Second-Order Time–Frequency Synthesis of Nonstationary Random Processes

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***Abstract*—We present time–frequency methods for the synthesis of finite-energy, nonstationary random processes. The energetic characteristics of the process to be synthesized are specified in a joint time–frequency domain via a time–frequency model function. The synthesis methods optimize the autocorrelation function of the process such that the process' Wigner–Ville spectrum is closest to the given model function. An optional signal subspace constraint allows the incorporation of additional properties such as bandlimitation and also permits the reformulation of the synthesis methods in a discrete-time setting. The synthesized process is expressed either in terms of an orthonormal basis of the constraint subspace or via its Karhunen–Loève expansion. An example involving the prolate spheroidal functions is given, and computer simulation results are provided.**

***Index Terms*— Nonstationary random processes, process synthesis, Wigner–Ville spectrum, time–frequency signal processing.**

I. Introduction

An intuitively appealing interpretation of a nonstationary random process is that the process "changes its spectral content with time." Often, this interpretation is in good agreement with the physical situation generating the process. While the definition of a "time-varying spectrum" for nonstationary processes is not unique, a specific definition known as the *Wigner–Ville spectrum* (WVS) [1], [2] is particularly attractive. The WVS is a distribution of the mean instantaneous power over frequency, or equivalently, a distribution of the mean energy over a joint time–frequency (TF) plane.

It is natural to exploit the advantages of the WVS representation not only for the *analysis* but also for the *synthesis* of processes. In many applications (e.g., testing algorithms for nonstationary signal processing), one is interested in generating nonstationary processes with specified second-order characteristics. Conventionally, this would require the specification of the process' autocorrelation function (ACF) which, in the nonstationary case, is a complicated, complex-valued two-dimensional function with little relation to physical intuition. Often, it would be easier and intuitively more meaningful to specify the time-varying spectrum (i.e., WVS) of a process. In this paper, therefore, we are looking for a method to generate the ACF and furthermore realizations of a process from a "specified WVS," i.e., from a user-defined TF model function which, in general, will not be a valid WVS of any process. The TF model function expresses the desired temporal evolution of the power spectral density. We shall start our discussion with a brief review of the WVS and a statement of the TF synthesis problem.

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A. The Wigner–Ville Spectrum

The Wigner–Ville spectrum (WVS) [1], [2] of a complex-valued, generally nonstationary random process $x(t)$ is defined as the expectation of the process' Wigner distribution [3], [4] or, equivalently, as[1]

$$\overline{W}_x(t,f) \triangleq \int_\tau R_x\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau \qquad (1)$$

where $R_x(t_1,t_2) = \mathcal{E}\{x(t_1)x^*(t_2)\}$ is the autocorrelation function (ACF) of the process $x(t)$, t and f denote time and frequency, respectively, and integrations are from $-\infty$ to ∞ unless explicitly specified otherwise. The WVS is a real-valued function which can be interpreted as a time-varying power spectral density since

$$\int_f \overline{W}_x(t,f)\, df = \mathcal{E}\{|x(t)|^2\}.$$

The WVS will reduce to the conventional power spectral density if the process $x(t)$ is wide-sense stationary. Alternatively, if the process' mean energy

$$\overline{E}_x = \int_t \mathcal{E}\{|x(t)|^2\}\, dt = \int_t R_x(t,t)\, dt$$

is finite, then the WVS can also be viewed as a distribution of $\overline{E}_x$ over the TF plane in the sense that

$$\int_t \int_f \overline{W}_x(t,f)\, dt\, df = \overline{E}_x.$$

The WVS provides strictly the same information as the ACF $R_x(t_1,t_2)$. Indeed, these two quantities can be derived from one another by means of (1) and the inversion of (1),

$$R_x(t_1,t_2) = \int_f \overline{W}_x\left(\frac{t_1+t_2}{2}, f\right) e^{j2\pi(t_1-t_2)f}\, df. \qquad (2)$$

However, the WVS typically yields a characterization of a process which is clearer and easier to interpret than the ACF from which it is derived, since both temporal and spectral features are displayed simultaneously. A simple example showing the advantages gained from a TF description by means of the

[1] The bar notation in $\overline{W}_x(t,f)$ emphasizes the expectation taken and also serves to distinguish the WVS of a process $x(t)$ from the process' Wigner distribution $W_x(t,f)$ which is random.

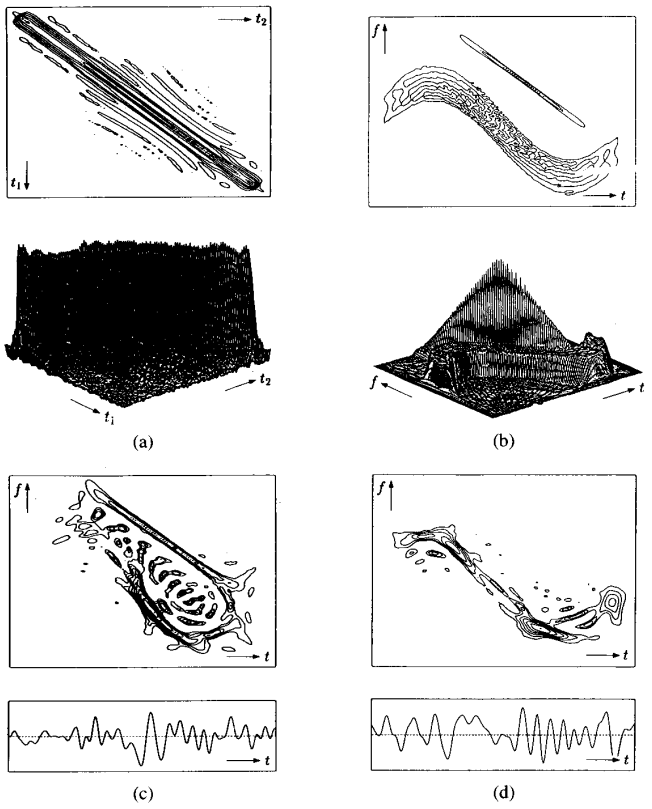

Fig. 1. Comparison of the ACF and the WVS for a nonstationary process consisting of two statistically orthogonal components. (a) Magnitude of ACF, (b) WVS, (c) and (d) real parts and Wigner distributions (incorporating a small amount of TF smoothing)[2] of two realizations of the process.

WVS is given in Fig. 1. The process under analysis consists of two statistically orthogonal components. The first component is a finite-duration bandpass noise with time-varying center frequency according to a sinusoidal modulation law. The second component is a finite-duration chirp (i.e., linear FM) signal which is deterministic except for a random amplitude. The process' TF structure is clearly shown in the WVS but not in the ACF.

B. TF Synthesis of Nonstationary Processes

We have seen above that the WVS is a "TF version" of the ACF which, in many cases of practical interest, has a more immediate relation to the physical reality generating the process than the ACF itself. We now consider the generation of a nonstationary process where our specifications are formulated in the TF plane. For example, we could specify a TF energy distribution that crudely resembles the WVS in Fig. 1(b), and look for the process whose WVS is closest to the specified TF energy distribution. The advantages gained from such a "TF specification" or "TF synthesis" are analogous to the advantages gained from a TF analysis (see Fig. 1).

[2] A judicious smoothing of the Wigner distribution is usually employed to reduce interference terms which would otherwise mask the TF structure of the signal under analysis [4]. Such a smoothing it typically unnecessary for the WVS of a random process, since here the interference terms are usually "averaged out" by the expectation operation [5].

As a first step, let us attempt to specify a WVS by prescribing a real-valued TF function ("model") $\widetilde{W}(t,f)$. There arises the question whether the model function $\widetilde{W}(t,f)$ is a valid WVS. In contrast to the special case of stationary processes, nonnegativity of $\widetilde{W}(t,f)$ is neither necessary nor sufficient for the "validity" of $\widetilde{W}(t,f)$ [6]. In fact, it will be made clear presently that typically a given model $\widetilde{W}(t,f)$ will not be a valid WVS, i.e., there will not exist a process $x(t)$ such that $W_x(t,f) = \widetilde{W}(t,f)$.

In this situation, it is natural to consider the process $\widehat{x}(t)$ whose WVS is *closest* to the TF model in a least square sense; this process is the solution to the minimization problem (synthesis problem)

$$\widehat{x}(t) \triangleq \arg\min_x \epsilon_x \tag{3}$$

with the *synthesis error* given by

$$\epsilon_x^2 = \|\widetilde{W} - W_x\|^2 = \int_t \int_f \left[\widetilde{W}(t,f) - W_x(t,f)\right]^2 dt\, df. \tag{4}$$

To make the definition (4) of the synthesis error ϵ_x meaningful, we have to require that the model $\widetilde{W}(t,f)$ be square-integrable, i.e., $\widetilde{W}(t,f) \in \mathcal{L}_2(\mathbb{R}^2)$.

Since the WVS $W_x(t,f)$ contains strictly the same information about the process $x(t)$ as the ACF $R_x(t_1,t_2)$, it is clear that properties of $x(t)$ not contained in the ACF do

not enter in the optimization criterion (i.e., the synthesis error ϵ_x). In fact, our synthesis method is a "second-order" method which generates not the process as such but its ACF, which means that other process characteristics (such as the mean $m_x(t) = \mathcal{E}\{x(t)\}$ or probability density functions) are left unspecified to a certain degree. The generation of specific realizations of $\widehat{x}(t)$ is based on additional assumptions and will be discussed in later sections. However, it is clear that this fundamental ambiguity of the synthesis result is removed if, for example, the process $x(t)$ is assumed *a priori* to be zero-mean and normally distributed.

C. Subspace-Constrained Synthesis

The TF synthesis of a random process can be made more flexible by including a signal subspace constraint in the formulation (3) of the synthesis problem. We here require that the realizations of the process $x(t)$ be elements of a given linear signal subspace $\mathcal{S} \subseteq \mathcal{L}_2(\mathbb{R})$ of the space $\mathcal{L}_2(\mathbb{R})$ of square-integrable (finite-energy) deterministic signals [7]; this will be briefly denoted as $x(t) \in \mathcal{S}$. The resulting subspace-constrained synthesis problem then reads

$$\widehat{x}(t) \triangleq \arg\min_{x \in \mathcal{S}} \epsilon_x. \tag{5}$$

The advantage of the subspace constraint $x(t) \in \mathcal{S}$ is that it allows us to enforce certain properties of the synthesis result $\widehat{x}(t)$. Choosing, for example, the signal subspace $\mathcal{S}$ to be the space of signals bandlimited in a given frequency band B, it is guaranteed that the synthesis result $\widehat{x}(t)$ is bandlimited in B. More generally, a "TF subspace" according to [8] can be used to enforce energetic concentration of $\widehat{x}(t)$ in a given region of the TF plane; this results in a "TF-selective" synthesis with implicit TF filtering. It is convenient to consider synthesis without a subspace constraint as in (3) (termed *global synthesis* in the following) as a special case of subspace-constrained synthesis (5): for global synthesis, $\mathcal{S} = \mathcal{L}_2(\mathbb{R})$. The resulting condition $x(t) \in \mathcal{L}_2(\mathbb{R})$ means that the synthesized process $\widehat{x}(t)$ has finite energy. Thus our synthesis approach is essentially suited for the generation of "random transients."

The synthesis problems (3) and (5) can be considered as stochastic versions of the well-known *TF signal synthesis problem* [9]–[12]

$$\widehat{x}(t) \triangleq \arg\min_{x \in \mathcal{S}} \|\widetilde{W} - W_x\| \tag{6}$$

where $x(t)$ is a *deterministic* signal and $W_x(t,f)$ denotes its Wigner distribution [3]. Here, $\widehat{x}(t)$ is defined as the deterministic signal whose Wigner distribution is closest to the TF model function $\widetilde{W}(t,f)$. In fact, it will be seen presently that the solutions to the two problems (5) and (6) are closely related.

D. Outline of Paper

The subsequent sections of this paper are organized as follows. In Section II, we derive a solution to the subspace-constrained synthesis problem (5) which is formulated in terms of an arbitrary orthonormal basis of the constraint signal subspace $\mathcal{S}$. In Section III, this solution will be reformulated in terms of a "natural" basis, which is shown to result in the Karhunen–Loève expansion [13] of the synthesized process $\widehat{x}(t)$. This basis is also best suited for generating specific realizations of the process. Another version of the synthesis method, which does not use a basis, is considered in Section IV. Section V gives an example of theoretical interest, where the model function and the subspace constraint restrict the synthesized process in both time domain and frequency domain; it is shown here that the synthesis solution involves the well-known prolate spheroidal functions [14]. Section VI considers the discrete-time implementation of the synthesis algorithms, and Section VII presents computer simulation results illustrating the performance and application of the synthesis methods.

II. Solving the Synthesis Problem

We now discuss the solution of the subspace-constrained synthesis problem (5).

A. Transformation into the ACF Domain

Equations (1) and (2) connecting the WVS and the ACF describe a unitary mapping which is known as the *Weyl correspondence* [15]–[17]. This mapping relates the "TF domain" and the "ACF domain." Using (1), the TF model $\widetilde{W}(t,f)$ can be written in the form of a WVS

$$\widetilde{W}(t,f) = \int_\tau \widetilde{R}\Big(t + \frac{\tau}{2}, t - \frac{\tau}{2}\Big) e^{-j2\pi f\tau}\, d\tau$$

where $\widetilde{R}(t_1,t_2)$ is an "ACF model" which is derived from the TF model $\widetilde{W}(t,f)$ by the inverse mapping (2)

$$\widetilde{R}(t_1,t_2) = \int_f \widetilde{W}\left(\frac{t_1+t_2}{2}, f\right) e^{j2\pi(t_1-t_2)f}\, df. \tag{7}$$

For a real-valued TF model $\widetilde{W}(t,f)$, the ACF model $\widetilde{R}(t_1,t_2)$ is a Hermitian function, i.e., $\widetilde{R}^*(t_2,t_1) = \widetilde{R}(t_1,t_2)$. Note, however, that in general $\widetilde{R}(t_1,t_2)$ will not be a valid ACF (it is a valid ACF if and only if $\widetilde{W}(t,f)$ is a valid WVS).

Since the mapping between the TF domain and the ACF domain is unitary, the synthesis error ϵ_x can be reformulated in the ACF domain as

$$\begin{aligned}\epsilon_x^2 &= \|\widetilde{W} - \overline{W}_x\|^2 = \|\widetilde{R} - R_x\|^2 \\ &= \int_{t_1}\int_{t_2} |\widetilde{R}(t_1,t_2) - R_x(t_1,t_2)|^2 dt_1\, dt_2.\end{aligned}$$

This has to be minimized subject to the subspace constraint $x(t) \in \mathcal{S}$

$$\widehat{x}(t) = \arg\min_{x \in \mathcal{S}} \|\widetilde{R} - R_x\|.$$

Thus we have shown that the synthesis problem (5) is equivalent to the optimal approximation of a Hermitian function $\widetilde{R}(t_1,t_2)$ by the ACF $R_x(t_1,t_2)$ of a process $x(t)$ whose realizations belong to a given signal space $\mathcal{S}$.

B. Process Expansion

Let us assume that the linear signal space $\mathcal{S}$ has dimension N and that an orthonormal basis $\{s_k(t)\}_{k=1}^N$ of $\mathcal{S}$ is available[3]. The subspace constraint $x(t) \in \mathcal{S}$ can then be taken into account by representing $x(t)$ as[4]

$$x(t) = \sum_{k=1}^{N} a_k s_k(t) = \boldsymbol{a}^T \boldsymbol{s}(t)$$

with the coefficient vector $\boldsymbol{a} = (a_k)$ and the basis vector $\boldsymbol{s}(t) = (s_k(t))$. For a given basis $\{s_k(t)\}_{k=1}^N$, the properties of $x(t)$ are now determined by the properties of the random variables (coefficients) a_k. For example, the ACF $R_x(t_1, t_2)$ is given by

$$\begin{aligned} R_x(t_1, t_2) &= \sum_{k=1}^{N}\sum_{l=1}^{N} R_a(k,l) s_k(t_1) s_l^*(t_2) \\ &= \boldsymbol{s}^H(t_2) \boldsymbol{R}_a^T \boldsymbol{s}(t_1) \end{aligned} \tag{8}$$

where $\boldsymbol{R}_a = \mathcal{E}\{\boldsymbol{a}\boldsymbol{a}^H\} = (R_a(k,l))$ with $R_a(k,l) = \mathcal{E}\{a_k a_l^*\}$ is the correlation matrix of the coefficients a_k, and the superscript ^{H} stands for conjugate transposition.

C. The Induced ACF-Domain Subspace

The linear signal subspace $\mathcal{S} \subseteq \mathcal{L}_2(\mathbb{R})$ can be shown to "induce" a linear ACF-domain subspace $\mathcal{S}_R \subseteq \mathcal{L}_2(\mathbb{R}^2)$ which, loosely speaking, consists of all linear combinations of outer signal products $x(t_1)y^*(t_2)$ with $x(t), y(t) \in \mathcal{S}$ [19], [10]. The outer products of all orthonormal basis functions $s_k(t), R_{kl}(t_1, t_2) \triangleq s_k(t_1)s_l^*(t_2)$, can be shown to constitute an orthonormal basis of the induced ACF-domain subspace $\mathcal{S}_R$. In fact, (8) describes the expansion of the ACF $R_x(t_1, t_2)$ in terms of this induced orthonormal basis $R_{kl}(t_1, t_2)$

$$R_x(t_1, t_2) = \sum_{k=1}^{N}\sum_{l=1}^{N} R_a(k,l) R_{kl}(t_1, t_2). \tag{9}$$

Thus from $x(t) \in \mathcal{S}$ it follows that $R_x(t_1, t_2) \in \mathcal{S}_R$, i.e., $R_x(t_1, t_2)$ is an element of the induced ACF-domain subspace $\mathcal{S}_R$. The ACF model function $\widetilde{R}(t_1, t_2)$, on the other hand, will not generally be an element of $\mathcal{S}_R$. However, it can be decomposed as $\widetilde{R}(t_1, t_2) = \widetilde{R}_S(t_1, t_2) + \widetilde{R}_\perp(t_1, t_2)$ (see Fig. 2) where the "projected ACF model" $\widetilde{R}_S(t_1, t_2)$ is the orthogonal projection of $\widetilde{R}(t_1, t_2)$ on $\mathcal{S}_R$, given by

$$\widetilde{R}_S(t_1, t_2) = \sum_{k=1}^{N}\sum_{l=1}^{N} \gamma_{kl} R_{kl}(t_1, t_2) \tag{10}$$

with

$$\begin{aligned} \gamma_{kl} = \langle \widetilde{R}, R_{kl} \rangle &= \int_{t_1}\int_{t_2} \widetilde{R}(t_1, t_2) R_{kl}^*(t_1, t_2)\, dt_1\, dt_2 \\ &= \int_{t_1}\int_{t_2} \widetilde{R}(t_1, t_2) s_k^*(t_1) s_l(t_2)\, dt_1\, dt_2 \end{aligned} \tag{11}$$

and $\widetilde{R}_\perp(t_1, t_2)$ is orthogonal on $\mathcal{S}_R$.

[3] In practical applications, N is always finite. However, for theoretical analyses it is of interest to allow an infinite dimension N. In this case, we have to assume that the space $\mathcal{S}$ be *separable* [18], which implies the existence of an orthonormal basis of $\mathcal{S}$.

[4] Boldface print denotes column vectors and matrices, and the superscript ^{T} stands for transposition.

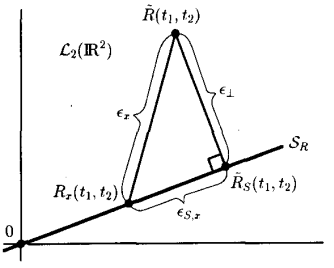

Fig. 2. Orthogonal decomposition of the ACF model $\widetilde{R}(t_1, t_2)$ and the synthesis error ϵ_x.

D. Error Decomposition

Using the Pythagorean theorem [18], the squared synthesis error can now be decomposed as (cf. Fig. 2)

$$\begin{aligned} \epsilon_x^2 &= \|\widetilde{R} - R_x\|^2 = \|(\widetilde{R}_S + \widetilde{R}_\perp) - R_x\|^2 \\ &= \|(\widetilde{R}_s - R_x) + \widetilde{R}_\perp\|^2 = \|\widetilde{R}_S - R_x\|^2 + \|\widetilde{R}_\perp\|^2 \\ &= \epsilon_{S,x}^2 + \epsilon_\perp^2. \end{aligned} \tag{12}$$

Since the "orthogonal" error component $\epsilon_\perp = \|\widetilde{R}_\perp\|$ does not depend on $x(t)$, there remains to minimize the error component $\epsilon_{S,x} = \|\widetilde{R}_S - R_x\|$. Using the expansions (9) and (10), $\epsilon_{S,x}$ can be reformulated in a "coefficient domain"

$$\begin{aligned} \epsilon_{S,x}^2 = \|\widetilde{R}_S - R_x\|^2 &= \left\| \sum_{k=1}^{N}\sum_{l=1}^{N} [\gamma_{kl} - R_a(k,l)] R_{kl} \right\|^2 \\ &= \sum_{k=1}^{N}\sum_{l=1}^{N} |\gamma_{kl} - R_a(k,l)|^2 = \|\boldsymbol{\Gamma} - \boldsymbol{R}_a\|_F^2, \end{aligned} \tag{13}$$

where $\boldsymbol{\Gamma} = (\gamma_{kl})$ is the matrix of the model's projection coefficients γ_{kl} and $\|\cdot\|_F$ denotes the Euclidian norm (Frobenius norm) of a matrix.

E. Optimization of the Correlation Matrix

It is easily checked that the "model matrix" $\boldsymbol{\Gamma}$ is Hermitian for a real-valued model $\widetilde{W}(t, f)$. The correlation matrix $\boldsymbol{R}_a$, on the other hand, is moreover positive semidefinite. Thus the minimization of (13) amounts to the optimal approximation of the Hermitian matrix $\boldsymbol{\Gamma}$ by a positive semidefinite matrix $\boldsymbol{R}_a$. It is shown in Appendix I that the solution to this problem is given by

$$\boldsymbol{R}_{\hat{a}} = \boldsymbol{\Gamma}_+ \tag{14}$$

where $\boldsymbol{\Gamma}_+$ denotes the positive part of the matrix $\boldsymbol{\Gamma}$,

$$\boldsymbol{\Gamma}_+ = \sum_{k=1}^{N_+} \lambda_k \boldsymbol{v}_k \boldsymbol{v}_k^H = \boldsymbol{V}\boldsymbol{\Lambda}_+\boldsymbol{V}^H. \tag{15}$$

Here, λ_k and $\boldsymbol{v}_k$ are the (real-valued) eigenvalues and the (orthonormal) eigenvectors of the Hermitian matrix $\boldsymbol{\Gamma}$, N_+ is the number of positive eigenvalues (the eigenvalues are assumed to be arranged in nonincreasing order, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$), $\boldsymbol{\Lambda}_+$ is the positive part of the diagonal eigenvalue matrix (containing the positive eigenvalues as the first N_+ diagonal elements, with the remaining diagonal elements zero), and $\boldsymbol{V}$ is the unitary matrix whose columns are the eigenvectors $\boldsymbol{v}_k$.

F. The Synthesis Algorithm

We shall now summarize the results derived above. We have shown that the solution $\hat{x}(t) \in \mathcal{S}$ to the subspace-constrained synthesis problem (5) is given by

$$\hat{x}(t) = \sum_{k=1}^{N} \hat{a}_k s_k(t) = \hat{\boldsymbol{a}}^T \boldsymbol{s}(t) \tag{16}$$

where $\boldsymbol{s}(t)$ is an arbitrary orthonormal basis of the constraint subspace $\mathcal{S}$, and the correlation matrix $\boldsymbol{R}_{\hat{a}}$ of the optimal coefficients $\hat{\boldsymbol{a}}$ can be constructed as follows:

1) The ACF model $\widetilde{R}(t_1, t_2)$ is derived from the TF model $\widetilde{W}(t, f)$ according to (7).
2) The model's projection coefficient matrix $\boldsymbol{\Gamma} = (\gamma_{kl})$ is calculated by means of (11).
3) The N_+ positive eigenvalues λ_k and the corresponding (normalized) eigenvectors $\boldsymbol{v}_k$ $(k = 1, \cdots, N_+)$ of $\boldsymbol{\Gamma}$ are determined.
4) The optimal correlation matrix is obtained as $\boldsymbol{R}_{\hat{a}} = \boldsymbol{\Gamma}_+$, where $\boldsymbol{\Gamma}_+$ is the positive part of $\boldsymbol{\Gamma}$ which is derived from the positive eigenvalues and the corresponding eigenvectors according to (15).

Due to (16), the problem of generating the process $\hat{x}(t)$ is reduced to the problem of generating the N random variables $\hat{a}_k$. These random variables are determined only with respect to their correlation matrix $\boldsymbol{R}_{\hat{a}}$. Accordingly, as shown by (8), also the resulting process $\hat{x}(t)$ is specified only with respect to its ACF. This means that other characteristics like the mean and probability density functions have still to be specified. This point will be further discussed in Section III. In particular, it will be shown how to generate realizations of $\hat{x}(t)$ for given coefficient correlation matrix $\boldsymbol{R}_{\hat{a}}$.

G. Energy of the Synthesis Result

It is easily shown that the mean energy

$$\overline{E}_{\hat{x}} = \int_t R_{\hat{x}}(t, t)\, dt$$

of $\hat{x}(t)$ is given by the trace of the coefficient correlation matrix $\boldsymbol{R}_{\hat{a}}$ which, due to (14) and (15), equals the sum of the positive eigenvalues of $\boldsymbol{\Gamma}$

$$\overline{E}_{\hat{x}} = \sum_{k=1}^{N} \mathcal{E}\{|\hat{a}_k|^2\} = \operatorname{tr}\{\boldsymbol{R}_{\hat{a}}\} = \operatorname{tr}\{\boldsymbol{\Gamma}_+\} = \sum_{k=1}^{N_+} \lambda_k.$$

We see that $\overline{E}_{\hat{x}}$ will be finite if and only if the sum of all positive eigenvalues λ_k is finite. In particular, this will always be the case for a finite-dimensional signal subspace $\mathcal{S}$ since here $N_+ \leq N < \infty$.

H. Residual Synthesis Error

With (12), the squared residual (minimum) synthesis error is $\epsilon^2_{x,\min} = \epsilon^2_{\hat{x}} = \epsilon^2_{S,\hat{x}} + \epsilon^2_{\perp}$. The "orthogonal" error component $\epsilon_{\perp} = \|\widetilde{R}_{\perp}\|$ is zero if and only if the ACF model $\widetilde{R}(t_1, t_2)$ belongs to the induced ACF-domain subspace $\mathcal{S}_R$. This will rarely occur except for the case of global synthesis where[5] $\mathcal{S} = \mathcal{L}_2(\boldsymbol{R})$. For the error component $\epsilon_{S,\hat{x}}$, we obtain with (13)–(15)

$$\begin{aligned} \epsilon^2_{S,\hat{x}} &= \|\boldsymbol{\Gamma} - \boldsymbol{\Gamma}_+\|_F^2 = \|\boldsymbol{V}(\boldsymbol{\Lambda} - \boldsymbol{\Lambda}_+)\boldsymbol{V}^H\|_F^2 \\ &= \|\boldsymbol{\Lambda} - \boldsymbol{\Lambda}_+\|_F^2 = \sum_{k=N_++1}^{N} \lambda_k^2 \end{aligned} \tag{17}$$

where the spectral decomposition of $\boldsymbol{\Gamma}$

$$\boldsymbol{\Gamma} = \sum_{k=1}^{N} \lambda_k \boldsymbol{v}_k \boldsymbol{v}_k^H = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^H \tag{18}$$

and the unitarity of $\boldsymbol{V}$ has been used. With (17), $\epsilon^2_{S,\hat{x}}$ is seen to be the sum of the squares of all *negative* eigenvalues of the matrix $\boldsymbol{\Gamma}$. Hence, the error component $\epsilon_{S,\hat{x}}$ is zero if any only if the matrix $\boldsymbol{\Gamma}$ is positive semidefinite, i.e., $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_+$. Positive semidefiniteness of $\boldsymbol{\Gamma}$ is thus recognized as a necessary and sufficient condition for the projected ACF model $\widetilde{R}_S(t_1, t_2)$ to be a valid ACF, $\widetilde{R}_S(t_1, t_2) = R_{\hat{x}}(t_1, t_2)$. We note that this "validity" of $\widetilde{R}_S(t_1, t_2)$ will always be satisfied (for arbitrary subspace $\mathcal{S}$) if the original TF model $\widetilde{W}(t, f)$ is a valid WVS.

I. Relation with Deterministic Signal Synthesis

The "stochastic" synthesis problem (5) is closely related to the "deterministic" signal synthesis problem (6). The solution to the latter problem is [10]

$$\hat{x}(t) = \sum_{k=1}^{N} \hat{a}_k s_k(t) = \hat{\boldsymbol{a}}^T \boldsymbol{s}(t),$$

with the (deterministic) coefficient vector $\hat{\boldsymbol{a}}$ given by

$$\hat{\boldsymbol{a}} = \sqrt{\lambda_1}\boldsymbol{v}_1 \tag{19}$$

where λ_1 is the largest eigenvalue of $\boldsymbol{\Gamma}$ (λ_1 is assumed to be nonnegative [10]) and $\boldsymbol{v}_1$ is the corresponding normalized eigenvector. The deterministic synthesis problem can be considered a special case of the stochastic synthesis problem: if the process $x(t)$ to be synthesized is constrained to be deterministic, then the coefficient correlation matrix $\boldsymbol{R}_a = \mathcal{E}\{\boldsymbol{a}\boldsymbol{a}^H\}$ equals the dyadic product $\boldsymbol{a}\boldsymbol{a}^H$, and the minimization of (13) thus reduces to the minimization of $\|\boldsymbol{\Gamma} - \boldsymbol{a}\boldsymbol{a}^H\|_F^2$ with respect of the vector $\boldsymbol{a}$, i.e., the optimal approximation of $\boldsymbol{\Gamma}$ by a dyadic product (rank-1 matrix) $\boldsymbol{a}\boldsymbol{a}^H$. The solution to this problem is given by $\hat{\boldsymbol{a}}\hat{\boldsymbol{a}}^H = \lambda_1 \boldsymbol{v}_1 \boldsymbol{v}_1^H$ or, equivalently, (19). A major difference between the solutions to the stochastic and deterministic problems is that the former involves all positive eigenvalues and associated eigenvectors of $\boldsymbol{\Gamma}$ whereas the latter involves only the largest positive eigenvalue and the associated eigenvector.

[5] If $\mathcal{S} = \mathcal{L}_2(\boldsymbol{R})$, then we have $\mathcal{S}_R = \mathcal{L}_2(\boldsymbol{R}^2)$. Since the TF model was assumed square-integrable, there is $\widetilde{W}(t, f) \in \mathcal{L}_2(\boldsymbol{R}^2)$ and thus also $\widetilde{R}(t_1, t_2) \in \mathcal{L}_2(\boldsymbol{R}^2)$. Combining with $\mathcal{S}_R = \mathcal{L}_2(\boldsymbol{R}^2)$, it is clear that $\widetilde{R}(t_1, t_2)$ is an element of the induced ACF-domain subspace $\mathcal{S}_R$.

III. Karhunen–Loève Representation

In the previous section, the solution $\hat{x}(t)$ to the subspace-constrained synthesis problem (5) has been formulated in terms of an arbitrary orthonormal basis $\boldsymbol{s}(t)$ of the constraint subspace $\mathcal{S}$. We shall now show that $\hat{x}(t)$ assumes a particularly simple form if a special, model-dependent basis of $\mathcal{S}$ is used. This corresponds to a diagonalization of the coefficient correlation matrix $\boldsymbol{R}_{\hat{a}}$ and will be seen to result in the Karhunen–Loève expansion of $\hat{x}(t)$. The practical advantage of this new formulation is that the generation of $\hat{x}(t)$ reduces to the generation of *uncorrelated* random variables.

A. Basis Transformation and Karhunen–Loève Expansion

The new basis, denoted by $\boldsymbol{u}(t)$, is related to the given basis $\boldsymbol{s}(t)$ by means of the unitary transformation

$$\boldsymbol{u}(t) = \boldsymbol{V}^T \boldsymbol{s}(t) \tag{20}$$

where $\boldsymbol{V}$ is the unitary eigenvector matrix of the model's projection coefficient matrix $\boldsymbol{\Gamma}$. Just as $\boldsymbol{s}(t), \boldsymbol{u}(t)$ is an orthonormal basis of $\mathcal{S}$. Thus $\hat{x}(t)$ can be expanded as

$$\hat{x}(t) = \sum_{k=1}^{N} \hat{c}_k u_k(t) = \hat{\boldsymbol{c}}^T \boldsymbol{u}(t) \tag{21}$$

where

$$\hat{\boldsymbol{c}} = \boldsymbol{V}^H \hat{\boldsymbol{a}} \tag{22}$$

due to (20) and (16). With (22) and (14), (15), the correlation matrix of the transformed coefficient vector $\hat{\boldsymbol{c}}$ is found to be the positive part of the eigenvalue matrix of $\boldsymbol{\Gamma}$.

$$\boldsymbol{R}_{\hat{c}} = \boldsymbol{V}^H \boldsymbol{R}_{\hat{a}} \boldsymbol{V} = \boldsymbol{V}^H \boldsymbol{\Gamma}_+ \boldsymbol{V} = \boldsymbol{\Lambda}_+.$$

This means that the correlation matrix $\boldsymbol{R}_{\hat{c}}$ is diagonal, i.e., the coefficients $\hat{c}_k$ are *orthogonal*:

$$R_{\hat{c}}(k,l) = \mathcal{E}\{\hat{c}_k \hat{c}_l^*\} = 0, \qquad \text{for } k \neq l \tag{23}$$

$$R_{\hat{c}}(k,k) = \mathcal{E}\{|\hat{c}_k|^2\} = \begin{cases} \lambda_k, & \text{for } 1 \leq k \leq N_+ \\ 0, & \text{for } N_+ + 1 \leq k \leq N. \end{cases} \tag{24}$$

Since $\mathcal{E}\{|\hat{c}_k|^2\} = 0$ for $N_+ + 1 \leq k \leq N$, the expansion (21) can be rewritten as

$$\hat{x}(t) = \sum_{k=1}^{N_+} \hat{c}_k u_k(t). \tag{25}$$

This shows that the effective "order" of $\hat{x}(t)$ is N_+, the number of positive eigenvalues of $\boldsymbol{\Gamma}$. Note that N_+ may be smaller than the subspace dimension N.

With $\hat{x}(t) = \hat{\boldsymbol{c}}^T \boldsymbol{u}(t)$ and $\boldsymbol{R}_{\hat{c}} = \boldsymbol{\Lambda}_+$, the ACF of $\hat{x}(t)$ is obtained as (cf. (8))

$$\begin{aligned} R_{\hat{x}}(t_1,t_2) &= \boldsymbol{u}^H(t_2) \boldsymbol{R}_{\hat{c}}^T \boldsymbol{u}(t_1) = \boldsymbol{u}^H(t_2) \boldsymbol{\Lambda}_+ \boldsymbol{u}(t_1) \\ &= \sum_{k=1}^{N_+} \lambda_k u_k(t_1) u_k^*(t_2). \end{aligned} \tag{26}$$

This is the spectral representation (or eigendecomposition) of the ACF $R_{\hat{x}}(t_1,t_2)$, and we conclude that the first N_+ basis signals $u_k(t)$ are eigenfunctions of the integral operator defined by the kernel $R_{\hat{x}}(t_1,t_2)$

$$\int_{t_2} R_{\hat{x}}(t_1,t_2) u_k(t_2)\, dt_2 = \lambda_k u_k(t_1), \qquad 1 \leq k \leq N_+. \tag{27}$$

This shows that (25) is in fact the Karhunen-Loève (KL) expansion [13] of $\hat{x}(t)$, the unique expansion in terms of orthonormal basis signals for which the expansion coefficients are orthogonal.

B. Process Generation

In previous sections, we have derived the ACF of $\hat{x}(t)$, but the actual generation of $\hat{x}(t)$ has not yet been discussed. The KL expansion (25)

$$\hat{x}(t) = \sum_{k=1}^{N_+} \hat{c}_k u_k(t)$$

is particularly advantageous for generating realizations of $\hat{x}(t)$. Clearly, the generation of $\hat{x}(t)$ reduces to the generation of N_+ *orthogonal* random variables $\hat{c}_k$ with given quadratic means $\mathcal{E}\{|\hat{c}_k|^2\} = \lambda_k$. The (linear) means $\mathcal{E}\{\hat{c}_k\}$ can be chosen arbitrarily apart from the constraint that the resulting mean of $\hat{x}(t)$ be consistent with the ACF $R_{\hat{x}}(t_1,t_2)$. To evaluate this constraint, we note that the KL expansion induces an analogous expansion of the mean of $\hat{x}(t)$

$$m_{\hat{x}}(t) = \mathcal{E}\{\hat{x}(t)\} = \sum_{k=1}^{N_+} m_{\hat{c}}(k) u_k(t) \tag{28}$$

where $m_{\hat{c}}(k) = \mathcal{E}\{\hat{c}_k\} = \langle m_{\hat{x}}, u_k \rangle$. Note that it follows from (28) that $m_{\hat{x}}(t)$ is itself an element of $\mathcal{S}$. The consistency of $m_{\hat{x}}(t)$ with $R_{\hat{x}}(t_1,t_2)$ is defined by the requirement that the autocovariance function

$$\begin{aligned} C_{\hat{x}}(t_1,t_2) &= \mathcal{E}\{[\hat{x}(t_1) - m_{\hat{x}}(t_1)][\hat{x}(t_2) - m_{\hat{x}}(t_2)]^*\} \\ &= R_{\hat{x}}(t_1,t_2) - m_{\hat{x}}(t_1) m_{\hat{x}}^*(t_2) \end{aligned} \tag{29}$$

be a positive semidefinite function. In Appendix II, it is shown that a necessary and sufficient condition for consistency is

$$\sum_{k=1}^{N_+} \frac{|m_{\hat{c}}(k)|^2}{\lambda_k} \leq 1. \tag{30}$$

A trivial but practically important special case where (30) is certainly satisfied is a zero-mean process, i.e., $m_{\hat{x}}(t) \equiv 0$ or equivalently $m_{\hat{c}}(k) = 0$. In this case, the coefficients $\hat{c}_k$ are uncorrelated in addition to being orthogonal. In the opposite case, we can always write $\hat{c}_k = \bar{c}_k + m_{\hat{c}}(k)$ where the random variables $\bar{c}_k$ are zero-mean and uncorrelated and the $m_{\hat{c}}(k)$ are the desired means. Thus in any case, the generation of the process $\hat{x}(t)$ reduces to the generation of N_+ zero-mean, uncorrelated random variables with prescribed variances.

The process synthesis described above is "second-order" in that the process $\hat{x}(t)$ is specified in terms of its ACF and mean, but not with respect to its probability density functions. In many cases, one is interested in a Gaussian process. Gaussianity of $\hat{x}(t)$ will be obtained if the coefficients $\hat{c}_k$

are Gaussian random variables[6]. For other desired probability density functions, such a simple recipe cannot be given.

IV. Basis-Free Method

The derivation of the synthesis result $\widehat{x}(t)$ in the KL form (25) is still based on an initial (arbitrary) orthonormal basis $\boldsymbol{s}(t)$ of the constraint signal subspace $\mathcal{S}$. While this approach may be computationally efficient (particularly so when the dimension N of the subspace $\mathcal{S}$ is small), a "basis-free" derivation of the KL expansion is of theoretical interest since it shows additional properties of the KL basis $\boldsymbol{u}(t)$.

With (10), the projected ACF model can be expressed as

$$\begin{aligned}\widetilde{R}_S(t_1,t_2) &= \sum_{k=1}^{N}\sum_{l=1}^{N}\gamma_{kl}s_k(t_1)s_l^*(t_2)\\ &= \boldsymbol{s}^H(t_2)\boldsymbol{\Gamma}^T\boldsymbol{s}(t_1) = \boldsymbol{s}^H(t_2)(\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^H)^T\boldsymbol{s}(t_1)\\ &= \boldsymbol{u}^H(t_2)\boldsymbol{\Lambda}\boldsymbol{u}(t_1) = \sum_{k=1}^{N}\lambda_k u_k(t_1)u_k^*(t_2) \qquad (31)\end{aligned}$$

where (18) and (20) have been used. Note that (31) is analogous to the expression (26) for $R_{\hat{x}}(t_1,t_2)$, the only difference being that, in general, $\widetilde{R}_S(t_1,t_2)$ contains also negative eigenvalues λ_k. Furthermore, it follows from (31) that

$$\int_{t_2}\widetilde{R}_S(t_1,t_2)u_k(t_2)dt_2 = \lambda_k u_k(t_1), \quad 1\le k\le N \qquad (32)$$

which is analogous to (27), again with the difference that (32) is valid also for the negative eigenvalues λ_k (if these exist). We conclude that the eigenvalues λ_k of $\boldsymbol{\Gamma}$ and the KL basis signals $u_k(t)$ equal the eigenvalues and eigenfunctions, respectively, of the integral operator defined by the kernel $\widetilde{R}_S(t_1,t_2)$.

A. Basis-Free Synthesis Algorithm

The above discussion demonstrates that the eigenvalues λ_k and eigenfunctions $u_k(t)$ are independent of the initial basis $\boldsymbol{s}(t)$ used in Section III. Moreover, it allows us to derive the KL expansion (25) of the synthesis result $\widehat{x}(t)$ without the use of the initial basis $\boldsymbol{s}(t)$, as explained in the following.

Suppose that an orthonormal basis $\boldsymbol{s}(t)$ of the constraint subspace $\mathcal{S}$ in not available; instead, $\mathcal{S}$ is characterized by its orthogonal projection operator $\boldsymbol{P}_{\mathcal{S}}$ with kernel $P_{\mathcal{S}}(t,t')$ [7], [18]. It can then the shown [19] that the orthogonal projection operator $\boldsymbol{P}_{\mathcal{S}_R}$ on the induced ACF-domain subspace $\mathcal{S}_R$ is given by the kernel $P_{\mathcal{S}_R}(t_1,t_2;t_1',t_2') = P_{\mathcal{S}}(t_1,t_1')P_{\mathcal{S}}^*(t_2,t_2')$. Hence, the projected ACF model $\widetilde{R}_S(t_1,t_2)$ can be derived from the original ACF model $\widetilde{R}(t_1,t_2)$ as

$$\begin{aligned}\widetilde{R}_S(t_1,t_2) &= \int_{t_1'}\int_{t_2'}P_{\mathcal{S}_R}(t_1,t_2;t_1',t_2')\widetilde{R}(t_1',t_2')\,dt_1'\,dt_2'\\ &= \int_{t_1'}\int_{t_2'}P_{\mathcal{S}}(t_1,t_1')P_{\mathcal{S}}^*(t_2,t_2')\widetilde{R}(t_1',t_2')\,dt_1'\,dt_2'.\end{aligned} \qquad (33)$$

[6]The generation of Gaussian random variables is discussed in [20]–[22].

The resulting "basis-free" synthesis algorithm can now be summarized as follows.

1) Transform the TF model $\widetilde{W}(t,f)$ into the ACF domain according to (7).
2) Calculate the projected ACF model $\widetilde{R}_S(t_1,t_2)$ by means of (33).
3) Find the positive eigenvalues λ_k and the corresponding normalized eigenfunctions $u_k(t)$ of $\widetilde{R}_S(t_1,t_2)$ by solving the eigenproblem (32) for $1\le k\le N_+$.
4) The synthesis result $\widehat{x}(t)$ is then given by (25), with the statistics of $\widehat{c}_k$ specified by (23), (24), and (30).

B. Global Synthesis

Global synthesis, i.e., synthesis without a subspace constraint, can be considered as subspace-constrained synthesis with $\mathcal{S} = \mathcal{L}_2(\mathbb{R})$. The computationally expensive projection step (33) can here be omitted. Indeed, since the TF model $\widetilde{W}(t,f)$ was assumed square-integrable, $\widetilde{W}(t,f)\in\mathcal{L}_2(\mathbb{R}^2)$, the ACF model $\widetilde{R}(t_1,t_2)$ is square-integrable as well, $\widetilde{R}(t_1,t_2)\in\mathcal{L}_2(\mathbb{R}^2)$. But $\mathcal{L}_2(\mathbb{R}^2)$ can be shown to be the induced ACF-domain space associated with $\mathcal{S} = \mathcal{L}_2(\mathbb{R})$ i.e., there is $\mathcal{S}_R = \mathcal{L}_2(\mathbb{R}^2)$. Hence, we have $\widetilde{R}(t_1,t_2)\in\mathcal{S}_R$, from which it follows that $\widetilde{R}_S(t_1,t_2) = \widetilde{R}(t_1,t_2)$.

V. An Example

In this section, we consider the problem of constructing a process that is strictly bandlimited in a given frequency band $|f| < F/2$ and also optimally concentrated in a given time interval $|t| < T/2$. This problem can be formulated as subspace-constrained TF synthesis with a model $\widetilde{W}(t,f)$ that is 1 inside $|t| < T/2$ and 0 outside

$$\widetilde{W}(t,f) = r_T(t) \triangleq \begin{cases}1, & \text{for } |t|<T/2\\ 0, & \text{for } |t|>T/2\end{cases}$$

and a constraint signal space $\mathcal{S}$ that is the subspace $\mathcal{S}_F$ of all signals bandlimited in $|f| < F/2$. The synthesis problem is the minimization of

$$\begin{aligned}\epsilon_x^2 = \|\widetilde{W}-\overline{W}_x\|^2 &= \int_{|t|<T/2}\int_f[1-\overline{W}_x(t,f)]^2\,dt\,df\\ &\quad+\int_{|t|>T/2}\int_f[\overline{W}_x(t,f)]^2\,dt\,df\end{aligned}$$

subject to the subspace constraint $x(t)\in\mathcal{S}_F$. Note that the model $\widetilde{W}(t,f) = r_T(t)$ requires that the synthesized process $\widehat{x}(t)$ feature optimal time concentration in the time interval $|t| < T/2$ while the subspace constraint $x(t)\in\mathcal{S}_F$ forces $\hat{x}(t)$ to be strictly bandlimited in the frequency band $|f| < F/2$.

We shall solve the above synthesis problem by means of the basis-free method discussed in Section IV. The orthogonal projection operator $\boldsymbol{P}_{\mathcal{S}_F}$ on $\mathcal{S}_F$ is the bandlimitation operator on $|f| < F/2$. Hence, its kernel $P_{\mathcal{S}_F}(t,t')$ is the impulse response of the ideal lowpass filter with cutoff frequency $F/2$, i.e, $P_{\mathcal{S}_F}(t,t') = h_F(t-t')$ with $h_F(t) = F\,\mathrm{sinc}\,(Ft)$, where

$$\mathrm{sinc}\,(\alpha) = \frac{\sin(\pi\alpha)}{\pi\alpha}.$$

Evaluation of (33) yields the following expression for the projected ACF model $\tilde{R}_S(t_1, t_2)$:

$$\tilde{R}_S(t_1, t_2) = \int_t h_F(t_1 - t) r_T(t) h_F(t - t_2)\, dt.$$

It is easily shown that this is the kernel of the composite operator $\boldsymbol{P}_{\mathcal{S}_F}\boldsymbol{P}_{\mathcal{S}_T}\boldsymbol{P}_{\mathcal{S}_F}$, where $\boldsymbol{P}_{\mathcal{S}_F}$ is the bandlimitation operator on $|f| < F/2$ and $\boldsymbol{P}_{\mathcal{S}_T}$ is the time-limitation operator on $|t| < T/2$ (or, in other words, the orthogonal projection operator on the subspace $\mathcal{S}_T$ of all signals time-limited in $|t| < T/2$). Thus the eigenequation (32) defining the eigenvalues λ_k and eigenfunctions $u_k(t)$ reads

$$(\boldsymbol{P}_{\mathcal{S}_F}\boldsymbol{P}_{\mathcal{S}_T}\boldsymbol{P}_{\mathcal{S}_F} u_k)(t) = \lambda_k u_k(t).$$

The eigenfunctions $u_k(t)$ of the operator $\boldsymbol{P}_{\mathcal{S}_F}\boldsymbol{P}_{\mathcal{S}_T}\boldsymbol{P}_{\mathcal{S}_F}$ are obviously bandlimited in the band $|f| < F/2$, i.e., there is $\boldsymbol{P}_{\mathcal{S}_F} u_k = u_k$; hence, the eigenequation can be rewritten as

$$(\boldsymbol{P}_{\mathcal{S}_F}\boldsymbol{P}_{\mathcal{S}_T} u_k)(t) = \lambda_k u_k(t)$$

which shows that λ_k and $u_k(t)$ are also the eigenvalues and eigenfunctions, respectively, of the operator[7] $\boldsymbol{P}_{\mathcal{S}_F}\boldsymbol{P}_{\mathcal{S}_T}$. The eigenfunctions of $\boldsymbol{P}_{\mathcal{S}_F}\boldsymbol{P}_{\mathcal{S}_T}$ are known as the *prolate spheroidal functions* $\psi_k^{(T,F)}(t)$ [14]. Hence, we obtain

$$u_k(t) = \psi_k^{(T,F)}(t), \qquad 0 \le k < \infty.$$

The prolate spheroidal functions $\psi_k^{(T,F)}(t)$ arise in the classical problem of maximizing the time concentration of a bandlimited deterministic signal. Specifically, the zero-order prolate spheroidal function $\psi_0^{(T,F)}(t)$ is the signal strictly bandlimited in $|f| < F/2$ which assumes a maximum part of its energy in the time interval $|t| < T/2$. It is shown in [14] that the eigenvalues satisfy $0 < \lambda_k < 1$. Hence, the number of positive eigenvalues is $N_+ = \infty$, and the synthesis result can finally be written as

$$\hat{x}(t) = \sum_{k=0}^{\infty} \hat{c}_k \psi_k^{(T,F)}(t)$$

where the coefficients $\hat{c}_k$ are orthogonal with quadratic means $\mathcal{E}\{|\hat{c}_k|^2\} = \lambda_k < 1$.

VI. Discrete-Time Formulation

The synthesis algorithms derived in the previous sections must be reformulated in a discrete-time setting if they are to be implemented on a digital computer.

A. The Discrete-Time WVS

For a discrete-time process $x(n)$, the WVS is defined as

$$\overline{W}_x(n, \theta) \triangleq 2 \sum_m R_x(n + m, n - m) e^{-j4\pi\theta m} \qquad (34)$$

with the ACF $R_x(n_1, n_2) = \mathcal{E}\{x(n_1)x^*(n_2)\}$. Here, n and m are integer time indices and θ is a normalized frequency variable. With respect to θ, the WVS is periodic with period $1/2$; this is in contrast to the Fourier transform of a discrete-time signal whose period is 1. Indeed, the WVS (in analogy to the discrete-time Wigner distribution of a deterministic signal [23]) suffers from aliasing effects unless the process $x(n)$ is a "halfband process," i.e., bandlimited in a "halfband" $|\theta - \theta_0| < 1/4$ with bandwidth $1/2$ and given (application-dependent) center frequency θ_0. Note that, in particular, $\theta_0 = 0$ corresponds to a halfband containing (among other signals) all real-valued signals oversampled by a factor 2 while $\theta_0 = 1/4$ corresponds to analytic signals.

Using the notation of Section I, the "no aliasing" condition can be phrased as $x(n) \in \mathcal{H}^{(\theta_0)}$, where the "halfband subspace" $\mathcal{H}^{(\theta_0)}$ is the linear space of all halfband signals with center frequency θ_0 [11]. We note that a halfband process $x(n) \in \mathcal{H}^{(\theta_0)}$ is uniquely specified by, e.g., the even-indexed samples $x(2\nu)$ since the odd-indexed samples can be derived from the even-indexed samples by means of the interpolation

$$x(2\nu + 1) = 2 \sum_{\nu'} h(2(\nu - \nu') + 1) x(2\nu') \qquad (35)$$

where

$$h(n) = \frac{1}{2} \operatorname{sinc}\left(\frac{n}{2}\right) e^{j2\pi\theta_0 n} \qquad (36)$$

is the impulse response of the idealized halfband filter with center frequency θ_0. This fact will be utilized in the discrete-time synthesis algorithms described below.

B. Discrete-Time Synthesis

The discrete-time synthesis problem is formulated as

$$\hat{x}(n) \triangleq \arg\min_{x \in \mathcal{S}} \epsilon_x$$

where

$$\epsilon_x^2 = \|\widetilde{W} - \overline{W}_x\|^2 = \sum_n \int_{\theta_0 - 1/4}^{\theta_0 + 1/4} [\widetilde{W}(n, \theta) - \overline{W}_x(n, \theta)]^2 d\theta. \qquad (37)$$

Here, $\widetilde{W}(n, \theta)$ is the (real-valued) model function which is defined on the halfband $|\theta - \theta_0| < 1/4$. The constraint space $\mathcal{S}$ is assumed to be a subspace of the halfband space $\mathcal{H}^{(\theta_0)}$, i.e., $\mathcal{S} \subseteq \mathcal{H}^{(\theta_0)}$. This assumption assures that the WVS of the synthesis result $\hat{x}(n) \in \mathcal{S}$ is nonaliased; furthermore, it permits the synthesis algorithms developed in previous sections to be reformulated in the discrete-time setting considered here. Indeed, the synthesis algorithms of Sections II–IV are based on the unitary mapping relating the TF domain and the ACF domain (cf. (1), (2)). In the discrete-time case, the mapping between TF domain and ACF domain (as given by (34)) is unitary only for halfband processes $x(n) \in \mathcal{H}^{(\theta_0)}$ [11]. Note that the condition $\mathcal{S} \subseteq \mathcal{H}^{(\theta_0)}$ disallows global synthesis; the place of global synthesis is here taken by the extreme case of halfband-constrained synthesis where $\mathcal{S} = \mathcal{H}^{(\theta_0)}$.

In the following, we summarize (without proof) two discrete-time synthesis algorithms. These algorithms are

[7] Note that, even though the eigenvalues and eigenfunctions of the operators $\boldsymbol{P}_{\mathcal{S}_F}\boldsymbol{P}_{\mathcal{S}_T}\boldsymbol{P}_{\mathcal{S}_F}$ and $\boldsymbol{P}_{\mathcal{S}_F}\boldsymbol{P}_{\mathcal{S}_T}$ are identical, the operators themselves are quite different (e.g., $\boldsymbol{P}_{\mathcal{S}_F}\boldsymbol{P}_{\mathcal{S}_T}\boldsymbol{P}_{\mathcal{S}_F}$ is a self-adjoint operator whereas $\boldsymbol{P}_{\mathcal{S}_F}\boldsymbol{P}_{\mathcal{S}_T}$ is not even normal).

analogous to the continuous-time basis method and basis-free method discussed in previous sections, apart from two differences [11]:

- they comprise an initial "halfband projection" of the TF model which assures the unitarity of the TF-domain/ACF-domain mapping, and
- they feature an implicit decimation by a factor 2 for the sake of increased efficiency (this is possible since $\mathcal{S} \subseteq \mathcal{H}^{(\theta_0)}$ and since, as mentioned above, halfband signals are fully characterized by, e.g., the even-indexed signal samples).

C. Basis Method

The discrete-time version of the basis method can be summarized as follows.

1) The model $\widetilde{W}(n,\theta)$ is projected on the induced TF-domain subspace corresponding to the halfband subspace $\mathcal{H}^{(\theta_0)}$; this projection amounts to the convolution [11]

$$\widetilde{W}_H(n,\theta) = \sum_{n'} W_h(n-n',\theta)\widetilde{W}(n',\theta), \quad \text{for } |\theta-\theta_0| < 1/4 \tag{38}$$

where $W_h(n,\theta) = (1-4|\theta-\theta_0|)\,\mathrm{sinc}\,[(1-4|\theta-\theta_0|)n]$ is the discrete-time Wigner distribution [23] of the halfband-filter impulse response $h(n)$. This convolution is seen to be a lowpass filtering with respect to n, where the bandwidth of the lowpass filter is $(1-4|\theta-\theta_0|)/2$ and thus depends on θ.

2) The projected model $\widetilde{W}_H(n,\theta)$ is transformed into the ACF domain according to

$$\tilde{R}_H(2\nu_1,2\nu_2) = \int_{\theta_0-1/4}^{\theta_0+1/4} \widetilde{W}_H(\nu_1+\nu_2,\theta)e^{j4\pi(\nu_1-\nu_2)\theta}\,d\theta. \tag{39}$$

This yields the ACF model for even-indexed time indices.

3) The model's projection coefficients are calculated as

$$\gamma_{kl} = 4\sum_{\nu_1}\sum_{\nu_2} \tilde{R}_H(2\nu_1,2\nu_2)s_k^*(2\nu_1)s_l(2\nu_2) \tag{40}$$

where $\{s_k(n)\}_{k=1}^N$ is an orthonormal basis of $\mathcal{S}$.

4) The synthesis result is given by

$$\hat{x}(n) = \sum_{k=1}^{N} \hat{a}_k s_k(n) = \hat{\boldsymbol{a}}^T \boldsymbol{s}(n)$$

where the correlation matrix $\boldsymbol{R}_{\hat{a}}$ of the expansion coefficients $\hat{a}_k$ is the positive part of the projection coefficient matrix $\boldsymbol{\Gamma} = (\gamma_{kl})$, i.e., $\boldsymbol{R}_{\hat{a}} = \boldsymbol{\Gamma}_+$. Alternatively, the synthesis result $\hat{x}(n)$ can be represented in terms of its KL expansion by transforming the basis $\boldsymbol{s}(n)$ as detailed in Section III.

D. Basis-Free Method

The first two steps of the basis-free method are identical with the first two steps of the basis method, but the remaining steps are different:

1) The projection of the model $\widetilde{W}(n,\theta)$ on the induced TF-domain halfband space is calculated according to (38).
2) The projected model $\widetilde{W}_H(n,\theta)$ is transformed into the ACF domain by means of (39).
3) The ACF model $\tilde{R}_H(2\nu_1,2\nu_2)$ is projected according to

$$\tilde{R}_S(2\nu_1,2\nu_2) = 4\sum_{\nu_1'}\sum_{\nu_2'} P_{\mathcal{S}}(2\nu_1,2\nu_1') \cdot P_{\mathcal{S}}^*(2\nu_2,2\nu_2')\tilde{R}_H(2\nu_1',2\nu_2') \tag{41}$$

where $P_{\mathcal{S}}(n_1,n_2)$ is the kernel of the orthogonal projection operator on $\mathcal{S}$.

4) The positive eigenvalues λ_k and the corresponding eigenvectors $u_k(\nu)$ of $\tilde{R}_S(2\nu_1,2\nu_2)$ are found by solving the eigenproblem

$$\sum_{\nu_2} \tilde{R}_S(2\nu_1,2\nu_2)u_k(\nu_2) = \lambda_k u_k(\nu_1), \quad 1 \le k \le N_+.$$

5) The even-indexed samples of the synthesis result $\widehat{x}(n)$ are given (in the KL form) as

$$\hat{x}(2\nu) = \sum_{k=1}^{N_+} \widehat{c}_k u_k(\nu) = \widehat{\boldsymbol{c}}^T \boldsymbol{u}(\nu)$$

where the coefficients $\widehat{c}_k$ are orthogonal with quadratic means $\mathcal{E}\{|\widehat{c}_k|^2\} = 2\lambda_k$.

6) The odd-indexed samples of $\widehat{x}(n)$ are obtained via the interpolation (35).

In practice, the WVS is discretized also with respect to the frequency variable θ. Both the lowpass filtering (38) and the inverse Fourier transform (39) can then be performed efficiently by means of fast Fourier transform techniques. We note that suboptimal, reduced-cost algorithms are obtained simply by omitting the initial projection step (38); the resulting process will then still be an element of the subspace $\mathcal{S}$ but it will no longer minimize the synthesis error (37). However, experiments have shown that the difference between this suboptimal process and the optimal solution is typically not dramatic.

E. Halfband-Constrained Synthesis

If the constraint subspace equals the total halfband space, $\mathcal{S} = \mathcal{H}^{(\theta_0)}$, the projection step (40) of the basis method can be omitted. In fact, if the orthonormal basis spanning $\mathcal{S} = \mathcal{H}^{(\theta_0)}$ is chosen as $s_k(n) = \sqrt{2}h(n-2k)$ (cf. (36)), then there is directly $\gamma_{kl} = 2\tilde{R}_H(2k,2l)$. Similarly, in the basis-free method we can omit the projection (41) since $\tilde{R}_S(2\nu_1,2\nu_2) = \tilde{R}_H(2\nu_1,2\nu_2)$.

VII. Simulation Results

This section presents computer simulation results illustrating the performance and application of TF synthesis. The discrete-time synthesis algorithms discussed in the previous section

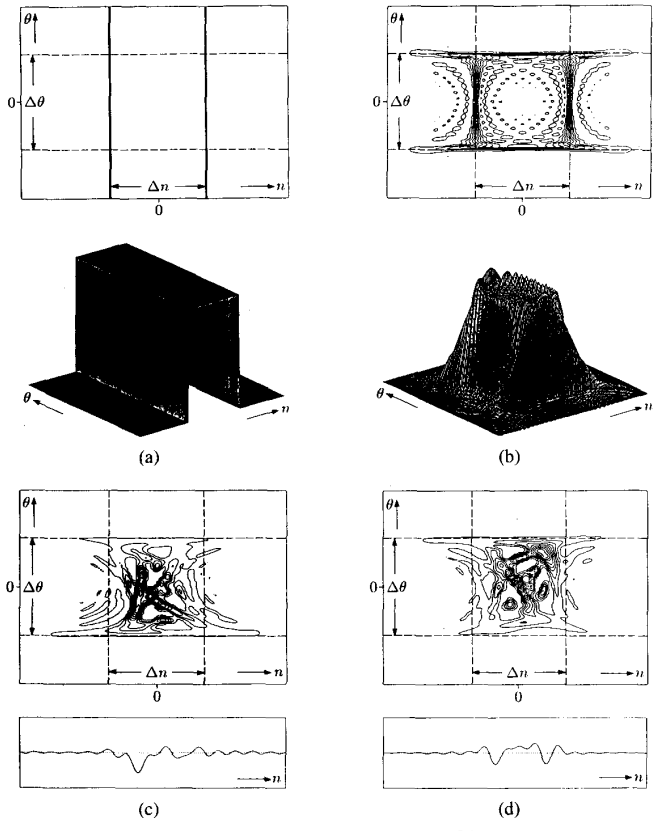

Fig. 3. Bandwidth-constrained synthesis from a time-limited plateau model. (a) TF model $\widetilde{W}(n,\theta)$ and frequency band corresponding to the constraint subspace $\mathcal{S}_F$. (b) WVS of the synthesized process $\hat{x}(n)$. (c) and (d) Real parts and smoothed Wigner distributions of two realizations of $\hat{x}(n)$.

were employed to obtain the KL eigenvalues and eigenfunctions of the process, from which individual process realizations (assuming zero mean and Gaussian distribution) were then generated as discussed in Section III.

Fig. 3 reconsiders the example discussed in Section V, namely, the synthesis of a process from a model that is strictly time-limited, under a subspace constraint enforcing strict bandlimitation of the process. The model $\widetilde{W}(n,\theta)$ and the frequency band corresponding to the constraint subspace $\mathcal{S}_F$ are shown in Fig. 3(a). In the discrete-time setting used for these computer simulations, the time-length parameter T corresponds to the discrete time length $\Delta n = 45$ and the bandwidth F corresponds to the normalized bandwidth $\Delta\theta = 1/4$, which is one half of the fundamental frequency period $1/2$. From the WVS of the synthesized process (see Fig. 3(b)) we verify that the process is indeed bandlimited in the frequency band corresponding to $\mathcal{S}_F$ (apart from small errors caused by the finite time support of all signals calculated) and well concentrated in the time interval defined by $\widetilde{W}(n,\theta)$. Two different realizations of the synthesized process (assuming zero mean and Gaussian distribution) are depicted in Fig. 3(c) and (d). Both the real parts of the signals and the signals' (smoothed) Wigner distributions are shown. It is seen that both realizations are strictly bandlimited in the correct frequency band and also well concentrated in the prescribed time interval.

The second example, shown in Fig. 4, illustrates the results of halfband-constrained, discrete-time synthesis for a model consisting of two plateaus with different heights, where the plateaus are nonoverlapping in the TF plane (see Fig. 4(a)). This is the TF model for a process consisting of two statistically orthogonal components, where the first component is concentrated inside an elliptic TF region and the second component is a burst of bandpass noise whose time-varying center frequency corresponds to a sinusoidal frequency modulation law. The WVS of the synthesized process, depicted in Fig. 4(b), is indeed quite similar to the model. Two realizations of the process are shown in Fig. 4(c) and (d); again, a zero-mean, normally distributed process was assumed. From the Wigner distributions of the realizations, it is verified that the realizations are well concentrated inside the model's TF support (the Wigner distribution components located outside the model's TF support are residual interference terms [4] which do not contain signal energy).

VIII. Conclusion

We have presented methods for the time–frequency (TF) synthesis of nonstationary random processes with finite energy. These methods allow a specification of the energetic (second-order) properties of a process in a joint TF domain. They are

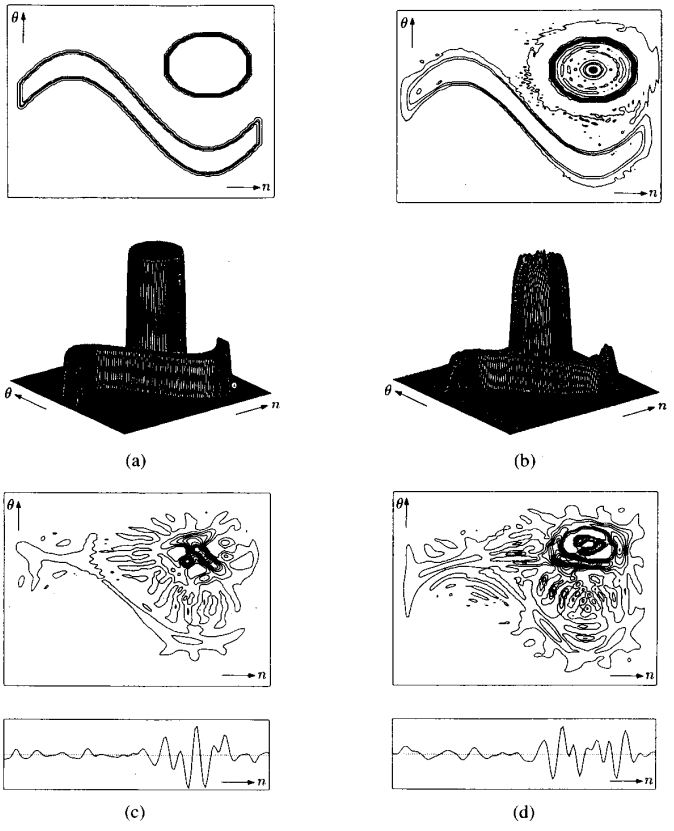

Fig. 4. Halfband-constrained synthesis from a two-component plateau model. (a) TF model $\widetilde{W}(n,\theta)$. (b) WVS of synthesized process $\hat{x}(n)$. (c) and (d) real parts and smoothed Wigner distributions of two realizations of $\hat{x}(n)$.

based on the energetic TF representation of processes by means of the Wigner–Ville spectrum (WVS). The autocorrelation function of the process is determined such that the WVS is closest to a specified TF energy distribution. The final result of this optimization is given by the Karhunen–Loève eigenvalues and eigenfunctions which allow an easy generation of realizations of the synthesized process.

A signal subspace constraint can be included which assures that the realizations of the synthesized process are elements of a prescribed linear signal subspace. By this, certain properties (e.g., bandlimitation) of the process can be enforced. Also, a particular signal subspace constraint (enforcing a "halfband limitation") allows the synthesis methods to be reformulated in a discrete-time setting.

It should be noted that the TF synthesis methods formulated here for the WVS can readily be generalized to other definitions of time-varying spectra or TF process representations provided that these are defined as the expectation of a *unitary* [19], [24] quadratic TF signal representation. Examples of unitary TF signal representations (other than the Wigner distribution underlying the WVS) are the Rihaczek distribution, ambiguity function, Altes–Marinovic Q-distribution, and Bertrand P_0 distribution [19], [24], [25]. The generalized synthesis methods are obtained by extending the results of this paper along the lines discussed in [10] for the deterministic case.

Appendix I
Positive Semidefinite Approximation of a Hermitian Matrix

The minimization problem to be solved (cf. (13)) is

$$\boldsymbol{R}_{\hat{a}} = \arg \min_{\mathbf{R}_a \geq \mathbf{0}} \|\boldsymbol{\Gamma} - \boldsymbol{R}_a\|_F^2 \tag{42}$$

where $\boldsymbol{\Gamma}$ is a Hermitian matrix and $\boldsymbol{R}_a$ is constrained to be positive semidefinite. Using the spectral representation of $\boldsymbol{R}_a$

$$\boldsymbol{R}_a = \sum_{k=1}^{N} \rho_k \boldsymbol{r}_k \boldsymbol{r}_k^H$$

where the eigenvalues ρ_k are nonnegative and the eigenvectors $\boldsymbol{r}_k$ are orthonormal, the squared error norm in (42) can be developed as

$$\begin{aligned}\|\boldsymbol{\Gamma} - \boldsymbol{R}_a\|_F^2 &= \|\boldsymbol{\Gamma}\|_F^2 + \|\boldsymbol{R}_a\|_F^2 - 2\langle \boldsymbol{\Gamma}, \boldsymbol{R}_a \rangle_F \\ &= \|\boldsymbol{\Gamma}\|_F^2 + \left\| \sum_{k=1}^{N} \rho_k \boldsymbol{r}_k \boldsymbol{r}_k^H \right\|_F^2 \\ &\quad - 2 \sum_{k=1}^{N} \rho_k \langle \boldsymbol{\Gamma}, \boldsymbol{r}_k \boldsymbol{r}_k^H \rangle_F \\ &= \|\boldsymbol{\Gamma}\|_F^2 + \sum_{k=1}^{N} \rho_k^2 - 2 \sum_{k=1}^{N} \rho_k \boldsymbol{r}_k^H \boldsymbol{\Gamma} \boldsymbol{r}_k \end{aligned} \tag{43}$$

where

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle_F = \sum_{i=1}^{N} \sum_{j=1}^{N} \boldsymbol{A}_{ij} \boldsymbol{B}_{ij}^*.$$

We first carry out the optimization with respect to the eigenvectors $\boldsymbol{r}_k$. For this, we have to maximize the last term of (43) subject to the normalization constraints[8] $\|\boldsymbol{r}_k\| = 1$ or, equivalently, $1 - \|\boldsymbol{r}_k\|^2 = 0$. Incorporating these normalization constraints via Lagrangian multipliers μ_k, the quantity to be maximized is

$$\sigma_r \triangleq \sum_{k=1}^{N} \rho_k \boldsymbol{r}_k^H \boldsymbol{\Gamma} \boldsymbol{r}_k + \sum_{k=1}^{N} \mu_k \left(1 - \|\boldsymbol{r}_k\|^2\right).$$

Setting the gradient of σ_r with respect to the ith eigenvector $\boldsymbol{r}_i$ equal to zero yields the following set of necessary conditions:

$$\boldsymbol{\Gamma} \boldsymbol{r}_i = \frac{\mu_i}{\rho_i} \boldsymbol{r}_i.$$

This shows that the $\boldsymbol{r}_i$ must be eigenvectors of $\boldsymbol{\Gamma}$. Denoting the eigenvectors of $\boldsymbol{\Gamma}$ by $\boldsymbol{v}_k$, the optimal correlation matrix $\boldsymbol{R}_{\hat{a}}$ can hence be written as

$$\boldsymbol{R}_{\hat{a}} = \sum_{k=1}^{N} \rho_k \boldsymbol{v}_k \boldsymbol{v}_k^H. \tag{44}$$

We next determine the optimal eigenvalues ρ_k of $\boldsymbol{R}_{\hat{a}}$. To this end, we insert (44) and the spectral representation of $\boldsymbol{\Gamma}$, (18), into the error in (42). Using the orthonormality of the eigenvectors $\boldsymbol{v}_k$ of $\boldsymbol{\Gamma}$, we obtain

$$\|\boldsymbol{\Gamma} - \boldsymbol{R}_{\hat{a}}\|_F^2 = \left\| \sum_{k=1}^{N} (\lambda_k - \rho_k) \boldsymbol{v}_k \boldsymbol{v}_k^H \right\|_F^2 = \sum_{k=1}^{N} (\lambda_k - \rho_k)^2 \tag{45}$$

where the λ_k are the (real-valued) eigenvalues of $\boldsymbol{\Gamma}$. The error (45) has to be minimized subject to the constraints $\rho_k \geq 0$ expressing the positive semidefiniteness of $\boldsymbol{R}_{\hat{a}}$. Obviously, the solution is given by

$$\rho_k = \begin{cases} \lambda_k, & \text{for} \quad 1 \leq k \leq N_+ \\ 0, & \text{for} \quad N_+ + 1 \leq k \leq N \end{cases} \tag{46}$$

where N_+ is the number of positive eigenvalues λ_k (it has been assumed that the eigenvalues λ_k are arranged such that $\lambda_k > 0$ for $1 \leq k \leq N_+$). With (46), the optimal correlation matrix $\boldsymbol{R}_{\hat{a}}$ is finally obtained as

$$\boldsymbol{R}_{\hat{a}} = \sum_{k=1}^{N_+} \lambda_k \boldsymbol{v}_k \boldsymbol{v}_k^H = \boldsymbol{\Gamma}_+$$

which is recognized as the positive part of $\boldsymbol{\Gamma}$.

[8] The orthogonality of the eigenvectors $\boldsymbol{r}_k$ need not be incorporated via an explicit constraint since, as will be seen presently, it will be given automatically.

APPENDIX II
ACF-COMPATIBILITY OF THE MEAN

Positive semidefiniteness of the autocovariance $C_{\hat{x}}(t_1, t_2)$ in (29) means that

$$Q_y \triangleq \int_{t_1} \int_{t_2} C_{\hat{x}}(t_1, t_2) y^*(t_1) y(t_2) dt_1 dt_2 \geq 0, \quad \text{for all } y(t).$$

In order to show that the condition (30) is necessary and sufficient for positive semidefiniteness of $C_{\hat{x}}(t_1, t_2)$, we insert (28) and (26) into (29), whereby, after straightforward manipulations, the following expression for the quadratic form Q_y is obtained:

$$Q_y = \sum_{k=1}^{N_+} \sum_{l=1}^{N_+} [\lambda_k \delta_{kl} - m_{\hat{c}}(k) m_{\hat{c}}^*(l)] \langle y, u_k \rangle^* \langle y, u_l \rangle.$$

Introducing the diagonal matrix $\boldsymbol{\Lambda}_+$ and the vectors $\boldsymbol{m}$ and $\boldsymbol{y}$ according to[9]

$$\begin{aligned} (\boldsymbol{\Lambda}_+)_{kl} &= \lambda_k \delta_{kl}, \quad 1 \leq k, l \leq N_+ \\ (\boldsymbol{m})_k &= m_{\hat{c}}(k) = \langle m_{\hat{x}}, u_k \rangle \\ (\boldsymbol{y})_k &= \langle y, u_k \rangle, \quad 1 \leq k \leq N_+ \end{aligned} \tag{47}$$

we can write Q_y as

$$Q_y = \boldsymbol{y}^H (\boldsymbol{\Lambda}_+ - \boldsymbol{m}\boldsymbol{m}^H) \boldsymbol{y}. \tag{48}$$

Since $\boldsymbol{\Lambda}_+$ is positive definite, it allows the factorization $\boldsymbol{\Lambda}_+ = \boldsymbol{\Lambda}_+^{1/2} \boldsymbol{\Lambda}_+^{1/2}$ with the matrix square root $\boldsymbol{\Lambda}_+^{1/2}$ given by $(\boldsymbol{\Lambda}_+^{1/2})_{kl} = \sqrt{\lambda_k} \delta_{kl}$. The inverse $\boldsymbol{\Lambda}_+^{-1/2}$ of $\boldsymbol{\Lambda}_+^{1/2}$ exists and is given by $(\boldsymbol{\Lambda}_+^{-1/2})_{kl} = (1/\sqrt{\lambda_k}) \delta_{kl}$. We can now reformulate (48) as

$$\begin{aligned} Q_y &= \boldsymbol{y}^H (\boldsymbol{\Lambda}_+ - \boldsymbol{m}\boldsymbol{m}^H) \boldsymbol{y} \\ &= \boldsymbol{y}^H \boldsymbol{\Lambda}_+^{1/2} (\boldsymbol{I} - \boldsymbol{\Lambda}_+^{-1/2} \boldsymbol{m}\boldsymbol{m}^H \boldsymbol{\Lambda}_+^{-1/2}) \boldsymbol{\Lambda}_+^{1/2} \boldsymbol{y} \\ &= \tilde{\boldsymbol{y}}^H (\boldsymbol{I} - \tilde{\boldsymbol{m}}\tilde{\boldsymbol{m}}^H) \tilde{\boldsymbol{y}} \\ &= \|\tilde{\boldsymbol{y}}\|_F^2 - |\tilde{\boldsymbol{m}}^H \tilde{\boldsymbol{y}}|^2 \end{aligned}$$

where

$$\tilde{\boldsymbol{y}} \triangleq \boldsymbol{\Lambda}_+^{1/2} \boldsymbol{y} \quad \text{and} \quad \tilde{\boldsymbol{m}} \triangleq \boldsymbol{\Lambda}_+^{-1/2} \boldsymbol{m}. \tag{49}$$

Applying Schwarz' inequality to the inner product $\tilde{\boldsymbol{m}}^H \tilde{\boldsymbol{y}}$, we obtain

$$\begin{aligned} Q_y &= \|\tilde{\boldsymbol{y}}\|_F^2 - |\tilde{\boldsymbol{m}}^H \tilde{\boldsymbol{y}}|^2 \geq \|\tilde{\boldsymbol{y}}\|_F^2 - \|\tilde{\boldsymbol{m}}\|_F^2 \|\tilde{y}\|_F^2 \\ &= \|\tilde{\boldsymbol{y}}\|_F^2 (1 - \|\tilde{\boldsymbol{m}}\|_F^2) \end{aligned}$$

which shows that

$$\|\tilde{\boldsymbol{m}}\|_F^2 \leq 1 \tag{50}$$

is a sufficient condition for $Q_y \geq 0$. On the other hand, we may choose a special $y(t)$ such that $\tilde{\boldsymbol{y}} = \tilde{\boldsymbol{m}}$, which gives

$$\begin{aligned} Q_y &= \|\tilde{\boldsymbol{y}}\|_F^2 - |\tilde{\boldsymbol{m}}^H \tilde{\boldsymbol{y}}|^2 = \|\tilde{\boldsymbol{m}}\|_F^2 - \|\tilde{\boldsymbol{m}}\|_F^4 \\ &= \|\tilde{\boldsymbol{m}}\|_F^2 (1 - \|\tilde{\boldsymbol{m}}\|_F^2). \end{aligned}$$

This shows that $\|\tilde{\boldsymbol{m}}\|_F^2 \leq 1$ is also a necessary condition for $Q_y \geq 0$.

[9] Note that $\boldsymbol{\Lambda}_+$ is different from the $\boldsymbol{\Lambda}_+$ defined by (15) in that the zero diagonal elements are left out and the matrix size is reduced accordingly.

With (47) and (49), the elements of $\tilde{\boldsymbol{m}}$ are $(\tilde{\boldsymbol{m}})_k = m_{\hat{c}}(k)/\sqrt{\lambda_k}, 1 \le k \le N_+$; hence, the necessary and sufficient condition (50) becomes

$$\sum_{k=1}^{N_+} \frac{|m_{\hat{c}}(k)|^2}{\lambda_k} \le 1$$

which is the condition (30) to be proved.

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