Time-Frequency Projection Filters and Time-Frequency Signal Expansions

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***Abstract*—We consider the problems of designing a linear, time-varying filter with a specified "time-frequency (TF) pass region" and of constructing an orthonormal basis for the parsimonious expansion of signals located in a given TF support region. These problems of *TF filtering* and *TF signal expansion* are reduced to the problem of designing a "TF subspace," i.e., a linear signal space comprising all signals located in a given TF region. Specifically, the TF filter is taken to be the orthogonal projection operator on the TF subspace. We present an optimum design of TF subspaces that is based on the *Wigner distribution of a linear signal space* that was recently introduced and is an extension of the well-known signal synthesis problem. The optimum TF subspace is shown to be an "eigenspace" of the TF region, and some properties of eigenspaces are discussed. The performance of TF projection filters and TF signal expansions is studied both analytically and via computer simulation.**

I. Introduction

IT IS often desirable to filter a signal contaminated by some interfering signal (e.g., noise) or to expand a signal into an orthonormal basis such that a minimum number of expansion coefficients are required. Appropriate solutions to these two problems depend on the signal model and the prior knowledge available.

A. Time-Frequency Filtering and Time-Frequency Expansions

In many situations involving nonstationary signals, it is advantageous to display a signal over a joint time-frequency (TF) plane using a TF signal representation [1]. In this paper, therefore, we shall assume that prior knowledge about the signal's *TF support* is available. We define the TF support of a signal $x(t)$ as the effective support of the signal's *Wigner distribution* (WD)[1] [1]–[3]

$$W_x(t,f) = \int_\tau x\left(t+\frac{\tau}{2}\right)x^*\left(t-\frac{\tau}{2}\right)e^{-j2\pi f\tau}\,d\tau \qquad (1)$$

(all integrals go from $-\infty$ to ∞ unless specified otherwise). The WD is a TF signal representation with particularly interesting properties. It is a real-valued function of time t and frequency f that can be interpreted (with some restrictions due to the uncertainty principle) as the signal's TF energy distribution [4], [3]. If the signal $x(t)$ is a (generally nonstationary) random process, then its TF support is defined as the effective support of the expected WD (which is known as the *Wigner-Ville spectrum*) [5], [6].

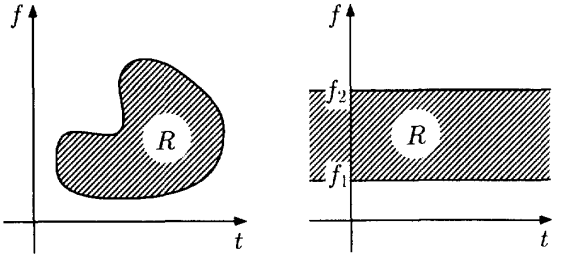

Fig. 1. Time-frequency pass region: (a) General; (b) corresponding to frequency band $B = [f_1, f_2]$.

The two problems considered in this paper are formally stated as follows:

- *TF filtering*: the construction of a filter with given "TF pass region" R (see Fig. 1(a)), i.e., a filter that passes all signals located inside[2] R but suppresses all signals located outside R.
- *TF expansion*: the construction of an orthonormal basis allowing the parsimonious expansion of all signals located in a given TF region R. The expansion coefficients represent the signal and can be used for further signal processing.

B. Time-Frequency Subspaces

As a motivation for our approach to solve both the TF filtering and TF expansion problems, we first assume that the joint TF localization is replaced by a pure frequency localization, i.e., the TF pass or support region R is formally replaced by a frequency interval (band) $B = [f_1, f_2]$. This is a special case of the situation considered previously. The TF region R now is an infinite strip running parallel to the time axis $R = (-\infty, \infty) \times B$ (see Fig. 1(b)). The theoretically appropriate solution to the filter problem is simply

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[1] This requires careful distinction between WD "signal terms" and WD "interference terms" [3]. By definition, the interference terms are not part of the TF support as they do not contain signal energy. In practice, the effective support of the WD, with the WD's interference terms essentially suppressed, can be estimated by calculating a smoothed WD (e.g., a spectrogram) and taking all TF points for which this smoothed WD is above a predefined threshold.

[2] We shall say that a signal is inside (outside) a given TF region R if the signal's TF support, which has been defined above, is inside (outside) R. Note that the TF support was defined as the *effective* support of the signal's WD; the WD may never be *strictly* contained inside a finite region R.

an idealized bandpass filter, i.e., a linear, time-invariant filter with frequency response $H(f) = 1$ for $f \in B$ and $H(f) = 0$ for $f \notin B$. Similarly, an appropriate signal basis is any orthonormal basis spanning the linear space $\mathcal{B}$ of all signals bandlimited in B (e.g., the basis of appropriately scaled and shifted sinc functions or a basis of prolate spheroidal wave functions [7], [8]). The two solutions are closely related since the idealized band-pass filter with pass-band B is the *orthogonal projection operator* on the linear space $\mathcal{B}$ of all signals bandlimited in B [9], [10]. *Thus, the linear signal space $\mathcal{B}$ provides the solution to both the TF filtering problem and the TF expansion problem.* Note that the space $\mathcal{B}$ fills out the TF region R energetically (all signals located inside R are elements of $\mathcal{B}$), but it does not have any energy outside R (signals outside R are orthogonal on $\mathcal{B}$).

Let us now return to a general TF region R as shown in Fig. 1(a). With the conceptual background developed above, we are able to formulate a unified approach to the solution of the TF filtering and TF signal expansion problems:

- We construct a "TF subspace" $\mathcal{S}_R$ corresponding to the TF region R, i.e., a linear signal space that fills out the TF region R energetically but has little or no energy outside R. Loosely speaking, $\mathcal{S}_R$ is "the linear space of all signals located inside the TF region R" [11], [12].
- The *TF filter* is the orthogonal projection operator on the TF subspace $\mathcal{S}_R$. This "TF projection filter" [12], [13] is a linear, generally time-varying system.
- The basis used for the *TF expansion* is any orthonormal basis spanning the TF subspace $\mathcal{S}_R$.

Thus, the problems of TF filtering and TF signal expansion have been reduced to the problem of designing a "TF subspace" $\mathcal{S}_R$ corresponding to a given TF region R. We would like to perform this design in an optimum manner. The definition of a meaningful optimality criterion obviously requires a means for characterizing the TF energy distribution (or TF localization) of a linear signal space. For this purpose, we shall use the *WD of a linear signal space* introduced in [14]. This results in a truly *joint TF design* of TF subspaces, i.e., a design whose optimality criterion is formulated in the TF plane.

The TF projection filter defined above is a linear, generally time-varying system that corresponds to an orthogonal projection operator. Linearity is obviously desirable, and the time-varying nature is dictated by the general shape[3] of the TF pass region R. The orthogonal projection structure corresponds to the fact that the filter is supposed to *pass* signals in some TF region and to *reject* signals in the rest of the TF plane. This structure then follows from two simple assumptions:

- The filter's output signal (if nonzero at all) is inside the TF pass region R. Thus, if it is passed once again through the filter, it should not be changed any more. This requires that the linear operator be idempotent, i.e., a *projection operator* [10].
- The part of the input signal $x(t)$ that is passed by the filter $\mathbf{H}$, i.e., the output signal $(\mathbf{H}x)(t)$, and the part that is rejected, i.e., the "error signal" $x(t) - (\mathbf{H}x)(t)$, are approximately TF disjoint. It follows from Moyal's formula [2] that two strictly TF disjoint signals are orthogonal. Orthogonality of $(\mathbf{H}x)(t)$ and $x(t) - (\mathbf{H}x)(t)$ requires that the projection operator be *orthogonal* [10].

[3] A linear time-*in*variant filter would only allow TF pass regions consisting of strips as in Fig. 1(b).

C. Related Work

Various schemes for *TF filtering* have been proposed previously [15]. A conceptually simple method consists of a masking of the signal's WD followed by signal synthesis [16]–[19] and results in a highly nonlinear overall filter whose performance has been shown to be potentially poor [20], [15], [21]. Replacing the WD by a linear TF representation (such as the short-time Fourier transform [22]–[24], the Gabor expansion [25], or the wavelet transform [26]) results in a linear filter whose performance is often satisfactory but is influenced by the window or wavelet used and is also restricted by a TF resolution tradeoff [15]. Other linear TF filter designs are based on the *Weyl symbol* [27], [13], [15] or the *WD of a linear system* [28]. Although a quantitative assessment of performance is difficult due to the uncertainty principle (specifically, due to the fact that no signal may be exactly contained in a finite TF region), the performance of the TF projection filter proposed here was generally observed in simulation studies to be as good as, or better than, the performance of other TF filtering methods [15].

A classical method for the construction of *TF subspaces* and *TF expansions* is based on the prolate spheroidal wave functions [7], [8]. The underlying TF regions are here restricted to rectangular shapes. A mathematical operator framework of TF-concentrated basis systems has been introduced in [29], however, without an explicit method for constructing the relevant operator for a given TF region. Finally, it is clear that any set of TF-concentrated functions that are "sufficiently dense" in a given TF region can be used as a (generally nonorthogonal) basis of a TF subspace. This includes sets of Gabor logons [29], [30] or wavelet functions. This approach, however, has certain drawbacks due to the limited concentration of the functions used and the necessity of orthogonalizing the set of functions.

D. Survey of Paper

The remaining sections are organized as follows. Section II reviews the TF analysis of linear signal spaces using the *WD of a linear signal space* and introduces the *TF localization error* as a quantity characterizing a space's TF localization. Section III formulates the optimum design of TF subspaces as a minimization of the TF localization error. The solution to this minimization problem is shown to be an "eigenspace" of the TF region. Section IV studies interesting properties of the eigenvalues, eigensignals, and eigenspaces of a TF region. Section V investigates the performance of TF projection filters and TF signal expansions with respect to passing/rejecting signals and noise. Finally, in Section VI, the method's discrete-time implementation is considered, and simulation results are presented.

II. Time-Frequency Localization of Signal Spaces

In this section, the TF representation of linear signal spaces using the *WD of a linear signal space* is reviewed, and two related quantities measuring a signal space's TF localization are proposed.

A *linear signal space* $\mathcal{S}$ is a collection of signals $s(t)$ such that any linear combination $c_1 s_1(t) + c_2 s_2(t)$ of two elements $s_1(t) \in \mathcal{S}$ and $s_2(t) \in \mathcal{S}$ is again an element of $\mathcal{S}$ [9], [10]. In this paper, we shall consider subspaces of the space $\mathcal{L}_2(\mathbb{R})$ of finite-energy signals so that $\mathcal{S}$ is equipped with inner product $\langle s_1, s_2 \rangle = \int_t s_1(t)\, s_2^*(t)\, dt$ and norm $\|s\| = \langle s, s \rangle^{1/2} = \left[\int_t |s(t)|^2\, dt\right]^{1/2}$.

The *orthogonal projection* $x_{\mathcal{S}}(t) \in \mathcal{S}$ of a signal $x(t) \in \mathcal{L}_2(\mathbb{R})$ on $\mathcal{S}$ can be written as

$$x_{\mathcal{S}}(t) = (\mathbf{S}x)(t) = \int_{t'} S(t,t')x(t')dt' = \sum_{k=1}^{N_{\mathcal{S}}} \langle x, s_k \rangle s_k(t) \tag{2}$$

where

$\mathbf{S}$	orthogonal projection operator of $\mathcal{S}$
$S(t,t')$	its kernel
$\{s_k(t)\}_{k=1}^{N_{\mathcal{S}}}$	orthonormal basis of $\mathcal{S}$
$N_{\mathcal{S}}$	dimension of $\mathcal{S}$.

The orthogonal projection operator can be interpreted as a linear, time-varying system (filter). It is idempotent ($\mathbf{S}^2 = \mathbf{S}$) and self-adjoint ($\mathbf{S}^+ = \mathbf{S}$, where $\mathbf{S}^+$ denotes the adjoint of $\mathbf{S}$) and can be expressed in terms of any orthonormal basis $\{s_k(t)\}_{k=1}^{N_{\mathcal{S}}}$ of $\mathcal{S}$ as

$$S(t,t') = \sum_{k=1}^{N_{\mathcal{S}}} s_k(t)\, s_k^*(t'). \tag{3}$$

A. The WD of a Linear Signal Space

The *WD of a linear signal space* $\mathcal{S}$ is defined by averaging the WD of a signal over all elements of the space [14]. It can be expressed as the sum of the WD's of all orthonormal basis signals $s_k(t)$

$$W_{\mathcal{S}}(t,f) = \sum_{k=1}^{N_{\mathcal{S}}} W_{s_k}(t,f). \tag{4}$$

Although the orthonormal basis of a space is not unique, $W_{\mathcal{S}}(t,f)$ is independent of the particular basis used in the above expression. Indeed, it follows with (1) and (3) that

$$W_{\mathcal{S}}(t,f) = \int_\tau S\Big(t + \frac{\tau}{2}, t - \frac{\tau}{2}\Big) e^{-j2\pi f\tau} d\tau = L_S(t,f) \tag{5}$$

which is the *Weyl symbol* $L_S(t,f)$ [27], [31]–[33] of the projection operator $\mathbf{S}$, and evidently independent of a basis.

The WD of a linear signal space describes the space's TF energy distribution in a similar manner as the WD of a single signal describes the signal's TF energy distribution. In particular

$$\int_t \int_f W_{\mathcal{S}}(t,f)dt\, df = N_{\mathcal{S}} \tag{6}$$

i.e., the integral of the WD over the entire TF plane equals the space's dimension that can be interpreted as the space's *energy*. In [14], several further interesting properties of the WD of a space are discussed, and some special spaces are considered. Specifically, the WD of the space $\mathcal{B}$ of all signals bandlimited in a frequency band $B = [f_1, f_2]$, which has been considered in Section I-B, is one in the strip $R = (-\infty, \infty) \times B$ and zero outside R (cf. Fig. 1(b)). Thus, the WD shows that the space $\mathcal{B}$ fills out R with energy but does not have any energy outside R.

B. The Concentration

We may ask how well a linear signal space is concentrated in a given TF region R, i.e., what part of the space's energy is contained in R. Since the total energy of a linear signal space $\mathcal{S}$ equals the integral of $W_{\mathcal{S}}(t,f)$ over the entire TF plane according to (6), it is natural to define the space's "energy content in R" as the integral of $W_{\mathcal{S}}(t,f)$ over the TF region R

$$E_{\mathcal{S}}^{(R)} \triangleq \int\int_{(t,f)\in R} W_{\mathcal{S}}(t,f)dt\, df.$$

Furthermore,[4] we define the *concentration* $\rho(\mathcal{S},R)$ *of a space* $\mathcal{S}$ *in the TF region* R as the ratio of the regional energy $E_{\mathcal{S}}^{(R)}$ and the total energy (dimension) $N_{\mathcal{S}}$ [11]

$$\rho(\mathcal{S},R) \triangleq \frac{E_{\mathcal{S}}^{(R)}}{N_{\mathcal{S}}} = \frac{\int\int_{(t,f)\in R} W_{\mathcal{S}}(t,f)dt\, df}{\int_t \int_f W_{\mathcal{S}}(t,f)dt\, df}.$$

The concentration can be expressed as the normalized inner product

$$\begin{aligned}\rho(\mathcal{S},R) &= \frac{1}{N_{\mathcal{S}}}\langle W_{\mathcal{S}}, I_R \rangle \\ &= \frac{1}{N_{\mathcal{S}}} \int_t \int_f W_{\mathcal{S}}(t,f)\, I_R(t,f)dt\, df\end{aligned}$$

with the indicator function (or "characteristic function") of the TF region R

$$I_R(t,f) \triangleq \begin{cases} 1, & (t,f) \in R \\ 0, & (t,f) \notin R. \end{cases}$$

An upper bound on the concentration $\rho(\mathcal{S},R)$ is provided by the *concentration inequality* [12]

$$\left|\rho(\mathcal{S},R)\right| \le \sqrt{\frac{A_R}{N_{\mathcal{S}}}} \tag{7}$$

where

$$\begin{aligned}A_R &\triangleq \int\int_{(t,f)\in R} dt\, df \\ &= \int_t \int_f I_R(t,f)dt\, df = \int_t \int_f [I_R(t,f)]^2\, dt\, df\end{aligned} \tag{8}$$

is the area of the TF region R. From the concentration inequality, it follows that *good TF concentration of a space,* i.e., $\rho(\mathcal{S},R) \approx 1$, *is possible only if the space's dimension is*

[4] Note, however, that $E_{\mathcal{S}}^{(R)}$ is not guaranteed to be positive or smaller than the total energy $N_{\mathcal{S}}$.

not larger than the TF region's area. In fact, as the dimension of a space grows, the space's TF support grows as well; for $N_S > A_R$, the space will "spill over" the TF region R, and the concentration will be poor.

C. The Localization Error

We recall the basic problem considered in this paper. Given a TF region R, we want to find a linear signal space $\mathcal{S}_R$ corresponding to R in the sense that it fills out R energetically while having little or no energy outside R. Naively speaking, we would like the WD of the space $\mathcal{S}_R$ to be one inside R and zero outside R. The deviation from this desired idealized behavior is characterized by the *localization error* $\epsilon(\mathcal{S}, R)$ *of the space* $\mathcal{S}$ *with respect to the TF region* R [12]

$$\epsilon^2(\mathcal{S}, R) \triangleq \int\int_{(t,f)\in R} [1 - W_{\mathcal{S}}(t,f)]^2 dt\, df + \int\int_{(t,f)\notin R} [0 - W_{\mathcal{S}}(t,f)]^2 dt\, df.$$

Using the indicator function $I_R(t,f)$, the squared localization error may be compactly written as

$$\epsilon^2(\mathcal{S}, R) = \|I_R - W_{\mathcal{S}}\|^2 = \int_t\int_f [I_R(t,f) - W_{\mathcal{S}}(t,f)]^2 dt\, df.$$

It is easily shown that the concentration $\rho(\mathcal{S}, R)$ and the localization error $\epsilon(\mathcal{S}, R)$ are related as

$$\epsilon^2(\mathcal{S}, R) = A_R + N_{\mathcal{S}}\,[1 - 2\,\rho(\mathcal{S}, R)]. \tag{9}$$

Combining (7) and (9), it follows that the localization error is bounded from below according to

$$\epsilon(\mathcal{S}, R) \geq \left|\sqrt{N_{\mathcal{S}}} - \sqrt{A_R}\right|. \tag{10}$$

This *localization inequality* [12] shows that *good localization of a space in a TF region (i.e., small localization error) is possible only if the space's dimension is approximately equal to the region's area.*

Although the concentration $\rho(\mathcal{S}, R)$ and the localization error $\epsilon(\mathcal{S}, R)$ are related according to (9), the concentration is sensitive only to a "spilling over" of space energy outside R, whereas the localization error is sensitive both to a spilling over of energy outside R *and* to energy gaps inside R.

III. Optimum Design of Time-Frequency Subspaces

Using the concepts developed in the previous section, the definition of meaningful optimization criteria for the design of a TF subspace is now straightforward. We want the TF subspace $\mathcal{S}_R$ to be optimally localized in the given TF region R in the sense that the space's WD is closest to one inside R and to zero outside R. According to this *criterion of minimum localization error* (MLE), the optimum TF subspace is defined as

$$\mathcal{S}_{R,\mathrm{MLE}} \triangleq \arg\min_{\mathcal{S}} \epsilon(\mathcal{S}, R) = \arg\min_{\mathcal{S}} \|I_R - W_{\mathcal{S}}\|.$$

An alternative, intuitively appealing optimality criterion is the *criterion of maximum concentration* (MC). However, the concentration is insensitive to energy gaps inside R. Indeed, the concentration inequality (7) shows that good concentration is favored by small dimension of the signal space. It will be seen later that the MC criterion always results in a 1-D subspace, which is generally not a desired result (according to the localization inequality (10), the space dimension should be approximately equal to the TF region's area). Therefore, the MC criterion is meaningful only if the space's dimension is fixed beforehand, $N_{\mathcal{S}} = N_0$, and we define the MC subspace *of given dimension* N_0 as

$$\mathcal{S}_{R,\mathrm{MC}}(N_0) \triangleq \arg\max_{N_{\mathcal{S}}=N_0} \rho(\mathcal{S}, R) = \arg\max_{N_{\mathcal{S}}=N_0} \langle W_{\mathcal{S}}, I_R\rangle.$$

Note that this MC criterion leaves the dimension $N_{\mathcal{S}} = N_0$ to be chosen, whereas the MLE criterion yields an optimum dimension.

From (9), it follows that for prescribed (i.e., fixed) dimension of the space $N_{\mathcal{S}} = N_0$, minimization of the localization error is equivalent to maximization of the concentration

$$\mathcal{S}_{R,\mathrm{MLE}}(N_0) = \mathcal{S}_{R,\mathrm{MC}}(N_0).$$

Here, $\mathcal{S}_{R,\mathrm{MLE}}(N_0)$ denotes the space with minimum localization error subject to a given space dimension N_0. Note, however, that we normally use the MLE criterion without fixing the dimension beforehand.

A. Eigenspaces of a Time-Frequency Region

The calculation of the MLE and MC subspaces is based on a fundamental decomposition of the indicator function $I_R(t,f)$ expressed by the following theorem[5] [34], [11], [32].

Theorem 1: The indicator function of a finite TF region R can be written as a linear combination of the WD's of orthonormal signals $u_k(t)$

$$I_R(t,f) = \sum_{k=1}^{\infty} \lambda_k W_{u_k}(t,f) \qquad \text{with } \lambda_k \in \mathbb{R}. \tag{11}$$

The real-valued expansion coefficients λ_k and the orthonormal signals $u_k(t)$ are the solutions to the eigenequation

$$\int_{t_2} H_R(t_1,t_2) u_k(t_2) dt_2 = \lambda_k u_k(t_1) \tag{12}$$

where the Hermitian kernel $H_R(t_1,t_2)$ is derived from the indicator function $I_R(t,f)$ as

$$H_R(t_1,t_2) = \int_f I_R\left(\frac{t_1+t_2}{2}, f\right) e^{j2\pi(t_1-t_2)f} df. \tag{13}$$

[5] We note that Theorem 1 remains valid if the indicator function $I_R(t,f)$ is replaced by an arbitrary real-valued, square-integrable TF function $\widetilde{W}(t,f)$; however, with a general $\widetilde{W}(t,f)$, there is no simple association to a TF region.

Proof: The indicator function $I_R(t,f)$ can be written as the Weyl symbol (cf. (5)) of some function $H_R(t_1,t_2)$

$$I_R(t,f) = L_{H_R}(t,f) = \int_\tau H_R\Big(t+\frac{\tau}{2}, t-\frac{\tau}{2}\Big)e^{-j2\pi f\tau}d\tau. \quad (14)$$

The mapping $H_R(t_1,t_2) \longrightarrow I_R(t,f)$ defined above is linear and unitary and is inverted by (13). Since $I_R(t,f)$ is square integrable (for finite R) and real valued, the function $H_R(t_1,t_2)$ is square integrable and Hermitian, i.e., $H_R^*(t_2,t_1) = H_R(t_1,t_2)$. Thus, $H_R(t_1,t_2)$ is the kernel of a self-adjoint, compact linear operator [10]. This guarantees the existence of the eigenexpansion

$$H_R(t_1,t_2) = \sum_{k=1}^{\infty} \lambda_k u_k(t_1)u_k^*(t_2)$$

with real-valued eigenvalues λ_k and orthonormal eigenfunctions $u_k(t)$ defined by (12). Inserting this eigenexpansion in (14) gives the expansion (11) of Theorem 1. □

The real-valued coefficients λ_k and the orthonormal signals $u_k(t)$ are completely determined by the TF region R; they will hence be called *eigenvalues and eigensignals of the TF region* R, respectively. Using these eigenvalues and eigensignals, we next define the eigenspaces of a TF region [11].

Definition: The N-dimensional eigenspace $\mathcal{U}_R^{(N)}$ of a TF region R is defined as the space spanned by the N dominant eigensignals of R, i.e., the N eigensignals $u_k(t)$ with largest eigenvalues λ_k.

In what follows, we assume that the eigenvalues are arranged in nonincreasing order $\lambda_k \geq \lambda_{k+1}$. Then, the N-dimensional eigenspace is spanned by the *first* N eigensignals $\mathcal{U}_R^{(N)} = \text{span}\{u_k(t)\}_{k=1}^N$.

B. Time-Frequency Subspaces with Optimum Concentration and Localization

The eigenspaces of a TF region provide the solution to both the MLE and MC optimization problems as is stated by the following theorem whose proof is outlined in the Appendix.

Theorem 2: The N_0-dimensional linear signal space minimizing the localization error $\epsilon(\mathcal{S},R)$ and maximizing the concentration $\rho(\mathcal{S},R)$ for a given TF region R is the eigenspace of R with dimension N_0,

$$\mathcal{S}_{R,\text{MLE}}(N_0) = \mathcal{S}_{R,\text{MC}}(N_0) = \mathcal{U}_R^{(N_0)}.$$

If the eigenvalues λ_k are arranged in nonincreasing order, the resulting localization error (the minimal localization error for the given dimension N_0) is related to the eigenvalues as

$$\epsilon^2(\mathcal{S},R)_{\min,N_0} = \epsilon^2(\mathcal{S},R)\Big|_{\mathcal{S}=\mathcal{U}_R^{(N_0)}} = \sum_{k=1}^{N_0}(1-\lambda_k)^2 + \sum_{k=N_0+1}^{\infty}\lambda_k^2 \quad (15)$$

and the resulting concentration (the maximal concentration for the given dimension N_0) is the arithmetic mean of the N_0 first (largest) eigenvalues of R

$$\rho(\mathcal{S},R)_{\max,N_0} = \rho(\mathcal{S},R)\Big|_{\mathcal{S}=\mathcal{U}_R^{(N_0)}} = \frac{1}{N_0}\sum_{k=1}^{N_0}\lambda_k. \quad (16)$$

Note that we have assumed a fixed dimension N_0 also for the MLE optimization problem. However, since (15) gives the residual localization error for any dimension N_0, the optimum value of N_0 can be determined by minimizing (15) with respect to N_0. It is seen by inspection that the optimum N_0 equals the number of all eigenvalues larger than $1/2$. This yields the final result for the MLE subspace.

Corollary: The linear signal space minimizing the localization error $\epsilon(\mathcal{S},R)$ for a given TF region R is the eigenspace of R with dimension N_R

$$\mathcal{S}_{R,\text{MLE}} = \mathcal{U}_R^{(N_R)}$$

where N_R denotes the number of eigenvalues larger than $1/2$. The residual (absolutely minimum) localization error is

$$\epsilon^2(\mathcal{S},R)_{\min} = \epsilon^2(\mathcal{S},R)\Big|_{\mathcal{S}=\mathcal{U}_R^{(N_R)}} = \sum_{k=1}^{N_R}(1-\lambda_k)^2 + \sum_{k=N_R+1}^{\infty}\lambda_k^2. \quad (17)$$

In contrast with the MLE criterion, the MC criterion cannot be used for determining an optimum dimension N_0 in a meaningful manner. However, based on the expression (16) of the concentration in terms of the eigenvalues λ_k, the dimension N_0 can easily be adjusted such that a prescribed concentration is approximately achieved or exceeded: We simply have to select the N_0 for which the arithmetic mean of the first N_0 eigenvalues is, respectively, closest to or larger than the concentration specified.

From (16) and the monotonicity $\lambda_k \geq \lambda_{k+1}$ of the eigenvalues, it is clear that the concentration achieved will decrease with increasing dimension N_0. The absolutely maximum concentration is obtained for $N_0 = 1$. Of course, a 1-D signal space is generally incapable of filling the region R energetically and thus does not solve our problem. On the other hand, selecting N_0 too large results in a space that "spills over" the TF region R, causing a small concentration value. We shall usually adopt the optimum dimension $N_0 = N_R$; the corresponding eigenspace $\mathcal{U}_R^{(N_R)}$ will then simply be called "the eigenspace of R" and will be denoted as $\mathcal{U}_R$. Similarly, the residual (absolutely minimum) localization error (17) will be called "the localization error associated with R" and will be denoted as ϵ_R:

$$\mathcal{U}_R \triangleq \mathcal{U}_R^{(N_R)}, \qquad \epsilon_R \triangleq \epsilon(\mathcal{U}_R,R).$$

We emphasize that $\mathcal{U}_R$, ϵ_R, and N_R are completely determined by the TF pass region R.

With the definition of the "idealized eigenvalues"

$$\tilde{\lambda}_k \triangleq \begin{cases} 1, & 1 \leq k \leq N_R \\ 0, & N_R+1 \leq k < \infty \end{cases}$$

the squared localization error ϵ_R can be compactly written as the deviation between the actual eigenvalues and the idealized

eigenvalues $\epsilon_R^2 = \sum_{k=1}^{\infty}(\tilde{\lambda}_k - \lambda_k)^2$. Note that the localization error would be zero if the eigenvalues λ_k assumed only the values 0 or 1; however, this idealized eigenvalue distribution is never obtained for a finite TF region R.

The MLE problem $\mathcal{S}_R = \arg\min_{\mathcal{S}} \|I_R - W_{\mathcal{S}}\|$ is the optimum approximation of a "nonvalid" TF function $I_R(t,f)$ ("nonvalid" in that $I_R(t,f)$ is not a valid WD of a space) by a valid WD of a space. This is analogous to the *signal synthesis problem* $s_{\text{opt}}(t) = \arg\min_s \|\widetilde{W} - W_s\|$, where a "nonvalid" TF function $\widetilde{W}(t,f)$ ("nonvalid" in that $\widetilde{W}(t,f)$ is not a valid WD of a signal) is approximated by a valid WD of a signal [16], [19]. In fact, the signal synthesis problem is essentially the space synthesis problem with $N_0 = 1$. In a similar manner, the MC problem $\mathcal{S}_R = \arg\max_{\mathcal{S}} \langle W_{\mathcal{S}}, I_R\rangle$ can be viewed as an extension of the problem of maximizing the TF concentration of a signal. The latter problem, which is essentially equivalent to the signal synthesis problem, has been studied in [35].

IV. Properties of Eigenspaces

It has been shown above that the optimum space resulting from the MLE or MC criterion is an eigenspace $\mathcal{U}_R^{(N)}$ of the TF region R. This space is spanned by a set of dominant eigensignals $u_k(t)$, and the dimension chosen or obtained depends on the distribution of the eigenvalues λ_k. In this section, we study some properties of the eigenvalues, eigensignals, and eigenspaces of a TF region.

A. Concentration Bounds

Analogous to the concentration of a signal space, we may define the *concentration* $\rho(x,R)$ *of a signal* $x(t)$ *in a TF region* R as the ratio of the signal's "energy content in R" and the signal's total energy $E_x = \|x\|^2 = \int_t \int_f W_x(t,f)dtdf$, where $W_x(t,f)$ denotes the signal's WD [35], [11]:

$$\rho(x,R) \triangleq \frac{E_x^{(R)}}{E_x} = \frac{\int\int_{(t,f)\in R} W_x(t,f)dtdf}{\int_t\int_f W_x(t,f)dtdf} = \frac{1}{E_x}\langle W_x, I_R\rangle.$$

It is easily shown that the concentration of a signal space $\mathcal{S}$ is the arithmetic mean of the concentrations of the space's orthonormal basis signals $s_k(t)$, $\rho(\mathcal{S},R) = \frac{1}{N_{\mathcal{S}}}\sum_{k=1}^{N_{\mathcal{S}}}\rho(s_k,R)$. We now formulate an interesting interpretation of the eigenvalues of a TF region [11].

Theorem 3: The concentrations of the eigensignals $u_k(t)$ equal the eigenvalues λ_k

$$\rho(u_k,R) = \langle W_{u_k}, I_R\rangle = \int\int_{(t,f)\in R} W_{u_k}(t,f)dtdf = \lambda_k.$$

We note, for later use, that this relation can be generalized as

$$\langle W_{u_k,u_l}, I_R\rangle = \int\int_{(t,f)\in R} W_{u_k,u_l}(t,f)dtdf = \lambda_k\delta_{kl} \quad (18)$$

where $W_{u_k,u_l}(t,f) = \int_\tau u_k\left(t+\frac{\tau}{2}\right)u_l^*\left(t-\frac{\tau}{2}\right)e^{-j2\pi f\tau}d\tau$ is the cross WD of the eigensignals $u_k(t)$ and $u_l(t)$, and δ_{kl} denotes the Kronecker delta symbol.

Proof: We shall show the general relation (18). Expanding $I_R(t,f)$ as in (11), applying Moyal's formula $\langle W_{x,y}, W_{a,b}\rangle = \langle x,a\rangle\langle y,b\rangle^*$ [2], and using the orthonormality of the eigensignals, we obtain

$$\begin{aligned}\langle W_{u_k,u_l}, I_R\rangle &= \sum_{m=1}^{\infty}\lambda_m\langle W_{u_k,u_l}, W_{u_m}\rangle \\ &= \sum_{m=1}^{\infty}\lambda_m\langle u_k,u_m\rangle\langle u_l,u_m\rangle^* \\ &= \sum_{m=1}^{\infty}\lambda_m\delta_{km}\delta_{lm} = \lambda_k\delta_{kl}.\end{aligned}$$

□

It has been shown in Section III-B that the concentration of the N-dimensional eigenspace $\mathcal{U}_R^{(N)}$ equals the arithmetic mean of the N first (i.e., largest) eigenvalues. Due to $\lambda_{k+1} \le \lambda_k$, this arithmetic mean is bounded from below by λ_N and from above by λ_1. Thus, we have the bounds

$$\lambda_N \le \rho\left(\mathcal{U}_R^{(N)}, R\right) \le \lambda_1.$$

The concentration of a signal space is the *average* concentration of all signals in the space. The next theorem [11] establishes concentration bounds for any specific signal of the eigenspace $\mathcal{U}_R^{(N)}$. These bounds are tight since they are achieved by $u_N(t)$ and $u_1(t)$, both of which are elements of $\mathcal{U}_R^{(N)}$.

Theorem 4: The concentration of any element $x(t)$ of the N-dimensional eigenspace $\mathcal{U}_R^{(N)}$ is bounded as $\lambda_N \le \rho(x,R) \le \lambda_1$.

Proof: Any $x(t) \in \mathcal{U}_R^{(N)}$ can be expanded as $x(t) = \sum_{k=1}^{N} x_k u_k(t)$, from which it follows that the WD of $x(t)$ can be written as $W_x(t,f) = \sum_{k=1}^{N}\sum_{l=1}^{N} x_k x_l^* W_{u_k,u_l}(t,f)$. The regional energy then becomes

$$\begin{aligned}E_x^{(R)} = \langle W_x, I_R\rangle &= \sum_{k=1}^{N}\sum_{l=1}^{N} x_k x_l^*\langle W_{u_k,u_l}, I_R\rangle \\ &= \sum_{k=1}^{N}\sum_{l=1}^{N} x_k x_l^*\lambda_k\delta_{kl} = \sum_{k=1}^{N}\lambda_k|x_k|^2\end{aligned}$$

where (18) has been used. With the energy of $x(t)$ being $E_x = \sum_{k=1}^{N}|x_k|^2$, the concentration of $x(t)$ is

$$\begin{aligned}\rho(x,R) = \frac{E_x^{(R)}}{E_x} &= \frac{\sum_{k=1}^{N}\lambda_k|x_k|^2}{\sum_{k=1}^{N}|x_k|^2} \\ &\ge \lambda_N\frac{\sum_{k=1}^{N}|x_k|^2}{\sum_{k=1}^{N}|x_k|^2} = \lambda_N.\end{aligned}$$

The upper bound follows by an analogous argument. □

The eigenspace $\mathcal{U}_R$, where $N = N_R$, is of particular interest. Here, $\lambda_N = \lambda_{N_R} > 1/2$; therefore

$$\rho(x,R) > 1/2 \quad \text{for all } x(t) \in \mathcal{U}_R \quad \text{and} \quad \rho(\mathcal{U}_R,R) > 1/2.$$

B. Asymptotic Properties of Eigenvalues and Eigensignals

The eigenvalue distribution obtained for a TF region R determines the optimum dimension N_R and the dimension necessary for achieving a desired space concentration. The eigenvalues also determine the localization error ϵ_R. An interesting property of the eigenvalues is stated in the next theorem.

Theorem 5: The sum of all eigenvalues and the sum of all squared eigenvalues are both equal to the area A_R

$$\sum_{k=1}^{\infty} \lambda_k = \sum_{k=1}^{\infty} \lambda_k^2 = A_R. \tag{19}$$

Proof: Using (8), (11), and Moyal's formula $\langle W_x, W_y \rangle = |\langle x, y \rangle|^2$ [2], we have

$$\begin{aligned} A_R &= \int_t \int_f I_R(t,f)dtdf = \sum_{k=1}^{\infty} \lambda_k \int_t \int_f W_{u_k}(t,f)dtdf \\ &= \sum_{k=1}^{\infty} \lambda_k \|u_k\|^2 = \sum_{k=1}^{\infty} \lambda_k; \\ A_R &= \int_t \int_f [I_R(t,f)]^2 dtdf = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \lambda_k \lambda_l \langle W_{u_k}, W_{u_l} \rangle \\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \lambda_k \lambda_l |\langle u_k, u_l \rangle|^2 = \sum_{k=1}^{\infty} \lambda_k^2. \end{aligned}$$ □

In [36], the asymptotic behavior of the eigenvalues and eigensignals is studied for a finite TF region R. It is shown that $|\lambda_k|$ decays as $\mathcal{O}(k^{-3/4})$ but not as $\mathcal{O}(k^{-(1+\epsilon)})$ for any $\epsilon > 0$ and that $\sum_k |\lambda_k| = \infty$. (Note, however, that $\sum_k \lambda_k = \sum_k \lambda_k^2 = A_R < \infty$.) Furthermore, the eigensignals are shown to have faster than exponential decay in both the time domain and the frequency domain; for any $a > 0$, there exist constants k_a and K_a such that $|u_k(t)| \leq k_a e^{-a|t|}$ for all t and $|U_k(f)| \leq K_a e^{-a|f|}$ for all f, where $U_k(f)$ is the Fourier transform of $u_k(t)$. Moreover, both $u_k(t)$ and $U_k(f)$ are analytic functions, i.e., all derivatives exist and are in $\mathcal{L}_2(\mathbb{R})$. These results show that the error caused by a temporal truncation or bandlimitation of the eigensignals $u_k(t)$ will be negligibly small if the length of the time-gating window, or the bandwidth of the filter, is chosen sufficiently large.

C. Invariance Properties of Eigenspaces

There exists a class of unitary, linear signal transforms $\mathbf{C}$ corresponding to an area-preserving affine TF coordinate transform $(t,f) \longrightarrow (\alpha t + \beta f - \tau, \gamma t + \delta f - \nu)$, where the parameters $\alpha, \beta, \gamma, \delta, \tau$, and ν depend on the signal transform $\mathbf{C}$, and the area preservation implies a transform determinant $D = \alpha\delta - \gamma\beta = 1$ [37]. The WD of a transformed signal $(\mathbf{C}x)(t)$ is related to the WD of the original signal $x(t)$ as

$$W_{\mathbf{C}x}(t,f) = W_x(\alpha t + \beta f - \tau, \gamma t + \delta f - \nu). \tag{20}$$

Important special cases include the following:

$$\begin{aligned} &(\mathbf{C}x)(t) = x(t-\tau)e^{j2\pi\nu t} \;\Rightarrow\; (t,f) \longrightarrow (t-\tau, f-\nu) \\ &(\mathbf{C}x)(t) = \sqrt{|a|}x(at) \;\Rightarrow\; (t,f) \longrightarrow (at, f/a) \\ &(\mathbf{C}x)(t) = e^{j\pi ct^2}x(t) \;\Rightarrow\; (t,f) \longrightarrow (t, f-ct) \\ &(\mathbf{C}x)(t) = \sqrt{|b|}e^{j\pi bt^2} * x(t) \;\Rightarrow\; (t,f) \longrightarrow (t-f/b, f) \\ &(\mathbf{C}x)(t) = \sqrt{|a|}(\mathcal{F}x)(at) \;\Rightarrow\; (t,f) \longrightarrow (-f/a, at) \end{aligned}$$

where $*$ and $\mathcal{F}$ denote convolution and Fourier transform, respectively. Since a general affine coordinate transform can be represented by composing some of the specific transforms listed above, the general operator $\mathbf{C}$ in (20) can be constructed for any set of parameters $\alpha, \beta, \gamma, \delta, \tau, \nu$ satisfying $D = 1$ [37].

If we subject the TF plane to an area-preserving coordinate transform $(t,f) \rightarrow (\alpha t + \beta f - \tau, \gamma t + \delta f - \nu)$, the TF region R maps into a new region $\tilde{R}$. Due to area preservation, $A_{\tilde{R}} = A_R$. The next theorem (cf. [31]) shows how the eigenspaces of the original region R and the new region $\tilde{R}$ are related.

Theorem 6: Let $\tilde{R}$ be the TF region obtained by subjecting a given TF region R to an affine TF coordinate transform $(t,f) \longrightarrow (\alpha t + \beta f - \tau, \gamma t + \delta f - \nu)$ with $\alpha\delta - \gamma\beta = 1$. Then, the eigenvalues of $\tilde{R}$ equal those of R, whereas the eigensignals of $\tilde{R}$ are derived from those of R by the unitary, linear signal transform $\mathbf{C}$ corresponding to the given TF coordinate transform

$$\tilde{\lambda}_k = \lambda_k, \qquad \tilde{u}_k(t) = (\mathbf{C}u_k)(t).$$

The WD of the eigenspace $\mathcal{U}_{\tilde{R}}^{(N)}$ of $\tilde{R}$ is related to the WD of the original eigenspace $\mathcal{U}_R^{(N)}$ as

$$W_{\mathcal{U}_{\tilde{R}}^{(N)}}(t,f) = W_{\mathcal{U}_R^{(N)}}(\alpha t + \beta f - \tau, \gamma t + \delta f - \nu).$$

Proof: Using (11) and (20), the indicator function of the transformed TF region $\tilde{R}$ becomes

$$\begin{aligned} I_{\tilde{R}}(t,f) &= I_R(\alpha t + \beta f - \tau, \gamma t + \delta f - \nu) \\ &= \sum_{k=1}^{\infty} \lambda_k W_{u_k}(\alpha t + \beta f - \tau, \gamma t + \delta f - \nu) \\ &= \sum_{k=1}^{\infty} \lambda_k W_{\mathbf{C}u_k}(t,f). \end{aligned}$$

Since $\mathbf{C}$ is unitary, the transformed eigensignals $(\mathbf{C}u_k)(t)$ are again orthonormal; therefore, λ_k and $(\mathbf{C}u_k)(t)$ are recognized as the eigenvalues and eigensignals, respectively, of $\tilde{R}$, i.e., $\tilde{\lambda}_k = \lambda_k$ and $\tilde{u}_k(t) = (\mathbf{C}u_k)(t)$. With (4), the relation between $W_{\mathcal{U}_{\tilde{R}}^{(N)}}(t,f)$ and $W_{\mathcal{U}_R^{(N)}}(t,f)$ is now shown as follows:

$$\begin{aligned} W_{\mathcal{U}_{\tilde{R}}^{(N)}}(t,f) &= \sum_{k=1}^{N} W_{\mathbf{C}u_k}(t,f) \\ &= \sum_{k=1}^{N} W_{u_k}(\alpha t + \beta f - \tau, \gamma t + \delta f - \nu) \\ &= W_{\mathcal{U}_R^{(N)}}(\alpha t + \beta f - \tau, \gamma t + \delta f - \nu). \end{aligned}$$ □

V. Time-Frequency Projections

Two major applications of the optimum TF subspaces (eigenspaces) considered in previous sections are TF projection filters and TF signal expansions as discussed in Section I. Both of these applications are based on the orthogonal projection on the TF region's eigenspace.

A. Time-Frequency Projection Filters

With (2), the orthogonal projection of a signal $x(t)$ on the eigenspace $\mathcal{U}_R$ is

$$(\mathbf{U}_R x)(t) = \int_{t'} U_R(t,t')x(t')dt' = \sum_{k=1}^{N_R} \langle x, u_k\rangle u_k(t) \quad (21)$$

where $U_R(t,t')$ denotes the kernel of the orthogonal projection operator $\mathbf{U}_R$ of the eigenspace $\mathcal{U}_R$. The first expression is the input-output relation of a *linear, time-varying system or filter* with impulse response $U_R(t,t')$. We call this filter the *TF projection filter with pass region* R. The second expression, which is particularly suited to parallel processing, involves N_R inner products $\langle x, u_k\rangle = \int_t x(t)u_k^*(t)dt$ of the input signal $x(t)$ with the first N_R eigensignals $u_k(t)$. The computational expense associated with (21) is proportional to the subspace dimension N_R and increases (via the inner products $\langle x, u_k\rangle$) with the effective durations of the eigensignals $u_k(t)$. With $N_R \approx A_R$, it follows that the computational expense of a TF projection filter increases with the area of the TF pass region. The effective durations of the eigensignals depend primarily on the length of the TF region R in the time direction.

The TF projection filter $\mathbf{U}_R$, which was defined in this paper as the orthogonal projection on the space with optimum TF localization in the pass region R, can be derived using two alternative optimization criteria. First, $\mathbf{U}_R$ is the linear, time-varying system $\mathbf{H}$ whose Weyl symbol $L_H(t,f) = \int_\tau H\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right)e^{-j2\pi f\tau}d\tau$ (cf. (5), (14)) is closest to the TF indicator function $I_R(t,f)$ subject to the side constraint that the system $\mathbf{H}$ be an orthogonal projection operator [15], [27]. Second, since the Weyl symbol of an orthogonal projection system equals the system's WD [28], this optimization may also be stated in terms of the WD of a system, instead of the system's Weyl symbol.

B. Time-Frequency Signal Expansions

The *TF signal expansion* amounts to forming the N_R expansion coefficients $x_k = \langle x, u_k\rangle$ with $k = 1, \ldots, N_R$. If the signal $x(t)$ is well concentrated in R, it will be represented by the coefficients x_k, which can then be used for further signal processing. Note that the number of expansion coefficients N_R roughly equals the area A_R of the TF region R. The signal can be reconstructed according to

$$\hat{x}(t) = \sum_{k=1}^{N_R} \langle x, u_k\rangle u_k(t) = (\mathbf{U}_R x)(t)$$

which yields the signal's orthogonal projection on $\mathcal{U}_R$ and is thus equal to the output (21) of the TF projection filter. The expansion error ($\|x - \mathbf{U}_R x\|$) incurred will be small for signals well concentrated in R. Often, we are also interested in rejecting any signal that is outside R (e.g., to suppress noise or parasitic signal components). The computational expense of a TF expansion is again determined by the dimension N_R (corresponding to the area A_R) and the effective eigensignal durations.

C. Pass/Reject Analysis

The "output" of both the TF projection filter and the TF signal expansion is the signal's orthogonal projection on the eigenspace $\mathcal{U}_R$. We want this orthogonal projection to pass signals located inside the TF pass region R but reject signals located outside R. Thus, the *normalized output energy*[6]

$$e(x,R) \triangleq \frac{\|\mathbf{U}_R x\|^2}{\|x\|^2} = \frac{E_{\mathbf{U}_R x}}{E_x}$$

should be nearly one for an input signal well inside the pass region R (i.e., $\rho(x,R) \approx 1$) and nearly zero for an input signal well outside the pass region R (i.e., $\rho(x,R) \approx 0$). Hence, the desired filter/expansion performance can be summarized as

$$e(x,R) \approx 1 \qquad \text{for } x(t) \text{ such that } \rho(x,R) \approx 1,$$
$$e(x,R) \approx 0 \qquad \text{for } x(t) \text{ such that } \rho(x,R) \approx 0.$$

Since $0 \le \|\mathbf{U}_R x\|^2 \le \|x\|^2$, the normalized output energy is bounded as $0 \le e(x,R) \le 1$, with $e(x,R) = 0$ if and only if $x(t)$ is orthogonal on $\mathcal{U}_R$, and $e(x,R) = 1$ if and only if $x(t) \in \mathcal{U}_R$. Furthermore, bounds on $e(x,R)$ involving $\rho(x,R)$ and ϵ_R are given by

$$\rho(x,R) - \epsilon_R \le e(x,R) \le \rho(x,R) + \epsilon_R.$$

Proof: Using $e(x,R) = \frac{1}{E_x}\langle W_x, W_{\mathcal{U}_R}\rangle$ [14] and Schwarz' inequality, we have

$$\begin{aligned}|e(x,R) - \rho(x,R)| &= \left|\frac{1}{E_x}\langle W_x, W_{\mathcal{U}_R}\rangle - \frac{1}{E_x}\langle W_x, I_R\rangle\right| \\ &= \frac{1}{E_x}\left|\langle W_x, W_{\mathcal{U}_R} - I_R\rangle\right| \\ &\le \frac{1}{E_x}\|W_x\|\|W_{\mathcal{U}_R} - I_R\| = \frac{1}{E_x}E_x\epsilon_R = \epsilon_R.\end{aligned}$$

□

These bounds can be simplified in two cases. If $x(t)$ is well inside R, then $\rho(x,R) \approx 1$, and we obtain $1-\epsilon_R \le e(x,R) \le 1$. If $x(t)$ is well outside R, then $\rho(x,R) \approx 0$, and we have $0 \le e(x,R) \le \epsilon_R$. In either case, the maximum deviation from the desired behavior $e(x,R) = 1$ or $e(x,R) = 0$ is given by the localization error ϵ_R. However, since ϵ_R is often on the order of one, these bounds are rather loose.

D. Noise Analysis

Often, the TF projection filter/TF expansion will be used to suppress additive noise, or noise suppression is at least a desired side effect. If $w(t)$ is zero-mean, stationary white noise with power spectral density $S_w(f) = \eta$, then the mean energy

[6] The normalized deviation between the input signal and the orthogonal projection, $\bar{e}(x,R) \triangleq \|x - \mathbf{U}_R x\|^2/\|x\|^2$ can also be considered. However, using the idempotency of $\mathbf{U}_R$, it is easily shown that $\bar{e}(x,R) = 1 - e(x,R)$.

of the orthogonal projection $(\mathbf{U}_R w)(t)$ is easily shown to be ($\mathcal{E}$ denotes expectation)

$$\mathcal{E}\{E_{\mathbf{U}_R w}\} = \eta N_R \tag{22}$$

and thus proportional to the dimension N_R. Since $N_R \approx A_R$, we get the rule of thumb that *the mean noise energy passed is roughly proportional to the area of the TF pass region.*

If the input signal $x(t) = s(t) + w(t)$ consists of a deterministic signal component $s(t)$ with energy E_s and white noise $w(t)$ with power spectral density $S_w(f) = \eta$, the orthogonal projection of $x(t)$ on $\mathcal{U}_R$ becomes $(\mathbf{U}_R x)(t) = (\mathbf{U}_R s)(t) + (\mathbf{U}_R w)(t)$. We then define the "output signal-to-noise ratio" as the ratio of the energy $E_{\mathbf{U}_R s}$ of the projected signal component and the mean energy $\mathcal{E}\{E_{\mathbf{U}_R w}\}$ of the projected noise. With $E_{\mathbf{U}_R s} = E_s e(s, R)$ and $\mathcal{E}\{E_{\mathbf{U}_R w}\} = \eta N_R$, this output SNR is

$$\text{SNR} \triangleq \frac{E_{\mathbf{U}_R s}}{\mathcal{E}\{E_{\mathbf{U}_R w}\}} = \frac{E_s e(s, R)}{\eta N_R}.$$

If $s(t)$ is well concentrated in R, there will be $(\mathbf{U}_R s)(t) \approx s(t)$ and thus $e(s, R) \approx 1$ so that

$$\text{SNR} \approx \frac{E_s}{\eta N_R}.$$

VI. Implementation and Simulation Results

A. Discrete-Time Implementation

A discrete-time algorithm for calculating the eigenspace $\mathcal{U}_R$ of a TF region R is summarized below. The difference from the continuous-time method (cf. (12) and (13)) is due to the introduction of a *halfband restriction*; a discrete-time version of the localization error $\epsilon(\mathcal{S}, R)$ is minimized under the constraint that $\mathcal{S}$ is a subspace of a "halfband space" such that the space's discrete-time WD [14] does not contain aliasing. The theoretical background is discussed in [18].

Let $I_R(n, \theta)$ be the indicator function of the TF region R. Here, n and θ are discrete time and normalized frequency, respectively. θ is constrained to a "halfband" $|\theta - \theta_0| < 1/4$ with given center frequency θ_0. The eigenspace $\mathcal{U}_R$ is now obtained as follows:

1) The indicator function $I_R(n, \theta)$ is low-pass filtered in the time direction [18]

$$\tilde{I}_R(n, \theta) = \sum_{n'} g_\theta(n - n') I_R(n', \theta) \quad \text{for} \quad |\theta - \theta_0| < 1/4$$

where $g_\theta(n) = (1 - 4|\theta - \theta_0|)\text{sinc}[(1 - 4|\theta - \theta_0|)n]$ with $\text{sinc}(\alpha) = \frac{\sin \pi\alpha}{\pi\alpha}$ is the impulse response of an idealized low-pass filter with θ-dependent cut-off frequency.

2) The low-pass filtered indicator function $\tilde{I}_R(n, \theta)$ is Fourier transformed according to

$$\tilde{H}_R(\nu_1, \nu_2) = 2 \times \int_{\theta_0 - 1/4}^{\theta_0 + 1/4} \tilde{I}_R(\nu_1 + \nu_2, \theta) e^{j4\pi(\nu_1 - \nu_2)\theta} d\theta.$$

3) The eigenvalues λ_k and the (normalized) eigenvectors $\mathbf{v}_k$ of the Hermitian matrix $\tilde{\mathbf{H}} = \left(\tilde{H}_R(\nu_1, \nu_2)\right)$ are calculated. Let $\{\mathbf{v}_k\}_{k=1}^{N_R}$ denote all eigenvectors with eigenvalues $\lambda_k > 1/2$.

4) The eigenspace of R is $\mathcal{U}_R = \text{span}\{u_k(n)\}_{k=1}^{N_R}$, where the basis signals $u_k(n)$ are obtained from the eigenvectors $\mathbf{v}_k$ as

$$u_k(2\nu) = \frac{1}{\sqrt{2}}(\mathbf{v}_k)_\nu, \; u_k(2\nu + 1) = 2 \sum_{\nu'} h\Big(2(\nu - \nu') + 1\Big) u_k(2\nu'), k = 1, \ldots, N_R.$$

Here, $h(n) = (1/2)\text{sinc}(n/2)e^{j2\pi\theta_0 n}$ is the impulse response of an idealized halfband filter with center frequency θ_0.

In practice, a discretization is also performed with respect to the frequency variable θ. Steps 1, 2, and 4 can then be performed efficiently by means of FFT techniques. We note that a suboptimal, reduced-cost algorithm is obtained simply by omitting Step 1 [18].

B. Simulation Results

A TF filtering experiment illustrating the performance of TF projection filters is shown in Fig. 2. The three-component input signal consists of a Gaussian signal and two windowed quadratic FM signals. The signal's time duration is 128 samples. Since the signal components overlap with respect to both time and frequency, they may not be separated using a simple time gating or a time-invariant filter.

In order to isolate the middle signal component, a TF pass region R was chosen as shown in Fig. 2(b), and the TF projection filter or, equivalently, the region's eigenspace was derived as detailed in Section III. From Fig. 2(c), it follows that six eigenvalues are larger than $1/2$; hence, the optimum space dimension is $N_R = 6$, which is close to the TF region's area $A_R = 5.3$. Fig. 2(d) shows the WD of the eigenspace $\mathcal{U}_R$. The residual localization error is $\epsilon_R = 0.87$. Fig. 2(e) depicts the output signal of the TF projection filter (i.e., the orthogonal projection of the three-component input signal on the eigenspace $\mathcal{U}_R$). A comparison with the true signal component shown in Fig. 2(f) demonstrates that the filtering was successful in the sense that the desired signal component is obtained with little distortion, whereas the other two components are well rejected. A quantitative performance characterization is given by the normalized output energies (see Section V-C), which are -0.02 dB for the desired signal component and -40.7 and -66.3 dB for the two other signal components.

Fig. 3 illustrates the application of TF projection filters to noise suppression. The filter's input signal is the FM signal component of Fig. 2(f) contaminated by halfband-filtered[7] white noise with a SNR of -3 dB. (The SNR is defined as the

[7] In order to avoid aliasing effects in the discrete-time WD (see Part II of [2]), all signals used are restricted to one half of the total spectral period of discrete-time signals.

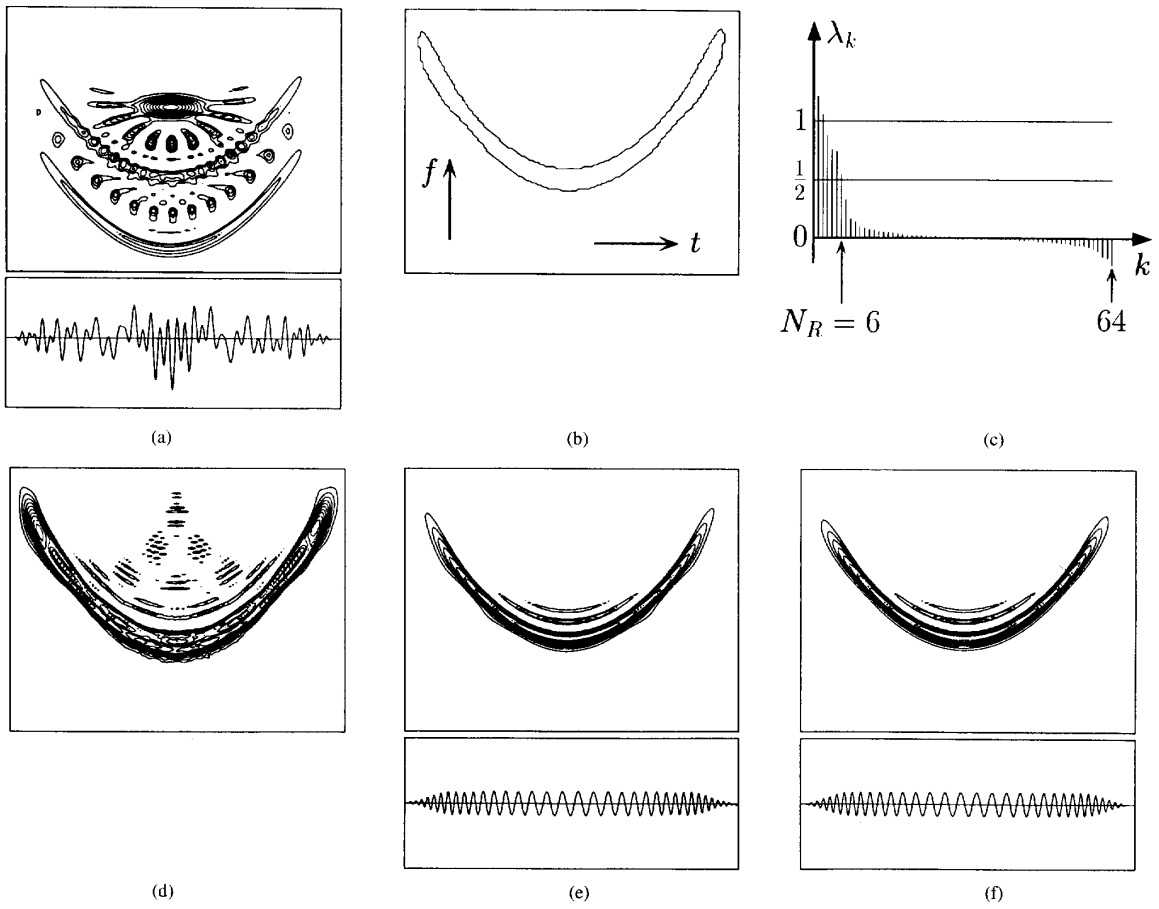

Fig. 2. Application of TF projection filter to signal separation: (a) Real part and (slightly smoothed) WD of three-component input signal; (b) TF pass region R; (c) eigenvalues of TF region R; (d) WD of eigenspace $\mathcal{U}_R$; (e) output of TF projection filter (projection of input signal on $\mathcal{U}_R$); (f) desired signal component.

ratio of the FM signal's energy and the noise energy.) The TF projection filter is the same as in the previous experiment. The output signal, shown in Fig. 3(b), is seen to be a reasonable estimate of the FM signal, apart from a parasitic amplitude modulation that is due to noise components located inside the pass region. The SNR of the output signal[8] is 6.9 dB. The overall SNR improvement achieved by the filter is 9.9 dB. Of course, this SNR improvement depends on the specific noise realization, which is random. In particular, the SNR improvement must be comparatively poor if a large part of the noise energy happens to fall inside the filter's pass region. Therefore, a more meaningful performance measure is the *mean* noise attenuation achieved by the filter (averaged over the entire noise ensemble), which is given by the ratio of the mean noise energies at the output and the input of the filter. It follows from (22) that the mean noise energy is proportional to the space dimension; hence, the noise attenuation factor is 6/64 since the eigenspace dimension is $N_R = 6$, and the dimension of the "total signal space" (which is the space of all discrete-time half-band signals of length 128) is 64. This corresponds to a mean noise attenuation of 10.28 dB.

Our last experiment, which is depicted in Fig. 4, is a noise suppression problem involving a natural signal, namely, two pitch periods of a voiced speech sound. Again, the signal length is 128 samples, and the SNR of the input signal is -3 dB. Note that the pass region is a multiple region. The area, dimension, and localization error are $A_R = 8.1$, $N_R = 9$, and $\epsilon_R = 1.4$, respectively. The SNR improvement achieved (defined as described above) is 7.4 dB. The mean noise attenuation is 9/64 or 8.52 dB.

The experiments presented are relevant not only to the performance of TF projection filters but also to that of the corresponding TF signal expansion since a TF signal expansion is based on the orthogonal projection on the TF region's eigenspace. Specifically, the number of expansion coefficients required equals the dimension N_R, and the total mean noise energy contained in the expansion coefficients equals the mean noise energy passed by the orthogonal projection and is thus proportional to N_R. It follows from $N_R \approx A_R$ that, for a parsimonious signal expansion and good noise suppression, A_R should be as small as possible. This implies a rather precise prior knowledge of the signal's TF support.

[8]This output SNR is based on the deviation from the *true* FM signal; it is different from the SNR considered in Section V-D in that it reflects both the residual noise passed by the filter and the distortion of the FM signal caused by the filter.

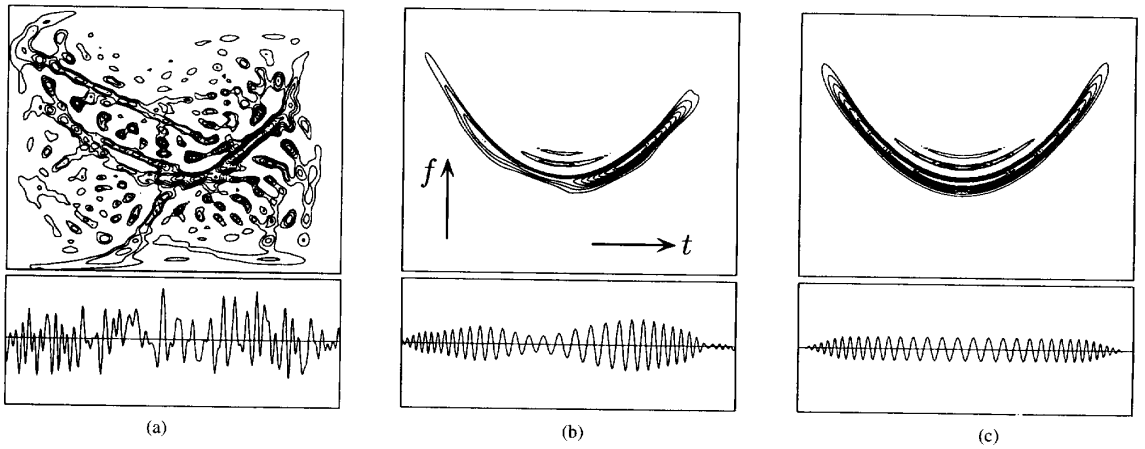

Fig. 3. Application of TF projection filter to noise suppression: (a) Real part and (slightly smoothed) WD of noisy input signal; (b) output of TF projection filter; (c) desired signal component.

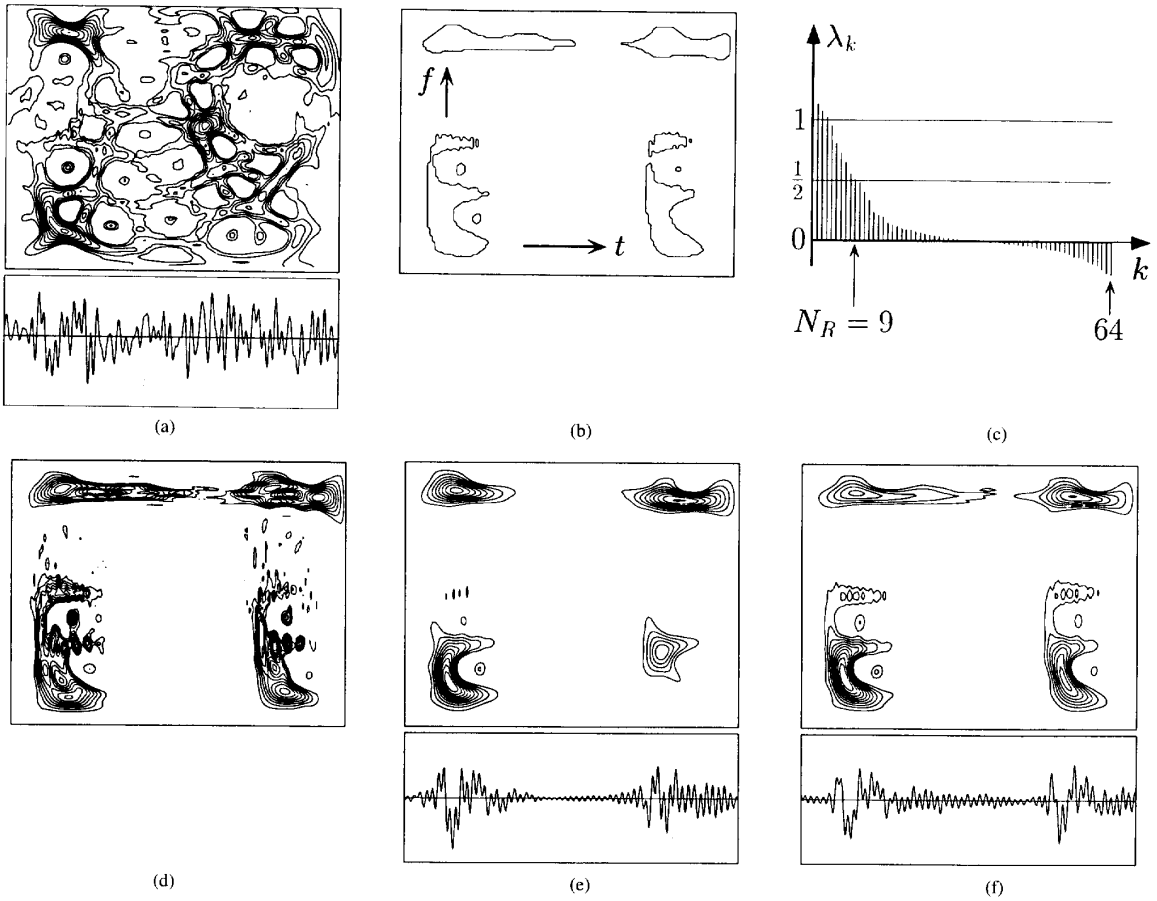

Fig. 4. Application of TF projection filter to noise suppression involving a speech signal: (a) Real part and (slightly smoothed) WD of noisy speech signal; (b) TF pass region R; (c) eigenvalues of TF region R; (d) WD of eigenspace $\mathcal{U}_R$; (e) output of TF projection filter (projection of noisy speech signal on $\mathcal{U}_R$); (f) desired speech signal.

VII. Conclusion

We have developed a coherent framework for the problems of time-frequency (TF) filtering and TF signal expansion. The solution of both problems has been based on the concept of a *TF subspace*, i.e., a linear signal space "comprising all signals located in a given TF region." We have presented an optimum design of TF subspaces, where the optimality criterion makes use of the WD of a linear signal space. This optimality criterion

is a joint TF criterion that, in contrast to classical approaches, does not restrict the shape of the TF region in any way. In particular, the TF region may be nonconvex or even a multiple region. The TF projection filters and TF signal expansions obtained were seen to perform quite well.

It should be noted that the choice of the WD as underlying TF representation is arbitrary to some degree. Indeed, the design of TF subspaces described in this paper can be based equally well on any other quadratic signal representation as long as this representation is *unitary*, i.e., satisfies Moyal's formula [38], [19]. An example is the family of *generalized WD's* [37]–[40], although there is little motivation for replacing the WD by some other member of this family [37], [15]. Of more practical interest are the *ambiguity function* [1], [41] or some unitary affine or hyperbolic time-frequency or time-scale representation such as the *Bertrand P_o- distribution* or the *Altes-Marinovic Q-distribution* [42]–[45].

The design of TF subspaces and TF projection filters can be extended in several other respects, which are briefly summarized in the following.

i) By the introduction of a *signal space constraint* [19], the TF subspace can be forced to be a subspace of some prescribed signal space, e.g., the space of all signals bandlimited in a given frequency band. Such a bandlimitation constraint is appropriate for a discrete-time implementation based on the discrete-time WD [18] (cf. Section VI-A).

ii) The *Weyl filter* [27], [15], [13] is a crude approximation of the TF projection filter. The Weyl filter does not have a projection structure, but its design does not require the solution of an eigenproblem.

iii) Both the TF projection filter and the Weyl filter can be extended to perfect-reconstruction *TF filter banks* [13].

We finally emphasize that the *orthogonal projection structure* assumed for the TF filter is appropriate only as long as the filtering task is to pass or reject signals in prescribed TF regions, i.e., to perform a "TF weighting" by weighting factors one (pass region) or zero (stop region). If some more general TF weighting characteristic is desired, then an orthogonal projection structure is no longer adequate, and a more general linear filter must be used. The design of such a TF filter may be based on linear TF signal representations like the short-time Fourier transform or the wavelet transform [22]–[26], [15], or on TF system representations like the Weyl symbol [27], [13], [15] or the WD of a linear system [28].

Appendix
Proof of Theorem 2

In this Appendix, we show that the signal space $\mathcal{S}$ maximizing the concentration $\rho(\mathcal{S}, R)$ for given space dimension $N_{\mathcal{S}} = N$ is the N-dimensional eigenspace $\mathcal{U}_R^{(N)}$. It suffices to determine an orthonormal basis $\{s_k(t)\}_{k=1}^N$ of $\mathcal{S}$. Since $N_{\mathcal{S}} = N$ is fixed, maximization of $\rho(\mathcal{S}, R)$ is equivalent to maximization of the spaces' regional energy, which equals the sum of the regional energies of all basis signals

$$E_{\mathcal{S}}^{(R)} = \sum_{k=1}^{N} E_{s_k}^{(R)} = \sum_{k=1}^{N} \langle W_{s_k}, I_R \rangle. \tag{23}$$

The eigensignals $u_k(t)$, which are suitably augmented if necessary, form an orthonormal basis of $\mathcal{L}_2(\mathbb{R})$. Hence, the basis signals can be expanded as

$$s_k(t) = \sum_{l=1}^{\infty} c_{kl} u_l(t). \tag{24}$$

Let us define the coefficient vectors $\mathbf{c}_k$ whose lth elements are c_{kl}. It is easily shown that orthonormality of the $s_k(t)$ implies orthonormality of the coefficient vectors $\mathbf{c}_k$, i.e., $\mathbf{c}_k^H \mathbf{c}_l = \delta_{kl}$ for $1 \le k, l \le N$, where ^{H} denotes complex conjugate transposition. Inserting (24) in (23), we obtain

$$\begin{aligned} E_{\mathcal{S}}^{(R)} &= \sum_{k=1}^{N} \left\langle \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} c_{kl} c_{km}^* W_{u_l, u_m}, I_R \right\rangle \\ &= \sum_{k=1}^{N} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} c_{kl} c_{km}^* \lambda_l \delta_{lm} = \sum_{k=1}^{N} \sum_{l=1}^{\infty} \lambda_l |c_{kl}|^2 \end{aligned} \tag{25}$$

where (18) has been used. This can also be written as the sum of quadratic forms

$$E_{\mathcal{S}}^{(R)} = \sum_{k=1}^{N} \mathbf{c}_k^H \mathbf{\Lambda} \mathbf{c}_k \tag{26}$$

where $\mathbf{\Lambda}$ is the diagonal eigenvalue matrix. We have to maximize (26) under the orthonormality constraint $\mathbf{c}_k^H \mathbf{c}_l = \delta_{kl}$. It suffices, however, to use a *normalization constraint* $\mathbf{c}_k^H \mathbf{c}_k = 1$ since, as will be seen presently, this automatically yields orthonormal vectors $\mathbf{c}_k$. Using Lagrange multipliers μ_k, the problem then amounts to the unconstrained maximization of

$$\begin{aligned} \sigma_c &= \sum_{k=1}^{N} \mathbf{c}_k^H \mathbf{\Lambda} \mathbf{c}_k + \sum_{k=1}^{N} \mu_k (1 - \mathbf{c}_k^H \mathbf{c}_k) \\ &= \sum_{k=1}^{N} \mathbf{c}_k^H (\mathbf{\Lambda} - \mu_k \mathbf{I}) \mathbf{c}_k + \sum_{k=1}^{N} \mu_k. \end{aligned}$$

Setting the gradient of σ_c with respect to $\mathbf{c}_i$ equal to zero yields the system of equations

$$\mathbf{\Lambda} \mathbf{c}_i = \mu_i \mathbf{c}_i, \quad 1 \le i \le N.$$

Hence, the coefficient vectors $\mathbf{c}_i$ must be normalized eigenvectors of $\mathbf{\Lambda}$. Assuming all eigenvalues λ_i to be distinct, the eigenvectors of the diagonal matrix $\mathbf{\Lambda}$ are the unit vectors[9] $\mathbf{e}_i$. Note that, as predicted, the orthonormality constraint is indeed satisfied. Let us adopt the first N eigenvectors so that $\mathbf{c}_i = \mathbf{e}_i$ for $1 \le i \le N$. This choice does not entail a loss of generality since the specific solution obtained depends on the ordering of the eigenvalues λ_i, which has not yet been fixed. Inserting $\mathbf{c}_i = \mathbf{e}_i$ in (24), we see that the basis signals are the first N

[9]The assumption of distinct eigenvalues is easily shown to be unnecessary. Indeed, if some eigenvalues are equal, the eigenvectors may be rotated versions of the unit vectors $\mathbf{e}_i$. Although this yields different basis signals, the space spanned by this rotated basis is identical to the space spanned by the unit vectors.

eigensignals of the TF region R

$$s_k(t) = \sum_{l=1}^{\infty} c_{kl} u_l(t) = \sum_{l=1}^{\infty} \delta_{kl} u_l(t) = u_k(t).$$

It now remains to be seen *which* eigensignals have to be taken. Inserting $\mathbf{c}_i = \mathbf{e}_i$ in (25), the regional energy achieved becomes the sum of the first N eigenvalues

$$E_S^{(R)} = \sum_{k=1}^{N} \sum_{l=1}^{\infty} \lambda_l |c_{kl}|^2 = \sum_{k=1}^{N} \sum_{l=1}^{\infty} \lambda_l \delta_{kl} = \sum_{k=1}^{N} \lambda_k.$$

This is maximized by arranging the eigenvalues such that the N first eigenvalues are the N largest eigenvalues. Hence, the optimum basis signals equal the N eigensignals corresponding to the N largest eigenvalues, or, equivalently, the optimum space is the N-dimensional eigenspace. The resulting (maximum) regional energy is then the sum of the N largest eigenvalues, which yields (16). Finally, (15) is easily derived from (16) with the use of (9) and (19).

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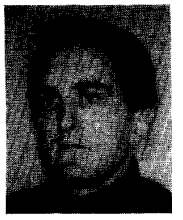

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