2\pi} \int_{\omega_a}^{\omega_b} \left[\int_{-\infty}^{\infty} W_f(t, \omega)\, \mathrm{d}t \right] \mathrm{d}\omega = \frac{1}{2\pi} \int_{\omega_a}^{\omega_b} |\, F(\omega)|^2\, \mathrm{d}\omega. \tag{2.40}$$

This relation shows that the integral of the WD of the signal f over the infinite horizontal strip $-\infty < t < \infty$, $\omega_a < \omega < \omega_b$ is equal to the energy contained in f in the frequency interval $\omega_a < \omega < \omega_b$. This relation is complementary to (2.32). Taking $\omega_a = -\infty$ and $\omega_b = \infty$ results again in eq. (2.33).

2.3.11. Moyal's formula

Moyal's formula [7]) is concerned with the integration of the product of two WD's. It reads

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{f_1,g_1}(t, \omega)\, W^*_{f_2,g_2}(t, \omega)\, \mathrm{d}t\, \mathrm{d}\omega = (f_1, f_2)\, (g_1, g_2)^* \tag{2.41}$$

and can be considered as the counterpart of Parseval's relation for WD's. Special cases of Moyal's formula can be obtained for suitable choices of f_1, f_2, g_1, g_2. In particular if we take all these signals equal to f we get

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f^2(t, \omega)\, \mathrm{d}t\, \mathrm{d}\omega = ||f||^4. \tag{2.42}$$

2.4. *Effects of time or band limitations on the WD*

2.4.1. Time-limited signals

If $f(t)$ and $g(t)$ are restricted to a finite time interval only and

$$f(t) = g(t) = 0 \qquad t < t_a \text{ or } t > t_b \tag{2.43}$$

then the WD is restricted to the same time interval, i.e.

$$W_{f,g}(t, \omega) = 0 \qquad t < t_a \text{ or } t > t_b, \ \forall \omega. \tag{2.44}$$

It should be noted that this property is a distinct difference from other time-frequency characterizations, which normally give contributions over a larger time interval than belonging to the signals themselves.

2.4.2. Frequency-limited signals

A similar conclusion as in 2.4.1 can be made if f and g are both band-limited. If

$$F(\omega) = G(\omega) = 0 \qquad \omega < \omega_a \text{ or } \omega > \omega_b \tag{2.45}$$

then from (2.13) it follows that

$$W_{f,g}(t, \omega) = 0 \qquad \omega < \omega_a \text{ or } \omega > \omega_b, \ \forall t. \tag{2.46}$$

2.4.3. Causal and analytic signals

In analogy with causal systems for which the impulse response vanishes for $t < 0$ signals with the property that

$$f(t) = 0 \qquad t < 0 \tag{2.47}$$

are sometimes called causal signals [11]). If f and g are two causal signals then it follows from (2.44) that

$$W_{f,g}(t, \omega) = 0 \qquad t < 0, \ \forall \omega. \tag{2.48}$$

Similarly a signal is called analytic [8]) if its spectrum vanishes for $\omega < 0$, i.e.

$$F(\omega) = 0 \qquad \omega < 0. \tag{2.49}$$

From (2.46) it then follows that if f and g are analytic signals then

$$W_{f,g}(t, \omega) = 0 \qquad \omega < 0, \ \forall t. \tag{2.50}$$

3. Examples

In this section some examples will be worked out to illustrate the concept of the WD.

3.1.

$$f(t) = \begin{cases} 1 & |t| < T \\ 0 & |t| > T, \end{cases}$$

$$W_f(t, \omega) = \begin{cases} \dfrac{2}{\omega} \sin 2\omega(T - |t|) & |t| < T \\ 0 & |t| > T. \end{cases} \tag{3.1}$$

This WD has a $\sin x/x$ shape with respect to frequency. The width of its main lobe becomes wider with increasing values of $|t|$. The WD attains its maximum at $(t, \omega) = (0, 0)$ and has the value $W_f(0, 0) = 4T$. Also it can be observed that W_f is negative in certain regions of the (t, ω) plane.

3.2.

$$f(t) = \begin{cases} e^{j\omega_0 t} & |t| < T \\ 0 & |t| > T. \end{cases}$$

The WD of this signal can most easily be obtained by making use of (2.18) and the result (3.1):

$$W_f(t, \omega) = \begin{cases} \dfrac{2}{\omega - \omega_0} \sin 2(\omega - \omega_0)(T - |t|) & |t| < T \\ 0 & |t| > T. \end{cases} \tag{3.2}$$

3.3.

$$f(t) = A\, e^{j\omega_0 t}, \ \forall\, t.$$

From the definition it follows that

$$\begin{aligned} W_f(t, \omega) &= \int_{-\infty}^{\infty} e^{-j\omega\tau} A\, e^{j\omega_0(t+\tau/2)} A^* \, e^{-j\omega_0(t-\tau/2)} \, d\tau \\ &= |A|^2 \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)\tau} \, d\tau = |A|^2 \, 2\pi\delta(\omega - \omega_0). \end{aligned} \tag{3.3}$$

This means that for this stationary signal the WD is independent of t and is confined to the line $\omega = \omega_0$.

3.4.

We take as our next example

$$f(t) = A_1 \, e^{j\omega_1 t}, \; g(t) = A_2 \, e^{j\omega_2 t}$$

and determine

$$\begin{aligned} W_{f,g}(t, \omega) &= \int_{-\infty}^{\infty} e^{-j\omega\tau} A_1 \, e^{j\omega_1 (t+\tau/2)} A_2^* \, e^{-j\omega_2 (t-\tau/2)} \, d\tau \\ &= A_1 \, A_2^* \, e^{j(\omega_1 - \omega_2)t} \, 2\pi\delta\left(\omega - \frac{\omega_1 + \omega_2}{2}\right). \end{aligned} \tag{3.4}$$

3.5.

This result can be used to evaluate the WD of a sinusoidal signal

$$f(t) = A \cos(\omega_0 \, t + \varphi) = \tfrac{1}{2} A \, e^{j\varphi} e^{j\omega_0 t} + \tfrac{1}{2} A \, e^{-j\varphi} e^{-j\omega_0 t}.$$

Using eqs (2.24), (3.3) and (3.4) we obtain

$$W_f(t, \omega) = |A|^2 \frac{\pi}{2} \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0) + 2\delta(\omega) \cos 2(\omega_0 \, t + \varphi) \right]. \tag{3.5}$$

We see that apart from stationary (time-independent) contributions at frequencies $\pm \omega_0$, a contribution at $\omega = 0$ occurs which fluctuates with a frequency $2\,\omega_0$. This contribution represents the nonstationarity in the signal caused by the fluctuations of the instantaneous power of the signal.

3.6.

$$f(t) = A \, e^{+j\alpha t^2/2}.$$

This is a so-called chirp, i.e. a signal, whose instantaneous frequency increases linearly with time. For this signal we find

$$\begin{aligned} W_f(t, \omega) &= |A|^2 \int_{-\infty}^{\infty} e^{-j\omega\tau} e^{j\,\alpha(t+\tau/2)^2/2} \, e^{-j\,\alpha(t-\tau/2)^2/2} \, d\tau \\ &= |A|^2 \, 2\pi \, \delta(\omega - \alpha t). \end{aligned} \tag{3.6}$$

This is an extremely satisfactory result which shows that for a chirp the WD is concentrated at any instant around the instantaneous frequency.

3.7.

$$f(t) = A \, e^{j(\omega_0 t + \beta \sin \omega_m t)}.$$

This signal represents another form of frequency modulation, this time with a sinusoidal input. For the WD we find

$$W_f(t, \omega) = |A|^2 \int_{-\infty}^{\infty} e^{-j\omega\tau} e^{j[\omega_0(t+\tau/2)+\beta \sin \omega_m(t+\tau/2)-\omega_0(t-\tau/2)-\beta \sin \omega_m(t-\tau/2)]} d\tau$$

$$= |A|^2 \int_{-\infty}^{\infty} e^{-j\omega\tau} e^{j(\omega_0\tau+2\beta \cos \omega_m t \sin \omega_m \tau/2)} d\tau. \tag{3.7}$$

Equation (3.7) allows the following interpretation: the WD of this FM signal at time t equals the spectrum of an FM signal with phase deviation $\beta' = 2\beta \cos \omega_m t$ and modulation frequency $\omega'_m = \omega_m/2$. Since we know that the spectrum of an FM signal with deviation β' and modulation frequency ω'_m is a line spectrum given by [10]).

$$\mathscr{F}[e^{j(\omega_0 t+\beta' \sin \omega'_m t)}](\omega) = 2\pi \sum_{l=-\infty}^{\infty} J_l(\beta')\, \delta(\omega - \omega_0 - l\omega'_m) \tag{3.8}$$

we find for the WD of eq. (3.7)

$$W_f(t, \omega) = |A|^2\, 2\pi \sum_{l=-\infty}^{\infty} J_l(2\beta \cos \omega_m t)\, \delta(\omega - \omega_0 - l\,\omega_m/2). \tag{3.9}$$

From equation (3.9) it follows that the WD shows a line structure with lines at $\omega_0 \pm l\omega_m/2$, i.e. separated by half of the input frequency. The contribution on each of these lines varies periodically in time with frequency ω_m.

This WD is sketched in fig. 1, for the case $\beta = 7$, for different instants taken over half a period of the modulation frequency ω_m. In this figure the dashed line indicates the instantaneous frequency of the signal, i.e. $\omega_0 + \beta\omega_m \cos \omega_m t$. We see from fig. 1 that in the neighbourhood of the instantaneous frequency the WD has all positive contributions, whereas at frequencies mirrored with respect to ω_0 the contributions are precisely as large but alternate in sign. The WD is totally concentrated when the instantaneous frequency passes through ω_0.

3.8.

As a final example we consider the Gaussian signal $f(t) = e^{-\alpha t^2}$. For this signal we have

$$W_f(t, \omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} e^{-[\alpha(t+\tau/2)^2+\alpha(t-\tau/2)^2]} d\tau$$

$$= \sqrt{\frac{2\pi}{\alpha}}\, e^{-2\alpha t^2} e^{-\omega^2/2\alpha}. \tag{3.10}$$

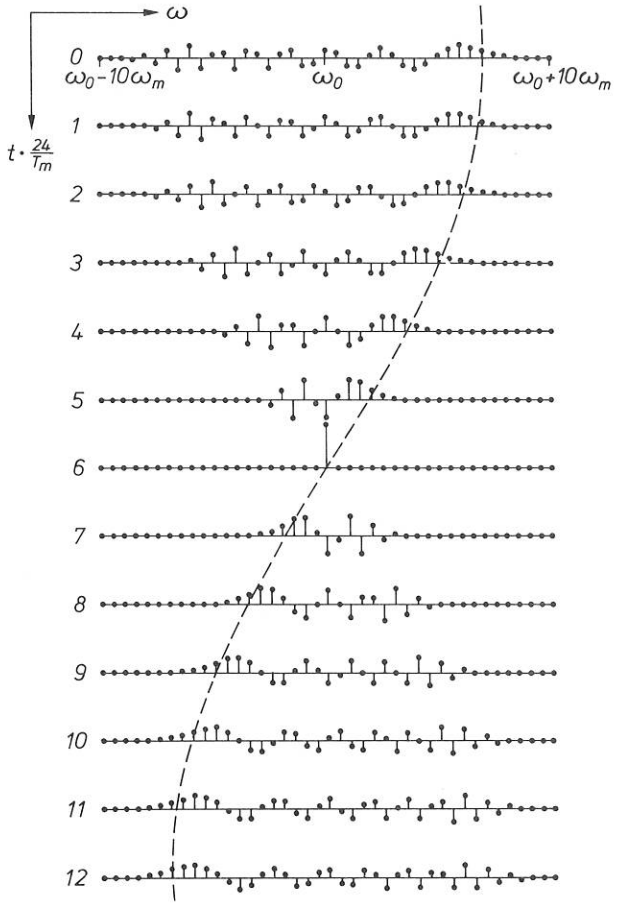

Fig. 1. Schematic representation of the WD of the FM signal $f(t) = \exp(j\omega_0 t + j\beta \sin \omega_m t)$ for $\beta = 7$. Indicated are slices of the WD taken at various instants over half a period of the modulation frequency. The dashed line indicates the instantaneous frequency $\omega_0 + \beta \omega_m \cos \omega_m t$.

The WD of this signal has the same shape in both the time and the frequency direction.

4. Effects of linear operations on the WD

In this section the effects of linear operations on the WD will be investigated. The two most important linear operations occurring in signal processing are filtering and modulation. Filtering corresponds in the time domain to a convolution and modulation to a multiplication. In the spectral domain this situation is reversed. This duality between the two operations will manifest itself clearly in their effects on the WD.

4.1. *Linear filtering*

If both signals f and g are processed by linear, time-invariant systems with impulse responses h_f and h_g, respectively, then the output signals of the systems are given by the convolution integrals

$$f_c(t) = (f * h_f)(t) = \int_{-\infty}^{\infty} f(\tau)\, h_f(t-\tau)\, d\tau, \tag{4.1.a}$$

$$g_c(t) = (g * h_g)(t) = \int_{-\infty}^{\infty} g(\tau)\, h_g(t-\tau)\, d\tau. \tag{4.1.b}$$

The WD of f_c and g_c is most easily determined in the frequency domain. The result is

$$W_{f_c,g_c}(t, \omega) = \int_{-\infty}^{\infty} W_{f,g}(\tau, \omega)\, W_{h_f,h_g}(t-\tau, \omega)\, d\tau. \tag{4.2}$$

W_{f_c,g_c} is therefore the convolution of $W_{f,g}$ and W_{h_f,h_g} in the time variable.

4.2. *Multiplication in the time domain*

Modulation of the signals f and g with carriers m_f and m_g, respectively, results in

$$f_m(t) = f(t)\, m_f(t), \tag{4.3.a}$$
$$g_m(t) = g(t)\, m_g(t). \tag{4.3.b}$$

The WD of f_m and g_m is then given by

$$W_{f_m,g_m}(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W_{f,g}(t, \eta)\, W_{m_f,m_g}(t, \omega-\eta)\, d\eta. \tag{4.4}$$

The WD W_{f_m,g_m} is therefore the convolution of $W_{f,g}$ and W_{m_f,m_g} in the frequency variable.

4.3. *Windowing in the time domain; the pseudo-Wigner distribution*

For computational purposes it will be necessary in general to weight the signals f and g by functions w_f and w_g, respectively, before evaluating the WD. These weighting functions are often called windows and will slide along the time axis with the instant t where the WD has to be evaluated. This means that, rather than considering the functions f and g, a family of functions f_t and g_t is considered given by

$$f_t(\tau) = f(\tau)\, \mathscr{S}_t\, w_f(\tau), \tag{4.5.a}$$
$$g_t(\tau) = g(\tau)\, \mathscr{S}_t\, w_g(\tau). \tag{4.5.b}$$

For each fixed t we can now compute the WD of the functions f_t and g_t and express it in terms of $W_{f,g}$ and the WD of the window functions w_f and w_g. According to (4.4) this yields

$$W_{f_t,g_t}(\tau, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W_{f,g}(\tau, \eta)\, W_{w_f,w_g}(\tau - t, \omega - \eta)\, \mathrm{d}\eta. \tag{4.6}$$

In this relation t appears as a parameter that indicates the position of the window as it slides over the time axis. It is therefore natural not to consider the whole family of WD's given by (4.6), but of each of the members of this family to consider only the values on the line $\tau = t$, i.e.

$$W_{f_t,g_t}(\tau, \omega)\Big|_{\tau=t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} W_{f,g}(t, \eta)\, W_{w_f,w_g}(0, \omega - \eta)\, \mathrm{d}\eta. \tag{4.7}$$

Considering this expression for different values of t we obtain

$$\tilde{W}_{f,g}(t, \omega) = W_{f_t,g_t}(\tau, \omega)\big|_{\tau=t} \tag{4.8}$$

which is a function of t and ω that resembles, *but in general is not*, a WD. This function will be called a pseudo-Wigner distribution (PWD) of f and g. It will be clear that the PWD of two functions depends on the windows w_f and w_g that are used, although this will not explicitly be indicated by the notation.

From equation (4.7) it follows that the PWD $\tilde{W}_{f,g}$ is equal to the original WD $W_{f,g}$ convoluted with respect to its frequency variable by a time-independent function. Considering for fixed t the WD as a function of ω, the effect of windowing on the WD can be interpreted as a filtering operation of this function with a filter with impulse response $W_{w_f,w_g}(0, \omega)$. (In comparison with conventional filtering the role of time is now played by ω.) If $w_f = w_g$ is a real even function then this impulse response equals

$$W_{w_f}(0, \omega) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}\omega\tau}\, w_f^2(\tau/2)\, \mathrm{d}\tau. \tag{4.9}$$

Hence the "transmission function" of this filter has the form of the square of the window, and will therefore always be of the low-pass type. The PWD of a function is therefore always a smoothed version with respect to frequency of the original WD. For example, a rectangular window with length T has the effect of an ideal low-pass filter that attenuates all variations of the WD in the ω-direction above $T/2$.

As an example of the influence of windowing on the WD, consider the FM signal $f(t)$ of example (3.7). If this signal is windowed with the Gaussian

window $w(t)$ of example (3.8), the PWD of the function will be

$$\tilde{W}_f(t, \omega) = \sqrt{\frac{2\pi}{\alpha}}\,|A|^2 \sum_{l=-\infty}^{\infty} J_l(2\beta \cos \omega_m t)\, e^{-(\omega-\omega_0-l\ \omega_m/2)^2/2\alpha}. \tag{4.10}$$

This PWD is sketched in fig. 2 for $\beta = 7$, $\alpha = 2.5\omega_m^2$ for the same values of t as in fig. 1. The effect of the low-pass filtering is clearly visible. It has the result that the highly varying contributions on the opposite side of ω_0 of the instantaneous frequency have vanished, and the PWD has become concentrated around this frequency.

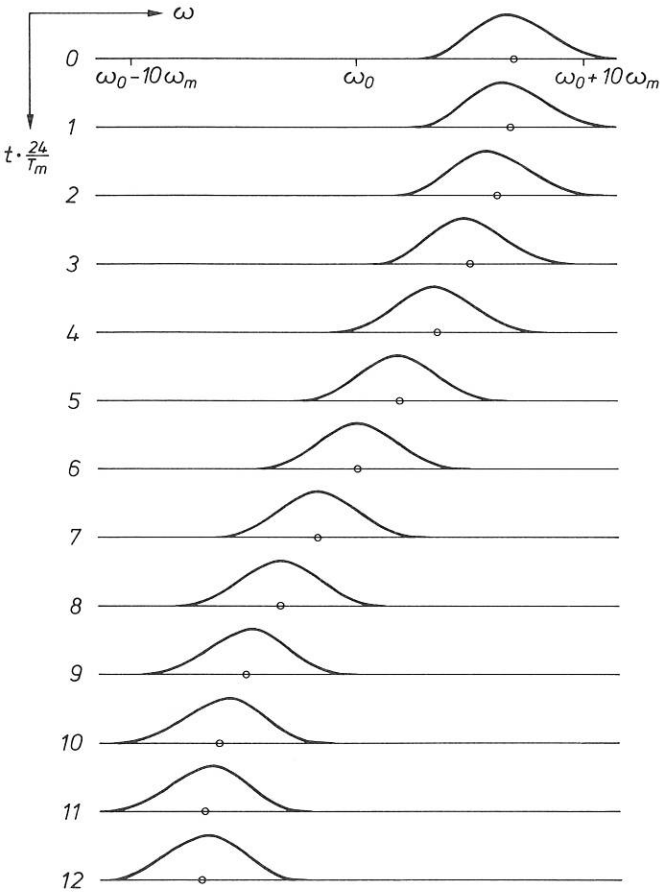

Fig. 2. Schematic representation of the PWD of the FM signal of fig. 1. The window was taken to be $w(t) = \exp(-\alpha t^2)$ with $\alpha = 2.5\,\omega_m^2$. The dots indicate the instantaneous frequency of the original FM signal.

5. Wigner distribution for analytic signals

The signals which are dealt with under normal circumstances are real. While this fact certainly facilitates the implementation of many signal-processing schemes, it involves on the other hand special peculiarities which complicate the analysis of these signals. A consequence of the realness of the signals is that their spectrum is always symmetric. Thus only one half of the spectrum contains information, while the other half increases the redundancy. This redundancy is eliminated by the use of the notion of the analytic signal.
The analytic signal is defined by [8])

$$f_a(t) = f(t) + \mathrm{j}\hat{f}(t), \tag{5.1}$$

where $\hat{f}$ is the Hilbert transform of f:

$$\hat{f}(t) = (\mathscr{H}f)(t) = \frac{1}{\pi}\,\mathrm{v.p.}\int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau}\,\mathrm{d}\tau. \tag{5.2}$$

(v.p. indicates the Cauchy principle value.)

The analytic signal has a spectrum given by

$$F_a(\omega) = \begin{cases} 2F(\omega) & \omega > 0 \\ F(0) & \omega = 0 \\ 0 & \omega < 0. \end{cases} \tag{5.3}$$

From (2.50) it therefore follows that

$$W_{f_a}(t, \omega) = 0 \qquad \omega < 0. \tag{5.4}$$

The relation between W_{f_a} and W_f can best be determined by the use of (2.13) and (2.14). For the definition of the WD in the frequency domain we have

$$W_{F_a}(\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{j}\xi t} F_a(\omega + \xi/2)\, F_a^*(\omega - \xi/2)\,\mathrm{d}\xi. \tag{5.5}$$

Using equation (5.3) we get

$$W_{F_a}(\omega, t) = \frac{2}{\pi}\int_{-2\omega}^{2\omega} \mathrm{e}^{\mathrm{j}\xi t} F(\omega + \xi/2)\, F^*(\omega - \xi/2)\,\mathrm{d}\xi, \quad \omega > 0. \tag{5.6}$$

Incorporating relation (2.34) leads to

$$W_{F_a}(\omega, t) = \frac{4}{\pi} \int_{-\infty}^{\infty} W_F(\omega, t - \tau) \frac{\sin 2\omega\tau}{\tau} \, d\tau, \quad \omega > 0 \tag{5.7}$$

and thus for W_{f_a}

$$W_{f_a}(t, \omega) = \begin{cases} \frac{4}{\pi} \int_{-\infty}^{\infty} W_f(t - \tau, \omega) \frac{\sin 2\omega\tau}{\tau} \, d\tau, & \omega > 0 \\ 0 & \omega < 0. \end{cases} \tag{5.8}$$

Equation (5.8) can be interpreted in the following way: the WD of the analytic signal at a fixed frequency value $\omega > 0$ can be obtained by considering $W_f(t, \omega)$ for this frequency value as a function of time and passing this time function through an ideal low-pass filter with cut-off frequency 2ω. This means that $W_{f_a}(t, \omega)$ with ω fixed is a time function whose highest frequency is at most 2ω.

As a simple example consider the sinusoidal signal of example (3.5). It's WD has three contributions, two of which at $\omega = \pm\omega_0$ are stationary, while the contribution at $\omega = 0$ is varying with a frequency $2\omega_0$. For this sinusoidal signal the analytic signal is $e^{j\omega_0 t}$, the WD of which was given in example (3.3). It can be seen that this WD contains only one stationary contribution at $\omega = \omega_0$. The stationary contribution in the original WD at $\omega = -\omega_0$ is suppressed since it occurs at a negative frequency, while the contribution at $\omega = 0$ is eliminated because it varies in time, which is prohibited at this frequency according to (5.8).

6. Global and local moments of the WD

6.1. *General remarks*

From the properties of the WD one might have gained the impression that the WD of a signal f can be interpreted as the energy distribution of f in time *and* frequency:

(a) The total energy in f is given by the integral of W_f over the whole (t, ω) plane, see (2.33).

(b) The energy in f contained in a certain interval of time (t_a, t_b) is given by the integral of W_f over the infinite strip $-\infty < \omega < \infty$, $t_a < t < t_b$; see (2.32).

(c) The energy in f contained in a certain frequency interval (ω_a, ω_b) is given by the integral of W_f over the infinite strip $-\infty < t < \infty$, $\omega_a < \omega < \omega_b$, see (2.40).

However, this interpretation of the WD of a signal must be used with care, because we have already seen in the examples that W_f can attain negative values, locally.

For a specific signal we can compute its WD and from an analysis of this function we can get an idea of how the energy in this signal is distributed in time and frequency. It may be concentrated in certain time or frequency intervals, or may be spread over the whole (t, ω) plane, etc.

It is possible, however, to characterize this distribution more specifically without giving the values of the WD for all (t, ω) values. For this we use the notion of (central) moments of the WD. These moments allow the specification of e.g. averages and the spread of the WD.

We can distinguish several different approaches. First a distinction can be made between local and global moments. The local moments are determined by considering the WD as a function of time for a fixed frequency ω or as a function of frequency for a fixed time t. The corresponding moments will then of course be dependent on frequency or time, respectively. Therefore these moments may still contain some information regarding variations in time or frequency. The global moments are found by integration over the whole plane, and are therefore independent of time and frequency.

A further distinction that can be made is that the moments can be considered not only for the WD, but also for its square. Taking the WD itself has the advantage that the WD is then considered truly as a distribution, but has the disadvantage that the resulting second order moments are not necessarily positive, since the WD is not always positive. This disadvantage can be overcome if instead of W_f the distribution of W_f^2 is considered. It turns out that in this case only the global moments have a simple interpretation, for which reason this approach will not be pursued in this paper. The corresponding discussion can be found in ref. 7.

6.2. *Central moments*

Let $K(x)$ be a function of a continuous variable x on $(-\infty, \infty)$ with

$$\int_{-\infty}^{\infty} K(x)\,\mathrm{d}x = m_0 > 0. \tag{6.1}$$

Because it was not assumed that $K(x)$ is positive for all values of x, the conclusion from (6.1) is that at least $K(x)$ has an average m_0 that is positive. To get an impression of the value distribution of $K(x)$ we consider the expression

$$\int_{-\infty}^{\infty} (x - x_0)^2\, K(x)\,\mathrm{d}x/m_0 \tag{6.2}$$

as a function of x_0 and seek to find the value of x_0 at which eq. (6.2) is minimized. The minimum is attained for x_0 equal to

$$m_1 = \int_{-\infty}^{\infty} x\,K(x)\,\mathrm{d}x/m_0 \tag{6.3}$$

and has the value

$$m_2 = \int_{-\infty}^{\infty} (x - m_1)^2\,K(x)\,\mathrm{d}x/m_0 = \int_{-\infty}^{\infty} x^2\,K(x)\,\mathrm{d}x/m_0 - m_1^2. \tag{6.4}$$

The values m_1 and m_2 may be interpreted as follows: m_1 is the centre of gravity of K, i.e. it is that value of x for which K will be in balance if K is considered as a mass distribution along a straight line (allowing negative masses for $K(x) < 0$). Similarly m_2 can be considered as the spread or variance of K (if $K(x)$ is non-negative), i.e. m_2 gives an indication about the spread of the masses along the real line and is small if K is mainly concentrated around $x = m_1$. If K is not always positive this interpretation of m_2 must be handled with care, because then m_2 too can become negative.

6.3. *Moments of the WD in the frequency variable*

In this and the following section we will apply the notion of moments to the WD. In this section the moments with respect to the frequency variable ω will be considered, while in the next section the moments with respect to the time variable t are discussed. Although there exists a large amount of similarity between the two concepts their interpretations have characteristic differences.

6.3.1. Local moments

If we fix the time we get for the average of the WD with respect to frequency

$$p_f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W_f(t, \omega)\,\mathrm{d}\omega = |f(t)|^2 \tag{6.5}$$

and we see that $p_f(t)$ is the instantaneous power of f, which is non-negative. This means that we can define the higher order moments only for those t for which $p_f(t) > 0$.

The first-order moment is then given by (see (6.3))

$$\Omega_f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega\,W_f(t, \omega)\,\mathrm{d}\omega/p_f(t) \tag{6.6}$$

and can be interpreted as the average frequency of the WD at time t. From

(A2) in the appendix it follows that

$$\Omega_f(t) = \mathrm{Im}\,\frac{f'(t)}{f(t)} = \mathrm{Im}\,\frac{\mathrm{d}}{\mathrm{d}t}\ln f(t). \tag{6.7}$$

It will be clear that for real-valued signals the average frequency is identically zero, since the WD is then an even function of ω. For real signals the average frequency as defined in (6.6) therefore provides no information. Let us therefore assume that $f(t)$ is complex-valued. Then $f(t)$ can be written in the form:

$$f(t) = v(t)\,\mathrm{e}^{\mathrm{j}\varphi(t)} \tag{6.8}$$

where $v(t)$ and $\varphi(t)$ are real functions. Using this representation of f we find from (6.7)

$$\Omega_f(t) = \varphi'(t). \tag{6.9}$$

In (6.8) $f(t)$ is given by its envelope $v(t)$ and phase $\varphi(t)$. Therefore we can conclude from (6.9) that the average frequency of the WD at time t is equal to the derivative of the phase. In section 5 the analytic signal was discussed as a particular example of a complex valued signal. If the analytic signal is written in the form of (6.8) then the derivative of the phase is called the *instantaneous frequency*[8,12]. From eq. (6.9) the remarkable fact follows that for these signals the average frequency of the WD is equal to the instantaneous frequency of the signal. But what is more, eq. (6.9) holds for any complex valued signal, and therefore opens the possibility to extend the definition of the instantaneous frequency to general complex valued signals.

Using (6.8) and (6.9) it follows that if $f(t)$ modulates a carrier $m(t)$ then the instantaneous frequency of the modulated signal is given by

$$\Omega_{f\cdot m}(t) = \Omega_f(t) + \Omega_m(t). \tag{6.10}$$

This is clear for a carrier of the form $m(t) = \mathrm{e}^{\mathrm{j}\omega_0 t}$, in which case it follows immediately from the frequency-shift property (2.18) of the WD, but according to (6.10) it holds for any function $m(t)$. In particular if $m(t)$ is a real function then $\Omega_m(t) = 0$ and (6.10) tells that the average frequency does not change. Recalling the definition of the PWD in section 4.3 it thus follows that the average frequency of the PWD is equal to that of the WD independent of the particular (real-valued) window that is chosen.

The local second-order moment with respect to ω is defined by (see (6.4))

$$m_f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\omega - \Omega_f(t)]^2\, W_f(t, \omega)\,\mathrm{d}\omega/p_f(t). \tag{6.11}$$

Since, as remarked before, W_f is not always positive $m_f(t)$ is not necessarily positive either and hence cannot simply be interpreted as the variance of the WD in the ω direction for a given time t. With the aid of (6.7) and (A3) in the appendix it follows that

$$m_f(t) = -\tfrac{1}{2}\,\mathrm{Re}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{f'(t)}{f(t)} \tag{6.12}$$

which in case $f(t)$ has the form of (6.8) can be written as

$$m_f(t) = -\tfrac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{v'(t)}{v(t)} = -\tfrac{1}{2}\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\ln|v(t)| \tag{6.13}$$

and thus only depends on the envelope of the signal. It can be seen from (6.13) that $m_f(t) \equiv 0$ if and only if

$$|v(t)| = A\,\mathrm{e}^{\gamma t} \tag{6.14}$$

for arbitrary A and γ.

Considering again the case of modulation it follows that

$$m_{f\cdot m}(t) = m_f(t) + m_m(t). \tag{6.15}$$

From this equation and eqs (4.5) and (4.8) it can be derived that the second-order moment $\tilde{m}_f(t)$ of the PWD $\tilde{W}_f$ is equal to

$$\tilde{m}_f(t) = m_f(t) + m_w(0). \tag{6.16}$$

As an example we may again consider the FM signal $f(t) = A\,\mathrm{e}^{+\mathrm{j}(\omega_0 t + \beta \sin \omega_m t)}$ from example 3.7, for which a sketch of a part of the WD was shown in fig. 1.

For this signal the average frequency $\Omega_f(t)$ at time t is equal to the instantaneous frequency:

$$\Omega_f(t) = \omega_0 + \beta\omega_m \cos \omega_m t \tag{6.17}$$

which has been indicated by the dashed line in fig. 1. The second-order moment $m_f(t)$ for this signal is equal to zero for all t, which follows simply from (6.14) with $\gamma = 0$. This may violate our intuition because inspection of fig. 1 shows us that for instants where $\Omega_f(t) \neq \omega_0$ the WD has non-zero contributions at frequencies that differ from $\Omega_f(t)$. If $m_f(t)$ could have been interpreted as the variance of the WD, it would certainly have had to be larger at $t = 0$ than at $t = T_m/2$. (If instead of W_f we had taken W_f^2 as the kernel in the definition of the moments, this would indeed have happened. In that case, however, the average frequency would have been equal to ω_0 for all t, since the square of the WD is symmetrical around this value.)

How then are we to interpret the result that $m_f(t) = 0, \forall t$? The clue is given by eq. (6.16). This equation tells us that if the FM signal is windowed, then the second-order moment of the PWD of the function is equal to that of the WD of the window at $t = 0$. For example for the gaussian window of example (3.8) we have $m_w(0) = \alpha$. This means that the second-order moment of the PWD of the FM signal is equal to α, $\forall t$. The effect of this windowing operation was shown in fig. 2 to make the PWD non-negative for almost all ω. Then this must necessarily mean that the PWD becomes concentrated around the mean frequency (which according to (6.21) is not affected by the window) with a spread equal to α for all time. It is interesting to see from fig. 2 that the spread of the PWD is indeed constant in time, and not larger for $t = 0$ than for $t = T_m/2$, as fig. 1 could have led us believe.

6.3.2. Global moments

Global moments of the WD are obtained by integration over the whole plane. They are therefore constants, characterizing the WD in a global sense. For the average of the WD we have

$$\bar{P}_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(t, \omega)\, \mathrm{d}t\, \mathrm{d}\omega = ||f||^2 = ||F||^2. \tag{6.18}$$

$\bar{P}_f$ is the total energy of the signal f, which is positive for all $f \not\equiv 0$, so that the global average frequency can be given by

$$\bar{\Omega}_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \omega\, W_f(t, \omega)\, \mathrm{d}t\, \mathrm{d}\omega / \bar{P}_f. \tag{6.19}$$

Recognizing the fact that the integration with respect to t can be performed first it follows directly from (2.37) that

$$\bar{\Omega}_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega\, |F(\omega)|^2\, \mathrm{d}\omega / ||F||^2. \tag{6.20}$$

Furthermore, using Parseval's relation and eq. (6.7), this can be rewritten as

$$\bar{\Omega}_f = \int_{-\infty}^{\infty} \Omega_f(t)\, |f(t)|^2\, \mathrm{d}t / ||f||^2. \tag{6.21}$$

The interpretation of these equations is that the global average frequency of the WD is on the one hand equal to the average frequency of the spectrum of

the signal and on the other hand to the weighted average of the instantaneous frequency of the signal, where the instantaneous power of the signal is used as the weighting function. The link between the latter two notions has been recognized before by Franks [12]), but it is interesting to see how naturally it arises in the context of the WD.

The global second-order moment with respect to the frequency variable is given by

$$\bar{m}_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\omega - \bar{\Omega}_f)^2 \, W_f(t, \omega) \, \mathrm{d}\omega \, \mathrm{d}t / \bar{P}_f \tag{6.22}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\omega - \bar{\Omega}_f)^2 \, |F(\omega)|^2 \, \mathrm{d}\omega / ||F||^2 \quad . \tag{6.23}$$

From (6.23) it follows that $\bar{m}_f$ is always non-negative and can be interpreted as the spread of the spectrum of the signal. Similarly as is done in (6.21) for the first-order moment, it is possible to relate $\bar{m}_f$ with the local moments according to

$$\bar{m}_f = \frac{\int_{-\infty}^{\infty} m_f(t) \, |f(t)|^2 \, \mathrm{d}t}{||f||^2} + \frac{\int_{-\infty}^{\infty} (\Omega_f(t) - \bar{\Omega}_f)^2 \, |f(t)|^2 \, \mathrm{d}t}{||f||^2} \quad . \tag{6.24}$$

Considering this form for the case of the FM signal of example (3.7) we find that the first term is zero, and that the spread $\bar{m}_f$ of the spectrum is due only to the variations of the instantaneous frequency.

6.4. *Moments of the WD in the time variable*

6.4.1. Local moments

For a fixed frequency ω the average of the WD with respect to the time is

$$P_f(\omega) = \int_{-\infty}^{\infty} W_f(t, \omega) \, \mathrm{d}t = |F(\omega)|^2 . \tag{6.25}$$

The average time at this frequency is given by the first-order moment

$$T_f(\omega) = \int_{-\infty}^{\infty} t \, W_f(t, \omega) \, \mathrm{d}t / P_f(\omega). \tag{6.26}$$

From (A5) in the appendix it follows that

$$T_f(\omega) = -\mathrm{Im} \frac{F'(\omega)}{F(\omega)} = -\mathrm{Im} \frac{\mathrm{d}}{\mathrm{d}\omega} \ln F(\omega). \tag{6.27}$$

Hence, if we write for $F(\omega)$

$$F(\omega) = A(\omega)\, e^{j\psi(\omega)}, \tag{6.28}$$

where $A(\omega)$ and $\psi(\omega)$ are real, we find for (6.27)

$$T_f(\omega) = -\psi'(\omega). \tag{6.29}$$

Thus the average time of the WD at frequency ω is equal to the negative of the derivative of the phase of the spectrum of the signal.

This relation leads to a peculiar result if $f(t)$ is taken to be the impulse response of a linear time-invariant system. In that case $F(\omega)$ is the transmission function of this system and the negative of the derivative of the phase is known as the *group delay* of the system [11]). Hence (6.29) has the interpretation that the average time of the WD of the impulse response of a linear system is equal to its group delay.

The local second-order moment with respect to t is defined by

$$M_f(\omega) = \int_{-\infty}^{\infty} (t - T_f(\omega))^2\, W_f(t, \omega)\, dt / P_f(\omega). \tag{6.30}$$

From (A6) in the appendix it follows that

$$M_f(\omega) = -\tfrac{1}{2}\, \mathrm{Re}\, \frac{d}{d\omega} \frac{F'(\omega)}{F(\omega)}$$

which in case $F(\omega)$ is of the form (6.28) can be rewritten as

$$M_f(\omega) = -\tfrac{1}{2} \frac{d}{d\omega} \frac{A'(\omega)}{A(\omega)} = -\tfrac{1}{2} \frac{d^2}{d\omega^2} \ln|A(\omega)| \tag{6.31}$$

and thus only depends on the amplitude of the spectrum of f.

6.4.2. Global moments

The global average $\bar{P}_f$ of the WD is given by (6.18). We are now concerned with the first-order global moment with respect to t, given by

$$\bar{T}_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t\, W_f(t, \omega)\, dt\, d\omega / \bar{P}_f. \tag{6.32}$$

Integrating first with respect to ω yields

$$\bar{T}_f = \int_{-\infty}^{\infty} t |f(t)|^2\, dt / ||f||^2. \tag{6.33}$$

Applying Parseval's theorem to (6.33) we obtain

$$\bar{T}_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} T_f(\omega) \, |F(\omega)|^2 \, \mathrm{d}\omega / ||F||^2 . \tag{6.34}$$

Similarly as was the case with the global average frequency (6.33) and (6.34) give two additional interpretations of the global average time. The first indicates that $\bar{T}_f$ is equal to the average time of the signal, the second that it is the weighted average of the group delay with the energy density spectrum as the weighting function.

As remarked before these concepts can also be applied to the impulse response of a linear system. In particular if a causal system is considered it follows directly from (6.33) that $\bar{T}_f > 0$. Therefore, although the group delay of a causal system is not necessarily positive, its weighted average (6.34) is non-negative. Because the weighting function in this case is the square of the modulus of the transmission function, the group delay has to be positive in the major part of the passband of the system.

The global second-order moment is equal to

$$\bar{M}_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (t - \bar{T}_f)^2 \, W_f(t, \omega) \, \mathrm{d}\omega \, \mathrm{d}t / \bar{P}_f \tag{6.35}$$

$$= \int_{-\infty}^{\infty} (t - \bar{T}_f)^2 \, |f(t)|^2 \, \mathrm{d}t / ||f||^2 \tag{6.36}$$

which has clearly a positive value.

It is related to the local moments according to

$$\bar{M}_f = \frac{\frac{1}{2\pi} \int_{-\infty}^{\infty} M_f(\omega) \, |F(\omega)|^2 \, \mathrm{d}\omega}{||F||^2} + \frac{\frac{1}{2\pi} \int_{-\infty}^{\infty} (T_f(\omega) - \bar{T}_f)^2 \, |F(\omega)|^2 \, \mathrm{d}\omega}{||F||^2} . \tag{6.37}$$

6.5. *Inequalities for the global second-order moments of the WD*

It is well known that a signal cannot simultaneously be arbitrarily concentrated in time and frequency. A signal concentrated in time must necessarily have a wide spectrum and vice versa. Since the WD contains both the time information and the frequency information of a signal, this fact must reflect also on the WD, and in particular on the global second-order moments $\bar{m}_f$ and $\bar{M}_f$. From Heisenberg's uncertainty inequality [5,7])

$$\| \mathscr{Q} f \| \; \| \mathscr{D} f \| \geq \tfrac{1}{2} \| f \|^2 \tag{6.38}$$

the shift property (2.17) of the WD and the definition of $\bar{m}_f$ and $\bar{M}_f$ we can derive [5])

$$T^2 \, \bar{m}_f + \frac{1}{T^2} \, \bar{M}_f \geq 2(\bar{m}_f \, \bar{M}_f)^{\frac{1}{2}} \geq 1 \qquad \forall \, T. \tag{6.39}$$

The inequality $T^2 \, \bar{m}_f + (1/T^2) \bar{M}_f \geq 1$ can be brought into the form

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\omega^2 \, T^2 + t^2/T^2) \, W_f(t, \omega) \, \mathrm{d}t \, \mathrm{d}\omega \geq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(t, \omega) \, \mathrm{d}t \, \mathrm{d}\omega \geq 0, \tag{6.40}$$

where use is made of the shift properties of the WD, eqs (2.17) and (2.18). This means that the WD cannot be totally concentrated in an ellipse with axes $(T, 1/T)$ in the (t, ω) plane.

In (6.40) we see that the WD is weighted by a weight function $\omega^2 T^2 + t^2/T^2$. This weight function accentuates remote contributions of the WD, so that (6.40) could still be satisfied by a WD that is extremely concentrated at the origin, but has a very small contribution for large values of t or ω. That such a concentration of the WD is not possible either follows from the inequality derived in ref. 5

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [1 - \mathrm{e}^{-(t^2/T^2 + \omega^2/\Omega^2)}] \, W_f(t, \omega) \, \mathrm{d}t \, \mathrm{d}\omega \geq \frac{1}{1 + \Omega T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(t, \omega) \, \mathrm{d}t \, \mathrm{d}\omega, \tag{6.41}$$

where the WD is weighted with a function that saturates to 1 for large values of t or ω.

Several different inequalities concerning the WD can be found in ref. 5.

7. Wigner distribution for band-limited functions

If the signals f and g are band-limited to ω_c, the spectra of both signals vanish outside the frequency band $|\omega| < \omega_c$

$$F(\omega) = G(\omega) = 0, \qquad |\omega| > \omega_c. \tag{7.1}$$

From section 2.4.2 we know, that in this case the WD vanishes also outside of this interval:

$$W_{f,g}(t, \omega) = 0, \qquad |\omega| > \omega_c. \tag{7.2}$$

To get an expression for the WD inside the interval, we start with eq. (2.21):

$$W_{f,g}(0, 0) = 2(f, \mathscr{R} g). \tag{7.3}$$

Both signals f and g can be represented by their samples taken at $t_k = kT$:

$$f(t) = \sum_{k=-\infty}^{\infty} f(kT) \frac{\sin \pi (t/T - k)}{\pi (t/T - k)} \quad \text{for} \quad T \leq \frac{\pi}{\omega_c} \tag{7.4}$$

and a similar interpolation formula holds also for g. Because of the following orthogonality relation [11])

$$\int_{-\infty}^{\infty} \frac{\sin \pi (t/T - n)}{\pi (t/T - n)} \frac{\sin \pi (t/T - k)}{\pi (t/T - k)} \, \mathrm{d}t = T \delta_{kn} \tag{7.5}$$

the inner product in (7.3) can be written as

$$W_{f,g}(0, 0) = 2T \sum_{k=-\infty}^{\infty} f(kT) g^*(-kT) \quad \text{for} \quad T \leq \frac{\pi}{\omega_c} . \tag{7.6}$$

With eq. (2.20) this result can now be used to evaluate the WD

$$W_{f,g}(t, \omega) = W_{\mathcal{S}_{-t}\mathcal{M}_{-\omega}f, \mathcal{S}_{-t}\mathcal{M}_{-\omega}g}(0, 0). \tag{7.7}$$

It should be observed, however, that the signals $\mathcal{S}_{-t}\mathcal{M}_{-\omega}f$ and $\mathcal{S}_{-t}\mathcal{M}_{-\omega}g$ are band-limited to the larger bandwidth $\omega_c + |\omega|$. For the application of eq. (7.6) it is therefore required to use the smaller sampling period $T' \leq \pi/(\omega_c + |\omega|)$. We have

$$(\mathcal{S}_{-t}\mathcal{M}_{-\omega}f)(\tau) = \mathrm{e}^{-\mathrm{j}\omega t} \mathrm{e}^{-\mathrm{j}\omega\tau} f(t + \tau), \tag{7.8}$$

where t is to be considered as a fixed parameter and τ is the running variable. As a result we get from (7.7)

$$\begin{aligned} W_{f,g}(t, \omega) &= 2T' \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}\omega t} \mathrm{e}^{-\mathrm{j}\omega kT'} f(t + kT') \mathrm{e}^{\mathrm{j}\omega t} \mathrm{e}^{-\mathrm{j}\omega kT'} g^*(t - kT') \\ &= 2T' \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}2\omega kT'} f(t + kT') g^*(t - kT') \\ &\qquad \text{for} \quad T' \leq \pi/(\omega_c + |\omega|). \end{aligned} \tag{7.9}$$

In principle this means that for different values of ω different sampling rates of f and g have to be used in (7.9). However, we know from (7.2) that ω can be restricted to the interval $|\omega| \leq \omega_c$ and consequently (7.9) can be used for all $|\omega| \leq \omega_c$ if $T' \leq \pi/2\omega_c$ is chosen. If f and g are sampled with at least twice their Nyquist rate, (7.9) represents for $|\omega| \leq \omega_c$ and all t the WD of f and g.

From (7.9) it is seen that for ω fixed the time dependence of the WD is contained in the products $f(t + kT') g^*(t - kT')$. Since f and g are band-limited

to ω_c these products are band-limited to $2\omega_c$. Therefore it can be concluded that $W_{f,g}(t, \omega)$ for fixed ω considered as a function of t is band-limited to $2\omega_c$, and may thus be sampled with $T'' \leq \pi/2\omega_c$ without causing any aliasing effects. Combining this fact with the remarks following eq. (7.9), we can choose

$$T' = T'' = T \leq \frac{\pi}{2\omega_c}$$

and obtain for the time samples of $W_{f,g}(t, \omega)$

$$W_{f,g}(nT, \omega) = 2T \sum_{k=-\infty}^{\infty} e^{-j2\omega kT} f[(n+k)T]\, g^*[(n-k)T], \qquad \text{for} \quad T \leq \frac{\pi}{2\omega_c}. \tag{7.10}$$

From these samples the WD can be recovered by using the interpolation formula as in (7.4):

$$W_{f,g}(t, \omega) = \sum_{n=-\infty}^{\infty} W_{f,g}(nT, \omega) \frac{\sin \pi(t/T - n)}{\pi(t/T - n)}, \qquad \text{for} \quad T \leq \frac{\pi}{2\omega_c}. \tag{7.11}$$

Remarks

It is not at all surprising that the WD of band-limited signals can be expressed in terms of the samples of these signals. In fact it is possible to do so even if we take the samples at any rate that is larger than the Nyquist rate. However, the corresponding relations will be much more complicated than (7.9) and (7.10). Sampling the WD in any case requires a rate larger than twice the Nyquist rate of the signal. The surprising thing is that if the signals are also sampled at this rate, then a simple relation like (7.10) results, which contains all the information that is contained in the WD of f and g. Relation (7.10) is very important, and will form the basis for the definition of a WD for discrete-time signals, which will be discussed in part II of this paper.

8. Conclusions

The Wigner distribution for continuous time signals and a number of its properties have been discussed. This bilinear signal transformation gives a mixed time-frequency characterization of the corresponding signal. It was shown to have some marked advantages over other time-frequency charac-

terizations. Noteworthy in this respect is that the WD of a time-limited signal is restricted to the same time interval, and similarly the WD of a band-limited signal has the same band limitation. Furthermore, integration of the WD over the frequency variable at a certain time yields the instantaneous power of the signal, and integration over the time at a certain frequency yields the energy density spectrum at this frequency.

The average frequency at a certain time that could be defined for the WD was shown to coincide with the well-known concept of instantaneous frequency. The latter, however, is usually only defined for harmonic signals with (slowly) varying frequency. Similarly the average time at a certain frequency of the WD was defined and shown to be equal to the group delay of the signal, and this too is normally only introduced for linear systems. The definition of these notions can thus be extended to more general signals by making use of the WD.

The WD of a harmonic signal with constant frequency was shown to have a contribution at this frequency only. More remarkable, however, is that the WD of a chirp signal, i.e. a signal with linearly increasing frequency, also gives a contribution only at the instantaneous frequency.

All these properties indicate the suitability of the WD for time-frequency analysis of a signal. Unfortunately the WD, as it is, is not very amenable for a hardware or software implementation. Its determination requires the signal to be known for all time, and requires the computation of a Fourier integral for every frequency of interest. The problems associated with the infinite time interval can be solved by using windows. This led to the definition of the pseudo-Wigner distribution which was shown to be a smoothed version of the WD with respect to the frequency variable.

The necessity of evaluating the Fourier integrals hints at a discretization of the signal in the time domain, so that DFT and in particular FFT techniques and the like will become applicable.

The conversion of the concept of the Wigner distribution to discrete-time signals is not trivial, and forms the subject of part II of this paper.

Philips Research Laboratories *Eindhoven, March 1980*

Appendix

In this appendix several relations for the WD are given. Their proofs can be found in ref. 7, or are easily derived from the properties of the WD.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} W_{f,g}(t,\omega)\,\mathrm{d}\omega = f(t)\,g^*(t) \tag{A1}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega\, W_{f,g}(t,\omega)\,\mathrm{d}\omega = \tfrac{1}{2}\,[(\mathscr{D}f)(t)\,g^*(t) + f(t)\,(\mathscr{D}g)^*(t)] \tag{A2}$$

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2\, W_{f,g}(t,\omega)\,\mathrm{d}\omega = \tfrac{1}{4}\,[&(\mathscr{D}^2 f)(t)\,g^*(t)\\ &+ 2(\mathscr{D}f)(t)\,(\mathscr{D}g)^*(t) + f(t)\,(\mathscr{D}^2 g)^*(t)]\end{aligned} \tag{A3}$$

$$\int_{-\infty}^{\infty} W_{f,g}(t,\omega)\,\mathrm{d}t = F(\omega)\,G^*(\omega) \tag{A4}$$

$$\int_{-\infty}^{\infty} t\, W_{f,g}(t,\omega)\,\mathrm{d}t = -\tfrac{1}{2}[(\mathscr{D}F)(\omega)\,G^*(\omega) + F(\omega)\,(\mathscr{D}G)^*(\omega)] \tag{A5}$$

$$\begin{aligned}\int_{-\infty}^{\infty} t^2\, W_{f,g}(t,\omega)\,\mathrm{d}t = \tfrac{1}{4}[&(\mathscr{D}^2 F)(\omega)\,G^*(\omega)\\ &+ 2(\mathscr{D}F)(\omega)\,(\mathscr{D}G)^*(\omega) + F(\omega)\,(\mathscr{D}^2 G)^*(\omega)]\end{aligned} \tag{A6}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{f,g}(t,\omega)\,\mathrm{d}t\,\mathrm{d}\omega = (f, g) \tag{A7}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t\, W_{f,g}(t,\omega)\,\mathrm{d}t\,\mathrm{d}\omega = (\mathscr{Q}f, g) \tag{A8}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \omega\, W_{f,g}(t,\omega)\,\mathrm{d}t\,\mathrm{d}\omega = (\mathscr{D}f, g) \tag{A9}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t^2\, W_{f,g}(t,\omega)\,\mathrm{d}t\,\mathrm{d}\omega = (\mathscr{Q}f, \mathscr{Q}g) \tag{A10}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \omega^2\, W_{f,g}(t,\omega)\,\mathrm{d}t\,\mathrm{d}\omega = (\mathscr{D}f, \mathscr{D}g). \tag{A11}$$

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