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Free Probability Based Capacity Calculation for MIMO Channels with Transmit or Receive Correlation

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***Abstract*— We apply tools from free probability theory to the asymptotic analysis of MIMO systems with spatial correlation at the transmitter or receiver side. The MIMO channel is modeled by a one-sided Kronecker model with exponential transmitter or receiver correlation matrix. We compute the asymptotic channel capacity based on a calculation of the asymptotic eigenvalue pdf. The dependence of the asymptotic channel capacity on the SNR, dimension ratio, and correlation parameter is studied. Finally, it is demonstrated that the asymptotic capacity closely approximates the simulated capacity of finite-dimensional MIMO channels already for moderate system dimensions.**

I. Introduction

Spatial correlations frequently occur in real multiple-input multiple-output (MIMO) channels and have a significant impact on the channel capacity [1]. Different spatial channel correlation models have been proposed, including product models such as the Kronecker model [2] or the more elaborate models introduced in [3] and [4].

The calculation of the capacity of a spatially correlated MIMO channel $\mathbf{H}$ can be based on the eigenvalues of the matrix $\mathbf{H}\mathbf{H}^H$. When stochastic models are used, the statistics of the (random) eigenvalues need to be known (e.g., [5]). Unfortunately, characterizing and manipulating the eigenvalue statistics is difficult in general.

On the other hand, asymptotic eigenvalue statistics of large random matrices can often be obtained more easily (e.g., [6–8]). This has led researchers to study communication systems in their large limit and compute asymptotic capacities [1, 3, 9, 10]. In the case of product channel models, the concepts of *asymptotic freeness* and *free multiplicative convolution* [8] were found to be useful for this task [3, 9]. In particular, in [9] the asymptotic capacity for a one-sided Kronecker channel model with a Hermitian Toeplitz correlation matrix has been calculated. The asymptotic probability density function of the eigenvalues (asymptotic eigenvalue pdf, abbreviated AEPDF) was not calculated in [9]; rather, a specific form of the AEPDF was *assumed* that simplified the capacity calculation.

Here, we consider a one-sided Kronecker channel model with "exponential" Toeplitz correlation matrix [5, 10]. We compute the asymptotic capacity based on a calculation of the AEPDF, thereby avoiding the assumption of an *ad hoc* AEPDF. The calculated AEPDF is found to be quite different from that assumed in [9].

Our paper is organized as follows. We first describe the system and channel correlation model and review the concepts of AEPDF and asymptotic capacity. In Section II, the AEPDF is calculated for the case of correlation on the receiver side. The asymptotic capacity is then computed and studied in Section III. In Section IV, we show how our results can be modified for the case of transmit correlation. Finally, in Section V we compare our asymptotic results to simulation results obtained for finite-dimensional MIMO systems.

A. System and Channel Correlation Model

We consider a MIMO channel with N_{T} transmit antennas and N_{R} receive antennas. The input-output relation is

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}\,,$$

with the N_{T} 1 input vector $\mathbf{x}$, the N_{R} 1 output vector $\mathbf{y}$ and noise vector $\mathbf{n}$, and the N_{R} N_{T} channel matrix $\mathbf{H}$. The total mean input power is denoted as $P \triangleq \mathbb{E}\{\|\mathbf{x}\|^2\}$. The noise is assumed iid circularly symmetric complex Gaussian with variance σ^2, i.e., $\mathbb{E}\{\mathbf{n}\mathbf{n}^H\} = \sigma^2 \mathbf{I}_{N_{\mathrm{R}}}$.

The random channel is modeled by the one-sided Kronecker model [1, 9, 10]

$$\mathbf{H} = \mathbf{\Theta}_{\mathrm{R}}^{1/2}\mathbf{G}\,. \tag{1}$$

Here, $\mathbf{G}$ is an N_{R} N_{T} matrix with iid circularly symmetric unit-variance complex Gaussian entries; $\mathbf{\Theta}_{\mathrm{R}}$ is the N_{R} N_{R} channel correlation matrix at the receiver side, i.e.,

$$\mathbf{\Theta}_{\mathrm{R}} \triangleq \frac{1}{N_{\mathrm{T}}}\,\mathbb{E}\{\mathbf{H}\mathbf{H}^H\}\,;$$

and $\mathbf{\Theta}_{\mathrm{R}}^{1/2}$ is the square root of $\mathbf{\Theta}_{\mathrm{R}}$ such that $\mathbf{\Theta}_{\mathrm{R}}^{H/2}\mathbf{\Theta}_{\mathrm{R}}^{1/2} = \mathbf{\Theta}_{\mathrm{R}}$. We model $\mathbf{\Theta}_{\mathrm{R}}$ as a Hermitian Toeplitz matrix with exponential entries $\mathbf{\Theta}_{\mathrm{R}\,i,j} = \rho^{|j-i|}$, $0 \leq \rho < 1$. This model has been used in [5, 10]; it is suited for modeling a uniform linear array. Note that $\mathrm{trace}(\mathbf{\Theta}_{\mathrm{R}}) = N_{\mathrm{R}}$. Our assumptions entail the following average normalization of the channel matrix:

$$\mathbb{E}\big\{\mathrm{trace}(\mathbf{H}\mathbf{H}^H)\big\} = \sum_{i=1}^{N_{\mathrm{R}}}\sum_{j=1}^{N_{\mathrm{T}}}\mathbb{E}\{|\mathbf{H}_{i,j}|^2\} = N_{\mathrm{T}}N_{\mathrm{R}}\,.$$

The model (1) only models the correlation at the receiver side. Alternatively, the analogous model

$$\mathbf{H} = \mathbf{G}\mathbf{\Theta}_{\mathrm{T}}^{1/2} \tag{2}$$

can be used to model the correlation at the transmitter side.

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B. Asymptotic Capacity

For a given channel $\mathbf{H}$, the channel capacity in the absence of channel state information at the transmitter is [11]

$$C = \log\left(\det\left(\mathbf{I} + \frac{\mathrm{SNR}}{\beta}\mathbf{W}\right)\right), \tag{3}$$

where

$$\mathrm{SNR} \triangleq \frac{P}{\sigma^2}, \qquad \beta \triangleq \frac{N_{\mathrm{T}}}{N_{\mathrm{R}}}, \qquad \mathbf{W} \triangleq \frac{1}{N_{\mathrm{R}}}\mathbf{H}\mathbf{H}^H.$$

The channel capacity can be written in terms of the eigenvalues $\lambda_k \geq 0$ of the N_{R} N_{R} matrix $\mathbf{W}$ as

$$C = \sum_{k=1}^{N_{\mathrm{R}}} \log\left(1 + \frac{\mathrm{SNR}}{\beta}\lambda_k\right). \tag{4}$$

We will use the *empirical cumulative distribution function (cdf) of the eigenvalues*, denoted $F_{\mathbf{W}}(x)$, which is defined as the percentage of eigenvalues that are $\leq x$:

$$F_{\mathbf{W}}(x) \triangleq \frac{|\{\lambda_k | \lambda_k \leq x\}|}{N_{\mathrm{R}}}.$$

Here, $|\mathcal{S}|$ denotes the size (cardinality) of the set $\mathcal{S}$. The *empirical eigenvalue pdf* is given by

$$f_{\mathbf{W}}(x) \triangleq \frac{dF_{\mathbf{W}}(x)}{dx} = \frac{1}{N_{\mathrm{R}}}\sum_{k=1}^{N_{\mathrm{R}}} \delta(x - \lambda_k).$$

Note that $F_{\mathbf{W}}(x)$ and $f_{\mathbf{W}}(x)$ are zero for $x < 0$. The capacity formula (4) can be written in terms of $f_{\mathbf{W}}(x)$:

$$C = N_{\mathrm{R}} \int_0^{\infty} \log\left(1 + \frac{\mathrm{SNR}}{\beta}x\right) f_{\mathbf{W}}(x)\, dx. \tag{5}$$

The channel $\mathbf{H}$ is random, and so are $f_{\mathbf{W}}(x)$ and C. However, for our channel model, when $N_{\mathrm{R}} \to \infty$ while $\beta = N_{\mathrm{T}}/N_{\mathrm{R}}$ is held constant, $f_{\mathbf{W}}(x)$ tends almost surely to a deterministic function $f_{\mathbf{W}}^{\infty}(x)$ that will be called the *asymptotic empirical eigenvalue pdf* (AEPDF). Hence, also the capacity per receive antenna C/N_{R} tends almost surely to a deterministic *asymptotic capacity per receive antenna* $C^{\infty}/N_{\mathrm{R}}$.

II. Asymptotic Eigenvalue PDF

We will now calculate the AEPDF $f_{\mathbf{W}}^{\infty}(x)$ for the receive correlation case defined by (1). With (1), we obtain

$$\mathbf{W} = \mathbf{\Theta}_{\mathrm{R}}^{1/2}\mathbf{V}\mathbf{\Theta}_{\mathrm{R}}^{H/2} \quad \text{with } \mathbf{V} \triangleq \frac{1}{N_{\mathrm{R}}}\mathbf{G}\mathbf{G}^H.$$

For two square matrices $\mathbf{A}$ and $\mathbf{B}$ of equal size, $\mathbf{AB}$ has the same eigenvalues as $\mathbf{BA}$. Thus, $\mathbf{W} = \mathbf{\Theta}_{\mathrm{R}}^{1/2}\mathbf{V}\mathbf{\Theta}_{\mathrm{R}}^{H/2}$ has the same eigenvalues as $\mathbf{V}\mathbf{\Theta}_{\mathrm{R}}^{H/2}\mathbf{\Theta}_{\mathrm{R}}^{1/2} = \mathbf{V}\mathbf{\Theta}_{\mathrm{R}}$, and hence

$$f_{\mathbf{W}}^{\infty}(x) = f_{\mathbf{V}\mathbf{\Theta}_{\mathrm{R}}}^{\infty}(x).$$

Now $\mathbf{V} = \frac{1}{N_{\mathrm{R}}}\mathbf{G}\mathbf{G}^H$ is a Wishart matrix [12], while $\mathbf{\Theta}_{\mathrm{R}}$ is a deterministic matrix having an AEPDF (as will be shown in Subsection II-B). Free probability theory then states that these matrices are *asymptotically free*, and thus the AEPDF of $\mathbf{V}\mathbf{\Theta}_{\mathrm{R}}$ is the free multiplicative convolution of the AEPDFs of $\mathbf{V}$ and $\mathbf{\Theta}_{\mathrm{R}}$. This can be calculated by means of the S-transform [8] because the S-transform $S_{\mathbf{W}}(z)$ of $f_{\mathbf{W}}^{\infty}(x) = f_{\mathbf{V}\mathbf{\Theta}_{\mathrm{R}}}^{\infty}(x)$ is the product of the S-transforms of $f_{\mathbf{V}}^{\infty}(x)$ and $f_{\mathbf{\Theta}_{\mathrm{R}}}^{\infty}(x)$ [8, 13]:

$$S_{\mathbf{W}}(z) = S_{\mathbf{V}}(z)\, S_{\mathbf{\Theta}_{\mathrm{R}}}(z). \tag{6}$$

In the following, therefore, we will calculate the AEPDFs of $\mathbf{V}$ and $\mathbf{\Theta}_{\mathrm{R}}$, compute their S-transforms, and finally determine the AEPDF of $\mathbf{W}$ through an inverse S-transform.

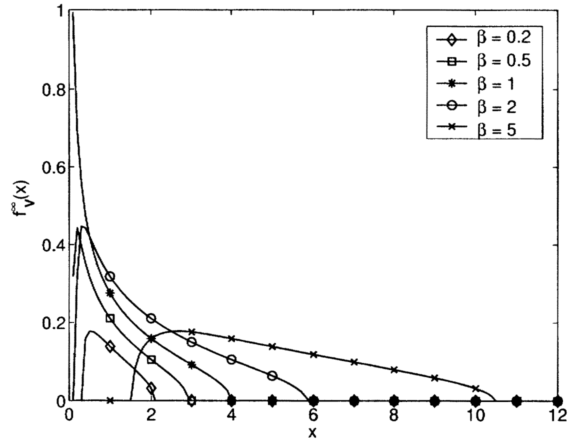

Fig. 1. Asymptotic eigenvalue pdf (without the Dirac component) of the matrix $\mathbf{V} = \frac{1}{N_{\mathrm{R}}}\mathbf{G}\mathbf{G}^H$ for different dimension ratios β.

A. AEPDF of $\mathbf{V}$ *and Its S-Transform*

The entries of the N_{R} N_{T} matrix $\mathbf{G}$ are iid complex Gaussian with zero mean and unit variance. The Marčenko-Pastur law [6, 7] then states that when $N_{\mathrm{R}} \to \infty$ while $\beta = N_{\mathrm{T}}/N_{\mathrm{R}} = \mathrm{const}$, the empirical eigenvalue pdf of the N_{R} N_{R} matrix $\mathbf{V} = \frac{1}{N_{\mathrm{R}}}\mathbf{G}\mathbf{G}^H$ tends almost surely to the AEPDF

$$f_{\mathbf{V}}^{\infty}(x) = h(x) + [1-\beta]^{+}\delta(x), \tag{7}$$

where

$$h(x) = \begin{cases} \frac{\sqrt{4\beta - (x-1-\beta)^2}}{2\pi x}, & (1-\sqrt{\beta})^2 < x < (1+\sqrt{\beta})^2 \\ 0, & \text{elsewhere} \end{cases}$$

and $[1-\beta]^{+}$ is $1-\beta$ for $\beta < 1$ and 0 for $\beta \geq 1$. The Dirac component $[1-\beta]^{+}\delta(x)$ in (7) accounts for zero eigenvalues of $\mathbf{V}$, which exist if $\beta < 1$ (i.e., $N_{\mathrm{R}} > N_{\mathrm{T}}$). The AEPDF $f_{\mathbf{V}}^{\infty}(x)$ is plotted in Fig. 1 for various values of β.

Next, we sketch the derivation of the S-Transform of $f_{\mathbf{V}}^{\infty}(x)$ given in [3]. We first calculate the Cauchy transform [8, 13] (which equals the Stieltjes transform [3] up to a sign factor)

$$G_{\mathbf{V}}(s) \triangleq \int_{-\infty}^{\infty} \frac{1}{s-x} f_{\mathbf{V}}^{\infty}(x)\, dx, \qquad s \in \mathbb{C}. \tag{8}$$

Inserting (7), we obtain

$$G_{\mathbf{V}}(s) = \frac{s + 1 - \beta \pm \sqrt{s^2 - 2(\beta+1)s + (\beta-1)^2}}{2s}.$$

Next, we compute the intermediate transform

$$\Upsilon_{\mathbf{V}}(s) \triangleq \frac{1}{s}G_{\mathbf{V}}\left(\frac{1}{s}\right) - 1 \tag{9}$$

$$= \frac{1-s-\beta s \pm \sqrt{1-2(\beta+1)s+(\beta+1)^2 s^2}}{2s}.$$

We then obtain the S-transform $S_{\mathbf{V}}(z)$ through the relation [3]

$$S_{\mathbf{V}}(z) = \frac{z+1}{z}\,\Upsilon_{\mathbf{V}}^{-1}(z), \tag{10}$$

where $\Upsilon_{\mathbf{V}}^{-1}(z)$ denotes the inverse of the function $\Upsilon_{\mathbf{V}}(s)$ (constructed by solving $\Upsilon_{\mathbf{V}}(s) = z$ for s). We obtain $\Upsilon_{\mathbf{V}}^{-1}(z) = z/(z^2 + z\beta + z + \beta)$ and thus

$$S_{\mathbf{V}}(z) = \frac{1}{z+\beta}. \tag{11}$$

B. AEPDF of Θ_{R} and Its S-Transform

It is known [14] that the eigenvalues of a Toeplitz matrix tend to the discrete Fourier transform of the first row as the size of the matrix tends to infinity. Hence, for large N_{R}, the eigenvalues μ_k of $\Theta_{\mathrm{R}} = \big(\rho^{|j-i|}\big)$ are approximately given by

$$\begin{aligned}\mu_k &\approx \sum_{i=-(N_{\mathrm{R}}-1)}^{N_{\mathrm{R}}-1} \rho^{|i|} e^{j2\pi k i/N_{\mathrm{R}}} \\ &= \frac{2\rho^{N_{\mathrm{R}}+1}\cos(2\pi k/N_{\mathrm{R}}) - 2\rho^{N_{\mathrm{R}}} + 1 - \rho^2}{1+\rho^2 - 2\rho\cos(2\pi k/N_{\mathrm{R}})} \\ &\approx \frac{1-\rho^2}{1+\rho^2 - 2\rho\cos(2\pi k/N_{\mathrm{R}})}, \qquad k = 1,\ldots,N_{\mathrm{R}}, \end{aligned} \tag{12}$$

where we used $\lim_{N_{\mathrm{R}}\to\infty} \rho^{N_{\mathrm{R}}} = 0$. These approximations are exact for $N_{\mathrm{R}} \to \infty$. Let us assume that N_{R} is even (a similar argument with the same final result exists for N_{R} odd). Due to $\mu_{N_{\mathrm{R}}-k} = \mu_k$, all μ_k except for $k = N_{\mathrm{R}}/2$ and N_{R} are obtained twice in (12). Thus, as far as the asymptotic empirical eigenvalue cdf/pdf is concerned, we can restrict to, e.g., the index range $k = N_{\mathrm{R}}/2+1,\ldots,N_{\mathrm{R}}$. Performing the index transformation $l \triangleq k - N_{\mathrm{R}}/2$ and using $\cos(x+\pi) = -\cos(x)$, we then obtain (the tilde indicates the different indexing)

$$\tilde{\mu}_l \approx \frac{1-\rho^2}{1+\rho^2+2\rho\cos(2\pi l/N_{\mathrm{R}})}, \qquad l = 1,\ldots,N_{\mathrm{R}}/2. \tag{13}$$

Solving for $l/(N_{\mathrm{R}}/2)$ yields

$$\frac{l}{N_{\mathrm{R}}/2} \approx \frac{1}{\pi}\cos^{-1}\!\left(\frac{1-\rho^2}{2\rho\tilde{\mu}_l} - \frac{1+\rho^2}{2\rho}\right).$$

Because in (13) the $\tilde{\mu}_l$ are indexed in nondecreasing order, $l/(N_{\mathrm{R}}/2)$ is equal (within our approximation) to the percentage of eigenvalues $\tilde{\mu}_i$ that are $\le \tilde{\mu}_l$. It follows that for $N_{\mathrm{R}} \to \infty$, $l/(N_{\mathrm{R}}/2)$ directly yields the asymptotic empirical eigenvalue cdf $F_{\Theta_{\mathrm{R}}}^{\infty}(x)$ when $\tilde{\mu}_l$ is replaced by x:

$$F_{\Theta_{\mathrm{R}}}^{\infty}(x) = \frac{1}{\pi}\cos^{-1}\!\left(\frac{1-\rho^2}{2\rho x} - \frac{1+\rho^2}{2\rho}\right).$$

The AEPDF is then obtained by differentiation:

$$f_{\Theta_{\mathrm{R}}}^{\infty}(x) = \begin{cases} \frac{1}{\pi x\sqrt{-x^2+2x\alpha-1}}, & \frac{1-\rho}{1+\rho} < x < \frac{1+\rho}{1-\rho} \\ 0, & \text{elsewhere,} \end{cases} \tag{14}$$

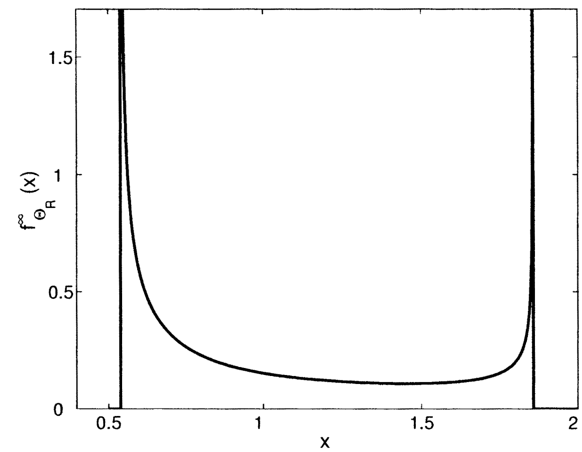

Fig. 2. Asymptotic eigenvalue pdf of the receive correlation matrix Θ_{R} for $\rho = 0.3$.

with

$$\alpha \triangleq \frac{1+\rho^2}{1-\rho^2}.$$

This function is plotted in Fig. 2 for $\rho = 0.3$. Our result (14) is quite different from the AEPDF $f_{\Theta_{\mathrm{R}}}^{\infty}(x)$ assumed in [9], even though the support interval $\big[\frac{1-\rho}{1+\rho}, \frac{1+\rho}{1-\rho}\big]$ is the same.

To calculate the S-Transform of $f_{\Theta_{\mathrm{R}}}^{\infty}(x)$, we first evaluate the intermediate transform (cf. (9) and (8))

$$\Upsilon_{\Theta_{\mathrm{R}}}(s) = \int_{-\infty}^{\infty} \frac{sx}{1-sx}\, f_{\Theta_{\mathrm{R}}}^{\infty}(x)\,dx, \qquad s \in \mathbb{C}.$$

Inserting (14), we obtain

$$\Upsilon_{\Theta_{\mathrm{R}}}(s) = \frac{s}{\sqrt{s^2 - 2s\alpha + 1}},$$

and thus

$$\Upsilon_{\Theta_{\mathrm{R}}}^{-1}(z) = \frac{\alpha z^2 \pm z\sqrt{\alpha^2 z^2 - (z^2-1)}}{z^2-1}.$$

Finally, the S-transform $S_{\Theta_{\mathrm{R}}}(z)$ results as (cf. (10))

$$S_{\Theta_{\mathrm{R}}}(z) = \frac{\alpha z \pm \sqrt{\alpha^2 z^2 - (z^2-1)}}{z-1}. \tag{15}$$

C. AEPDF of $\mathbf{W}$

Inserting (11) and (15) into (6), we finally obtain the S-transform of $\mathbf{W} = \frac{1}{N_{\mathrm{R}}}\mathbf{H}\mathbf{H}^H$ as

$$S_{\mathbf{W}}(z) = \frac{\alpha z \pm \sqrt{\alpha^2 z^2 - (z^2-1)}}{(z+\beta)(z-1)}.$$

The AEPDF of $\mathbf{W}$ can now be computed by inverting the S-transform according to the relation [3, 13]

$$f_{\mathbf{W}}^{\infty}(x) = \lim_{y\to 0^+} \Im\{G_{\mathbf{W}}(x+jy)\}. \tag{16}$$

It remains to calculate $G_{\mathbf{W}}(s)$ from $S_{\mathbf{W}}(z)$. We first note that (cf. (10))

$$\Upsilon_{\mathbf{W}}^{-1}(z) = \frac{z}{z+1}\,S_{\mathbf{W}}(z)$$

$$= \frac{\alpha z^2 \pm z\sqrt{\alpha^2 z^2 - (z^2-1)}}{(z+\beta)(z^2-1)} . \qquad (17)$$

Now we know that (cf. (9)) $\Upsilon_{\mathbf{W}}(s) = \frac{1}{s} G_{\mathbf{W}}\big(\frac{1}{s}\big) - 1$ or, equivalently,

$$\Upsilon_{\mathbf{W}}\Big(\frac{1}{s}\Big) = s\, G_{\mathbf{W}}(s) - 1 .$$

To calculate $\Upsilon_{\mathbf{W}}\big(\frac{1}{s}\big)$ from $\Upsilon_{\mathbf{W}}^{-1}(z)$ in (17), we have to solve the equation $\Upsilon_{\mathbf{W}}^{-1}(z) = \frac{1}{s}$ for z. This equation reads

$$\frac{(z+\beta)^2(z^2-1)}{s^2} - z^2\left[\frac{2\alpha(z+\beta)}{s} - 1\right] = 0 .$$

Inserting $z = \Upsilon_{\mathbf{W}}\big(\frac{1}{s}\big) = s\,G_{\mathbf{W}}(s) - 1$ finally yields the following fourth-order equation in $G_{\mathbf{W}}(s) = G$:

$$c_4 G^4 + c_3 G^3 + c_2 G^2 + c_1 G + c_0 = 0 , \qquad (18)$$

where

$$\begin{aligned}
c_0 &= -2\alpha s(\beta-1) + s^2\\
c_1 &= s\big[-2(\beta-1)^2 + 4\alpha(\beta-1)s - 2\alpha s - 2s^2\big]\\
c_2 &= s^2\big[(\beta-1)^2 - 4(\beta-1) - 2\alpha(\beta-1)s + 4\alpha s + s^2\big]\\
c_3 &= s^3\big[2(\beta-1) - 2 - 2\alpha s\big]\\
c_4 &= s^4 .
\end{aligned}$$

Solving (18) for each value of s (e.g., by means of the Ferrari-Cardano formulae) yields $G_{\mathbf{W}}(s)$ in a pointwise fashion.

The AEPDF $f_{\mathbf{W}}^{\infty}(x)$ is finally obtained according to (16) by determining the imaginary part of $G_{\mathbf{W}}(s)$ for $s = x \in \mathbb{R}$. In Fig. 3, $f_{\mathbf{W}}^{\infty}(x)$ is plotted for different values of β and ρ. It can be seen that the percentage of small eigenvalues increases for growing ρ, which correctly indicates the fact that the channel matrix gets worse conditioned for stronger correlation.

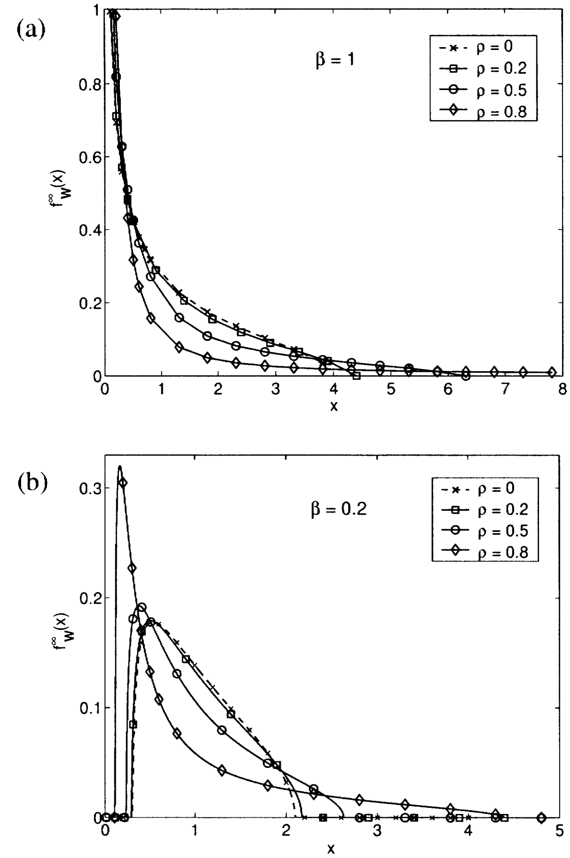

Fig. 3. Asymptotic eigenvalue pdf of the matrix $\mathbf{W} = \frac{1}{N_{\mathrm{R}}}\mathbf{H}\mathbf{H}^H$ for (a) $\beta = 1$ and (b) $\beta = 0.2$, for different correlation parameters ρ.

III. Asymptotic Capacity

The asymptotic capacity per receive antenna, $C^{\infty}/N_{\mathrm{R}}$, is obtained by substituting in (5) $f_{\mathbf{W}}^{\infty}(x)$ for $f_{\mathbf{W}}(x)$:

$$\frac{C^{\infty}}{N_{\mathrm{R}}} = \int_0^{\infty} \log\left(1 + \frac{\mathrm{SNR}}{\beta}\, x\right) f_{\mathbf{W}}^{\infty}(x)\, dx . \qquad (19)$$

This integral can be evaluated numerically. We briefly discuss the dependence of $C^{\infty}/N_{\mathrm{R}}$ on the SNR, the dimension ratio $\beta = N_{\mathrm{T}}/N_{\mathrm{R}}$, and the correlation parameter ρ.

Fig. 4 shows $C^{\infty}/N_{\mathrm{R}}$ versus the SNR for $\beta = 1$ and different values of ρ. As expected, $C^{\infty}/N_{\mathrm{R}}$ increases for growing SNR but decreases for growing ρ. Qualitatively similar results for this correlation model were obtained in [9] in spite of the different AEPDF $f_{\mathbf{\Theta}_{\mathrm{R}}}^{\infty}(x)$ used there.

In Fig. 5, $C^{\infty}/N_{\mathrm{R}}$ is plotted versus β for high SNR (30 dB) and different values of ρ. We see that for $\beta \leq 1$, $C^{\infty}/N_{\mathrm{R}}$ grows approximately linearly with β, hence is proportional to N_{T}. For $\beta > 1$, the $C^{\infty}/N_{\mathrm{R}}$ curves grow less rapidly and then effectively stay constant, which implies that $C^{\infty}/N_{\mathrm{R}}$ is proportional to N_{R}. Combining these two results, we conclude that for high SNR $C^{\infty}/N_{\mathrm{R}}$ is roughly proportional to $\min\{N_{\mathrm{T}}, N_{\mathrm{R}}\}$. Such a behavior has previously been reported for uncorrelated channels (e.g., [15]).

Finally, Fig. 6 shows the dependence of $C^{\infty}/N_{\mathrm{R}}$ on ρ for SNR = 30 dB and different values of β. This dependence is quite weak up to a certain ρ (which is higher for smaller β). However, ultimately $C^{\infty}/N_{\mathrm{R}}$ decreases rapidly for growing ρ. This behavior is consistent with the results of [5], where a similar channel correlation model was used and it was found that the impact of spatial channel correlation is only significant for $\rho > 0.5$. Note that for small β (i.e., $N_{\mathrm{R}} \gg N_{\mathrm{T}}$), $C^{\infty}/N_{\mathrm{R}}$ is approximately constant over almost the entire ρ range.

IV. Transmit Correlation

We next consider the case of transmit correlation, in which the channel matrix—now denoted as $\widetilde{\mathbf{H}}$—is modeled by $\widetilde{\mathbf{H}} = \mathbf{G}\mathbf{\Theta}_{\mathrm{T}}^{1/2}$ (cf. (2)). The matrix $\widetilde{\mathbf{W}} = \frac{1}{N_{\mathrm{R}}}\widetilde{\mathbf{H}}\widetilde{\mathbf{H}}^H$ determining the capacity according to (3) is obtained as $\widetilde{\mathbf{W}} = \frac{1}{N_{\mathrm{R}}}\mathbf{G}\mathbf{\Theta}_{\mathrm{T}}\mathbf{G}^H$. Our results can easily be modified for this case. $\widetilde{\mathbf{W}}$ has the same nonzero eigenvalues as $\mathbf{\Theta}_{\mathrm{T}}\widetilde{\mathbf{V}}$, with $\widetilde{\mathbf{V}} \triangleq \frac{1}{N_{\mathrm{R}}}\mathbf{G}^H\mathbf{G}$, but its size is $1/\beta$ times that of $\mathbf{\Theta}_{\mathrm{T}}\widetilde{\mathbf{V}}$. This implies that

$$f_{\widetilde{\mathbf{W}}}^{\infty}(x) = \beta f_{\mathbf{\Theta}_{\mathrm{T}}\widetilde{\mathbf{V}}}^{\infty}(x) + (1-\beta)\delta(x) . \qquad (20)$$

The AEPDF $f_{\mathbf{\Theta}_{\mathrm{T}}\widetilde{\mathbf{V}}}^{\infty}(x)$ can be calculated via its S-transform, which is given by

$$S_{\mathbf{\Theta}_{\mathrm{T}}\widetilde{\mathbf{V}}}(z) = S_{\widetilde{\mathbf{V}}}(z)\, S_{\mathbf{\Theta}_{\mathrm{T}}}(z) .$$

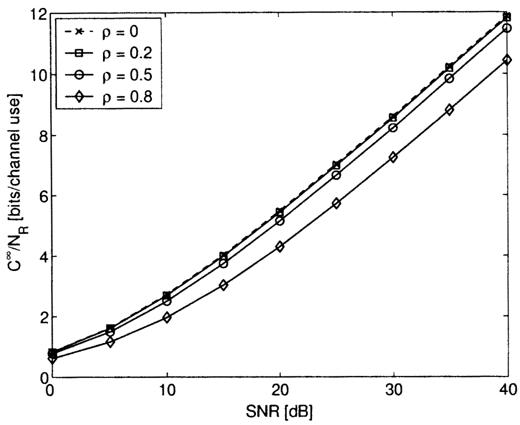

Fig. 4. Asymptotic capacity per receive antenna versus SNR for dimension ratio $\beta = 1$ and different correlation parameters ρ.

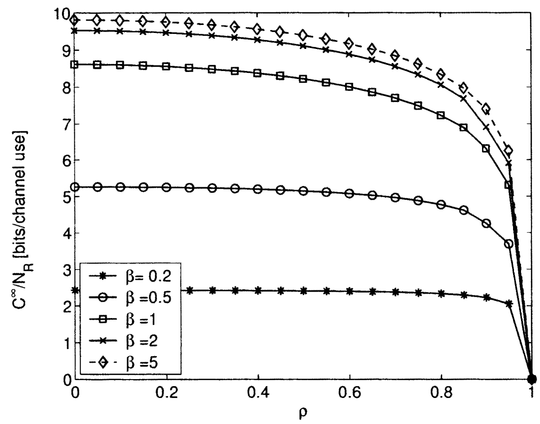

Fig. 6. Asymptotic capacity per receive antenna versus the correlation parameter ρ for SNR $= 30\,$dB and different dimension ratios β.

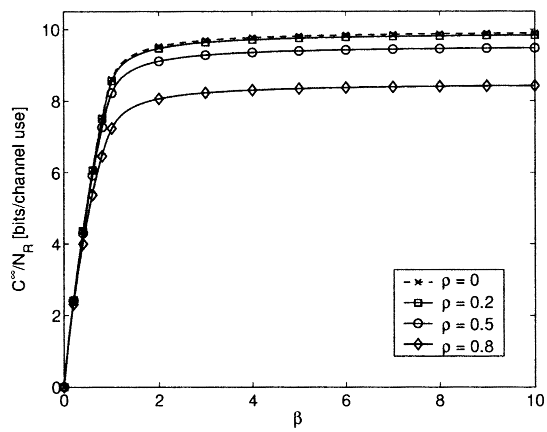

Fig. 5. Asymptotic capacity per receive antenna versus the dimension ratio β for SNR $= 30\,$dB and different correlation parameters ρ.

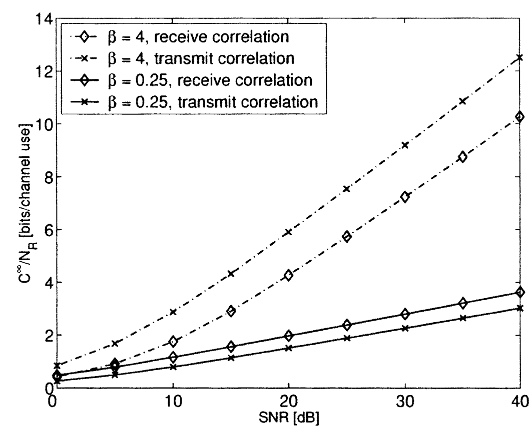

Fig. 7. Asymptotic capacity per receive antenna versus SNR—comparison of transmit correlation case and receive correlation case for $\beta = 0.25$ and $\beta = 4$, at $\rho = 0.8$.

Since $\mathbf{\Theta}_{\mathrm{T}}$ uses the same exponential model as $\mathbf{\Theta}_{\mathrm{R}}$, $f^{\infty}_{\mathbf{\Theta}_{\mathrm{T}}}(x)$ and $S_{\mathbf{\Theta}_{\mathrm{T}}}(z)$ are again given by (14) and (15), respectively.

Applying the Marčenko-Pastur law [6, 7] to $\widetilde{\mathbf{V}} = \frac{1}{N_{\mathrm{R}}}\mathbf{G}^H\mathbf{G}$, it is seen that $f^{\infty}_{\widetilde{\mathbf{V}}}(x)$ is related to $f^{\infty}_{\mathbf{V}}(x)$ in (7) as

$$f^{\infty}_{\widetilde{\mathbf{V}}}(x) \,=\, \frac{1}{\beta}\, f^{\infty}_{\mathbf{V}}(x) \,+\, \Big(1-\frac{1}{\beta}\Big)\,\delta(x)\,.$$

This implies that the intermediate transform is $\Upsilon_{\widetilde{\mathbf{V}}}(s) = \frac{1}{\beta}\Upsilon_{\mathbf{V}}(s)$ and, further, that the S-transform is[1]

$$S_{\widetilde{\mathbf{V}}}(z;\beta) \,=\, \frac{1}{\beta}\, S_{\mathbf{V}}\Big(z;\frac{1}{\beta}\Big)\,.$$

Recalling that $S_{\mathbf{\Theta}_{\mathrm{T}}}(z) = S_{\mathbf{\Theta}_{\mathrm{R}}}(z)$, we then obtain (cf. (6))

$$\begin{aligned} S_{\mathbf{\Theta}_{\mathrm{T}}\widetilde{\mathbf{V}}}(z;\beta) \,&=\, S_{\widetilde{\mathbf{V}}}(z;\beta)\, S_{\mathbf{\Theta}_{\mathrm{T}}}(z) \,=\, \frac{1}{\beta}\, S_{\mathbf{V}}\Big(z;\frac{1}{\beta}\Big)\, S_{\mathbf{\Theta}_{\mathrm{R}}}(z) \\ &=\, \frac{1}{\beta}\, S_{\mathbf{W}}\Big(z;\frac{1}{\beta}\Big)\,. \end{aligned}$$

[1] We temporarily use the notation $S_{\widetilde{\mathbf{V}}}(z;\beta)$ etc. to denote the S-transform of $f^{\infty}_{\widetilde{\mathbf{V}}}(x)$ etc. for dimension ratio β.

From this relation, it follows that

$$f^{\infty}_{\mathbf{\Theta}_{\mathrm{T}}\widetilde{\mathbf{V}}}(x;\beta) \,=\, \frac{1}{\beta}\, f^{\infty}_{\mathbf{W}}\Big(\frac{x}{\beta};\frac{1}{\beta}\Big)\,.$$

Inserting this into (20), the AEPDF for transmit correlation finally results as

$$f^{\infty}_{\widetilde{\mathbf{W}}}(x;\beta) \,=\, f^{\infty}_{\mathbf{W}}\Big(\frac{x}{\beta};\frac{1}{\beta}\Big) + (1-\beta)\,\delta(x)\,. \tag{21}$$

Inserting (21) into (19), we finally see that the asymptotic capacity for the transmit correlation case, $\widetilde{C}^{\infty}$, can easily be derived from the asymptotic capacity for the receive correlation case, C^{∞}, by a simple transformation of SNR and β:

$$\widetilde{C}^{\infty}(\mathrm{SNR},\beta) \,=\, \beta\, C^{\infty}\Big(\frac{\mathrm{SNR}}{\beta},\frac{1}{\beta}\Big)\,.$$

This is identical to the correspondence relation presented in [9]. Note that $\widetilde{C}^{\infty} = C^{\infty}$ for $\beta = 1$. Fig. 7 compares $\widetilde{C}^{\infty}/N_{\mathrm{R}}$ and $C^{\infty}/N_{\mathrm{R}}$ for $\beta = 0.25$ and $\beta = 4$, at $\rho = 0.8$. It is seen that $\widetilde{C}^{\infty}/N_{\mathrm{R}} < C^{\infty}/N_{\mathrm{R}}$ for $\beta = 0.25$ whereas $\widetilde{C}^{\infty}/N_{\mathrm{R}} > C^{\infty}/N_{\mathrm{R}}$ for $\beta = 4$. Thus, the correlation has

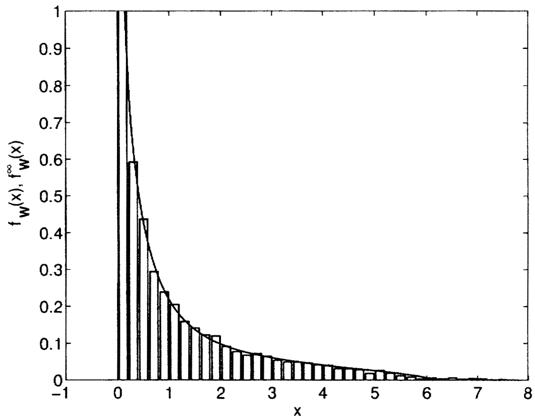

Fig. 8. Normalized histogram representing $f_{\mathbf{W}}(x)$ for $N_{\mathrm{T}} = N_{\mathrm{R}} = 8$ compared to the AEPDF $f_{\mathbf{W}}^{\infty}(x)$ for $\beta = 1$, at $\rho = 0.5$.

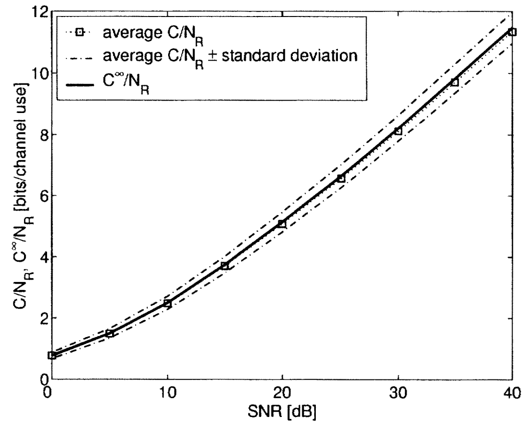

Fig. 9. Capacity per receive antenna C/N_{R} (average as well as average $\pm$ standard deviation) for $N_{\mathrm{T}} = N_{\mathrm{R}} = 8$ and asymptotic capacity per receive antenna $C^{\infty}/N_{\mathrm{R}}$ for $\beta = 1$ versus SNR, at $\rho = 0.5$.

a stronger capacity-reducing effect on the side with fewer antennas. A similar behavior was observed in [9].

V. Simulation Results for Finite Dimension

We computed the empirical eigenvalue pdf $f_{\mathbf{W}}(x)$ and the channel capacity C for finite-dimensional MIMO systems and compared them to their asymptotic counterparts $f_{\mathbf{W}}^{\infty}(x)$ and C^{∞}. Samples of the channel $\mathbf{H}$ were generated according to the model described in Subsection I-A (receive correlation case) for $\beta = 1$ (i.e., $N_{\mathrm{T}} = N_{\mathrm{R}}$) and $\rho = 0.5$.

Fig. 8 shows $f_{\mathbf{W}}(x)$, represented as a normalized histogram that was calculated from 1000 different samples of $\mathbf{H}$, for $N_{\mathrm{T}} = N_{\mathrm{R}} = 8$. The AEPDF $f_{\mathbf{W}}^{\infty}(x)$, again for $\beta = 1$ and $\rho = 0.5$, is also shown and is seen to provide an excellent approximation to $f_{\mathbf{W}}(x)$.

An analogous comparison is presented in Fig. 9 for the channel capacity, again for $N_{\mathrm{T}} = N_{\mathrm{R}} = 8$ and $\rho = 0.5$. The average and the average $\pm$ standard deviation of the C/N_{R} results obtained for 1000 different samples of $\mathbf{H}$ are compared to $C^{\infty}/N_{\mathrm{R}}$ as a function of the SNR. Note that the average can be interpreted as an estimate of the ergodic channel capacity per receive antenna. It is seen that $C^{\infty}/N_{\mathrm{R}}$ is an excellent approximation to the averaged C/N_{R}. It is also a good approximation to the individual C/N_{R} results because the standard deviation is fairly small.

VI. Conclusions

We computed the asymptotic channel capacity of MIMO channels with spatial correlation at the transmitter or receiver side, using a one-sided Kronecker model with "exponential" transmitter or receiver correlation matrix. An explicit calculation of the asymptotic eigenvalue pdf avoided the assumption of an *ad hoc* asymptotic eigenvalue pdf. We studied the dependence of the asymptotic channel capacity on the SNR, dimension ratio, and correlation parameter. In particular, we found that the asymptotic capacity per receive antenna is roughly proportional to the minimum of the numbers of receive and transmit antennas. We also observed that moderate spatial correlation has only a weak impact on asymptotic capacity. Simulation results for finite-dimensional MIMO channels showed that our asymptotic capacity results provide a good approximation to the actual capacity already for moderate system dimensions.

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