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ON THE TRANSPOSITION OF LINEAR TIME-VARYING DISCRETE-TIME NETWORKS AND ITS APPLICATION TO MULTIRATE DIGITAL SYSTEMS

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Abstract

Time-varying discrete-time networks are considered and their description by means of a transmission function is given. Such a description can be applied to discrete-time networks which contain e.g. modulators and subsystems operating at different sampling rates. Two forms of Tellegen's theorem are derived for these networks. Each of these forms suggests a definition of transposition, called hermitian transpose and generalized transpose respectively. The generalized transpose can be seen as a generalization of the transposition concept defined for time-invariant networks which it includes as a special case. For networks with real parameters the two transposition concepts are the same, but hermitian transposition has certain advantages for systems with complex parameters. A transposition theorem is discussed that relates the transmission function of either form of transpose network to that of the original network. As an application of this theorem a sensitivity analysis is given. Finally an extension of the foregoing theory is discussed for networks containing both analogue and digital parts.

1. Introduction

The complexity of a digital system for signal processing depends inter alia on the number of arithmetical operations that must be performed per unit of time. This number is proportional to the sampling rate at which the system operates. One of the aims in the design of a digital system is therefore to set the sampling rate at its lowest possible value. On the other hand it is known that a digital signal with a sampling rate $f_s = 1/T$ can only uniquely represent frequencies up to $f_s/2$ so that f_s must be higher than twice the highest frequency occurring in the signal. If the whole digital system operates at the same sampling rate this rate is determined by the highest frequency component that will ever be present in the system.

A more economical use of the arithmetical units can often be made by using different sampling rates for different parts of the system. Each sampling rate can then be adapted to the spectral content of the signals to be processed in the corresponding subsystem. Several such multi-rate processing systems have recently been proposed in the literature [1-7]). The increase or decrease of the sampling rate that is necessary to interconnect the various subsystems can be implemented very easily if the sampling rates are related by integer factors [1]). Introduction of the sampling rate increase (SRI) or sampling rate decrease (SRD) does not affect the linearity but makes the system time-variant as will be shown in sec. 2. It is therefore clear that many implementations of digital signal-processing schemes are time-varying discrete-time systems.

In this paper a description of linear discrete-time systems is given that takes into account these time variations. In section 2 the concepts of impulse response and transmission function are introduced. In section 3 two forms of Tellegen's theorem for these linear time-varying discrete-time systems are derived. Each of these forms suggests a definition of transposition that will be called hermitian transpose and generalized transpose respectively. For systems with real parameters the two forms of transposition are the same. The generalized transpose generalizes the concept of transposition as defined for time-invariant systems [8]), which it contains as a special case *). The hermitian transpose has the advantage that it yields the same result when applied to a network with complex parameters or to the real implementation of it. These forms of transposition are discussed in sec. 4, resulting in a transposition theorem that relates the transmission function of both forms of transpose networks to that of the original network.

Transposition of time-invariant networks leaves the transmission function the same and thus offers an alternative implementation of a transmission function [8]). Transposition when applied to time-varying systems yields in general a different transmission function, as can be expected from the fact that the input and output sampling rates of a system and its transpose need not be the same. The transpose system implements what can be called the complementary operation of that performed by the original system. Due to this property, transposition naturally arises in system analysis and synthesis and may lead to efficient designs of systems performing such complementary operations once the implementation of the original operation has been found. As an example, it is shown in sec. 5 that transposition of a decimator leads to an interpolator and of a modulator to a demodulator and vice versa.

Tellegen's theorem can also be used to obtain expressions for the sensitivity of the transmission function of a system to changes in the parameters of the network. These expressions will be derived in sec. 6. Finally in sec. 7 it will be shown that the concepts introduced before can be extended to incorporate networks containing both analogue and digital elements.

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*) This is the reason that we have preferred to speak of transposition rather than of duality, which is more customary in control theory [9]), or of adjointness, which is used in mathematics [10]).

2. Description of linear time-varying discrete-time systems

Every linear discrete-time system with one input and one output can be described by an impulse response $h(n,m)$, which is the response of the system to the input signal $x(n) = u(n-m)$, where $u(n)$ is the unit sample sequence *):

$$u(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0. \end{cases} \tag{1}$$

The output $y(n)$ for an arbitrary input signal $x(n)$ is given by

$$y(n) = \sum_{m=-\infty}^{\infty} h(n,m)\, x(m). \tag{2}$$

For time-invariant systems $h(n,m)$ depends only on the difference $n - m$ and thus takes the simpler form

$$h(n,m) = \tilde{h}(n-m), \tag{3}$$

which makes (2) a discrete convolution. A frequency domain description of a linear discrete-time system can be obtained by means of the Fourier transform for discrete-time signals:

$$X(\theta) = \sum_{n=-\infty}^{\infty} x(n) \exp(-\mathrm{j}n\theta) \tag{4}$$

with inverse transform

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\theta) \exp(\mathrm{j}n\theta)\, \mathrm{d}\theta. \tag{5}$$

In these expressions θ is a relative frequency which is related to the actual frequency ω by

$$\theta = \omega T, \tag{6}$$

where T is the sampling period of $x(n)$.

The transmission function $H(\theta,\xi)$ of a system with impulse response $h(n,m)$ is defined by **)

*) In contrast to the conventional notation we use $u(n)$ for the unit sample sequence and reserve the symbol δ for the Dirac function, which will be used later.

**) The transmission function so defined is the Fourier transform of the frequency-response function that usually is used in the analysis of linear time-varying systems [11,12]. It is the discrete-time analogon of the bi-frequency system function introduced by Zadeh [13].

$$H(\theta,\xi) \triangleq \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h(n,m) \exp\,[-\mathrm{j}(n\theta - m\xi)]. \tag{7}$$

For a system with a real impulse response it follows from (7) that

$$H(\theta,\xi) = H^*(-\theta,-\xi), \tag{8}$$

where the asterisk denotes complex conjugation.

With the above definition of the transmission function the relation between output spectrum $Y(\theta)$ and input spectrum $X(\theta)$ takes the form

$$Y(\theta) = \int_{-\pi}^{\pi} H(\theta,\xi)\, X(\xi)\, \mathrm{d}\xi. \tag{9}$$

In general the sampling period of $y(n)$ may differ from that of $x(n)$ and will be denoted by T_2 and T_1 respectively. From the frequency relation specified by eq. (6) it then follows that $X(\omega T_1)$ is the value of the input spectrum at frequency ω and $Y(\Omega T_2)$ the value of the output spectrum at frequency Ω. In a linear system these values are linearly related, and eq. (9) states that the proportionality factor is precisely $H(\Omega T_2,\omega T_1)$.

As an example the decrease in sampling rate by an integer factor N will be considered, which has the input–output relation [1])

$$y(n) = x(nN) \qquad \forall\, n. \tag{10}$$

Comparison of (10) with (2) yields

$$h(n,m) = u(nN - m). \tag{11}$$

Since $h(n,m)$ given by eq. (11) is not of the form of eq. (3) it must be concluded that the sampling rate decrease is a time-varying element. Its transmission function can be obtained from eq. (7):

$$H(\theta,\xi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp\,[-\mathrm{j}n(\theta - N\xi)]. \tag{12}$$

The right-hand side can be rewritten using the identity [14])

$$\sum_{n=-\infty}^{\infty} \exp\,(-\mathrm{j}n\theta) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\theta - 2k\pi). \tag{13}$$

This expression is a Dirac pulse-train with period 2π and will frequently occur in our analysis. Therefore we introduce the following function which apart from a scale factor is equal to the "shah"-function used by Bracewell [15]):

$$\text{Ш}(\theta) \overset{\triangle}{=} \sum_{k=-\infty}^{\infty} \delta(\theta - 2\pi k). \tag{14}$$

Similarly as $\delta(\omega)$ in the case of continuous-time signals, $\text{Ш}(\theta)$ occurs in the spectral analysis of discrete-time signals, taking account of the periodicity of the spectra of these signals. Manipulation with this function is very similar to that with the δ function.
From (14) and (12) it follows that

$$H(\theta,\xi) = \text{Ш}(\theta - N\xi). \tag{15}$$

This leads to a relation between input and output spectrum of the form

$$Y(\theta) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{\theta - 2\pi k}{N}\right) \tag{16}$$

using the fact that $\text{Ш}(\theta - N\xi) = 0$ if $N\xi \neq \theta - 2\pi k$. In terms of the actual frequencies the expression is

$$Y(\omega T_2) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\omega T_1 - k\frac{2\pi}{N}\right). \tag{17}$$

This is illustrated in fig. 1 for $N = 3$. Here, and in all subsequent examples, only the fundamental interval $-\pi \leqslant \theta \leqslant \pi$ of each of the spectra is depicted, and since $T_2 = NT_1$ the corresponding lengths of the frequency intervals are different for the two spectra. The various relations and the symbols used for

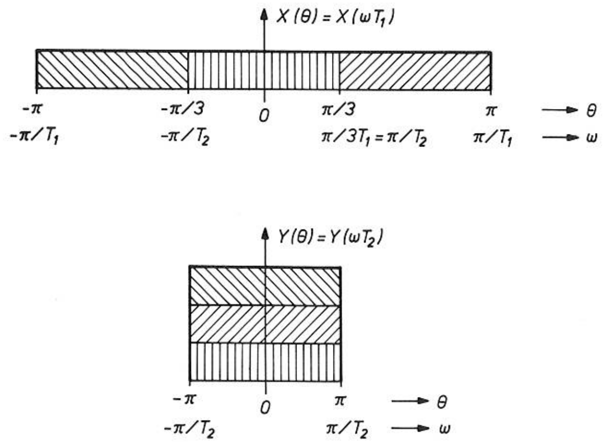


Fig. 1. Input spectrum and output spectrum of a sampling rate decrease for $N = 3$.

this SRD and for several other elements are summarized in table I, where $\tilde{H}(\theta)$ and $\Phi(\theta)$ are the Fourier transforms of $\tilde{h}(n)$ and $\varphi(n)$ respectively.

TABLE I

Time-domain and frequency-domain description of elements of linear time-varying discrete-time systems

operation	symbol	time-domain description		frequency-domain description	
		imp. resp. $h(n,m)$	input - output	transm. funct. $H(\theta,\xi)$	input - output
time-invariant	$x \to \boxed{\tilde{H}} \to y$; T_1, $T_2=T_1$	$\tilde{h}(n-m)$	$y(n)=\tilde{h}(n)*x(n)=\sum_{m=-\infty}^{\infty}\tilde{h}(n-m)\,x(m)$	$\tilde{H}(\theta)\,\text{Ш}(\theta-\xi)$	$Y(\theta)=\tilde{H}(\theta)X(\theta)$
modulation	$x \to \otimes \to y$, $\varphi(n)$; T_1, $T_2=T_1$	$\varphi(n)\,u(n-m)$	$y(n)=x(n)\varphi(n)$	$\frac{1}{2\pi}\Phi(\theta-\xi)$	$Y(\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi(\theta-\xi)X(\xi)\,d\xi$
● cosine		$\cos(n\theta_c)\,u(n-m)$	$y(n)=x(n)\cos(n\theta_c)$	$\frac{1}{2}\text{Ш}(\theta-\xi-\theta_c)+\frac{1}{2}\text{Ш}(\theta-\xi+\theta_c)$	$Y(\theta)=\frac{1}{2}X(\theta-\theta_c)+\frac{1}{2}X(\theta+\theta_c)$
● sine		$\sin(n\theta_c)\,u(n-m)$	$y(n)=x(n)\sin(n\theta_c)$	$\frac{1}{2j}\text{Ш}(\theta-\xi-\theta_c)-\frac{1}{2j}\text{Ш}(\theta-\xi+\theta_c)$	$Y(\theta)=\frac{1}{2j}X(\theta-\theta_c)-\frac{1}{2j}X(\theta+\theta_c)$
sampling rate decrease (SRD)	$x \to \boxed{\downarrow N} \to y$; T_1, $T_2=NT_1$	$u(nN-m)$	$y(n)=x(nN)$	$\text{Ш}(\theta-N\xi)$	$Y(\theta)=\frac{1}{N}\sum_{k=0}^{N-1}X\left(\frac{\theta-2\pi k}{N}\right)$
sampling rate increase (SRI)	$x \to \boxed{\uparrow N} \to y$; T_1, $T_2=T_1/N$	$u(n-mN)$	$y(n)=\begin{cases} x(n/N) & n=0,\pm N,\ldots \\ 0 & \text{elsewhere}\end{cases}$	$\text{Ш}(N\theta-\xi)$	$Y(\theta)=X(N\theta)$

The transmission function as introduced in eq. (7) can be used in much the same way as is conventionally done for time-invariant networks. For example the transmission function H of a cascade of two systems H_1 and H_2 as shown in fig. 2 can be expressed in terms of the individual transmission functions according to

$$H(\theta,\xi) = \int_{-\pi}^{\pi} H_2(\theta,\eta)\, H_1(\eta,\xi)\, \mathrm{d}\eta. \tag{18}$$

From eq. (18) it can be seen that the order in which the transmission functions

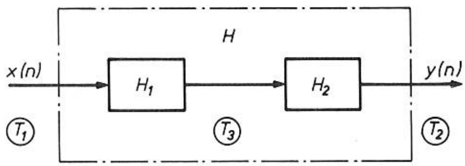


Fig. 2. Cascade of two discrete-time systems.

occur is of importance. An interchange of the two is in general not possible. Since expressions of the form (18) will frequently occur, the following shorthand notation is introduced:

$$H_2(\theta,\cdot)\bullet H_1(\cdot,\xi) \stackrel{\triangle}{=} \int_{-\pi}^{\pi} H_2(\theta,\eta)\, H_1(\eta,\xi)\, \mathrm{d}\eta. \tag{19}$$

Without ambiguity this notation can be extended to situations where the functions depend only on one variable. For example eq. (9) in this notation reads

$$Y(\theta) = H(\theta,\cdot)\bullet X(\cdot). \tag{9'}$$

3. Tellegen's theorem for time-varying networks

What nowadays is referred to as Tellegen's theorem [16,17]) is actually a very general network principle. The theorem is derived starting from an identity, that, as indicated by Penfield et al. [17]), may be expressed in various different forms. This degree of freedom makes the theorem very powerful since it allows its formulation to be adapted to particular classes of networks or to the problems under investigation. Fettweis [18]) has shown that to signal-flow networks the difference form of Tellegen's theorem is applicable. Also in this formulation there still remains a large amount of freedom in the precise form, a fact that can be used to advantage. In this paper we will give two formulations of the difference form of Tellegen's theorem. These forms are derived with the intent to generalize the concept of transposition to linear time-varying systems. Two different definitions of transposition will result as is discussed in sec. 4, which together with Tellegen's theorem lead to a transposition theorem applicable to arbitrary linear discrete-time signal-flow networks.

To this end let us consider such a network S having a certain topology. It consists of a set of I nodes connected by oriented branches. To each node i there corresponds a node variable w_i. Two types of signals are distinguished entering each node: x_i representing source variables and v_{ij} representing the output signal of the branch connecting node j to node i. This is illustrated in fig. 3, where output signals y_i are also indicated in the way proposed by Fettweis [18]). For each of the nodes the following equation holds

$$w_i(n) = x_i(n) + \sum_{j=1}^{I} v_{ij}(n) \qquad i = 1, \ldots, I. \tag{20}$$

It should be recalled that different sampling periods are allowed in various parts of the system, and, to deal with the most general situation, sampling periods T_i will be associated with each of the nodes as indicated in fig. 3. Of course the trivial assumption has been made that all signals entering a

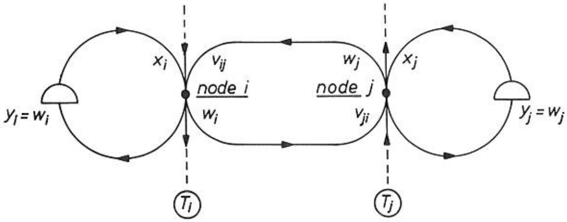


Fig. 3. Flow-graph representation of a discrete-time network. Shown are two nodes with their interconnections and sources.

specific node have the same sampling period. The Fourier transform of eq. (20) gives

$$W_i(\theta) = X_i(\theta) + \sum_{j=1}^{I} V_{ij}(\theta) \qquad i = 1, \ldots, I. \tag{21}$$

Tellegen's theorem relates variables of two different networks S and S' having the same topology. The variables in S' will be denoted by primed symbols and satisfy relations similar to (20) and (21).

The forms of Tellegen's theorem that we aim at can be derived from the following two identities.

$$\int_{-\pi}^{\pi} \sum_{i=1}^{I} [W_i(\theta)\, W_i'^*(-\theta) - W_i'^*(-\theta)\, W_i(\theta)]\, \mathrm{d}\theta = 0, \tag{22}$$

and

$$\int_{-\pi}^{\pi} \sum_{i=1}^{I} [W_i(\theta)\, W_i'(\theta) - W_i'(\theta)\, W_i(\theta)]\, \mathrm{d}\theta = 0. \tag{23}$$

First it can be remarked that if all $w_i'(n)$, the node variables in S', are real then (22) and (23) are the same, and thus differences can only be expected in networks with complex signals. Indeed in sec. 4 it will be shown that two different forms of transposition theorem result for networks with complex parameters from the two different forms of Tellegen's theorem. Secondly, a comparison with the derivation of Tellegen's theorem, as given by Fettweis, reveals that both (22) and (23) differ from Fettweis' formulation in that both expressions are integrated over a fundamental interval of θ. In this way account is taken of the fact that in a time-varying system frequency components of a signal at a certain node may be transferred to other frequencies during the transmission from one node to the other. It also makes the derivation more "symmetrical" in the sense that a similar derivation in the time domain is possible after applying Parseval's equality to (22) or (23), but this time domain form will not be given here. Only the derivation of the first form will be given explicitly since that of the second form follows the same lines.

From eq. (21) applied once to W_i and once to W_i' it follows that

$$\int_{-\pi}^{\pi} \sum_{i=1}^{I} \sum_{j=1}^{I} [W_i(\theta)\, V_{ij}'^*(-\theta) - W_i'^*(-\theta)\, V_{ij}(\theta)]\, \mathrm{d}\theta$$
$$+ \int_{-\pi}^{\pi} \sum_{i=1}^{I} [W_i(\theta)\, X_i'^*(-\theta) - W_i'^*(-\theta)\, X_i(\theta)]\, \mathrm{d}\theta = 0. \tag{24}$$

This is a form of Tellegen's theorem that holds for any discrete-time network, whether linear or not. If the network is linear, then in accordance with sec. 2 impulse responses $f_{ij}(n,m)$ with corresponding transmittances $F_{ij}(\theta,\xi)$ may be associated with the branches in S such that

$$V_{ij}(\theta) = F_{ij}(\theta, \cdot\,) \bullet W_j(\,\cdot\,), \qquad i,j = 1, \ldots, I.$$

A similar relation holds for the primed variables. Using these relations eq. (24) can be rewritten as

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sum_{i=1}^{I} \sum_{j=1}^{I} W_i(\theta)\, F_{ij}'^*(-\theta,\xi)\, W_j'^*(\xi)\, \mathrm{d}\theta\, \mathrm{d}\xi$$
$$- \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sum_{i=1}^{I} \sum_{j=1}^{I} W_i'^*(-\theta)\, F_{ij}(\theta,\xi)\, W_j(\xi)\, \mathrm{d}\theta\, \mathrm{d}\xi \tag{25}$$
$$+ \int_{-\pi}^{\pi} \sum_{i=1}^{I} [W_i(\theta)\, X_i'^*(-\theta) - W_i'^*(-\theta)\, X_i(\theta)]\, \mathrm{d}\theta = 0.$$

Now the order of the summations in the first double sum can be reversed and the integration variables θ and ξ replaced by ξ and $-\theta$ respectively to yield the desired result. With this derivation and the analogous derivation starting from eq. (23) the following theorem has been proved.

THEOREM 1 (Tellegen's theorem). In every two linear discrete-time networks S and S' with the same topology, the spectra of the signals satisfy the relations

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sum_{i=1}^{I} \sum_{j=1}^{I} W_i'^*(-\theta)\, W_j(\xi)\, [F_{ji}'^*(-\xi,-\theta) - F_{ij}(\theta,\xi)]\, \mathrm{d}\theta\, \mathrm{d}\xi$$
$$+ \int_{-\pi}^{\pi} \sum_{i=1}^{I} [W_i(\theta)\, X_i'^*(-\theta) - W_i'^*(-\theta)\, X_i(\theta)]\, \mathrm{d}\theta = 0 \tag{26}$$

and

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sum_{i=1}^{I} \sum_{j=1}^{I} W_i'(\theta)\, W_j(\xi)\, [F_{ji}'(\xi,\theta) - F_{ij}(\theta,\xi)]\, \mathrm{d}\theta\, \mathrm{d}\xi$$
$$+ \int_{-\pi}^{\pi} \sum_{i=1}^{I} [W_i(\theta)\, X_i'(\theta) - W_i'(\theta)\, X_i(\theta)]\, \mathrm{d}\theta = 0. \tag{27}$$

4. Transposition of linear time-varying networks

Transposition or flow-graph reversal is a well-known procedure for giving time-invariant networks a different structure while leaving the transmission between input and output unchanged [8,18]). Invariance of the transmission function for flow-graph reversal cannot be expected for time-varying systems since input and output may operate at different sampling rates. A high input rate and low output rate will become a low input rate and high output rate after flow-graph reversal and vice versa. Therefore a different definition of transposition is required. Two such definitions are suggested by the two forms of Tellegen's theorem derived in sec. 3. They will be denoted by hermitian transpose and generalized transpose respectively.

Let S be a linear discrete-time network with I nodes and branch transmittances $\{F_{ij}(\theta,\xi)\}_{i,j=1}^{I}$ where F_{ij} is the transmittance of the branch that connects node j to node i.

Definition 1. The hermitian transpose of S is a linear discrete-time network S^{H} with the same topology as S and in which node j is connected to node i by a branch with transmittance

$$F_{ij}^{\mathrm{H}}(\theta,\xi) = F_{ji}^{*}(-\xi,-\theta) \qquad i,j = 1,\ldots,I \tag{28}$$

with corresponding impulse response

$$f_{ij}^{\mathrm{H}}(n,m) = f_{ji}^{*}(-m,-n) \qquad i,j = 1,\ldots,I. \tag{29}$$

Definition 2. The generalized transpose of S is a linear discrete-time network S^{T} with the same topology as S and in which node j is connected to node i by a branch with transmittance

$$F_{ij}^{\mathrm{T}}(\theta,\xi) = F_{ji}(\xi,\theta) \qquad i,j = 1,\ldots,I \tag{30}$$

with corresponding impulse response

$$f_{ij}^{\mathrm{T}}(n,m) = f_{ji}(-m,-n) \qquad i,j = 1,\ldots,I. \tag{31}$$

It can be seen from eqs (29) and (31) that both forms of transposition preserve causality. From eq. (8) it follows that in the case of networks where f_{ij} is real for all i and j the two definitions coincide and thus $S^{\mathrm{H}} = S^{\mathrm{T}}$, but in the case of systems with complex parameters they generally differ. The hermitian transpose then has an important advantage over the generalized transpose. To see this, consider the network S of which the branch connecting node j to node i is depicted in fig. 4*a*. In the hermitian transpose network S^{H} and the generalized

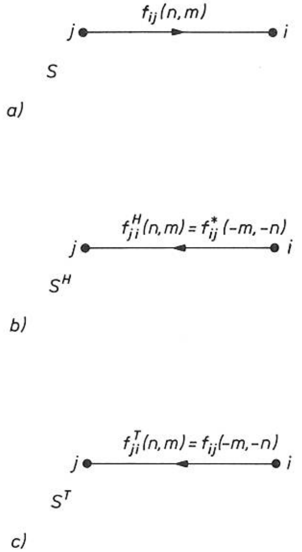


Fig. 4

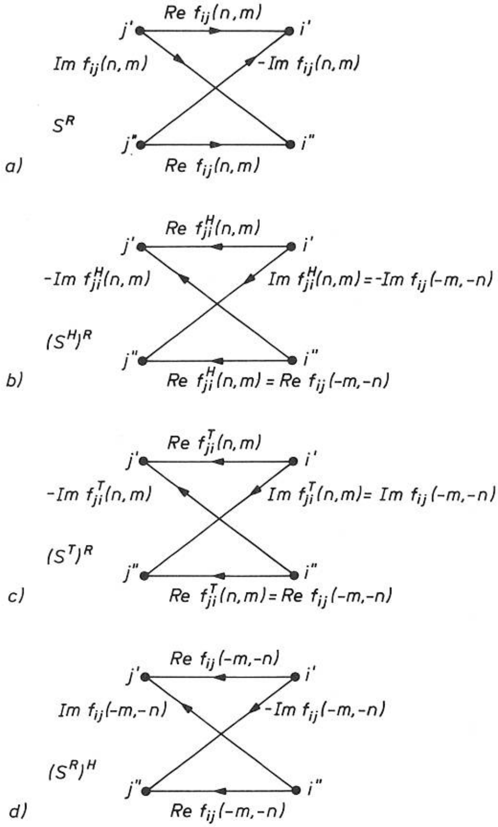


Fig. 5

Fig. 4. Flow-graph representation of connections between two nodes in various networks; (*a*) original, (*b*) hermitian transpose, (*c*) generalized transpose.

Fig. 5. Flow-graph of networks implementing the complex transmittances of fig. 4; (*a*) original, (*b*) hermitian transpose, (*c*) generalized transpose, (*d*) transpose of the network of fig. 5*a*.

transpose network S^T node i is connected to node j as indicated in figs 4*b* and 4*c* respectively. A practical realization of S will be a system S^R such that to every node i in S there correspond two nodes i' and i'' in S^R with node variables

$$w_{i'}{}^{R}(n) = \mathrm{Re}\, w_i(n), \tag{32}$$

$$w_{i''}{}^{R}(n) = \mathrm{Im}\, w_i(n). \tag{33}$$

The transmissions from nodes j' and j'' to nodes i' and i'' are characterized by the impulse responses

$$f_{i'j'}{}^{\mathrm{R}}(n,m) = \quad \mathrm{Re}\, f_{ij}(n,m), \tag{34}$$

$$f_{i''j'}{}^{\mathrm{R}}(n,m) = \quad \mathrm{Im}\, f_{ij}(n,m), \tag{35}$$

$$f_{i'j''}{}^{\mathrm{R}}(n,m) = -\mathrm{Im}\, f_{ij}(n,m), \tag{36}$$

$$f_{i''j''}{}^{\mathrm{R}}(n,m) = \quad \mathrm{Re}\, f_{ij}(n,m), \tag{37}$$

as shown in fig. 5*a*. The networks S^{H} and S^{T} will similarly have associated with them a network with real parameters that implement the complex transmissions as shown in figs 5*b* and 5*c*. These networks will be indicated by $(S^{\mathrm{H}})^{\mathrm{R}}$ and $(S^{\mathrm{T}})^{\mathrm{R}}$ respectively. Since the network S^{R} is also a linear discrete-time network, transposition can be applied to it. Clearly, since S^{R} has only real parameters, its hermitian and generalized transposes will be the same and can be indicated by $(S^{\mathrm{R}})^{\mathrm{H}}$ or $(S^{\mathrm{R}})^{\mathrm{T}}$ as desired. The nodes i', i'' and j', j'' of this network and the corresponding connections are shown in fig. 5*d*. It can be seen by comparison that

$$(S^{\mathrm{R}})^{\mathrm{H}} = (S^{\mathrm{H}})^{\mathrm{R}} \tag{38}$$

but

$$(S^{\mathrm{R}})^{\mathrm{T}} \neq (S^{\mathrm{T}})^{\mathrm{R}}, \tag{39}$$

which means that different systems result when generalized transposition is applied to a network with complex parameters, or to its practical realization.

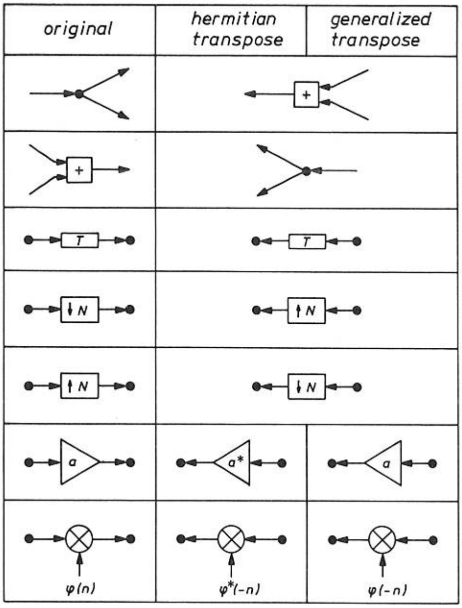
Applying hermitian or generalized transposition to a discrete-time system first of all implies flow-graph reversal, as indicated by the reversal of indices in eqs (28) to (31). This means that branch points in the system become summation points and vice versa. Besides this flow reversal the elements must be replaced by elements having the specified transmittance. From inspection of table I it can be seen, for example, that an SRD must be replaced by an SRI and vice versa. Table II summarizes the necessary replacements. From the definition of S^{H} and S^{T} it follows that $(S^{\mathrm{H}})^{\mathrm{H}} = (S^{\mathrm{T}})^{\mathrm{T}} = S$, which means that if an element must be replaced by an other upon transposition then after transposition this latter element must be replaced by the first one, so that two such elements are always mutually transposed.

The usefulness of the definitions given above becomes clear when we apply Tellegen's theorem to the network and its transpose. Applying the first form (eq. (26)) to S and S^{H} we see that the double sum vanishes and thus

$$\int_{-\pi}^{\pi} \sum_{i=1}^{I} [W_i(\theta)\, X_i^{\mathrm{H}*}(-\theta) - W_i^{\mathrm{H}*}(-\theta)\, X_i(\theta)]\, \mathrm{d}\theta = 0. \tag{40}$$

TABLE II

Elements in the original discrete-time system S and the corresponding elements in the transposed systems S^{H} and S^{T}



A similar form holds for S and S^{T}. Equation (40) is a generalization of the interreciprocity relation as defined in ref. 18.

Now assume that S is excited by a single input $x_a(n)$ incident on node a, and consider as output the signal $y_b(n) = w_b(n)$ on node b. The transmission from input to output is characterized by a transmission function $H_{ba}(\theta,\xi)$ such that according to eq. (9′)

$$Y_b(\theta) = H_{ba}(\theta, \cdot\,) \bullet X_a(\,\cdot\,). \tag{41}$$

Due to the reversal of the signal flow the hermitian transpose system S^{H} will have a single input $x_b{}^{\mathrm{H}}(n)$ incident on node b and output $y_a{}^{\mathrm{H}}(n) = w_a{}^{\mathrm{H}}(n)$. The spectra of these signals are related by

$$Y_a{}^{\mathrm{H}}(\theta) = H_{ab}{}^{\mathrm{H}}(\theta, \cdot\,) \bullet X_b{}^{\mathrm{H}}(\,\cdot\,). \tag{42}$$

Inserting eqs (41) and (42) in the interreciprocity relation (eq. (40)) and observing that $X_i(\theta) = 0, i \neq a$, and $X_i{}^{\mathrm{H}}(\theta) = 0$, $i \neq b$, the first part of the transposition theorem follows immediately. The second part results from a similar reasoning but applied to S and S^{T}.

THEOREM 2 (Transposition theorem). If a linear discrete-time system S realizes a transmission function $H_{ba}(\theta,\xi)$ between input node a and output node b then

(1) its hermitian transpose system S^{H} realizes a transmission function $H_{ab}{}^{\mathrm{H}}(\theta,\xi)$ between input node b and output node a given by

$$H_{ab}{}^{\mathrm{H}}(\theta,\xi) = H_{ba}{}^{*}(-\xi,-\theta), \tag{43a}$$

(2) its generalized transpose system S^{T} realizes a transmission function $H_{ab}{}^{\mathrm{T}}(\theta,\xi)$ between input node b and output node a given by

$$H_{ab}{}^{\mathrm{T}}(\theta,\xi) = H_{ba}(\xi,\theta). \tag{43b}$$

The implications of these properties will be clarified by means of some examples in sec. 5, but it may be noted here that in general the transmission function will not be invariant upon either type of transposition. Therefore transposition does not merely provide an alternative implementation of a certain transmission function, as in the time-invariant case, but rather it yields an implementation of a system that performs a complementary operation.

A property of both types of transposition is that it changes neither the number of multipliers nor the rate at which these multipliers operate. This observation leads to the following corollary, which clearly shows the impact of these forms of transposition on a hardware implementation.

Corollary. If a linear discrete-time system S that realizes the transmission function $H(\theta,\xi)$ is optimized with respect to multiplication rate, then

(1) S^{H} is an optimal realization with respect to multiplication rate of the transmission function

$$H^{\mathrm{H}}(\theta,\xi) = H^{*}(-\xi,-\theta).$$

(2) S^{T} is an optimal realization with respect to multiplication rate of the transmission function

$$H^{\mathrm{T}}(\theta,\xi) = H(\xi,\theta).$$

5. Applications

To clarify the concepts of transposition introduced in sec. 4 a number of examples will be given. We start with an implementation of a decimation-in-time Fast Fourier Transform (FFT) algorithm, of which an 8-point version is shown in fig. 6. Since this system is time-invariant the generalized transposition

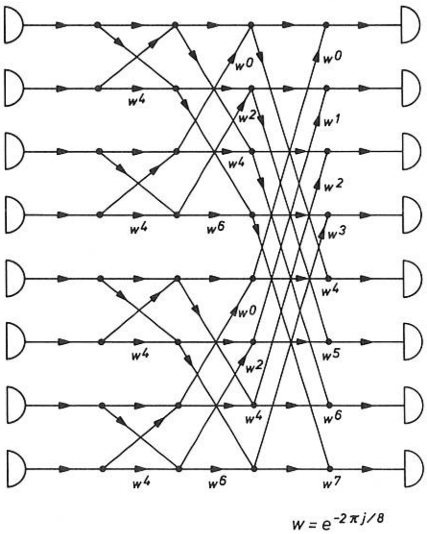


Fig. 6. Flow-graph of a decimation-in-time FFT algorithm for 8 points.

coincides with the conventional transposition and, as is well known, leads to a decimation-in-frequency FFT algorithm (ref. 8, sec. 6.3.2). Since the implementation of the FFT in fig. 6 has complex parameters, the hermitian transpose will be different from the generalized transpose. From the transposition theorem it follows that the hermitian transpose implements the inverse discrete Fourier transform with a decimation-in-frequency FFT algorithm.

The system in the next example has real parameters. Without ambiguity we then use the term transposition. Figure 7*a* depicts an implementation of a decimator [3]) derived from a FIR filter with linear phase and sampling rate decrease. Use is made of the symmetry of the impulse response to reduce the multiplication rate. The transmission function of the system is

$$H_{ba}(\theta,\xi) = \tilde{H}(\xi) \sqcup\!\sqcup (\theta - N\xi), \tag{44}$$

where

$$\tilde{H}(\xi) = \Big(h_0 + 2\sum_{k=1}^{M} h_k \cos k\xi\Big) \exp(-\mathrm{j}M\xi).$$

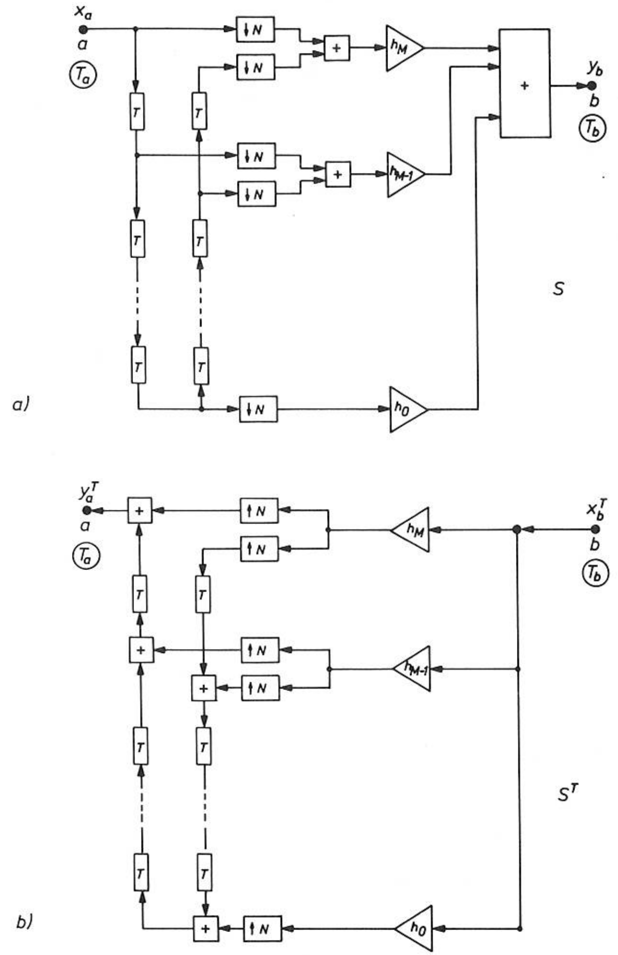


Fig. 7. (*a*) Implementation of a linear phase FIR filter for sampling rate reduction (decimator). (*b*) Transpose of the decimator. This structure realizes an interpolator.

If $X_a(\theta)$ is the input spectrum the output spectrum equals

$$Y_b(\theta) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{H}\left(\frac{\theta - 2\pi k}{N}\right) X_a\left(\frac{\theta - 2\pi k}{N}\right). \tag{45}$$

In such a decimator $\tilde{H}$ has a low-pass characteristic with cut-off frequency at $\omega = \pi/T_b$. A schematic representation of $\tilde{H}(\theta)$ and the spectra is given in figs 8*a*, *b* and *c*. The transpose of this decimator is shown in fig. 7*b* and according to the transposition theorem it has the transmission function

$$H_{ab}{}^{\mathrm{T}}(\theta,\xi) = \tilde{H}(\theta) \sqcup\!\sqcup (\xi - N\theta). \tag{46}$$

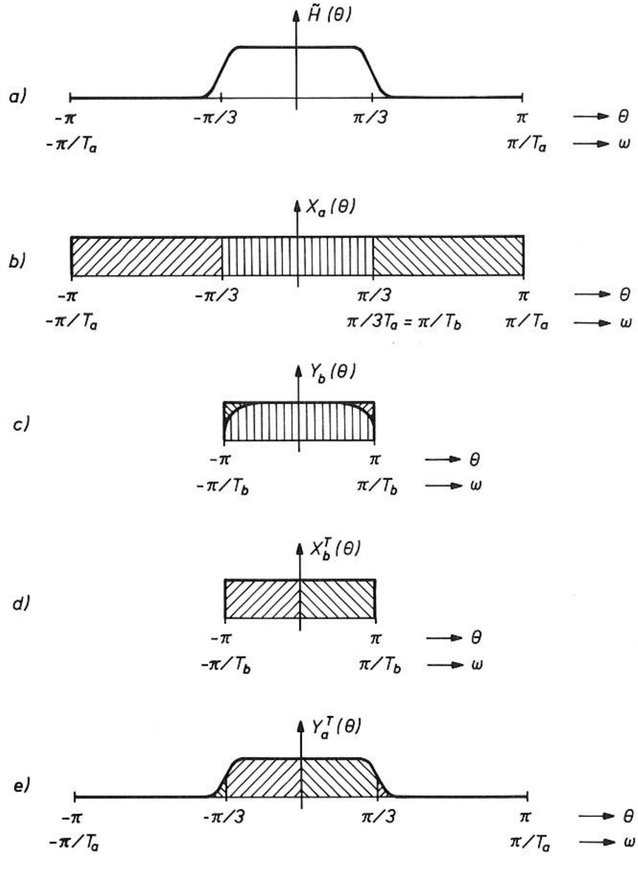


Fig. 8. Spectra of various signals of systems in figs 7*a* and *b*; (*a*) characteristic of the function $\tilde{H}$, (*b*) input spectrum of decimator, (*c*) output spectrum of decimator, (*d*) input spectrum of interpolator, (*e*) output spectrum of interpolator.

Excitation of this transpose system with an input signal with spectrum $X_b^{\mathrm{T}}(\theta)$ gives the output spectrum

$$Y_a^{\mathrm{T}}(\theta) = \tilde{H}(\theta)\, X_b^{\mathrm{T}}(N\theta) \tag{47}$$

as shown in figs 8*d* and 8*e*. It can be concluded that this transpose system is an interpolator [2-5]. In fact this system is a particular implementation of an interpolating FIR filter that was previously proposed by Bellanger and Bonnerot [19]. It is important to note that use is again made of the symmetry of the impulse response of the FIR filter to reduce the number of multiplications, in contrast to conventional implementations of an interpolator [3-6]. This is an immediate consequence of the corollary in sec. 4.

The following example concerns a Weaver single-sideband modulator shown in fig. 9*a*. In this system $\tilde{H}$ is a low-pass filter with cut-off frequency at $\pi/2T_a$,

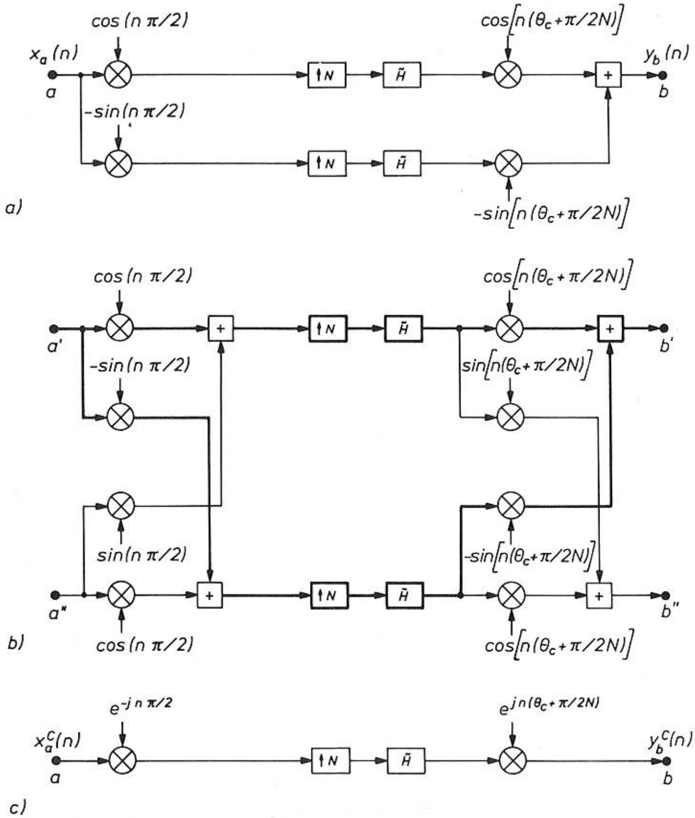


Fig. 9. (*a*) Weaver single-sideband modulator. (*b*) Modulator of fig. 9*a* extended with additional input and output. (*c*) Complex representation of the modulator of fig. 9*b*.

as shown in fig. 10*a*. For the input spectrum in fig. 10*b* the output spectrum of fig. 10*c* results. Extending this modulator as shown in fig. 9*b* we obtain a real implementation of the complex system shown in fig. 9*c*. Of course for real input signals the real part of the output signal of the complex modulator is the same as the output of the modulator in fig. 9*a*. The transmission function of the complex modulator equals

$$H_{ba}{}^{\mathrm{C}}(\theta,\xi) = \tilde{H}(\theta - \theta_{\mathrm{c}} - \pi/2N) ⧢ [\xi - N(\theta - \theta_{\mathrm{c}})] \tag{48}$$

and a sketch of the output spectrum $Y_b{}^{\mathrm{C}}(\theta)$ is given in fig. 10*d*, assuming the input spectrum of fig. 10*b*. Due to the form of the output spectrum the complex system may be called an upper sideband modulator. The hermitian transpose of this system is obtained by flow-graph reversal and changing the elements as prescribed by table II. In this case all elements remain the same except for the SRI, which is replaced by an SRD. The transmission function of the hermitian transpose system is

$$H_{ab}{}^{\mathrm{CH}}(\theta,\xi) = \tilde{H}(\xi + \theta_{\mathrm{c}} + \pi/2N) ⧢ [\theta - N(\xi + \theta_{\mathrm{c}})]. \tag{49}$$

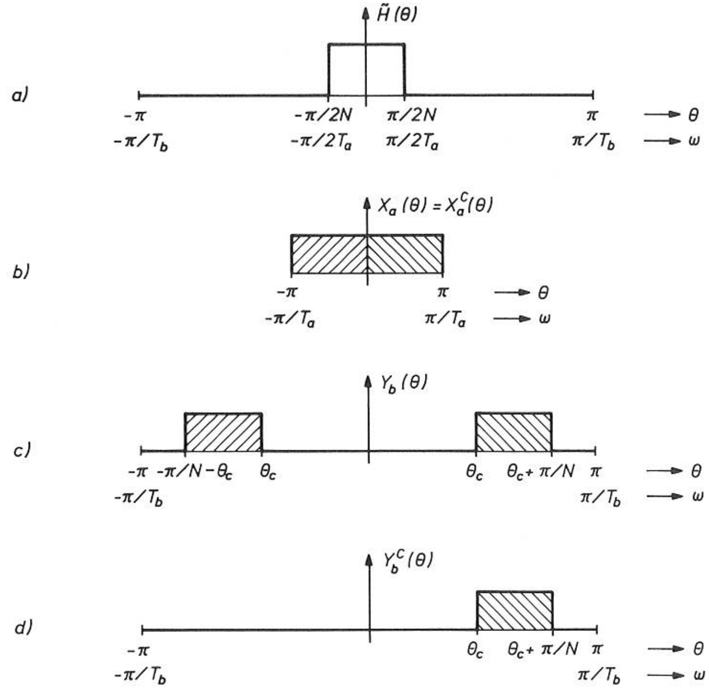


Fig. 10. (*a*) Transmission function of filter $\widetilde{H}$ in fig. 9. (*b*) Input spectrum of the modulators in fig. 9. (*c*) Output spectrum of the Weaver modulator of fig. 9*a*. (*d*) Output spectrum of the complex modulator of fig. 9*c*.

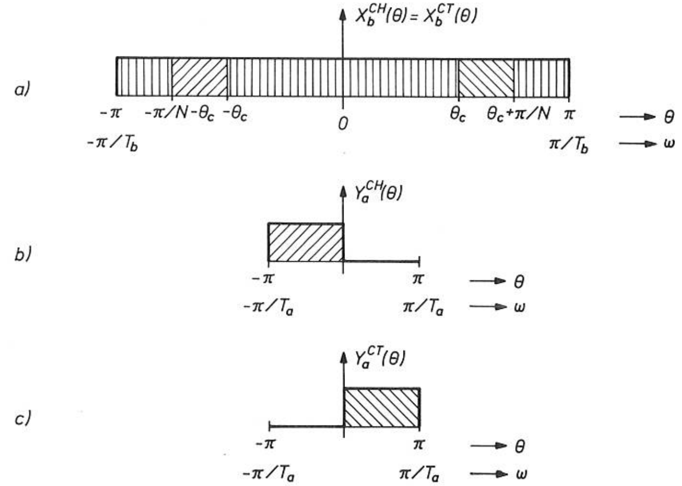


Fig. 11. (*a*) Input spectrum of the transposes of the modulators of fig. 9. (*b*) Output spectrum of the hermitian transpose of the modulator of fig. 9*c*. (*c*) Output spectrum of the generalized transpose of the modulator of fig. 9*c*.

For the input spectrum $X_b^{\mathrm{CH}}(\theta)$ of fig. 11*a* the output spectrum is sketched in fig. 11*b*, and the hermitian transpose can be seen to be a lower sideband demodulator. In accordance with the discussion in sec. 4, the transpose of the modulator of fig. 9*b* will be a real implementation of the hermitian transpose of the complex modulator, and when only real parts are considered the transpose of the Weaver modulator of fig. 9*a* results and is a single-sideband demodulator. If we apply generalized transposition to the complex modulator of fig. 9*c*, then not only must we replace the SRI by an SRD but we must also replace n by $-n$ in the modulation function. This yields the transmission function

$$H_{ab}{}^{\mathrm{CT}}(\theta,\xi) = \tilde{H}(\xi - \theta_{\mathrm{c}} - \pi/2N) ⧢ [\theta - N(\xi - \theta_{\mathrm{c}})]. \tag{50}$$

The output spectrum of this system is shown in fig. 11*c* and it can be concluded that the generalized transpose of an upper sideband modulator is an upper sideband demodulator.

As a final example the TDM–FDM translator discussed in ref. 7 can be mentioned, which in its most general form contains complex-valued signal processing operations. The structure of the corresponding FDM–TDM translator is found by applying hermitian transposition to the TDM–FDM system. Generalized transposition too will yield an FDM–TDM translator but, for a complex system, will have a slightly different implementation.

6. Sensitivity analysis

An important application of Tellegen's theorem is the derivation of formulae for the sensitivity of transmission functions to changes of system parameters [8,18]. Such formulae are important for determining the influence of parameter quantization [8]) on the system characteristics. Moreover, as indicated by Jackson [20]) there is a close relation between the coefficient sensitivity of a network and its roundoff noise resulting from signal quantization.

Completely analogous to the method given by Fettweis, a sensitivity formula for time-varying discrete-systems can be derived from any of the two forms of Tellegen's theorem as stated in sec. 3. Here only the result of the derivation will be given.

Consider a system S as depicted in fig. 12*a*. The input and output nodes are labeled a and b respectively, and the transmission function between these nodes is $H_{ba}(\theta,\xi)$. We assume a transmittance $F_{ji}(\theta,\xi)$ in the branch connecting node i to node j, and want to determine the influence of changes of F_{ji} on H_{ba}. To this end we introduce a system S' which is identical to S except for the branch that connects node i to node j, which has a transmittance $F_{ji}(\theta,\xi) + \Delta F_{ji}(\theta,\xi)$. Denoting the transmittances in S' by primed variables, the following expression for the variation of H_{ba} can be derived:

$$\begin{aligned}\Delta H_{ba}(\theta,\xi) &\triangleq H_{ba}'(\theta,\xi) - H_{ba}(\theta,\xi)\\ &= H_{bj}(\theta,\cdot) \bullet \Delta F_{ji}(\cdot,\cdot) \bullet H_{ia}'(\cdot,\xi),\end{aligned} \tag{51}$$

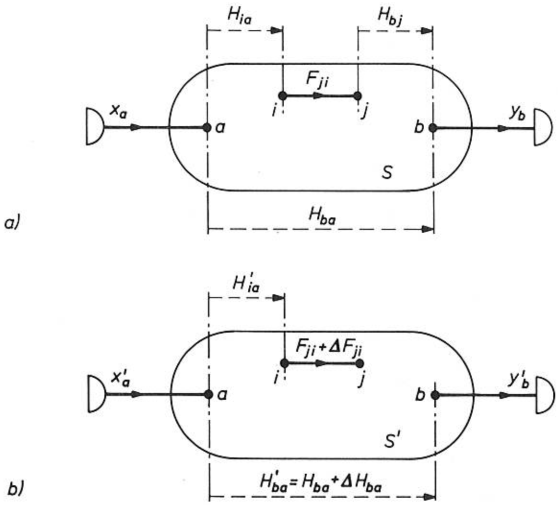


Fig. 12. (*a*) Discrete-time system S realizing the transmission function H_{ba}. (*b*) Discrete-time system S' obtained from S by perturbing the transmittance F_{ji} by ΔF_{ji}.

where the notation introduced in sec. 2 is used. Equation (51) describes a cascade of three subsystems as shown in fig. 13a. Since H_{ia}' too is a transmission function of the perturbed system S', we may likewise write

$$H_{ia}'(\theta,\xi) = H_{ia}(\theta,\xi) + \Delta H_{ia}(\theta,\xi), \tag{52}$$

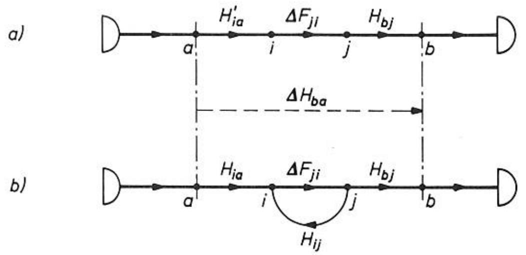


Fig. 13. (a) Flow-graph of the cascade that realizes the variation ΔH_{ba} of the transmission function H_{ba} due to a change ΔF_{ji} in F_{ji} in the system S. (b) Alternative network that realizes ΔH_{ba} and only contains transmission functions of S.

where $\Delta H_{ia}(\theta,\xi)$ is given by (51) but with index b replaced by i. It therefore follows that

$$H_{ia}'(\theta,\xi) = H_{ia}(\theta,\xi) + H_{ij}(\theta,\cdot) \bullet \Delta F_{ji}(\cdot,\cdot) \bullet H_{ia}'(\cdot,\xi) \tag{53}$$

and thus $H_{ia}'(\theta,\xi)$ is the solution of this integral equation. The interpretation is that $H_{ia}'(\theta,\xi)$ may be constructed by a cascade of H_{ia} and a feedback loop with transmission function ΔF_{ji} and H_{ij} as shown in fig. 13b. Apart from ΔF_{ji} this latter system only contains transmission functions of the unperturbed system S. If each of the transmission functions in eq. (53) describes a time-invariant system, then the cascade operation becomes a simple multiplication and the well-known relation for large scale variations [8,21]) follows immediately *). Such an explicit relation does not exist in the general time-varying case, but repeated substitution of (53) into (51) leads to the Neumann series

$$\Delta H_{ba}(\theta,\xi)$$
$$= H_{bj}(\theta,\cdot) \bullet [\Delta F_{ji}(\cdot,\cdot) + \Delta F_{ji}(\cdot,\cdot) \bullet H_{ij}(\cdot,\cdot) \bullet \Delta F_{ji}(\cdot,\cdot) + \ldots] \bullet H_{ia}(\cdot,\xi). \tag{54}$$

The terms in the brackets are related to network sensitivities of increasing order [21]). In particular the first term

$$H_{bj}(\theta,\cdot) \bullet \Delta F_{ji}(\cdot,\cdot) \bullet H_{ia}(\cdot,\xi) \tag{55}$$

*) Such a simplification is also possible if both ΔF_{ji} and H_{ij} are transmission functions corresponding to time-invariant impulse responses.

gives the first-order variation of H_{ba} to changes in the transmittance F_{ji}. For the specific case that only a constant multiplier with coefficient λ_{ji} connects node i to node j this expression yields the sensitivity

$$\frac{\partial H_{ba}(\theta,\xi)}{\partial \lambda_{ji}} = H_{bj}(\theta,\cdot) \bullet H_{ia}(\cdot,\xi) \tag{56}$$

which very much resembles the familiar relation for time-invariant systems [8,18].

Finally, in eq. (56) the transmission function $H_{bj}(\theta,\xi)$ may be replaced by $H_{jb}{}^{\mathrm{T}}(\xi,\theta)$, which is the transmission from node b to node j in the generalized transpose system S^{T}. With this modification the sensitivity of H_{ba} with respect to all network coefficients can be determined by analysing once the original network (to obtain all $H_{ia}(\theta,\xi)$) and once its transpose (to obtain all $H_{jb}{}^{\mathrm{T}}(\xi,\theta)$) [18].

7. Extension to systems with continuous-time and discrete-time signals

The foregoing discussion can easily be extended to networks that contain both continuous-time and discrete-time signals. In such systems we must allow elements with analogue inputs and digital outputs or vice versa. Such elements have impulse responses of the form $h_{\mathrm{da}}(n,\tau)$ and $h_{\mathrm{ad}}(t,m)$ with corresponding input–output relations

$$y_{\mathrm{d}}(n) = \int_{-\infty}^{\infty} h_{\mathrm{da}}(n,\tau)\, x_{\mathrm{a}}(\tau)\, \mathrm{d}\tau \tag{57}$$

and

$$y_{\mathrm{a}}(t) = \sum_{m=-\infty}^{\infty} h_{\mathrm{ad}}(t,m)\, x_{\mathrm{d}}(m) \tag{58}$$

respectively. Natural candidates for such elements are the ideal A/D converter (in which amplitude quantization effects are disregarded) and the idealized D/A converter (that produces weighted δ functions), which have the impulse responses $\delta(nT-\tau)$ and $\delta(t-mT)$ respectively, where T is the sampling period of the devices. The transmission functions of these hybrid elements are now defined by

$$H_{\mathrm{da}}(\theta,\Omega) \stackrel{\triangle}{=} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} h_{\mathrm{da}}(n,\tau) \exp\left[-\mathrm{j}(n\theta-\Omega\tau)\right] \mathrm{d}\tau \tag{59}$$

for analogue input and digital output, and

$$H_{\mathrm{ad}}(\omega,\xi) \stackrel{\triangle}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h_{\mathrm{ad}}(t,m) \exp\left[-\mathrm{j}(\omega t-m\xi)\right] \mathrm{d}t \tag{60}$$

for digital input and analogue output. Applying these definitions we find that the transmission functions of the A/D converter and the D/A converter are given by $⊔⊔\, (\theta - \Omega T)$ and $⊔⊔\, (\omega T - \xi)$ respectively. The corresponding spectral relations can be obtained from

$$Y_\mathrm{d}(\theta) = \int_{-\infty}^{\infty} H_\mathrm{da}(\theta,\Omega)\, X_\mathrm{a}(\Omega)\, \mathrm{d}\Omega \tag{61}$$

and

$$Y_\mathrm{a}(\omega) = \int_{-\pi}^{\pi} H_\mathrm{ad}(\omega,\xi)\, X_\mathrm{d}(\xi)\, \mathrm{d}\xi. \tag{62}$$

With the transmission functions thus defined it is easy to modify Tellegen's theorem and the transposition theorem in such a way that hybrid systems containing both continuous-time and discrete-time subsystems can also be dealt with. It then follows that the A/D converter and the D/A converter are mutually transposed.

8. Conclusions

Two forms of Tellegen's theorem have been derived that are applicable to any pair of linear discrete-time networks with the same topology. Both forms of this theorem suggested a definition of transposition, which were called hermitian transposition and generalized transposition respectively, thus generalizing the transposition concept that hitherto only applied to time-invariant systems.

Next a transposition theorem was given that relates the transmission function of a linear network to that of its hermitian or generalized transpose. In contrast to the time-invariant case where the transpose has the same transmission function as the original network, the transposes in the time-varying case realize functions that in a sense are complementary to that of the original network. As examples it was shown that the hermitian transpose of an FFT implementation performs the inverse transform, transposition of a decimator yields an interpolator and a modulator became a demodulator after transposition. Therefore transposition offers a simple and effective way to derive implementations of systems that realize such a complementary operation once the implementation of the original operation is known. In particular when the original system has been optimized with respect to multiplication rate the transpose will automatically be optimal in this sense too.

Sensitivity formulae were derived that make it possible to compute the influence of small- and large-scale changes of network elements on the transmission function of a system. Finally an extension was discussed to networks that consist of both continuous-time and discrete-time parts.

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