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THE WIGNER DISTRIBUTION — A TOOL FOR TIME-FREQUENCY SIGNAL ANALYSIS

PART III: RELATIONS WITH OTHER TIME-FREQUENCY SIGNAL TRANSFORMATIONS

by T. A. C. M. CLAASEN and W. F. G. MECKLENBRÄUKER

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1. Introduction

The previous parts of this paper have primarily dealt with the description of the Wigner distribution and its properties. The aim of this part is to discuss the connection of the Wigner distribution with other time-frequency signal transformations. First the relation between the Wigner distribution and the ambiguity function is investigated. A comparison of some of the fundamental properties of these two transformations will be made.

Next, in section 3, a class of time-frequency signal transformations is considered. This class has been discussed before in the context of quantum mechanics by Cohen [1,2]). Each member of this class is uniquely determined by a two-dimensional kernel function. The class is rather broad and contains, apart from the Wigner distribution itself, other known representations such as the one proposed by Rihaczek [3]) and the spectrogram [4,5]) as will be shown in sec. 4. Furthermore a number of properties of such a transformation will be considered that are of interest in the context of signal analysis. In this application these transformations are used to represent the distribution of the signal energy in time and frequency. All properties that are investigated derive their importance from this interpretation. The restrictions these properties impose on a particular representation will be discussed. Although the discussion will be restricted to continuous-time signals, the results are easily extendable to discrete-time signals, using the methods described in part II. Moreover, only auto-Wigner distributions are considered since only these can be interpreted as an energy distribution.

2. Wigner distribution and ambiguity function

As was already remarked in part I of this paper, the definitions of the Wigner distribution (WD) and the ambiguity function (AF) look very much alike at first glance. In this section we will discuss the relations between these two time-frequency signal representations, and point out the similarities and the differences. The ambiguity function has been widely used in the context of radar and sonar, and its properties are very well understood [6,7,8]. The discussion here will therefore be focussed only on those properties that are of importance for a better understanding of the WD.

As before, we denote the signal by $f(t)$ and its spectrum by $F(\omega)$. Introducing the functions $\gamma_f(t, \tau)$ by

$$\gamma_f(t, \tau) = f(t + \tau/2)\, f^*(t - \tau/2) \tag{2.1}$$

and $\Gamma_f(\xi, \omega)$ by

$$\Gamma_f(\xi, \omega) = F(\omega + \xi/2)\, F^*(\omega - \xi/2), \tag{2.2}$$

the WD can be written as

$$W_f(t, \omega) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}\omega\tau}\, \gamma_f(t, \tau)\, \mathrm{d}\tau \tag{2.3a}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{j}\xi t}\, \Gamma_f(\xi, \omega)\, \mathrm{d}\xi. \tag{2.3b}$$

Similarly the AF is given by [8]

$$A_f(\xi, \tau) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}\xi t}\, \gamma_f(t, \tau)\, \mathrm{d}t \tag{2.4a}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{j}\omega\tau}\, \Gamma_f(\xi, \omega)\, \mathrm{d}\omega. \tag{2.4b}$$

Relations (2.3) and (2.4) are represented in fig. 1, where the operator $\mathscr{F}_t$ represents the Fourier transform with respect to the variable t, and similarly for the other variables.

It follows immediately that the WD and the AF are related by the two-dimensional Fourier transform

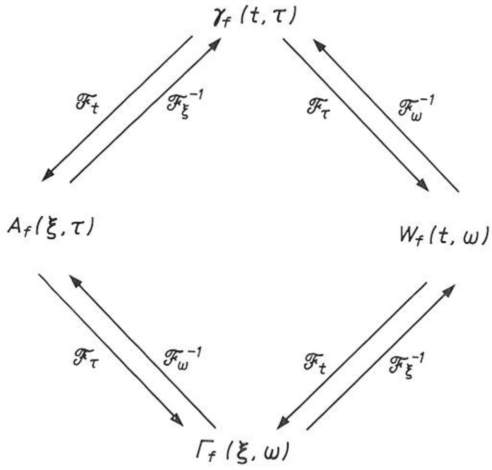

Fig. 1. Diagram indicating the relation between the signal, the spectrum, the Wigner distribution and the ambiguity function.

$$A_f(\xi, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}(\xi t - \omega \tau)} \, W_f(t, \omega) \, \mathrm{d}t \, \mathrm{d}\omega. \tag{2.5}$$

In general eq. (2.5) indicates that the WD and the AF are different signal representations. A noteworthy difference is for example that the WD of any signal is real, while the AF, even of a real-valued signal, will in general be complex. However, for signals that are even or odd functions of time, i.e.

$$f(t) = \pm f(-t) \tag{2.6}$$

the AF and the WD are the same up to scale factors, i.e. [9])

$$W_f(t, \omega) = \pm 2A_f(2\omega, 2t). \tag{2.7}$$

As an example the chirp signal

$$f(t) = \mathrm{e}^{\mathrm{j}\alpha t^2/2} \tag{2.8}$$

has WD and AF given by

$$W_f(t, \omega) = A_f(\omega, t) = 2\pi \, \delta(\omega - \alpha t). \tag{2.9}$$

The differences between the two signal representations become most pronounced if shifts in time or frequency of the signal are considered. While such shifts lead to corresponding shifts in the WD, the effect on the AF is a phase

TABLE I

	signal	spectrum	Wigner distribution	ambiguity function
original	$f(t)$	$F(\omega)$	$W_f(t, \omega)$	$A_f(\xi, \tau)$
time shift	$f(t-t_0)$	$F(\omega)\,\mathrm{e}^{-\mathrm{j}\omega t_0}$	$W_f(t-t_0, \omega)$	$A_f(\xi, \tau)\,\mathrm{e}^{-\mathrm{j}\xi t_0}$
frequency shift	$f(t)\,\mathrm{e}^{\mathrm{j}\omega_0 t}$	$F(\omega-\omega_0)$	$W_f(t, \omega-\omega_0)$	$A_f(\xi, \tau)\,\mathrm{e}^{\mathrm{j}\omega_0 \tau}$
filtering	$\int_{-\infty}^{\infty} f(u)\,h(t-u)\,\mathrm{d}u$	$F(\omega)\,H(\omega)$	$\int_{-\infty}^{\infty} W_f(u, \omega)\,W_h(t-u, \omega)\,\mathrm{d}u$	$\int_{-\infty}^{\infty} A_f(\xi, u)\,A_h(\xi, \tau-u)\,\mathrm{d}u$
modulation	$f(t)\,m(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\eta)\,M(\omega-\eta)\,\mathrm{d}\eta$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} W_f(t, \eta)\,W_m(t, \omega-\eta)\,\mathrm{d}\eta$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} A_f(\eta, \tau)\,A_m(\xi-\eta, \tau)\,\mathrm{d}\eta$

factor only, as can be seen from table I. This illustrates clearly that the interpretation of the time and frequency variables of the WD corresponds to that of the signal, while this is not the case for the AF. In fact the magnitude of the AF is completely insensitive to shifts in time or frequency of the signal. In table I, the effects of filtering and modulation on the WD and the AF, respectively, have been indicated too. Although the corresponding relations look very similar, their effect on the two functions will in general be very different, as already shown for the two special cases discussed above.

3. A generalized class of time-frequency signal representations

In part I the WD was introduced in a rather axiomatic way and was shown to have some interesting properties. Such an approach leaves the question unanswered whether the WD is the only such representation, i.e. whether other representations exist that have similar or even more desirable properties. This question will be addressed in some detail in this section.

The starting point is a generalized class of time-frequency signal representations that was introduced by Cohen [1,2]). This class is given by

$$C_f(t,\omega;\Phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{j}(\xi t - \tau\omega - \xi u)}\, \Phi(\xi,\tau) f(u+\tau/2) f^*(u-\tau/2)\,\mathrm{d}u\,\mathrm{d}\tau\,\mathrm{d}\xi, \tag{3.1}$$

where Φ is at present an arbitrary kernel function, determining the particular representation in the class. Using the definition of the ambiguity function of f it is easy to see that (3.1) can be rewritten as

$$C_f(t,\omega;\Phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{j}(\xi t - \omega\tau)}\, \Phi(\xi,\tau)\, A_f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{3.2}$$

Alternatively C_f can be expressed in terms of the WD according to

$$C_f(t,\omega;\Phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(t-\tau,\omega-\xi)\, W_f(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi. \tag{3.3}$$

where

$$\varphi(t,\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{j}(\xi t - \omega\tau)}\, \Phi(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{3.4}$$

The interpretation of eq. (3.3) is that all members of this class of signal representations can be obtained by a linear transformation of the WD characterized by the kernel $\varphi(t, \omega)$ that is related to the kernel $\Phi(\xi, \tau)$ by a two-dimensional Fourier transformation according to (3.4).

Clearly the identity kernel

$$\Phi_{\mathrm{W}}(\xi, \tau) = 1 \qquad \forall\, \xi, \tau \tag{3.5a}$$

or

$$\varphi_{\mathrm{W}}(\tau, \xi) = 2\pi\, \delta(\tau)\, \delta(\xi) \tag{3.5b}$$

leads to the WD. Allowing the kernel Φ to depend on t and ω and setting

$$\Phi_{\mathrm{A}}(\xi, \tau; t, \omega) = 2\pi\, \delta(\tau - t)\, \delta(\xi - \omega) \tag{3.6a}$$

or

$$\varphi_{\mathrm{A}}(\tau, \xi; t, \omega) = \mathrm{e}^{\mathrm{j}(\omega\tau - \xi t)} \tag{3.6b}$$

the AF is obtained:

$$A_f(\omega, t) = C_f(t, \omega; \Phi_{\mathrm{A}}). \tag{3.7}$$

In the remaining part of this section several properties of time-frequency signal representations are stated that are desirable in signal analysis. Each of these properties has associated with it one or more constraints on the kernels. In this way it is possible to analyse the properties of a particular representation in a systematic way once its kernel is given. The various constraints will henceforth be labelled "Ck" and the corresponding property "Pk".

The first two properties to be discussed are that shifts in time or frequency of the signal result in corresponding shifts of the representation:

P1: $$C_{\mathscr{S}_{t_0}f}(t, \omega; \Phi) = C_f(t - t_0, \omega; \Phi) \tag{3.8}$$

P2: $$C_{\mathscr{M}_{\omega_0}f}(t, \omega; \Phi) = C_f(t, \omega - \omega_0; \Phi), \tag{3.9}$$

where the shift operator $\mathscr{S}_{t_0}$ and the modulation operator $\mathscr{M}_{\omega_0}$ are defined in sec. 2.1. in part I. Since the WD satisfies both of these properties, it is easily seen from (3.3) that these properties hold for any $C_f(t, \omega; \Phi)$ if the kernel Φ is independent of t or ω. Using the notation of eq. (3.6) we get the constraints

C1: $$\Phi(\xi, \tau; t, \omega) \quad \text{does not depend on } t, \tag{3.10}$$

C2: $$\Phi(\xi, \tau; t, \omega) \quad \text{does not depend on } \omega. \tag{3.11}$$

Henceforth only representations with kernels that are independent of both t and ω will be considered. This excludes, for example, the ambiguity function whose kernel does not satisfy either of the constraints C1 and C2. Both properties are essential if we wish the time and frequency variables of the signal representation to correspond to those of the signal and its spectrum, respectively.

The next two properties of interest have already been considered by Cohen [1]), namely

P3:
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} C_f(t,\omega;\Phi)\,\mathrm{d}\omega = |f(t)|^2 \tag{3.12}$$

and

P4:
$$\int_{-\infty}^{\infty} C_f(t,\omega;\Phi)\,\mathrm{d}t = |F(\omega)|^2. \tag{3.13}$$

These properties are obtained by representations with kernels satisfying the constraints

C3:
$$\Phi(\xi,0) = 1 \qquad \forall\ \xi \tag{3.14}$$

C4:
$$\Phi(0,\tau) = 1 \qquad \forall\ \tau. \tag{3.15}$$

Properties P3 and P4 are attractive in view of the desire to interpret the time-frequency representation as an energy distribution of the signal over time and frequency. P3 states that the integral over all frequencies at time t is equal to the instantaneous power, while P4 states similarly that the integral over all time at a certain frequency ω is equal to the value of the energy density spectrum at that frequency. An immediate consequence is that if either of the two properties is satisfied then the integral over the whole (t,ω)-plane

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_f(t,\omega;\Phi)\,\mathrm{d}t\,\mathrm{d}\omega = \|f\|^2 \tag{3.16}$$

is equal to the signal energy.

Apart from the WD for which P3 and P4 were demonstrated in part I to hold, the representations obtained by taking

$$\Phi_\alpha(\xi,\tau) = \mathrm{e}^{\mathrm{j}\alpha\xi\tau} \tag{3.17}$$

have these properties as well. For $\alpha = \frac{1}{2}$ this kernel leads to the representation proposed by Rihaczek [3]):

$$C_f(t,\omega;\Phi_{\frac{1}{2}}) = f^*(t)\,F(\omega)\,\mathrm{e}^{\mathrm{j}\omega t}. \tag{3.18}$$

This representation has the charm of being simple, but, in general, it is complex valued which is not very convenient in view of our interpretation of C_f as an energy distribution. If we wish the representations to be real, i.e.

P5: $$C_f(t,\omega;\Phi) = C_f^*(t,\omega;\Phi) \tag{3.19}$$

then we must put the constraint

C5: $$\Phi(\xi,\tau) = \Phi^*(-\xi,-\tau) \tag{3.20}$$

on the kernel of the representation.

It would of course be even more desirable for such a representation to be positive for all time and frequency, because then it could truly be interpreted as an energy distribution over time and frequency. However, such an interpretation would even then be questionable, as Heisenberg's uncertainty relation prohibits an arbitrarily sharp frequency discrimination from being possible in an arbitrarily short period of time.

Moreover, as will be shown at the end of this section, the positivity requirement is incompatible with other requirements to be introduced subsequently, and which in our view are more preferable than positivity. In any case suitable averages of the WD, incorporating values over a portion of the (t,ω) plane that has a dimension that is in accordance with Heisenberg's uncertainty relation, can be shown to yield positive values [10]). This means that locally negative values of such a representation need not necessarily be disturbing.

A kernel related to that of Rihaczek, but satisfying all constraints C1 to C5 is given by

$$\tilde{\Phi}_\alpha(\xi,\tau) = \cos(\alpha\xi\tau) \tag{3.21}$$

of which the particular case $\alpha = \frac{1}{2}$ has been considered by Cohen [2]). For this value of α the corresponding representation is equal to the real part of the Rihaczek representation of eq. (3.18).

In section 6 of part I it was shown that the WD has two more properties that are of importance in signal analysis. Firstly, the average frequency $\Omega_f(t)$ of the distribution at a certain time is equal to the instantaneous frequency of the signal. It should be remembered that this holds for complex-valued signals, and that the instantaneous frequency of a real-valued signal is defined as that of the corresponding analytic signal. Secondly, the average time $T_f(\omega)$ at a certain frequency is equal to the group delay. These properties of the generalized distributions can be expressed by

P6: $$\frac{\int\limits_{-\infty}^{\infty} \omega\, C_f(t,\omega;\Phi)\,\mathrm{d}\omega}{\int\limits_{-\infty}^{\infty} C_f(t,\omega;\Phi)\,\mathrm{d}\omega} = \Omega_f(t) = \mathrm{Im}\,\frac{\mathrm{d}}{\mathrm{d}t}\ln f(t), \tag{3.22}$$

$$\text{P7:} \qquad \frac{\int_{-\infty}^{\infty} t\, C_f(t,\omega;\Phi)\,\mathrm{d}t}{\int_{-\infty}^{\infty} C_f(t,\omega;\Phi)\,\mathrm{d}t} = T_f(\omega) = -\operatorname{Im}\frac{\mathrm{d}}{\mathrm{d}\omega}\ln F(\omega), \tag{3.23}$$

Property P6 holds if the following contraints are met:

$$\text{C6:} \qquad \begin{aligned} &\Phi(\xi, 0) = 1 && \forall\, \xi \\ &\frac{\partial}{\partial\tau}\Phi(\xi,\tau)|_{\tau=0} = 0 && \forall\, \xi. \end{aligned} \tag{3.24}$$

Similarly, to have P7 we must require:

$$\text{C7:} \qquad \begin{aligned} &\Phi(0, \tau) = 1 && \forall\, \tau \\ &\frac{\partial}{\partial\xi}\Phi(\xi,\tau)|_{\xi=0} = 0 && \forall\, \tau. \end{aligned} \tag{3.25}$$

It should be observed that the first parts in C6 and C7 are identical to the constraints C3 and C4 respectively.

While the kernel $\tilde{\Phi}_\alpha$ in (3.21) satisfies all these conditions, the kernel Φ_α in (3.17) does not. This means that the Rihaczek representation does not have properties P6 and P7, but its real part does.

Finally we will consider the finite support properties. By this it is meant that for a signal which has a finite extent (support) in time or frequency its representation has the same finite support in the corresponding variable. Hence

P8: if $\qquad f(t) = 0 \qquad |t| > T \qquad$ (3.26)

then $\qquad C_f(t,\omega;\Phi) = 0 \qquad |t| > T, \qquad$ (3.27)

P9: while if $\qquad F(\omega) = 0 \qquad |\omega| > \Omega \qquad$ (3.28)

then $\qquad C_f(t,\omega;\Phi) = 0 \qquad |\omega| > \Omega. \qquad$ (3.29)

Property P8 leads to a constraint on the kernel Φ of the form

$$\text{C8:} \qquad \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{j}\xi t}\,\Phi(\xi,\tau)\,\mathrm{d}\xi = 0 \qquad |\tau| < 2|t|. \tag{3.30}$$

This constraint is equivalent to the requirement that $\Phi(\xi, \tau)$ must be an entire function of ξ of exponential type [7]). More specifically this requires that functions $A(\tau)$ and $\sigma(\tau)$ exist such that $A(\tau)$ is bounded,

$$\sigma(\tau) \leq |\tau|/2 \tag{3.31}$$

and

$$|\Phi(\xi, \tau)| < A(\tau)\, e^{\sigma(\tau)|\xi|} \tag{3.32}$$

for all complex values of ξ. Similarly P9 leads to

$$\text{C9:} \qquad \int_{-\infty}^{\infty} e^{-j\omega\tau}\, \Phi(\xi, \tau)\, d\tau = 0 \qquad |\xi| < 2|\omega| \tag{3.33}$$

which is equivalent to the requirement that Φ must be an entire function of τ of exponential type. Hence, functions $B(\xi)$ and $\varrho(\xi)$ must exist such that $B(\xi)$ is bounded,

$$\varrho(\xi) \leq |\xi|/2 \tag{3.34}$$

and

$$|\Phi(\xi, \tau)| < B(\xi)\, e^{\varrho(\xi)|\tau|} \tag{3.35}$$

for all complex values of τ.

As an example we may consider the kernel Φ_α from (3.17) which satisfies C8 and C9 only if $|\alpha| \leq \frac{1}{2}$. The same conclusion holds for the kernel $\tilde{\Phi}_\alpha(\xi, \tau) = \cos(\alpha\xi\tau)$ as well. With this restriction on α all representations based on this kernel satisfy each of the properties P1 to P9 discussed so far. For $\alpha = 0$ the WD is obtained and for $\alpha = \frac{1}{2}$ the real part of the Rihaczek representation results. It is interesting to note that such a variety of time-frequency signal representations share this large number of desirable properties.

To conclude this section it will be indicated that the properties P7 and P8 are incompatible with the requirement that the representation is nonnegative for all time and frequency values. One of the ways to see this is to consider a causal signal, i.e. a signal that vanishes identically for negative times. Property P8 guarantees that C_f too will vanish for negative t. Assuming C_f to be nonnegative, it follows from property P7 that for such a signal the group delay has to be nonnegative for all frequencies. This contradicts the well known fact that causal signals can have a negative group delay for certain frequencies [11]).

Similarly it is possible to prove that properties P6 and P9 cannot hold for signal representations $C_f(t, \omega)$ that are nonnegative. This follows from the fact that analytic signals may have negative values for the instantaneous frequency during certain periods of time. Observations related to the above have been made by Friberg in the context of radiance functions in optics [12]).

4. Investigation of methods for spectral analysis

In this section various methods for spectral analysis of signals will be investigated in relation to the Wigner distribution. First in sec. 4.1 it will be indicated that the spectrogram of the signal [4,5]) is a special case of the time-frequency representations introduced in sec. 3. Secondly in sec. 4.2 this fact is also shown for the pseudo-Wigner distribution that was introduced in part I. Finally it is shown that the output signal of a spectrum analyser that uses a chirp signal to sweep the signal past a fixed tuned narrow-band filter can be described elegantly by means of the WD.

4.1. *The spectrogram*

Spectrograms are used extensively for time-frequency analysis of speech and other signals [4,5,13]). They are two-dimensional density functions that are derived from the signal and are considered to represent the energy distribution of the signal in time and frequency. The aim of this section is to show that these spectrograms are special cases of the signal representations discussed in sec. 3, and hence are a weighted version in time and frequency of the WD (see eq. (3.3)). We will concentrate on the case that the spectrogram is obtained by a short-time Fourier transform [14]). It has been shown in the literature that different techniques for obtaining the spectrogram, such as by means of a filterbank, yield exactly the same result [15]).

The short-time Fourier transform requires the use of a sliding window $h(t)$ and computation of the Fourier transform of the windowed signal. Although in the bulk of the applications these computations are performed on discrete-time versions of the signals, so that computationally efficient algorithms for the discrete Fourier transform can be applied, we will restrict the discussion to the case of continuous-time signals. The results of part II of this paper give the means for the necessary modifications to deal with discrete-time signals.

Letting $f(\tau)$ denote the signal whose spectrogram is to be determined and $h(\tau)$ the window function, windowing generates the signal

$$f_t(\tau) = f(\tau)\, h(\tau - t), \tag{4.1}$$

where t is the time instant indicating the position of the window on the time axis.

The short-time Fourier transform (SFT) is given by

$$F_t(\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}\omega\tau} f(\tau)\, h(\tau - t)\, \mathrm{d}\tau. \tag{4.2}$$

The spectrogram $S_f(t, \omega)$ is obtained by taking the magnitude squared of this SFT for all possible positions of the window, i.e. by considering t as a running variable:

$$S_f(t, \omega) = |F_t(\omega)|^2. \tag{4.3}$$

Using eq. (I.2.37), eq. (4.3) can be written as

$$S_f(t, \omega) = \int_{-\infty}^{\infty} W_{f_t}(\tau, \omega)\, \mathrm{d}\tau. \tag{4.4}$$

The Wigner distribution W_{f_t} of the windowed signal is given by eq. (I.4.6), which yields

$$S_f(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(\tau, \eta)\, W_h(\tau - t, \omega - \eta)\, \mathrm{d}\tau\, \mathrm{d}\eta. \tag{4.5}$$

Interpreting this equation in the sense of eq. (3.3) it is seen that $S_f(t, \omega)$ is a member of the class of representations given by (3.1) and is generated by the kernels

$$\varphi_s(\tau, \xi) = W_h(-\tau, \xi) \tag{4.6}$$

or

$$\Phi_s(\xi, \tau) = A_h(-\xi, \tau) \tag{4.7}$$

i.e. by the WD or the AF of the window function.

The fact that the spectrogram is a representation generated by a kernel that is itself a WD excludes certain properties discussed in sec. 3 for this representation. Clearly, independent of the window that is used the spectrogram has the properties P1, P2 and P5, but properties P3 and P4 can never be obtained since

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_f(t, \omega)\, \mathrm{d}\omega = \int_{-\infty}^{\infty} |f(\tau)|^2 |h(\tau - t)|^2\, \mathrm{d}\tau \tag{4.8}$$

and

$$\int_{-\infty}^{\infty} S_f(t, \omega)\, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\eta)|^2 |H(\omega - \eta)|^2\, \mathrm{d}\eta, \tag{4.9}$$

where $H(\omega)$ is the spectrum of the window.

Instead of obtaining the instantaneous power in (4.8) one obtains the average power over the duration of the window with the square of the window as weighting function. A similar conclusion holds for the energy density spectrum in (4.9). Furthermore, since by definition S_f is nonnegative, it cannot simultaneously have properties P7 and P8 nor P6 and P9 as was shown in sec. 3. In fact none of these properties is satisfied by the spectrogram. The average frequency of the spectrogram $\Omega_{S_f}(t)$ is given by

$$\Omega_{S_f}(t) = \frac{\int\limits_{-\infty}^{\infty} \Omega_f(\tau)|f(\tau)|^2|h(\tau - t)|^2 \,\mathrm{d}\tau}{\int\limits_{-\infty}^{\infty} |f(\tau)|^2|h(\tau - t)|^2 \,\mathrm{d}\tau}, \tag{4.10}$$

i.e. it is an average of the instantaneous frequency of the signal with the instantaneous power of the windowed signal as the weighting function. Similarly the average time of the spectrogram is equal to

$$T_{S_f}(\omega) = \frac{\int\limits_{-\infty}^{\infty} T_f(\eta)|F(\eta)|^2|H(\omega - \eta)|^2 \,\mathrm{d}\eta}{\int\limits_{-\infty}^{\infty} |F(\eta)|^2|H(\omega - \eta)|^2 \,\mathrm{d}\eta}. \tag{4.11}$$

Hence P6 and P7 cannot be obtained.

Additionally, condition C8 requires $\gamma_h(t, \tau)$ to vanish for $|t| > |\tau|/2$, which in particular for $\tau = 0$ requires $h(t) = 0,\ t \neq 0$. It is obvious that no window fulfils this condition and hence P8 is never obtained. Similarly C9 would require $H(\omega) = 0,\ \omega \neq 0$, which is not realizable.

In conclusion, of the properties stated in sec. 3, only the shift properties and the realness are satisfied by the spectrogram of a signal. As regards the other properties, an average of the desired quantities is obtained taken over the duration of the window. This means that for signals for which these quantities do not vary appreciably over the effective length of the window (short-time stationary signals) the spectrogram provides useful information. If this short-time stationarity cannot be guaranteed, then it may be advantageous to use the pseudo-Wigner distribution (PWD) which is better able to cope with non-stationarities, as will be shown in the following section.

4.2. *The pseudo-Wigner distribution*

In part I the pseudo-Wigner distribution was introduced as the concatenation of slices of the WD of the windowed signal $f_t(\tau)$. It can be computed from the signal according to

$$\tilde{W}_f(t, \omega) = \int\limits_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}\omega\tau} f(t + \tau/2)\, f^*(t - \tau/2)\, h(\tau/2)\, h^*(-\tau/2) \,\mathrm{d}\tau. \tag{4.12}$$

Moreover it was shown that this PWD is related to the WD by

$$\tilde{W}_f(t, \omega) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} W_f(t, \eta)\, W_h(0, \omega - \eta) \,\mathrm{d}\eta, \tag{4.13}$$

where as before W_h is the WD of the window. Clearly therefore the PWD is a member of the class of representations of eq. (3.3) as well. The corresponding kernels are

$$\varphi_P(\tau, \xi) = \delta(\tau)\, W_h(0, \xi) \tag{4.14}$$

and

$$\Phi_P(\xi, \tau) = h(\tau/2)\, h^*(-\tau/2). \tag{4.15}$$

Thus the constraints C1, C2, C3 and C5 are satisfied for all windows and hence the PWD has properties P1, P2, P3 and P5. C6 is satisfied for any window that has an instantaneous frequency that vanishes at $\tau = 0$ ($\Omega_h(0) = 0$) which in particular is the case for a real-valued window. In that case the average frequency of the PWD is equal to the instantaneous frequency of the signal.

Property P7 does not hold, however, because the first part of constraint C7 is not satisfied. In fact, the average time of the PWD can be computed to be given by

$$T_P(\omega) = \frac{\int_{-\infty}^{\infty} T_f(\eta)|F(\eta)|^2\, W_h(0, \omega - \eta)\, \mathrm{d}\eta}{\int_{-\infty}^{\infty} |F(\eta)|^2\, W_h(0, \omega - \eta)\, \mathrm{d}\eta}. \tag{4.16}$$

Comparing this result with the corresponding expression for the spectrogram (4.11) we observe that in both cases an average of the group delay is obtained but the weighting functions are different.

Finally, it follows from (4.15) that P8 holds, but P9 does not, so that the finite support property is only fulfilled with respect to the time variable.

4.3. *Spectrum analyser using a chirp signal*

A classical way to obtain a plot of the spectral content of an assumedly stationary signal is to sweep the signal past a fixed tuned narrow-band filter by modulating it with a chirp signal. A simplified version of such a configuration is sketched in fig. 2, where for ease of analysis it is assumed that a complex chirp signal and a lowpass filter are used. It will be shown now that this spectral analysis method can easily be interpreted in terms of the WD of the signal, which is the case even if the signal is not stationary.

In the scheme of fig. 2 it is assumed that the input $f(t)$ modulates a complex carrier with instantaneous frequency $-\alpha t$:

$$x(t) = f(t)\, \mathrm{e}^{-\mathrm{j}\alpha t^2/2}. \tag{4.17}$$

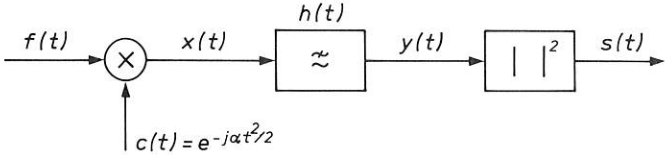

Fig. 2. Schematic diagram of a spectrum analyser using a chirp signal (wobbler).

The WD of this signal can be obtained from eq. (I.4.4) and (I.3.6) to yield

$$W_x(t, \omega) = W_f(t, \omega + \alpha t). \tag{4.18}$$

The lowpass filter has the impulse response $h(t)$ which has a WD $W_h(t, \omega)$. Using eq. (I.4.2) we find for the WD of the output signal $y(t)$

$$W_y(t, \omega) = \int_{-\infty}^{\infty} W_f(\tau, \omega + \alpha\tau)\, W_h(t - \tau, \omega)\, \mathrm{d}\tau. \tag{4.19}$$

The square of the magnitude of the output signal can be expressed as an integral of its WD according to eq. (I.2.29)

$$\begin{aligned} s(t) = |y(t)|^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} W_y(t, \omega)\, \mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(\tau, \omega + \alpha\tau)\, W_h(t - \tau, \omega)\, \mathrm{d}\tau\, \mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(\tau, \eta)\, W_h(t - \tau, \eta - \alpha\tau)\, \mathrm{d}\tau\, \mathrm{d}\eta. \end{aligned} \tag{4.20}$$

This means that the output signal of this spectral analyser can again be described by an averaging of the WD of the input signal in time and frequency.

A nice interpretation of the averaging is obtained by rewriting eq. (4.20) according to

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(\tau, \eta)\, W_h[t - \tau, \alpha(t - \tau) - (\omega - \eta)]\, \mathrm{d}\tau\, \mathrm{d}\eta \big|_{\omega = \alpha t}. \tag{4.21}$$

We see that $s(t)$ is obtained by a convolution of the WD $W_f(t, \omega)$ of the input signal with the kernel

$$\varphi(t, \omega) = W_h(t, \alpha t - \omega) \tag{4.22}$$

and taking the values on the line $\omega = \alpha t$.

If the impulse response of the low-pass filter $h(t)$ is real-valued, then the WD W_h has its main support in the cross-hatched area indicated in fig. 3*a*. According to (4.22) this means that the kernel in the convolution has its support along the line $\omega = \alpha t$ as indicated in fig. 3*b* by the cross-hatched region. The convolution in (4.21) for a fixed value of t then means an averaging of the WD of the input signal over the area cross-hatched in fig. 3*c*. Inspection of this figure shows that the frequency resolution of the method is of the order of $(2\pi B + \alpha/B)$ while the time resolution is of the order $1/B$, where B is the bandwidth of the low-pass filter. To obtain an acceptable resolution in the frequency α has to be taken sufficiently small, but this makes the analyser rather slow. A practical figure is $\alpha \cong \pi B^2/10$ (ref. 16).

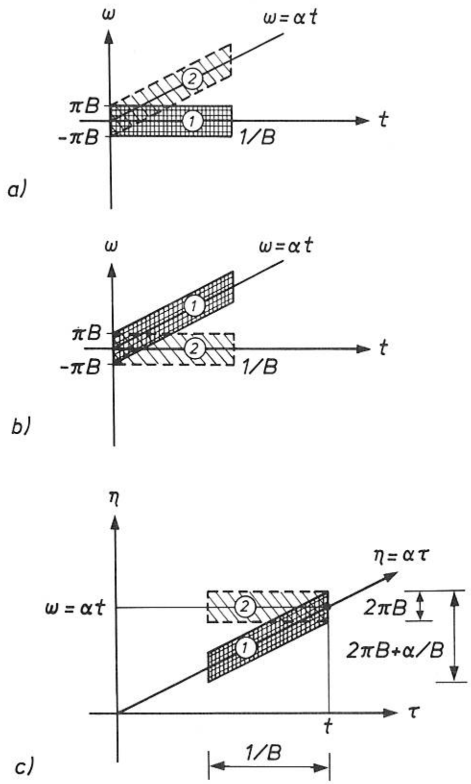

Fig. 3. Illustration of the moving averaging process of the Wigner distribution that takes place in a spectrum analyser. (*a*) Region of support of the WD of the filter in the spectrum analyser. (*b*) Region of support of the kernel of the convolution. (*c*) Region of the averaging that takes place in the convolution. With increasing time this region moves along the line $\eta = \alpha\tau$. The regions labelled 1 correspond to the case of a real-valued filter, while the regions labelled 2 are applicable in case of a complex-valued filter with impulse response given by eq. (4.23).

It is possible to make the frequency resolution independent of the scanning parameter α by using a filter with the complex-valued impulse response

$$h(t) = h'(t)\, \mathrm{e}^{\mathrm{j}\alpha t^2/2}, \tag{4.23}$$

where $h'(t)$ is as before the real-valued impulse response of a low-pass filter. The relation between the WD of $h(t)$ and that of $h'(t)$ is given by

$$W_h(t, \omega) = W_{h'}(t, \omega - \alpha t) \tag{4.24}$$

according to (I.4.4). This means that if the WD of $h'(t)$ has its support in the cross-hatched region labelled 1 in fig. 3*a* then the support of $W_h(t, \omega)$ is the region labelled 2 in this figure. The support of the kernel φ is then centred around the line $\omega = 0$ which means that the convolution in eq. (4.22) is an averaging over the area labelled 2 in fig. 3*c*. Clearly in this case the frequency resolution is $2\pi B$. However, the filter has now become complex and must be adapted to the scan speed α of the analyser.

It can be seen that by decreasing B the region of averaging becomes longer in the t-direction and narrower in the ω-direction, which means that for stationary signals a better estimate of the energy density spectrum is obtained. As already shown by Papoulis [7]) in the limiting case $B = 0$ we get

$$s(t) = |F(\alpha t)|^2. \tag{4.25}$$

Figures similar to fig. 3 have often been used to explain symbolically the working principle of a spectrum analyser. The description given in this section shows that this figure can be interpreted in a much more concrete way in terms of the WD in view of eq. (4.20).

5. Conclusion

In this paper an attempt has been made to place the Wigner distribution in the perspective of time-frequency signal representations and spectral analysis methods. Generally it has been shown that all such representations that aim at giving an energy distribution over time and frequency can be obtained as a weighted average of the Wigner distribution. In particular this has been shown for the spectrogram, and also for the output of commonly used spectrum analysers that use a chirp signal. It was indicated that a number of useful properties of the WD could be secured also for other representations if the corresponding weighting function (kernel) fulfils certain conditions. It then appeared that apart from the WD a number of representations exist for which the kernel satisfies all constraints and hence have all the properties that were considered.

For the spectrogram it was shown that the corresponding kernel could not fulfil most of these requirements. This results in a smearing in both time and frequency of the information contained in the WD if the spectrogram is computed for nonstationary signals. A better situation is obtained by using the pseudo-Wigner distribution, which, like the spectrogram, can be computed from windowed data, but introduces no smearing in the time direction. This latter representation is not always positive, however.

On the other hand it was shown that positivity of such a representation excludes it from having properties that are desirable for the extraction of instantaneous information like the intantaneous power, instantaneous frequency, and the finite support property. If such instantaneous characteristics should be obtainable from such a time-frequency representation then we have to accept negative values for it. This seems to be the only way to accommodate Heisenberg's uncertainty relation.

After completion of this manuscript we became aware of the paper by Flandrin and Escudie [17]) who report results which are closely related to those described in this part of our paper.

Philips Research Laboratories *Eindhoven, May 1980*

REFERENCES

1) L. Cohen, Generalized phase-space distribution functions, J. of Math. Phys. **7**, 781-786, 1966.
2) H. Margenau and L. Cohen, Probabilities in Quantum mechanics, Quantum theory and reality, ch. 4, M. Bunge (ed.), Springer-Verlag, Berlin, 1967, pp. 71-89.
3) A. W. Rihaczek, Signal energy distribution in time and frequency, IEEE Trans. on Inf. Th. **IT-14**, 369-374, 1968.
4) A. V. Oppenheim, Speech spectrograms using the fast Fourier transform, IEEE Spectr., August 1970, pp. 57-62.
5) L. R. Rabiner and R. W. Schafer, Digital processing of speech signals, Prentice-Hall, Inc., 1978.
6) D. E. Vackman, Sophisticated signals and the uncertainty principle, Springer Publ. Co., Inc., New York, 1958.
7) A. Papoulis, Signal analysis, McGraw-Hill, New York, 1977.
8) L. E. Franks, Signal theory, Prentice Hall, Englewood Cliffs, 1969.
9) J. H. McClellan, privaté communication.
10) N. G. de Bruijn, Uncertainty principles in Fourier analysis, in Inequalities, O. Shisha (ed.), Academic Press, New York, 1967, pp. 57-71.
11) L. Weinberg, Network analysis and synthesis, McGraw-Hill, New York, 1962.
12) A. T. Friberg, On the existence of a radiance function for finite planar sources of arbitrary states of coherence, J. Opt. Soc. Am. **69**, 192-198, 1979.
13) R. K. Potter, G. A. Kopp and H. C. Green, Visible speech, D. van Nostrand Co., Inc., 1947.
14) M. R. Portnoff, Time-frequency representation of digital signals and systems based on short-time Fourier analysis, IEEE Trans. on Acoustics, Speech, and Signal Processing **ASSP-28**, 55-69, 1980.
15) J. B. Allen and L. R. Rabiner, A unified approach to short-time Fourier analysis and synthesis, Proc. IEEE **65**, 1558-1564, 1977.
16) K. Küpfmüller, Die Systemtheorie der elektrischen Nachrichtenübertragung, Hirzel Verlag, Stuttgart, 1968.
17) P. Flandrin and B. Escudie, Time and frequency representation of finite energy signals: A physical property as a result of an Hilbertian condition, Signal Processing **2**, 93-100, 1980.