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THE WIGNER DISTRIBUTION — A TOOL FOR TIME-FREQUENCY SIGNAL ANALYSIS

PART II: DISCRETE-TIME SIGNALS

by T. A. C. M. CLAASEN and W. F. G. MECKLENBRÄUKER

Abstract

In this second part of the paper the Wigner distribution is adapted to the case of discrete-time signals. It is shown that most of the properties of this time-frequency signal representation carry over directly to the discrete-time case, but some others cause problems. These problems are associated with the fact that in general the Wigner distribution of a discrete-time signal contains aliasing contributions. It is indicated that these aliasing components will not be present if the signal is either oversampled by a factor of at least two, or is analytic.

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1. Introduction

In part I of this paper [1]) the Wigner distribution (WD) of continuous-time signals was discussed, and it was shown that this function has some very interesting properties. The determination of this distribution function requires, like the spectrum, an integral of the Fourier type to be evaluated. Ideally this requires the signal to be known for all time, but in practice windowing techniques can be used to relax this requirement. The effects of windowing on the WD were discussed in part I.

In general two different approaches can be distinguished to compute these Fourier-type integrals. The first is by means of analogue signal processing, and recently optical signal processing methods have been proposed for determining suitable approximations to the WD [2]). The second approach is based on digital signal processing. This opens the way to apply computationally efficient methods for evaluating the discrete Fourier transform, but requires the concept of the Wigner distribution to be transferred to the case of discrete-time signals. This is the aim of this part of the paper.

As can be expected, the WD for discrete-time signals shows much similarity with that for continuous-time signals, but in some respects it has characteristic differences.

To emphasize the similarities and point out the differences we will try to follow as closely as possible the same lines as in part I, and give comments only on those results that differ from that of the continuous-time counterpart.

Also the numbering of the equations is made such that corresponding equations have the same number. This has the consequence that sometimes equation numbers are not successive if equations have been deleted, and that equations that do not occur in part I have a special numbering. If reference is made to an equation in part I the equation is given the prefix I.

All sections, except sec. 7, have the same topic and heading as in part I. Section 7, which in part I deals with the WD of band-limited signals, now deals with the WD of finite duration sequences. Equations in this section do not correspond in general with an equation of part I.

2. The Wigner distribution for discrete-time signals

2.1 *Preliminaries*

In this paper we consider in general complex valued, discrete-time signals $f(n)$, $f \in \mathrm{C}$, $n \in \mathrm{Z}$ for which the (Fourier) spectrum is defined by [3])

$$F(\theta) = (\mathscr{F}_{\mathrm{d}} f)(\theta) = \sum_{n=-\infty}^{\infty} f(n)\, \mathrm{e}^{-\mathrm{j}n\theta}. \tag{2.1.a}$$

The inverse transform is given by

$$f(n) = (\mathscr{F}_{\mathrm{d}}^{-1} F)(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta)\, \mathrm{e}^{\mathrm{j}n\theta}\, \mathrm{d}\theta. \tag{2.1.b}$$

Inner products are defined for the signals and spectra by

$$(f, g) = \sum_{n=-\infty}^{\infty} f(n)\, g^*(n) \tag{2.2.a}$$

and

$$(F, G) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta)\, G^*(\theta)\, \mathrm{d}\theta \tag{2.2.b}$$

respectively. Norms and Parseval's relation are then the same as in eqs (I.2.3) and (I.2.4) respectively.

The following operators will be used.
The shift operator for the signals

$$(\mathscr{S}_{\mathrm{k}} f)(n) = f(n-k), \qquad k \in \mathrm{Z} \tag{2.5.a}$$

and for the spectrum

$$(\mathscr{S}_{\xi} F)(\theta) = F(\theta - \xi), \qquad \xi \in \mathrm{R}, \tag{2.5.b}$$

(complex) modulation in the time domain

$$(\mathscr{M}_\xi f)(n) = f(n)\,\mathrm{e}^{\mathrm{j}n\xi} \qquad \xi \in \mathrm{R} \tag{2.6.a}$$

and in the frequency domain

$$(\mathscr{M}_n F)(\theta) = F(\theta)\,\mathrm{e}^{\mathrm{j}n\theta} \qquad n \in \mathrm{Z}\,, \tag{2.6.b}$$

differentiation of the spectrum

$$(\mathscr{D}F)(\theta) = \frac{1}{\mathrm{j}} F'(\theta), \tag{2.7}$$

multiplication by the running variable

$$(\mathscr{Q}f)(n) = nf(n), \tag{2.8}$$

time reversal

$$(\mathscr{R}f)(n) = f(-n). \tag{2.9}$$

There are several different ways to link analogue and digital signals and systems, and hence a variety of ways to define a discrete-time version of the Wigner distribution. What one would like with such a definition is
(1) to obtain a simple concept;
(2) to retain as many as possible of the properties of the WD of continuous-time signals;
(3) to find a simple relation between the discrete-time and continuous-time WD's for discrete-time signals that are obtained by sampling of analogue signals.
The definition which, in our opinion, best matches these requirements is the one suggested by eq. (I.7.10).

2.2. *Definition of the Wigner distribution*

The cross-Wigner distribution of two discrete-time signals $f(n)$ and $g(n)$ is defined by

$$W_{f,g}(n,\theta) = 2 \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}2k\theta} f(n+k)\, g^*(n-k). \tag{2.10}$$

The auto-Wigner distribution of a signal is then given by

$$W_f(n,\theta) = \mathrm{W}_{f,f}(n,\theta) = 2 \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}2k\theta} f(n+k) f^*(n-k). \tag{2.11}$$

Both functions will be called a Wigner distribution (WD). Aiming at obtaining a relation similar to (I.2.13) the WD for the spectra must be defined by

$$W_{F,G}(\theta, n) = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{j2n\xi} F(\theta + \xi)\, G^*(\theta - \xi)\, d\xi \tag{2.12}$$

so that

$$W_{F,G}(\theta, n) = W_{f,g}(n, \theta). \tag{2.13}$$

2.3. *Properties of the Wigner distribution*

For the WD as defined in (2.10) the following properties hold.

2.3.0. Periodicity

The WD is a function of the discrete variable n and the continuous variable θ. With respect to the latter variable the function is periodic with period π

$$W_{f,g}(n, \theta) = W_{f,g}(n, \theta + \pi), \qquad \forall\, n, \theta. \tag{b.1}$$

The period π is distinct from the period 2π that all spectra of discrete-time signals have. This discrepancy could have been avoided by deleting the factor 2 in the exponent in eq. (2.10), but this has the disadvantage that frequency components in f at θ will occur at 2θ in the WD.

2.3.1. Symmetry

$$W_{f,g}(n, \theta) = W^*_{g,f}(n, \theta) \tag{2.14}$$

$$W_f(n, \theta) = W_f^*(n, \theta) \qquad \text{is real,} \tag{2.15}$$

$$W_f(n, \theta) = W_{f^*}(n, -\theta). \tag{2.16}$$

2.3.2. Time shift

$$W_{\mathscr{S}_k f,\, \mathscr{S}_k g}(n, \theta) = W_{f,g}(n - k, \theta). \tag{2.17}$$

2.3.3. Modulation

$$W_{\mathscr{M}_\xi f,\, \mathscr{M}_\xi g}(n, \theta) = W_{f,g}(n, \theta - \xi). \tag{2.18}$$

2.3.4. Inner product

From the definition of the WD and that of the inner product it follows that

$$W_{f,g}(0, 0) = 2(f, \mathscr{R} g) \tag{2.21}$$

which, combined with (2.17) and (2.18), yields

$$W_{f,g}(n, \theta) = 2(\mathscr{S}_{-n}\, \mathscr{M}_{-\theta} f,\ \mathscr{R}\, \mathscr{S}_{-n}\, \mathscr{M}_{-\theta}\, g). \tag{2.22}$$

2.3.5. Sum formula

$$W_{f_1+g_1, f_2+g_2}(n, \theta) = W_{f_1,g_1}(n, \theta) + W_{f_1,g_2}(n, \theta) + W_{f_2,g_1}(n, \theta) + W_{f_2,g_2}(n, \theta) \tag{2.23}$$

and in particular

$$W_{f+g}(n, \theta) = W_f(n, \theta) + W_g(n, \theta) + 2\,\mathrm{Re}\, W_{f,g}(n, \theta). \qquad (2.24)$$

2.3.6. Multiplication by n or $e^{j2\theta}$

$$2n\, W_{f,g}(n, \theta) = W_{\mathcal{Q}f,g}\,(n, \theta) + W_{f,\mathcal{Q}g}\,(n, \theta). \qquad (2.25)$$

The relation corresponding with (I.2.26), i.e. the multiplication by the frequency variable, cannot simply be copied. Because the WD is a periodic function with period π, the multiplication of the WD by a function of θ can only be expressed in terms of WD's if this function is periodic with this period as well. The most elementary such function is $e^{j2\theta}$, yielding

$$e^{j2\theta}\, W_{f,g}(n, \theta) = W_{\mathcal{S}_{-1}f,\, \mathcal{S}_1 g}(n, \theta). \qquad (2.26)$$

2.3.7. Inverse transform in the time

The WD evaluated at frequency $\theta/2$ can be considered as the Fourier transform of the sequence $2f(n+k)\,g^*(n-k)$ with fixed n. Hence, from the inverse transform relation (2.1.b) we find

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{jk\theta}\, W_{f,g}(n, \theta/2)\, d\theta = 2f(n+k)\, g^*(n-k).$$

This can be written in the form

$$\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j(n_1 - n_2)\theta}\, W_{f,g}\left(\frac{n_1 + n_2}{2}, \theta\right) d\theta = f(n_1)\, g^*(n_2), \qquad (2.27)$$

where $(n_1 + n_2)/2$ must be an integer.

Three special cases are:

(i) $n_1 = n_2 = n$, yielding

$$\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} W_{f,g}(n, \theta)\, d\theta = f(n)\, g^*(n) \qquad (2.28)$$

and in particular

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} W_f(n, \theta)\,\mathrm{d}\theta = |f(n)|^2. \tag{2.29}$$

This shows that the integral over one period of the WD in its frequency variable is equal to the instantaneous signal power.

(ii) $n_1 = 2n,\ n_2 = 0$ gives

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{j}2n\theta}\, W_{f,g}(n, \theta)\,\mathrm{d}\theta = f(2n)\, g^*(0). \tag{2.30a}$$

(iii) $n_1 = 2n - 1,\ n_2 = 1$ gives

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{j}2(n-1)\theta}\, W_{f,g}(n, \theta)\,\mathrm{d}\theta = f(2n-1)\, g^*(1). \tag{2.30b}$$

Recovery of the signals from the WD up to constant factors is thus possible, but requires a different procedure for the even and odd numbered samples.

2.3.8.

Summation of eq. (2.28) over n yields

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\pi/2}^{\pi/2} W_{f,g}(n, \theta)\,\mathrm{d}\theta = (f, g) \tag{2.31}$$

which has the same special case as (I.2.33).

2.3.9. Recovery of the spectrum

Using the fact that the WD of the signals is equal to that of the corresponding spectra (eq. (2.13)) it follows from (2.12) that the WD can be considered as the inverse Fourier transform of $F(\theta + \xi)\, G^*(\theta - \xi)$ considered as a function of ξ for fixed θ, and evaluated at $2n$. This means that only the even numbered samples of this function are available, from which the function $F(\theta + \xi)\, G^*(\theta - \xi)$ cannot simply be recovered in general. Nevertheless it holds that

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}2n\xi}\, W_{f,g}(n, \theta) = F(\theta + \xi)\, G^*(\theta - \xi) + F(\theta + \xi + \pi)\, G^*(\theta - \xi + \pi) \tag{2.34}$$

which means that an aliased version can be reobtained from the WD. Equation (2.34) can be rewritten as

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{j}n(\theta_1-\theta_2)}\, W_{f,g}[n, (\theta_1 + \theta_2)/2] = F(\theta_1)\, G^*(\theta_2) + F(\theta_1 + \pi)\, G^*(\theta_2 + \pi). \tag{2.35}$$

Again three special cases are of importance.

(i) $\theta_1 = \theta_2 = \theta$, yielding

$$\sum_{n=-\infty}^{\infty} W_{f,g}(n, \theta) = F(\theta)\, G^*(\theta) + F(\theta + \pi)\, G^*(\theta + \pi) \tag{2.36}$$

and in particular

$$\sum_{n=-\infty}^{\infty} W_f(n, \theta) = |F(\theta)|^2 + |F(\theta + \pi)|^2. \tag{2.37}$$

The aliasing term in (2.37) is necessary to make both sides of the equation periodic with period π. This term will cause some problems later on in various equations. There are two important situations, however, where the aliasing does not cause any problems, because in these situations the spectrum $F(\theta)$ occupies only an interval of length π and is zero in the remaining part of one period of the spectrum. The first case is that of an "oversampled" signal, i.e. a signal with a band-limitation to less than $\pi/2$. Such a signal can be obtained either from an analogue signal by sampling it with a sampling frequency that is larger than twice the Nyquist rate or by interpolation of a discrete signal by a factor 2 (ref. 4). It should be remarked that the definition of the WD was inspired by a formula that was derived for this case (eq. (I.7.10)) and it therefore need not surprise us that for such a signal the WD behaves in the same way as that of an analogue signal.

A second situation where only one of the terms on the right hand side in eq. (2.37) differs from zero arises if we consider analytic signals, the spectrum of which vanishes for negative values of θ over the period of the spectrum. The WD of these signals will be discussed in sec. 5.

(ii) $\theta_1 = \theta$, $\theta_2 = 0$, which gives

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{j}n\theta}\, W_{f,g}(n, \theta/2) = F(\theta)\, G^*(0) + F(\theta + \pi)\, G^*(\pi). \tag{2.38a}$$

(iii) $\theta_1 = \theta + \pi,\ \theta_2 = 0$ which gives

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{j}n\theta} (-1)^n\, W_{f,g}\left(n, \frac{\theta+\pi}{2}\right) = F(\theta+\pi)\, G^*(0) + F(\theta)\, G^*(\pi). \tag{2.38b}$$

If $|G(0)| \neq |G(\pi)|$ these two equations permit the recovery of the spectrum of f. For interpolated or analytic signals only one of the two equations suffices for recovering the spectrum.

2.3.10.

Integration of (2.36) with respect to θ yields

$$\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \sum_{n=-\infty}^{\infty} W_{f,g}(n, \theta)\, \mathrm{d}\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta)\, G^*(\theta)\, \mathrm{d}\theta = (F, G) \tag{2.39}$$

which is the same result as (2.31).

2.3.11. Moyal's formula

The discrete-time analogon of Moyal's formula is somewhat more complicated than the original version (eq. (I.2.40)). It reads

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \sum_{n=-\infty}^{\infty} W_{f_1,g_1}(n, \theta)\, W^*_{f_2,g_2}(n, \theta)\, \mathrm{d}\theta \\ &= (f_1, f_2)\, (g_1, g_2)^* + (f_1, \mathcal{M}_\pi f_2)\, (g_1, \mathcal{M}_\pi g_2)^* && (2.40a) \\ &= (F_1, F_2)\, (G_1, G_2)^* + (F_1, \mathcal{S}_\pi F_2)\, (G_1, \mathcal{S}_\pi G_2)^*. && (2.40b) \end{aligned}$$

The operation $\mathcal{M}_\pi$ in (2.40a) is defined by (2.6) and changes the sign of the odd numbered samples, which is equivalent to a shift of the spectrum over π, expressed by the operation $\mathcal{S}_\pi$ in (2.40b).

2.4. *Effects of time- and band-limitations on the WD*

2.4.1. Time-limited signals

If both f and g are time-limited (finite duration) signals, i.e.

$$f(n) = g(n) = 0, \qquad n < n_a \quad \text{or} \quad n > n_b \tag{2.43}$$

then

$$W_{f,g}(n, \theta) = 0, \qquad n < n_a \quad \text{or} \quad n > n_b. \tag{2.44}$$

2.4.2. Band-limited signals

In the definition of the WD no restriction on the spectra of the signals was assumed, but discrete-time signals have a spectrum that is periodic with period 2π. On the other hand, the WD was found to be periodic with period π, and as discussed in the previous subsection this causes aliasing to occur, i.e. frequency components that are π apart have the same influence on the WD. In fact we have

$$W_{f,g}(n, \theta) = W_{\mathcal{M}_{\pi}f,\, \mathcal{M}_{\pi}g}\,(n, \theta). \tag{b.2}$$

If, however, the signal spectrum is nonzero only over an interval of length less than π, then aliasing will not occur. Thus if there exist θ_a and θ_b such that

$$F(\theta) = G(\theta) = 0, \qquad \theta_a < \theta < \theta_b \tag{2.45}$$

and

$$\theta_b - \theta_a > \pi \tag{b.3}$$

then

$$W_{f,g}(n, \theta) = 0, \qquad \theta_a < \theta < \theta_b - \pi. \tag{2.46}$$

This is illustrated in fig. 1, where the shaded areas indicate the frequency regions where the corresponding functions differ from zero.

The two cases that we discussed before, i.e. oversampled signals and analytic signals, have $\theta_a = \pi/2$, $\theta_b = 3\pi/2$ and $\theta_a = \pi$, $\theta_b = 2\pi$, respectively, and hence have $\theta_b - \theta_a = \pi$ so that generally the WD of these signals has no gap in the frequency domain, but no aliasing either.

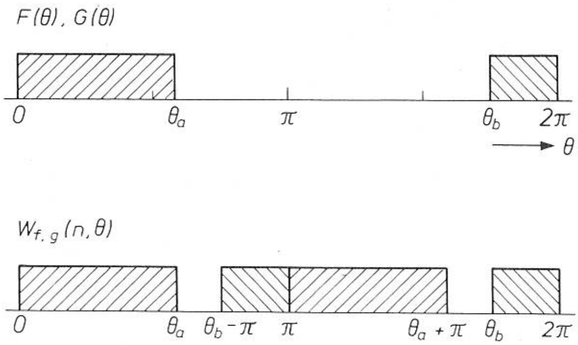

Fig. 1. Illustration of the areas where the WD has zero contributions in the frequency direction, if the spectra have no contributions in the range $\theta_a < \theta < \theta_b$.

2.5. *The WD of sampled analogue signals*

If $f(n)$ and $g(n)$ are obtained by sampling the analogue signals $f_a(t)$ and $g_a(t)$ respectively with a sampling period T, then the WD of f and g can be related to that of f_a and g_a according to

$$W_{f,g}(n,\theta) = \frac{1}{T} \sum_{k=-\infty}^{\infty} W_{f_a,g_a}\left(nT, \frac{\theta + k\pi}{T}\right). \tag{b.4}$$

This formula resembles very much the relation between the input and output spectra of a uniform sampler (eq. (1.28) in ref. 3), but differs in one important respect, namely that the folding (aliasing) in the frequency occurs around π/T rather than $2\pi/T$. This means that to avoid aliasing in the WD the signals must be sampled at twice the Nyquist rate at least. For signals that are sampled at this rate eq. (b.4) reduces to

$$W_{f,g}(n, \omega T) = \frac{1}{T} W_{f_a,g_a}(nT, \omega), \qquad |\omega| < \frac{\pi}{2T}. \tag{b.5}$$

3. Examples

Most of the examples given in part I carry over rather straightforwardly to the discrete-time case. Therefore we will discuss only two simple examples here

3.1.

$$f(n) = \begin{cases} 1 & |n| < N \\ 0 & |n| \geq N, \end{cases} \tag{3.1}$$

for which the WD is given by

$$W_f(n,\theta) = \begin{cases} 2\,\dfrac{\sin\,[2\theta\,(N - |n| + \frac{1}{2})]}{\sin\,\theta} & |n| < N \\ 0 & |n| \geq N \end{cases} \tag{3.2}$$

3.2.

$$f(n) = A\,\mathrm{e}^{\mathrm{j}\alpha n^2/2},$$

i.e. the discrete-time version of a chirp signal. This signal has the WD

$$W_f(n,\theta) = |A|^2\, 2\pi \sum_k \delta(\theta - \alpha n - k\pi) \tag{3.6}$$

which has been plotted in fig. 2 for the case that α is non-rational. In this figure each dot represents a δ-function. This chirp signal has only one frequency component lying at $n\alpha \bmod \pi$ in a period of its WD.

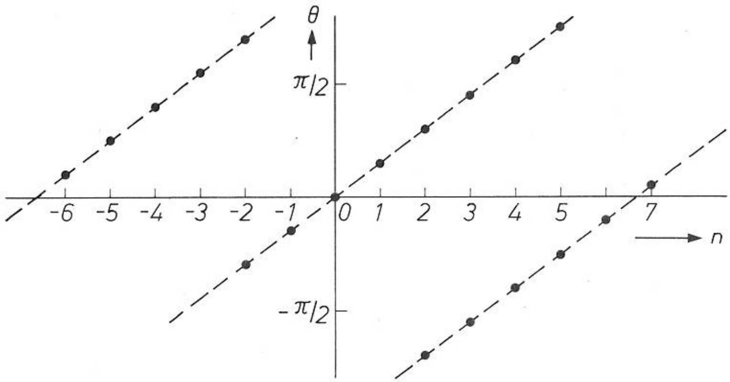

Fig. 2. Wigner distribution of a discrete-time chirp signal.

4. Effects of linear operations on the WD

4.1. *Linear filtering*

In this section the effect of filtering of the signals f and g on their WD will be described. In general, i.e. if neither f and g nor the filters are band-limited to $\pi/2$, a relation similar to (I.4.2) cannot be obtained. If we consider both the outputs of the original filters, i.e.

$$f_c(n) = (f * h_f)(n) = \sum_k f(n-k)\, h_f(k) \tag{4.1.a}$$

$$g_c(n) = (g * h_g)(n) = \sum_k g(n-k)\, h_g(k) \tag{4.1.b}$$

and the outputs of two complementary filters, obtained by changing the sign of the odd numbered samples of the impulse responses h_f and h_g, i.e.

$$f_c^0(n) = (f * h_f^0)(n) = \sum_k f(n-k)\,(-1)^k\, h_f(k) \tag{d.1.a}$$

$$g_c^0(n) = (g * h_g^0)(n) = \sum_k g(n-k)\,(-1)^k\, h_g(k) \tag{d.1.b}$$

we can derive that

$$W_{f_c,g_c}(n,\theta) + W_{f_c^0,g_c^0}(n,\theta) = \sum_k W_{f,g}(k,\theta)\, W_{h_f,h_g}(n-k,\theta). \tag{4.2}$$

From (d.1.a) and (d.1.b) it follows that the spectra of the output of the complementary filters are given by

$$F_c^0(\theta) = F(\theta)\, H_f(\theta + \pi) \tag{d.2.a}$$

$$G_c^0(\theta) = \mathrm{G}(\theta)\, H_g(\theta + \pi). \tag{d.2.b}$$

Hence if f and g and also h_f and h_g are band-limited to $\pi/2$ or less, the outputs of these complementary filters are zero, in which case eq. (4.2) gets the simpler form

$$W_{f_c, g_c}(n, \theta) = \sum_k W_{f,g}(k, \theta)\, W_{h_f, h_g}(n - k, \theta). \tag{4.2}$$

4.2. *Multiplication in the time domain*

If f and g modulate the carriers m_f and m_g respectively we get

$$f_m(n) = f(n)\, m_f(n) \tag{4.3.a}$$

$$g_m(n) = g(n)\; m_g(n). \tag{4.3.b}$$

The WD of the modulated signals is given by

$$W_{f_m, g_m}(n, \theta) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} W_{f,g}(n, \xi)\, W_{m_f, m_g}(n, \theta - \xi)\, \mathrm{d}\xi. \tag{4.4}$$

4.3. *Windowing in the time domain; the pseudo-Wigner distribution*

For computational purposes windowing has to be applied in the discrete-time case as well in order to evaluate the WD. Therefore sliding windows are applied to both signals f and g, yielding

$$f_n(\nu) = f(\nu)\, \mathscr{S}_n\, w_f(\nu) = f(\nu)\, w_f(\nu - n) \tag{4.5.a}$$

$$g_n(\nu) = g(\nu)\, \mathscr{S}_n\, w_g(\nu) = g(\nu)\, w_g(\nu - n). \tag{4.5.b}$$

The corresponding WD according to (4.4) is equal to

$$W_{f_n, g_n}(\nu, \theta) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} W_{f,g}(\nu, \xi)\, W_{w_f, w_g}(\nu - n, \theta - \xi)\, \mathrm{d}\xi. \tag{4.6}$$

Again considering this WD only at the instant $\nu = n$ the pseudo-Wigner distribution for discrete-time signals is obtained

$$\tilde{W}_{f,g}(n, \theta) = W_{f_n, g_n}(\nu, \theta)\Big|_{\nu = n}$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} W_{f,g}(n, \xi)\, W_{w_f, w_g}(0, \theta - \xi)\, \mathrm{d}\xi. \tag{4.7}$$

5. The Wigner distribution for analytic signals

A discrete-time analytic signal is of the form [3])

$$f_a(n) = f(n) + \mathrm{j}\hat{f}(n), \tag{5.1}$$

where $f(n)$ is real and $\hat{f}(n)$ is the discrete Hilbert transform of f, defined by

$$\hat{f}(n) = (\mathscr{H}_d f)(n) = \sum_{m \neq n} f(m) \frac{\sin^2 \pi (m-n)/2}{\pi (m-n)/2}. \tag{5.2}$$

This means that the spectrum of the analytic signal is given by

$$F_a(\theta) = \begin{cases} 2F(\theta) & 0 < \theta < \pi \\ F(\theta) & \theta = 0 \\ 0 & -\pi < \theta < 0. \end{cases} \tag{5.3}$$

Equation (5.2) describes the ideal case, but the corresponding filter is not realizable. Realizable approximations are described in the literature, however, and approximations of analytic signals are therefore easily obtainable by means of digital signal processing [5,6]). Here we will restrict the discussion to the ideal case in which the analytic signal is related to the original signal by a linear filtering with the transmission function

$$H(\theta) = \begin{cases} 2 & 0 < \theta < \pi \\ 1 & \theta = 0 \\ 0 & -\pi < \theta < 0. \end{cases} \tag{e.1}$$

Apart from the usefulness of the analytic signal for describing single-side-band modulation systems and in other system and signal analysis applications, the analytic signal plays an important role in the framework of the WD. As was demonstrated already in sec. 2, the spectral occupancy of the analytic signal as given by (5.3), taking nonzero values only over an interval of length π, assures that the WD of the analytic signal does not contain aliasing components.

We can now make a comparison between the two types of signals that have a WD without aliasing, i.e. the interpolated signal and the analytic signal. Both can be obtained from the original signal by a linear filtering operation. This is shown in fig. 3, where the boxes with an arrow indicate sampling rate increase and decrease by a factor of 2 (ref. 7). The interpolated signal is real if the original signal is real, but has the double rate, while the analytic signal is complex but has the same rate as the original signal. The number of real valued samples per second is therefore the same for both signals. Also, in general, the complexity of the low-pass filtering to obtain the interpolated

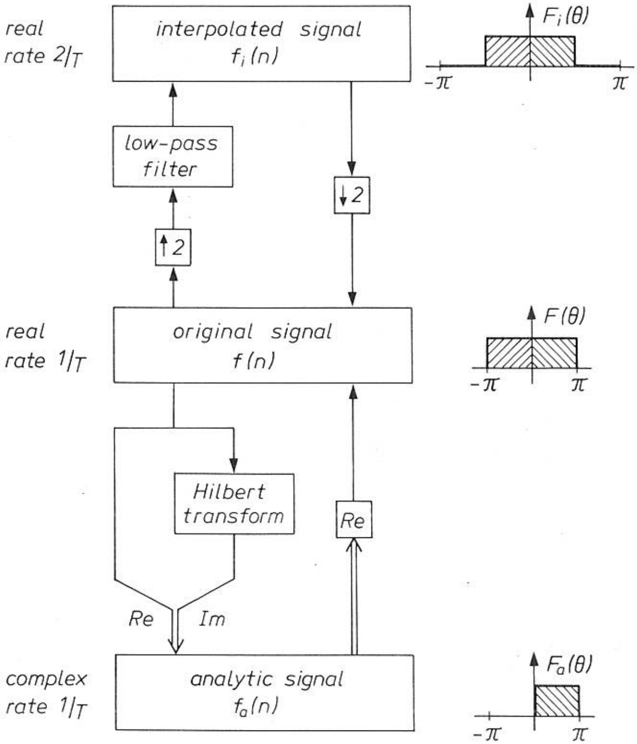

Fig. 3. Relations between a discrete-time signal, the corresponding interpolated version and the corresponding analytic signal.

signal is quite comparable to that of the Hilbert transformer if the same attenuation in the stopband is required.

6. Global and local moments of the WD

6.1. *General remarks*

In this section local and global central moments of the WD for discrete-time signals are discussed. These moments allow a coarse characterization of the WD without the need to evaluate it. For discrete-time signals, however, we are confronted with the fact that the WD has a different behaviour with respect to its time and frequency variable, i.e. its time variable is discrete, while it is a periodic function of the continuous frequency variable. This will require some special care in defining the different moments.

6.2. *Central moments*

6.2.1. Central moments of a periodic function

Let $K(\theta)$ be a periodic function with period π and a positive mean given by

$$\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} K(\theta)\, \mathrm{d}\theta = M_0 > 0. \tag{6.1.a}$$

In part I the moments were obtained by the minimization of a weighted integral of the function K. Pursuing the same approach, we have to search for an expression that satisfies the following conditions.

(1) Weighting functions that it contains should be periodic with period π.

(2) It should have a unique minimum.

(3) For positive valued functions K the value of the minimum should be smaller if $K(\theta)$ is more concentrated.

From a number of candidates for such an expression that satisfy these criteria we will choose the following one

$$I(\theta_0) = \frac{\int_{-\pi/2}^{\pi/2} \sin^2(\theta - \theta_0)\, K(\theta)\, \mathrm{d}\theta}{\int_{-\pi/2}^{\pi/2} \cos^2(\theta - \theta_0)\, K(\theta)\, \mathrm{d}\theta}. \tag{6.2.a}$$

This choice is rather arbitrary and a justification for it can only be found in the results that are obtained and that satisfy our expectations reasonably well. However, replacing in the denominator $\cos^2(\theta - \theta_0)$ by 1 does not have a dramatic influence on the results. In fact the position of the minimum will not be changed at all by this replacement.

The expression (6.2a) has a unique minimum in the interval under the condition (6.1.a) at $\theta_0 = \theta_K$, where

$$\theta_K = \tfrac{1}{2} \arg \left[\int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{j}2\theta} K(\theta)\, \mathrm{d}\theta \right] \tag{6.3a}$$

and the value of the minimum is given by

$$I(\theta_K) = \frac{M_0 - \left| \dfrac{1}{\pi} \displaystyle\int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{j}2\theta} K(\theta)\, \mathrm{d}\theta \right|}{M_0 + \left| \dfrac{1}{\pi} \displaystyle\int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{j}2\theta} K(\theta)\, \mathrm{d}\theta \right|}. \tag{6.4a}$$

From (6.1a) and (6.4a) it follows that

$$-1 < I(\theta_K) \leq 1 \tag{f.1}$$

and if, moreover, $K(\theta)$ is positive then

$$\left| \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{j2\theta} K(\theta)\, d\theta \right| \leq M_0 \tag{f.2}$$

so that

$$0 \leq I(\theta_K) \leq 1. \tag{f.3}$$

The lower bound is only attained if $K(\theta)$ is fully concentrated in the point θ_K, i.e. $K(\theta) = \delta(\theta - \theta_K)$, $|\theta| \leq \pi/2$.

6.2.2. *Central moments for discrete-time functions*

Let $k(n)$ be a function of the discrete variable $n \in \mathrm{Z}$ with

$$\sum_n k(n) = m_0 > 0. \tag{6.1b}$$

In this case the central moments are found by minimization of the expression

$$i(n_0) = \sum_n (n - n_0)^2\, k(n) \,/\, m_0. \tag{6.2b}$$

In principle it is possible to restrict n_0 to Z, but this is not necessary and it is easily seen that the location of the minimum is then not unique. Minimizing (6.2b) for $n_0 \in \mathrm{R}$ yields a minimum at $n_0 = n_k$, with

$$n_k = \sum_n n\, k(n) \,/\, m_0 \tag{6.3b}$$

and the minimum equals

$$i(n_k) = \sum_n (n - n_k)^2\, k(n) \,/\, m_0 = \sum_n n^2\, k(n) \,/\, m_0 - n_k^2. \tag{6.4b}$$

6.3. *Moments of the WD in the frequency variable*

6.3.1. Local moments

For a fixed time n the WD is a periodic function of θ with period π, for which we know from (2.29) that

$$\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} W_f(n, \theta)\, d\theta = |f(n)|^2 = p_f(n) \geq 0. \tag{6.5}$$

Hence, for values of n for which $f(n) \neq 0$ we can define the average frequency $\theta_f(n)$ of the WD at time n as the first-order moment with respect to the frequency of the WD. From (6.3a) we find

$$\theta_f(n) = \tfrac{1}{2}\arg\left[\int_{-\pi/2}^{\pi/2} e^{j2\theta}\, W_f(n, \theta)\, d\theta\right]. \tag{6.6}$$

From (A2) in the appendix it follows that

$$\theta_f(n) = \tfrac{1}{2}\arg\left(f(n+1)\, f^*(n-1)\right). \tag{6.7}$$

For real valued signals this value is zero or $\pi/2$ and contains therefore no information. Considering complex valued signals $f(n)$ can be written in the form

$$f(n) = v(n)\, e^{j\varphi(n)}, \tag{6.8}$$

where $v(n)$ and $\varphi(n)$ are real functions. Using this representation of f we find from (6.7)

$$\theta_f(n) = \left[\frac{\varphi(n+1) - \varphi(n-1)}{2}\right] \bmod \pi \tag{6.9}$$

where $[\cdot] \bmod \pi$ means that the expression must be evaluated modulo π. (Arbitrarily it has been assumed that θ_f takes values in the interval $(0, \pi)$, but any other interval of length π could have been taken).

In the continuous-time case the counterpart of eq. (6.9) has the interpretation that the average frequency of the WD is equal to the instantaneous frequency of the signal. For discrete-time signals we are faced with the problem that a notion like the instantaneous frequency is not defined. Both the forward difference

$$(\Delta_f\, \varphi)(n) = [\varphi(n+1) - \varphi(n)] \bmod 2\pi \tag{f.4}$$

and the backward difference

$$(\Delta_b\, \varphi)(n) = [\varphi(n) - \varphi(n-1)] \bmod 2\pi \tag{f.5}$$

are candidates for such a definition, but suffer from the fact that they are asymmetrical. (If for example $\varphi(n) = \cos n\theta_m$ then one would like the instantaneous frequency to be zero in a maximum of $\varphi(n)$ like $n = 0$).

Equation (6.9) yields the result that the average frequency of the WD is the arithmetic mean of the forward and backward differences, taken modulo π because the period of the WD is equal to π. Thus

$$\theta_f(n) = \left[\frac{(\Delta_f\, \varphi)(n) + (\Delta_b\, \varphi)(n)}{2}\right] \bmod \pi. \tag{f.6}$$

For arbitrary complex valued signals the "instantaneous frequency" can assume values over an interval of length 2π. Taking the average modulo π in (f.6) reflects the inability of the WD to discriminate between frequencies in the interval $(0, \pi)$ and those in the interval $(\pi, 2\pi)$ because of the aliasing that occurs (see sec. 2). If, however, the signal f is either band-limited to $\pi/2$ by interpolation or is an analytic signal, then the instantaneous frequency is also constrained to an interval of length π; see table I. In both cases the symmetrical difference

$$(\Delta_s \varphi)(n) = \left[\frac{\varphi(n+1) - \varphi(n-1)}{2}\right] \bmod 2\pi \tag{f.7}$$

is a suitable candidate for the instantaneous frequency. Since this frequency must lie in the intervals indicated in table I, a restriction modulo π instead of modulo 2π is possible. From equations (f.6) and (f.7) it then follows that the average frequency of the WD is again equal to the instantaneous frequency of the signal.

TABLE I

Intervals of the instantaneous frequency for various complex valued types of signals

$f(n)$	interval of instantaneous frequency
arbitrary	$(-\pi, \pi)$
interpolated	$(-\pi/2, \pi/2)$
analytic	$(0, \pi)$

The local second-order moment with respect to θ is

$$m_f(n) = \frac{p_f(n) - \left|\dfrac{1}{2\pi}\displaystyle\int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{j}2\theta}\, W_f(n,\theta)\,\mathrm{d}\theta\right|}{p_f(n) + \left|\dfrac{1}{2\pi}\displaystyle\int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{j}2\theta}\, W_f(n,\theta)\,\mathrm{d}\theta\right|}. \tag{6.11}$$

This can be expressed in terms of the signal according to

$$m_f(n) = \frac{|f(n)|^2 - |f(n+1)f^*(n-1)|}{|f(n)|^2 + |f(n+1)f^*(n-1)|}. \tag{6.12}$$

6.3.2. Global moments

The global average of the WD is equal to

$$\bar{P}_f = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \sum_n W_f(n, \theta)\, \mathrm{d}\theta = ||f||^2 = ||F||^2. \tag{6.18}$$

The global average frequency is given by

$$\bar{\theta}_f = \tfrac{1}{2} \arg \left[\int_{-\pi/2}^{\pi/2} \sum_n \mathrm{e}^{\mathrm{j}2\theta}\, W_f(n, \theta)\, \mathrm{d}\theta \right] \tag{6.19}$$

$$= \tfrac{1}{2} \arg \left[\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{j}2\theta}\, |F(\theta)|^2\, \mathrm{d}\theta \right], \tag{6.20}$$

where use is made of (2.37) and the periodicity of $\mathrm{e}^{\mathrm{j}2\theta}$ to derive the last equation. Using Parseval's relation yields

$$\bar{\theta}_f = \tfrac{1}{2} \arg \left[\sum_n f(n+1) f^*(n-1)\right]$$

$$= \tfrac{1}{2} \arg \left[\sum_n \mathrm{e}^{\mathrm{j} \arg(f(n+1)f^*(n-1))}\, |f(n+1) f^*(n-1)|\right]$$

$$= \tfrac{1}{2} \arg \left[\sum_n \mathrm{e}^{\mathrm{j}2\theta_f(n)}\, |f(n+1) f^*(n-1)|\right], \tag{6.21}$$

where the latter expression follows from (6.7). Therefore the global average frequency of the WD is on the one hand equal to the average frequency of the spectrum of f (eq. (6.20)) and on the other is the weighted average of the instantaneous frequency, where $|f(n+1) f^*(n-1)|$ is the weighting function.

The global second-order moment is given by

$$\bar{m}_f = \frac{\bar{P}_f - \left| \dfrac{1}{2\pi} \displaystyle\int_{-\pi/2}^{\pi/2} \sum_n \mathrm{e}^{\mathrm{j}2\theta}\, W_f(n, \theta)\, \mathrm{d}\theta \right|}{\bar{P}_f + \left| \dfrac{1}{2\pi} \displaystyle\int_{-\pi/2}^{\pi/2} \sum_n \mathrm{e}^{\mathrm{j}2\theta}\, W_f(n, \theta)\, \mathrm{d}\theta \right|} \tag{6.22}$$

which can be written in the form

$$\bar{m}_f = \frac{\int_{-\pi}^{\pi} \sin^2(\theta - \bar{\theta}_f) \, |F(\theta)|^2 \, \mathrm{d}\theta}{\int_{-\pi}^{\pi} \cos^2(\theta - \bar{\theta}_f) \, |F(\theta)|^2 \, \mathrm{d}\theta}. \tag{6.23}$$

6.4. *Moments of the WD in the time variable*

6.4.1. Local moments

For a fixed frequency θ the average of the WD over the time is

$$P_f(\theta) = \sum_n W_f(n, \theta) = |F(\theta)|^2 + |F(\theta + \pi)|^2 \geq 0. \tag{6.25}$$

The average time is given by the first-order moment

$$T_f(\theta) = \sum_n n \, W_f(n, \theta) \big/ P_f(\theta). \tag{6.26}$$

It is rather straightforward to derive an expression in which $T_f(\theta)$ is expressed in terms of the spectrum of the signal. For the general case this expression is somewhat complicated and not easy to interpret. This is due to the presence of aliasing in the WD of arbitrary signals. If the discussion is restricted to signals with a spectrum that occupies an interval of length π only, then the aliasing terms vanish. Therefore considering interpolated or analytic signals the expression simplifies to

$$T_f(\theta) = -\operatorname{Im} \frac{F'(\theta)}{F(\theta)} = -\operatorname{Im} \frac{\mathrm{d}}{\mathrm{d}\theta} \ln F(\theta), \tag{6.27}$$

which holds for $\theta \in (-\pi/2, \pi/2)$ in the case of interpolated signals and $\theta \in (0, \pi)$ for analytic signals.

Writing $F(\theta)$ in terms of amplitude and phase

$$F(\theta) = A(\theta) \, \mathrm{e}^{\mathrm{j}\psi(\theta)} \tag{6.28}$$

this yields

$$T_f(\theta) = -\psi'(\theta) \tag{6.29}$$

with the same restriction for θ as in (6.27).

For these signals we obtain again the result that the average time of the WD is equal to the group delay of the signal.

The second-order moment is given by

$$M_f(\theta) = \sum_n [n - T_f(\theta)]^2 \, W_f(n, \theta) \big/ P_f(\theta). \tag{6.30}$$

For interpolated or analytic signals this can be brought into the form

$$M_f(\theta) = -\tfrac{1}{2}\,\mathrm{Re}\,\frac{\mathrm{d}}{\mathrm{d}\theta}\,\frac{F'(\theta)}{F(\theta)}$$

$$= -\tfrac{1}{2}\,\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\,\ln|A(\theta)|\,. \tag{6.31}$$

6.4.2. Global moments

The global average time of the WD is given by

$$\bar{T}_f = \sum_n \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} n\, W_f(n,\theta)\,\mathrm{d}\theta \Big/ \bar{P}_f \tag{6.32}$$

$$= \sum_n n\,|f(n)|^2 \big/ \|f\|^2. \tag{6.33}$$

For interpolated or analytic signals this can be written as

$$\bar{T}_f = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_f(\theta)\,|F(\theta)|^2\,\mathrm{d}\theta \Big/ \|F\|^2 \tag{6.34}$$

which means that $\bar{T}_f$ is equal to the weighted average of the group delay of the signal.

The global second-order moment with respect to the time is equal to

$$\bar{M}_f = \sum_n \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} (n - \bar{T}_f)^2\, W_f(n,\theta)\,\mathrm{d}\theta \Big/ \bar{P}_f \tag{6.35}$$

$$= \sum_n (n - \bar{T}_f)^2\,|f(n)|^2 \big/ \|f\|^2 \geq 0. \tag{6.36}$$

6.5. *Inequality for the global second-order moments*

The global second-order moments of the WD of a continuous-time signal satisfy a Heisenberg type of uncertainty relation, given by (I.6.39). Such an inequality generally does not hold for the moments of the WD of discrete-time signals. This can simply be demonstrated by the example $f(n) = \delta(n)$ for which $\bar{m}_f = 1$, $\bar{M}_f = 0$. For functions that are band-limited to $\pi/2$ an equality of this type can be derived, however, using the inequality

$$\frac{\int_{-\pi/2}^{\pi/2} \sin^2\theta\,|F(\theta)|^2\,\mathrm{d}\theta}{\int_{-\pi/2}^{\pi/2} \cos^2\theta\,|F(\theta)|^2\,\mathrm{d}\theta} \geq \frac{4}{\pi^2}\,\frac{\int_{-\pi/2}^{\pi/2} \theta^2\,|F(\theta)|^2\,\mathrm{d}\theta}{\int_{-\pi/2}^{\pi/2} |F(\theta)|^2\,\mathrm{d}\theta}\,. \tag{f.7}$$

Hence, using the same arguments that led to (I.6.39) we obtain

$$\bar{m}_f + \bar{M}_f \geq 2\,(\bar{m}_f\,\bar{M}_f)^{\frac{1}{2}} \geq 2/\pi.$$

7. Wigner distribution of finite duration signals

If the signals f and g have a finite duration, i.e. if

$$f(n) = g(n) = 0 \qquad |n| \geq L \tag{7.1}$$

then it follows from (2.44) that

$$W_{f,g}(n,\theta) = 0 \qquad |n| \geq L. \tag{7.2}$$

As remarked in sec. 2, the WD $W_{f,g}(n,\theta/2)$ can be interpreted as the Fourier transform of the sequence $f(n+k)\,g^*(n-k)$ considered as a function of k for fixed n. It is known that the Fourier transform of a finite duration sequence can be recovered from its samples taken at a sufficient number of equidistant points on the interval $(-\pi,\pi)$, by means of interpolation. The interpolation formula is [3])

$$W_{f,g}(n,\theta/2) = \sum_{m=-N+1}^{N-1} W_{f,g}(n, m\theta_M/2)\,\frac{\sin M(\theta-\theta_M)/2}{M\sin(\theta-\theta_M)/2}\,, \tag{7.3}$$

where

$$\theta_M = 2\pi/M \tag{7.4}$$

and

$$M = 2N - 1 \tag{7.5}$$

with M taken larger than or equal to the duration of the sequence. For the sequence $f(n+k)\,g^*(n-k)$ this means that

$$N \geq L - |n|\,. \tag{7.6}$$

To compute the WD of two finite duration sequences it therefore suffices to determine the samples occurring in (7.3). These samples are given by

$$W_{f,g}\left(n, m\,\frac{\pi}{M}\right) = 2 \sum_{k=-N+1}^{N-1} \mathrm{e}^{-\mathrm{j}km\,(2\pi/M)} f(n+k)\, g^*(n-k). \qquad (7.7)$$

The right-hand side of eq. (7.7) can be interpreted as the M-point discrete Fourier transform (DFT) of the sequence $f(n+k)\,g^*(n-k)$, $k \in (-N+1, \ldots, N-1)$ for fixed n (see ref. 3). This means that the evaluation of the WD of a finite sequence is equivalent to determining one DFT for each value of n, which can be done efficiently using fast transform algorithms like the FFT [3]) or Winograd algorithm [8]).

The above reasoning can also be applied to find a computationally efficient way to determine the pseudo-Wigner distribution that was introduced in section 4.3. From its definition it follows that it can be computed according to

$$\tilde{W}_{f,g}(n, \theta) = 2 \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}2k\theta}\, w(k)\, f(n+k)\, w^*(-k)\, g^*(n-k). \qquad (7.8)$$

If the windows have a duration $2L-1$, i.e. if

$$w(k) = 0 \qquad |k| \geq L \qquad (7.9)$$

then the same reasoning as before can be applied to show that the PWD can be determined from the samples

$$\tilde{W}_{f,g}\left(n, m\,\frac{\pi}{M}\right) = 2 \sum_{k=-L+1}^{L-1} \mathrm{e}^{-\mathrm{j}km\,(2\pi/M)}\, w(k)\, w^*(-k)\, f(n+k)\, g^*(n-k) \qquad (7.10)$$

with

$$M = 2L - 1, \qquad (7.11)$$

and using the interpolation eq. (7.3). This means that to compute the PWD at a certain instant n, a DFT of length at least $M = 2L-1$ must be computed. If $w(k)\,w^*(-k)$ is stored, then for each n such a computation requires $2M$ multiplications to determine $w(k)\,w^*(-k)\,f(n+k)\,g^*(n-k)$ and $\frac{1}{2}M \log_2 M$ to perform the DFT by means of an FFT, where it is assumed that its length is taken a power of 2. Exploiting the fact that the PWD of any function is real it is possible to compute simultaneously the values of the PWD for two distinct values of n using only one FFT. This results in a further reduction of the computational complexity by a factor of 2.

8. Conclusions

The concept of the Wigner distribution has been introduced for the case of discrete-time signals, and has been shown to share most of the properties of the WD for continuous-time signals. Deviations from this behaviour were observed at some points, however, due to the fact that the WD cannot discriminate between high and low frequencies because of aliasing components that are present in the general case. It was shown that these aliasing terms vanish in two important cases, namely in the case of oversampling by a factor of at least 2, and in the case of analytic signals. Oversampling can be realised if the discrete-time signal is obtained by sampling an analogue signal at a rate of at least twice the Nyquist rate. It can also be accomplished by means of an interpolation process applied to a digital signal. For such signals the average frequency and average time that were defined for the WD could be given the same interpretation as for the continuous-time WD. This led to a useful definition of the instantaneous frequency of a discrete-time signal, which was then again equal to the average frequency of the WD. The average time of the WD was shown to be equal to the group delay of the signal. Finally it was shown that the WD of a finite duration sequence at a fixed time n could be determined from a finite set of samples taken in the frequency interval $(-\pi, \pi)$. These samples of the WD can be determined from the signals by means of a discrete Fourier transform, so that efficient algorithms like the FFT can be used. A particular example of such a WD is the pseudo-Wigner distribution (PWD) which was defined for arbitrary (infinite duration) sequences by means of windowing. This PWD was shown to be a filtered version of the WD of the signals, where filtering takes place in the frequency variable. If finite length windows are used this PWD is not only computable, but can be computed with a complexity of the order of $\frac{1}{2}M(\log_2 M + 4)$, where M is the length of the window. This complexity is of the same order of magnitude as that of the short-time Fourier transform, for which the PWD forms a viable alternative.

The relation between these two notions will be clarified in part III of this paper, where different time-frequency signal representations will be considered and will be related to the WD.

Philips Research Laboratories *Eindhoven, May 1980*

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Appendix

In this appendix several relations for the moments of the WD will be stated without proof. The proofs follow easily by using the properties of the WD given in sec. 2.3.

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} W_{f,g}(n,\theta)\,\mathrm{d}\theta = f(n)\,g^*(n) \tag{A1}$$

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{j}2\theta} W_{f,g}(n,\theta)\,\mathrm{d}\theta = f(n+1)\,g^*(n-1) \tag{A2}$$

$$\sum_n W_{f,g}(n,\theta) = \sum_{k=0}^{1} F(\theta+k\pi)\,G^*(\theta+k\pi) \tag{A4}$$

$$\sum_n n\,W_{f,g}(n,\theta) = -\tfrac{1}{2}\sum_{k=0}^{1}\,[(\mathscr{D}F)(\theta+k\pi)\,G^*(\theta+k\pi) + F(\theta+k\pi)\,(\mathscr{D}G)^*(\theta+k\pi)] \tag{A5}$$

$$\sum_n n^2\,W_{f,g}(n,\theta) = \tfrac{1}{4}\sum_{k=0}^{1}\,[(\mathscr{D}^2F)(\theta+k\pi)\,G^*(\theta+k\pi) + 2(\mathscr{D}F)(\theta+k\pi)\,(\mathscr{D}G)^*(\theta+k\pi) + F(\theta+k\pi)\,(\mathscr{D}^2G)^*(\theta+k\pi)] \tag{A6}$$

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\sum_n W_{f,g}(n,\theta)\,\mathrm{d}\theta = (f,g) \tag{A7}$$

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\sum_n n\,W_{f,g}(n,\theta)\,\mathrm{d}\theta = (\mathscr{Q}f,g) \tag{A8}$$

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\sum_n \mathrm{e}^{\mathrm{j}2\theta} W_{f,g}(n,\theta)\,\mathrm{d}\theta = (\mathscr{S}_{-1}f,\mathscr{S}_1 g) \tag{A9}$$

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\sum_n n^2\,W_{f,g}(n,\theta)\,\mathrm{d}\theta = (\mathscr{Q}f,\mathscr{Q}g) \tag{A10}$$